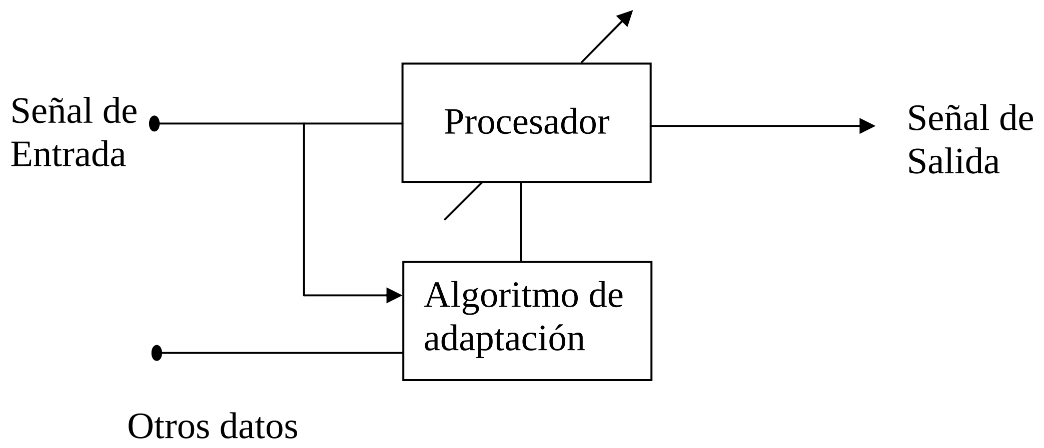
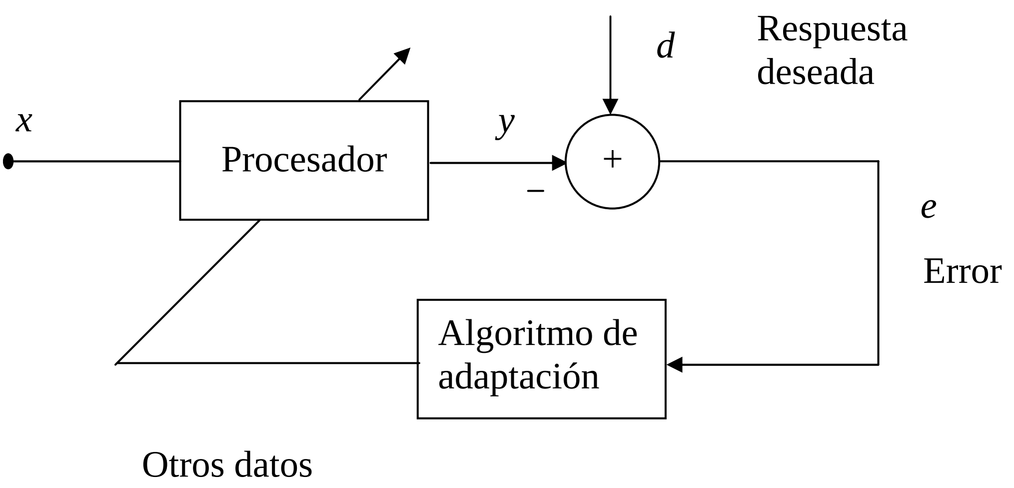
# Procesamiento Adaptativo de Señales

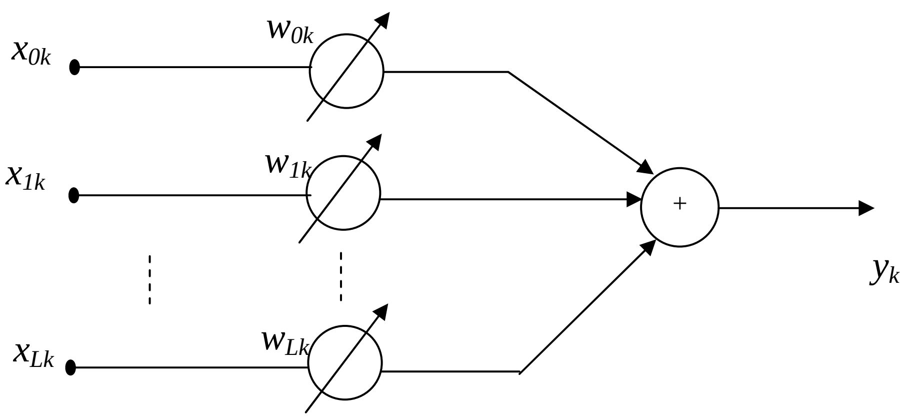
## Adaptación de un sistema a lazo abierto



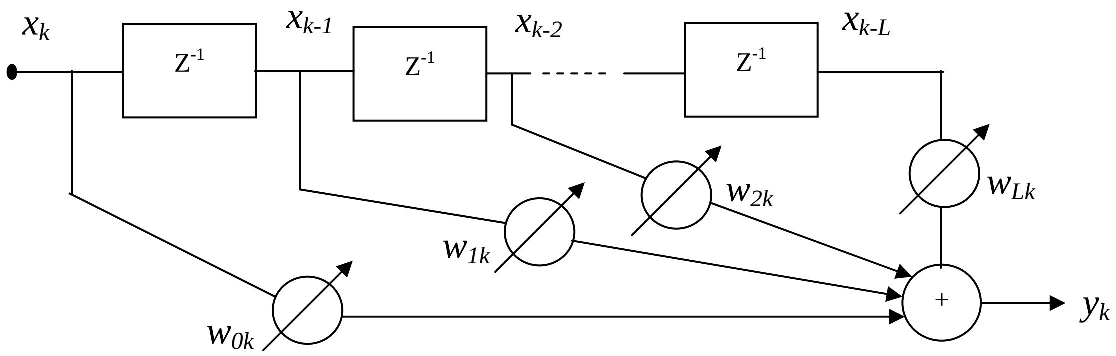

## Adaptación de un sistema a lazo cerrado




## Combinador lineal adaptativo



- Múltiples entradas $\quad x_k = [x_{0k}\ x_{1k} \ldots\ x_{lk}]^T$
- Entrada única $\quad x_k = [x_k\ x_{k-1} \ldots\ x_{k-l}]^T$




Entrada única:

$$y_k = \sum_{l=0}^{L} w_{lk} x_{k-l}$$

Entradas Múltiples:

$$y_k = \sum_{l=0}^{L} w_{lk} x_{lk}$$

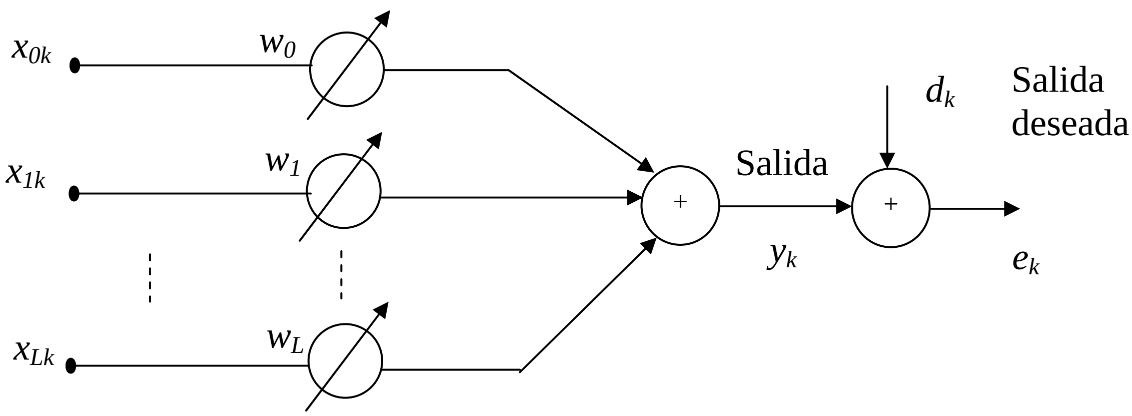
Ahora si:

$$W_k = \begin{bmatrix} w_{0k} & w_{1k} & \ldots & w_{Lk} \end{bmatrix}^T$$

Tenemos que:

$$y_k = X_k^T W_k = W_k^T X_k$$

<u>Respuesta deseada y error</u>



$$e_k = d_k - y_k = d_k - X_k^T W_k = d_k - W^T X_k$$

$$e_k^2 = d_k^2 + W^T X_k X_k^T W - 2 d_k X_k^T W$$

Se supone que $e_k$, $d_k$ y $X_k$ son estacionarios. Calculando la esperanza:

$$E\left[e_k^2\right] = E\left[d_k^2\right] + W^T E\left[X_k X_k^T\right] W - 2E\left[d_k X_k^T\right] W$$

Definiendo

$$R = E\left[X_k X_k^T\right] = E\left[\begin{bmatrix} x_{0k} \\ x_{1k} \\ \vdots \\ x_{Lk} \end{bmatrix} \begin{bmatrix} x_{0k} & x_{1k} & \cdots & x_{Lk} \end{bmatrix}\right] = E\begin{bmatrix} x_{0k}^2 & x_{0k}x_{1k} & \cdots & x_{0k}x_{Lk} \\ x_{1k}x_{0k} & x_{1k}^2 & \cdots & x_{1k}x_{Lk} \\ \vdots & \vdots & \ddots & xx \\ x_{Lk}x_{0k} & x_{Lk}x_{1k} & \cdots & x_{Lk}^2 \end{bmatrix}$$

$$P = E\left[d_k X_k\right] = E\begin{bmatrix} d_k x_{0k} & d_k x_{1k} & \cdots & d_k x_{Lk} \end{bmatrix}^T$$

Se obtiene el Error Cuadrático Medio:

$$MSE = \xi = E\left[e_k^2\right] = E\left[d_k^2\right] + W^T R W - 2P^T W$$

## Gradiente y Error Cuadrático Medio

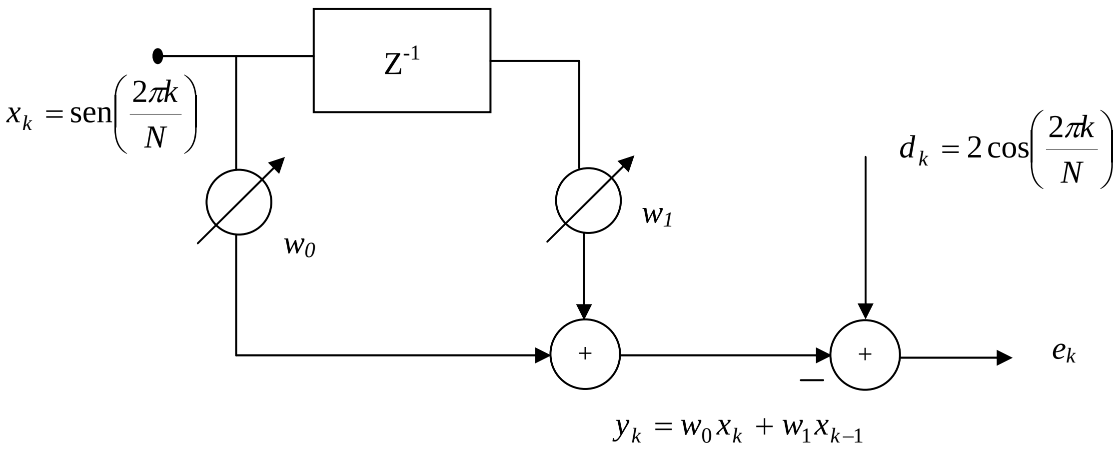
$$\nabla = \frac{\partial \xi}{\partial W} = \begin{bmatrix} \dfrac{\partial \xi}{\partial w_{0k}} & \dfrac{\partial \xi}{\partial w_{1k}} & \cdots & \dfrac{\partial \xi}{\partial w_{Lk}} \end{bmatrix}^T = 2RW - 2P$$

Para obtener el mínimo:

$$\nabla = 0 = 2RW^* - 2P$$

$$W^{*} = R^{-1}P \qquad \text{y} \qquad \xi_{MIN} = E\left[d_k^2\right] - P^T W^{*}$$

## Ejemplo de superficie de minimización




$$R = E\left[X_k X_k^T\right] = E\begin{bmatrix} x_k^2 & x_k x_{k-1} \\ x_{k-1}x_k & x_{k-1}^2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2}\cos\frac{2\pi}{N} \\ \frac{1}{2}\cos\frac{2\pi}{N} & \frac{1}{2} \end{bmatrix}$$

$$P = E\left[d_k X_k^T\right] = \left[d_k x_k \quad d_k x_{k-1}\right]^T = \left[0 \quad -\operatorname{sen}\frac{2\pi}{N}\right]$$

$$MSE = \xi = E\left[e_k^2\right] = E\left[d_k^2\right] + W^T R W - 2P^T W$$

$$\xi = 2 + \left[w_0 \quad w_1\right]\begin{bmatrix} \frac{1}{2} & \frac{1}{2}\cos\frac{2\pi}{N} \\ \frac{1}{2}\cos\frac{2\pi}{N} & \frac{1}{2} \end{bmatrix}\begin{bmatrix} w_0 \\ w_1 \end{bmatrix} - 2$$

$$\xi = 2 + \frac{1}{2}\left(w_0^2 + w_1^2\right) + w_1 w_2 \cos\frac{2\pi}{N} + 2w_1 \operatorname{sen}\frac{2\pi}{N}$$

$$\nabla = 0 = 2RW^{*} - 2P$$

$$\nabla = \begin{bmatrix} 1 & \cos\frac{2\pi}{N} \\ \cos\frac{2\pi}{N} & 1 \end{bmatrix}\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} - \begin{bmatrix} 0 \\ -2\operatorname{sen}\frac{2\pi}{N} \end{bmatrix}$$

$$W^* = \begin{bmatrix} w_0^* & w_1^* \end{bmatrix}^T = \begin{bmatrix} 2\cot g\,\frac{2\pi}{N} & -2\cos\sec\frac{2\pi}{N} \end{bmatrix}^T$$

Para N=5, los valores que obtenemos son:

$$W^* = \begin{bmatrix} w_0^* & w_1^* \end{bmatrix}^T = \begin{bmatrix} 0 & -2 \end{bmatrix}^T$$

# El algoritmo de Cuadrados Mínimos (LMS)

En el combinador lineal adaptativo, podemos representar el error de salida como:

$$e_k = d_k - X_k^T W_k$$

Siendo:

$X_k^T$ vector de datos de entrada en el instante *k*.

$W_k^T$ Vector de pesos en el instante *k*.

Se puede estimar el gradiente a partir del error cuadrático:

$$\hat{\nabla}_k = \begin{bmatrix} \dfrac{\partial e_k^2}{\partial w_0} \\ \vdots \\ \dfrac{\partial e_k^2}{\partial w_L} \end{bmatrix} = 2e_k \begin{bmatrix} \dfrac{\partial e_k}{\partial w_0} \\ \vdots \\ \dfrac{\partial e_k}{\partial w_1} \end{bmatrix} = -2e_k X_k$$

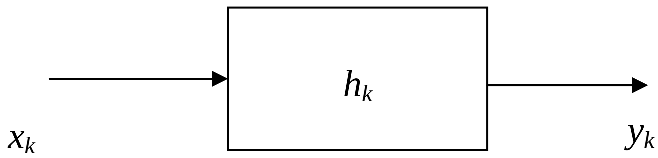
Teniendo en cuenta el método del descenso mas escalonado, el vector de pesos se puede representar como:

$$W_{k+1} = W_k + \mu\left(-\nabla_k\right)$$

$$W_{k+1} = W_k + 2\mu e_k X_k$$

Esto es válido sólo para el combinador lineal adaptativo.

# Filtros Recursivos Adaptativos

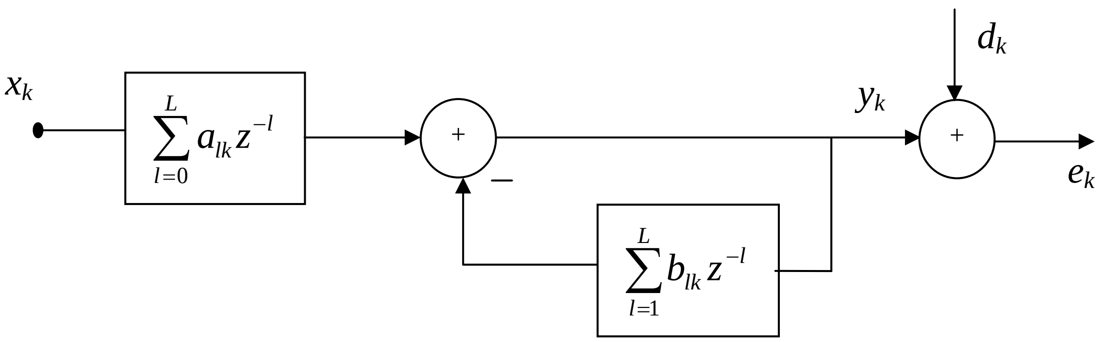



$$y_k = x_k * h_k$$

$$yk = \sum_{n=0}^{L} a_n x_{k-n} + \sum_{n=1}^{L} b_n y_{k-n}$$

Transformando por Z ambos miembros y separando la transferencia del filtro, obtenemos:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{n=0}^{L} a_n z^{-n}}{1 - \sum_{n=1}^{L} b_n z^{-n}} = \frac{A(z)}{1 - B(z)}$$




Ahora bien, se introduce a *k* como parámetro en los coeficientes de los polinomios ya que varían en cada iteración, entonces:

$$A_k(z) = \sum_{l=0}^{L} a_{lk} z^{-l} \quad \text{y} \quad B_k(z) = \sum_{l=0}^{L} b_{lk} z^{-l}$$

$a_{lk}$ y $b_{lk}$ son dependientes de *k* ya que se ajustan iteración a iteración.

$$y_k = \sum_{n=0}^{L} a_{nk} x_{k-n} + \sum_{n=1}^{L} b_{nk} y_{k-n}$$

Definiendo los siguientes vectores:

$$W_k = \begin{bmatrix} a_{0k} & a_{1k} & a_{2k} & \cdots & a_{lk} & b_{1k} & b_{2k} & \cdots & b_{Lk} \end{bmatrix}^T$$

$$U_k = \begin{bmatrix} x_k & x_{k-1} & x_{k-2} & \cdots & x_{k-L} & y_{k-1} & y_{k-2} & \cdots & y_{k-L} \end{bmatrix}^T$$

Podemos escribir el error de la siguiente manera:

$$e_k = d_k - y_k = d_k - \sum_{n=0}^{L} a_{nk} x_{k-n} - \sum_{n=0}^{L} b_{nk} y_{k-n}$$

$$e_k = d_k - W_k^T U_k$$

Ahora, teniendo en cuenta el algoritmo de LMS:

$$W_{k+1} = W_k - \mu \hat{\nabla}_k$$

$$\hat{\nabla}_k = \frac{\partial e_k^2}{\partial W_k} = 2 e_k \frac{\partial e_k}{\partial W_k}$$

$$\hat{\nabla}_k = 2 e_k \begin{bmatrix} \frac{\partial e_k}{\partial a_{0k}} & \frac{\partial e_k}{\partial a_{1k}} & \cdots & \frac{\partial e_k}{\partial a_{Lk}} & \frac{\partial e_k}{\partial b_{1k}} & \frac{\partial e_k}{\partial b_{2k}} & \cdots & \frac{\partial e_k}{\partial b_{Lk}} \end{bmatrix}^T$$

$$\hat{\nabla}_k = -2 e_k \begin{bmatrix} \frac{\partial y_k}{\partial a_{0k}} & \frac{\partial y_k}{\partial a_{1k}} & \cdots & \frac{\partial y_k}{\partial a_{Lk}} & \frac{\partial y_k}{\partial b_{1k}} & \frac{\partial y_k}{\partial b_{2k}} & \cdots & \frac{\partial y_k}{\partial b_{Lk}} \end{bmatrix}^T$$

Definimos:

$$\alpha_{nk} = \frac{\partial y_k}{\partial a_{nk}} = x_{k-n} + \sum_{l=1}^{L} b_{lk} \frac{\partial y_{k-l}}{\partial a_{nk}} = x_{k-n} + \sum_{l=1}^{L} b_{lk} \alpha_{n,k-l}$$

$$\beta_{nk} = \frac{\partial y_k}{\partial b_{nk}} = y_{k-n} + \sum_{l=1}^{L} b_{lk} \frac{\partial y_{k-l}}{\partial b_{nk}} = y_{k-n} + \sum_{l=1}^{L} b_{lk} \beta_{n,k-l}$$

Con esta nueva definición, tenemos que:

$$\hat{\nabla}_k = -2e_k \left[ \alpha_{0k} \quad \alpha_{1k} \quad \cdots \quad \alpha_{Lk} \quad \beta_{1k} \quad \beta_{2k} \quad \cdots \quad \beta_{Lk} \right]^T$$

Finalmente:

$$W_{k+1} = W_k - M \hat{\nabla}_k$$

Siendo:

$$M = diag\left[ \mu \quad \cdots \quad \mu V_1 \quad \cdots \quad \mu V_L \right] \text{ con } \quad 0 < \mu < \frac{1}{\text{máx}\, \lambda_R}$$

Resumiendo,el método de LMS para filtros recursivos IIR es:

$$y_k = \sum_{n=0}^{L} a_{nk} x_{k-n} + \sum_{n=1}^{L} b_{nk} y_{k-n} = W_k^T U_k$$

$$W_k = \left[ a_{0k} \quad a_{1k} \quad a_{2k} \quad \cdots \quad a_{lk} \quad b_{1k} \quad b_{2k} \quad \cdots \quad b_{Lk} \right]^T$$

$$U_k = \left[ x_k \quad x_{k-1} \quad x_{k-2} \quad \cdots \quad x_{k-L} \quad y_{k-1} \quad y_{k-2} \quad \cdots \quad y_{k-L} \right]^T$$

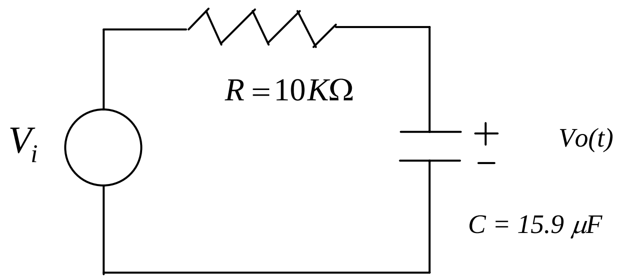
$$\alpha_{nk} = \frac{\partial y_k}{\partial a_{nk}} = x_{k-n} + \sum_{l=1}^{L} b_{lk} \alpha_{n,k-l}$$

$$\beta_{nk} = \frac{\partial y_k}{\partial b_{nk}} = y_{k-n} + \sum_{l=1}^{L} b_{lk} \beta_{n,k-l}$$

$$\hat{\nabla}_k = -2(d_k - y_k)\left[ \alpha_{0k} \quad \alpha_{1k} \quad \cdots \quad \alpha_{Lk} \quad \beta_{1k} \quad \beta_{2k} \quad \cdots \quad \beta_{Lk} \right]^T$$

$$W_{k+1} = W_k - M \hat{\nabla}_k$$

# Implementación de un filtro pasa bajos recursivo




La transferencia del sistema en el plano transformado S es:

$H(s) = \frac{V_o(s)}{V_i(s)} = \frac{1}{1 + sRC} = \frac{1}{1 + \frac{s}{s_p}}$ siendo $s_p = \frac{1}{RC}$, la frecuencia de corte del filtro pasabajos RC.

Si utilizamos la transformación bilineal para encontrar el filtro digital equivalente, tenemos que:

$$s = \frac{2}{T_s}\frac{1 - z^{-1}}{1 + z^{-1}} \quad \Rightarrow$$

$$H(z) = \frac{s_p T_s + s_p T_s z}{(s_p T_s - 2) + (s_p T_s + 2)z} = \frac{s_p T_s + s_p T_s z^{-1}}{(s_p T_s + 2) + (s_p T_s - 2)z^{-1}}$$

$$H(z) = \frac{a_0 + a_1 z^{-1}}{1 + b_1 z^{-1}}, \quad \text{siendo:} \quad \begin{cases} a_0 = \frac{s_p T_s}{s_p T_s + 2} \\ a_1 = \frac{s_p T_s}{s_p T_s + 2} \\ b_1 = \frac{s_p T_s - 2}{s_p T_s + 2} \end{cases}$$

La implementación del LMS será la siguiente:

$$W_k = [a_{0k} \quad a_{1k} \quad b_{1k}]^T$$

$$U_k = [x_k \quad x_{k-1} \quad y_{k-1}]^T$$

$$\alpha_{0k} = x_k + \sum_{l=1}^{1} b_{lk}\alpha_{0,k-l} = x_k + b_{1k}\alpha_{0,k-1}$$

$$\alpha_{1k} = x_{k-1} + \sum_{l=1}^{1} b_{lk}\alpha_{1,k-l} = x_{k-1} + b_{1k}\alpha_{1,k-1}$$

$$\beta_{1k} = y_{k-1} + \sum_{l=1}^{1} b_{lk}\beta_{1,k-l} = y_{k-1} + b_{1k}\beta_{1,k-1}$$

$$d_k = a_0 x_k + a_1 x_{k-1} - b_1 d_{k-1}$$

$$y_k = a_{0k} x_k + a_{1k} x_{k-1} + b_1 d_{k-1}$$

$$\hat{\nabla}_k = -2(dk - yk)\begin{bmatrix}\alpha_{0k} & \alpha_{1k} & \beta_{1k}\end{bmatrix}^T$$

$$W_{k+1} = W_k - M\,\hat{\nabla}_k$$

$$M = diag\begin{bmatrix}0.05 & 0.005 & 0.0025\end{bmatrix}$$

Condiciones iniciales de estimación:

$$\begin{cases} k = 0 \\ yk = 0, \quad a_{0k} = a_{1k} = B_{1k} = 0, \quad \alpha_{0k} = \alpha_{1k} = \beta_{1k} = 0 \end{cases}$$

Resultados:

$$\begin{cases} W = \begin{bmatrix}0.1586 & 0.1586 & -0.6828\end{bmatrix}^T \\ W^* = \begin{bmatrix}0.1586 & 0.1586 & 0.6828\end{bmatrix}^T \end{cases}$$