Proofs

Wednesday, September 10, 2025 1:31 PM

Types of statements

Propositions (prop)

- claim posed in the form of a statement that must be proven/disproven

Theorems (thm)

- particularily important prop.

Lemmas

- "helper", used to prove a thesis

Corollary

- an "obvious" result of another prop.

ex. 1) Prove $\forall \theta \mathbb{R}$, $\sin(3\theta) = 3\sin\theta - 4\sin^3\theta$

Warnings: do not chose a particular value for x
do not assume the statement in its proof

Prove, $\sin(3\theta) = 3\sin\theta - 4\sin^3\theta$

Let $\theta \in \mathbb{R}$

$$
\begin{aligned}
\sin(2\theta+\theta) &= \sin 2\theta\cos\theta + \cos 2\theta \sin\theta \\
&= 2\sin\theta\cos^2\theta + (1-2\sin^2\theta)\sin\theta \\
&= 2\sin\theta(1-\sin^2\theta) + \sin\theta - 2\sin^3\theta \\
&= 2\sin\theta - 2\sin^3\theta + \sin\theta - 2\sin^3\theta \\
\sin(3\theta) &= 3\sin\theta - 4\sin^3\theta
\end{aligned}
$$

Prove $\max(a, b) = \dfrac{x+y+|x-y|}{2}$

Case 1: let $x \geq y$

$$
\begin{aligned}
\frac{x+y+|x-y|}{2} &= \frac{x+y+x-y}{2} \\
&= \frac{2x}{2} \\
&= x \\
&= \max(x, y)
\end{aligned}
$$

Case 2: let $x < y$

$$
\begin{aligned}
x+y+|x-y| &= \frac{x+y-x+y}{2} \\
&\triangleright \frac{2y}{2} \\
&= y \\
&= \max(x, y)
\end{aligned}
$$

ex. 2) Let $x, y \in \mathbb{R}$ [illegible] x^4 [illegible] $^2 \geq 5x^2$ [illegible] $3y^2$

Prove that $\forall x, y \in \mathbb{R}$, $x^4+x^2y+y^2 \geq 5x^2y-3y^2$

$x^4+x^2y+y^2-5x^2y+3y^2 \geq 0$

$x^4-4x^2y+4y^2 \geq 0$

$(x^2-2y)^2 \geq 0$

not proof

Let $x,y \in \mathbb{R}$,

$(x^2-2y)^2 \geq 0$

$x^4-4x^2y+4y^2 \geq 0$

$x^4+x^2y-5x^2y+y^2+3y^2 \geq 0$

$\underline{x^4+x^2y+y^2 \geq 5x^2y-3y^2}$

proof

Proving existentially quantified statements

- $\exists x \in S$,
- may tak

1) Find an example
2) Show its in the domain
3) Show it satisfies $P(x)$

ex) prove there is a perfect square k such t

$2k^2-31k-16=0$

$2k^2-32k+k-16=0$

$2k(k-16)+(k-16)=0$

not a proof

$(2k+1)(k-16)=0$

$k=-\frac{1}{2},\ k=16$

Proof

Let $k=16$.

$k=4^2$, so k is a perfect square

$k^2-\frac{31}{2}k = 16^2-\frac{31}{2}\cdot 16 = 256-248=8$

Disproving $\exists$ means proving $\forall$

ex) disprove $\exists x \in \mathbb{R},\ \cos 2x+\sin 2x=3$

$\Leftrightarrow$ prove: $\forall x \in \mathbb{R},\ \cos 2x+\sin 2x \neq 3$

$\cos 2x \le 1$

$\sin 2x \le 1$

$\cos 2x+\sin 2x \le 2 < 3$

Proving nested quantifiers

ex. prove $\forall x \in \mathbb{R},\ \exists y \in \mathbb{R},\ x^3-y^3=1$

$x^3-1=y^3$

$$y = \sqrt[3]{x^3 - 1}$$

Let $x \in \mathbb{R}$, consider $y = \sqrt[3]{x^3 - 1}$

then $x^3 - (\sqrt[3]{x^3-1})^3 = x^3 - (x^3 - 1)$
$= 1$

Proving implications

1. assume hypothesis
2. use it to prove conclusion

Warning: do not assume the conclusion

ex) prove that $\forall k \in \mathbb{Z}$, if k^5 is a perfect square then $9k^{19}$ is also a perfect square

Let $k^5 = a^2$, where $a \in \mathbb{Z}$,

$$9k^{19} = 9(a^2)^{19} = 9a^{38}$$
$$= 3^2 a^{38}$$
$$= 3^2(a^{17})^2$$
$$= (3a^{17})^2$$

$-(\check{\smile}\,\ddot{\omega}\,)$

Prove that if m is an even integer, then $7m^2+4$ is also an even integer

Proof: Let m be an even int,

then $\exists n \in \mathbb{Z}$, $m = 2n$

then $7m^2+4 = 7(2n)^2+4 = 2(14n^2+2)$