MAT 308: Differential Equations with Linear Algebra

Spring 2025

1 Introduction to Differential Equations

- Differential equations describe how quantities change over time.
- Found in physics, engineering, and many sciences (e.g., Newton's laws, Schrödinger's equation).
- Types of differential equations:
  - **Ordinary Differential Equations (ODEs)**: Involves ordinary derivatives (e.g., $y' + y = x$).
  - **Partial Differential Equations (PDEs)**: Involves partial derivatives (e.g., Heat equation $\frac{\partial u}{\partial t} = -\frac{\partial^2 u}{\partial x^2}$).

2 Review of Calculus

2.1 Limits and Continuity

- A function $f(x)$ is continuous if $\lim_{x\to a} f(x) = f(a)$.
- Important rules:
  - Sum Rule: $\lim_{x\to a}(f(x) + g(x)) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$.
  - Product Rule: $\lim_{x\to a}(f(x)g(x)) = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)$.

2.2 Derivatives

- Definition: $f'(x) = \lim\limits_{h\to 0} \frac{f(x+h)-f(x)}{h}$.
- Common derivative rules:
  - Power Rule: $\frac{d}{dx} x^n = nx^{n-1}$.
  - Chain Rule: $\frac{d}{dx} f(g(x)) = f'(g(x))g'(x)$.

2.3 Integration

- **Definite integral**: $\int_a^b f(x)dx$ gives area under the curve.
- **Fundamental Theorem of Calculus**: $\int_a^b f(x)dx = F(b) - F(a)$ where $F'(x) = f(x)$.

3 First-Order Differential Equations

3.1 Definition

- A first-order ODE is an equation of the form $y' = F(x, y)$.
- Example: $y' = -y/x$.

3.2 Direction Fields

- Shows the slope of the function at various points.
- Helps visualize solution curves.

3.3 Separation of Variables

- Used for equations of the form $y' = f(x)g(y)$.
- Steps:
  1. Rewrite as $\frac{dy}{g(y)} = f(x)dx$.
  2. Integrate both sides.
  3. Solve for y.
- Example: $y' = ky$

$$\frac{dy}{y} = kdx$$
$$\int \frac{1}{y}dy = \int kdx$$
$$\ln|y| = kx + C$$
$$y = Ce^{kx}$$

4 Existence and Uniqueness of Solutions

- A solution exists if $F(x, y)$ is continuous.
- A unique solution exists if $\frac{\partial F}{\partial y}$ is also continuous.

5 Numerical Methods: Euler's Method

- Used to approximate solutions when an exact solution is hard to find.
- Given $y' = F(x, y)$ and initial value y_0:
  1. Choose step size h.
  2. Compute $y_{n+1} = y_n + hF(x_n, y_n)$.

6 Linear Differential Equations

6.1 Definition

- A first-order linear equation has the form:

$$y' + g(x)y = f(x).$$

- If $f(x) = 0$, the equation is homogeneous.

6.2 Solving Using Integrating Factor

- Multiply by integrating factor $\mu(x) = e^{\int g(x)dx}$.
- The equation becomes:

$$\frac{d}{dx}(y\mu) = f(x)\mu.$$

- Integrate both sides to solve for y.

6.3 Example

Solve $y' - xy = x$:

- Identify $g(x) = -x$, so integrating factor is:

$$\mu(x) = e^{\int -x dx} = e^{-x^2/2}.$$

- Multiply by $\mu(x)$:

$$e^{-x^2/2}y' - xe^{-x^2/2}y = xe^{-x^2/2}.$$

- Recognizing the left side as a derivative:

$$\frac{d}{dx}(ye^{-x^2/2}) = xe^{-x^2/2}.$$

- Integrate both sides:

$$ye^{-x^2/2} = -e^{-x^2/2} + C.$$

- Solve for y:

$$y = -1 + Ce^{x^2/2}.$$