Lucas Rhode
Homework 1
Due: Wednesday, February 5
MAT 308, Spring 2025

Problem 1(a)

Given:

$$\frac{d^2s}{dt^2} = \sin t, \quad s(0) = 1, \quad \left.\frac{ds}{dt}\right|_{t=0} = 1.$$

Step 1: Integrate once.

$$\frac{d^2s}{dt^2} = \sin t \quad \Longrightarrow \quad \frac{ds}{dt} = \int \sin t\, dt = -\cos t + C_1.$$

Step 2: Use initial condition on $s'(0) = 1$.

$$s'(0) = -\cos(0) + C_1 = -1 + C_1 = 1 \quad \Longrightarrow \quad C_1 = 2.$$

So

$$s'(t) = -\cos t + 2.$$

Step 3: Integrate again.

$$s(t) = \int (-\cos t + 2)\, dt = -\sin t + 2t + C_2.$$

Step 4: Use $s(0) = 1$.

$$s(0) = -\sin(0) + 2 \cdot 0 + C_2 = C_2 = 1.$$

Hence

$$\boxed{s(t) = -\sin t + 2t + 1.}$$

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Problem 1(b)

Given:

$$y' = xe^x, \quad y(0) = 1.$$

Step 1: Integrate.

$$\frac{dy}{dx} = xe^x \quad \Longrightarrow \quad y = \int xe^x\,dx + C.$$

Use integration by parts ($u = x$, $dv = e^x dx$):

$$\int xe^x\,dx = xe^x - \int e^x\,dx = xe^x - e^x = e^x(x-1).$$

So

$$y(x) = e^x(x-1) + C.$$

Step 2: Use $y(0) = 1$.

$$y(0) = e^0(0-1) + C = -1 + C = 1 \quad \Longrightarrow \quad C = 2.$$

Hence

$$\boxed{y(x) = e^x(\,x-1\,) + 2.}$$

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Problem 1(c)

Given:

$$\frac{du}{dv} = v(1-v), \quad u(0) = 1.$$

Step 1: Integrate.

$$\frac{du}{dv} = v(1-v) = v - v^2 \quad \Longrightarrow \quad u(v) = \int (v - v^2)\, dv + C.$$

$$\int (v - v^2)\, dv = \frac{v^2}{2} - \frac{v^3}{3}.$$

So

$$u(v) = \frac{v^2}{2} - \frac{v^3}{3} + C.$$

Step 2: Use $u(0) = 1$.

$$u(0) = 0 - 0 + C = 1 \quad \Longrightarrow \quad C = 1.$$

Hence

$$\boxed{u(v) = \frac{v^2}{2} - \frac{v^3}{3} + 1.}$$

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Problem 1(d)

Given:

$$\frac{d^3z}{dt^3} = 1, \quad z(0) = 0, \quad z'(0) = 0, \quad z''(0) = 0.$$

Step 1: Integrate once.

$$\frac{d^3z}{dt^3} = 1 \quad \Longrightarrow \quad \frac{d^2z}{dt^2} = \int 1\, dt = t + C_1.$$

Use $z''(0) = 0$:

$$z''(0) = 0 = 0 + C_1 \quad \Longrightarrow \quad C_1 = 0.$$

So $z''(t) = t$.

Step 2: Integrate again.

$$z'(t) = \int t\, dt = \frac{t^2}{2} + C_2.$$

Use $z'(0) = 0$:

$$z'(0) = 0 = 0 + C_2 \quad \Longrightarrow \quad C_2 = 0.$$

So $z'(t) = \frac{t^2}{2}$.

Step 3: Integrate once more.

$$z(t) = \int \frac{t^2}{2}\, dt = \frac{1}{2} \cdot \frac{t^3}{3} + C_3 = \frac{t^3}{6} + C_3.$$

Use $z(0) = 0$:

$$z(0) = 0 = \frac{0^3}{6} + C_3 \quad \Longrightarrow \quad C_3 = 0.$$

Hence

$$\boxed{z(t) = \frac{t^3}{6}.}$$

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Problem 1(e)

Given:

$$\frac{dx}{dt} = \arctan(t), \quad x(0) = 0.$$

Step 1: Integrate.

$$x(t) = \int \arctan(t)\, dt + C.$$

Compute $\displaystyle\int \arctan(t)\, dt$.

Use integration by parts. Let $u = \arctan(t)$, so $du = \frac{1}{1+t^2}\, dt$. Let $dv = dt$, so $v = t$. Then

$$\int \arctan(t)\, dt = t\, \arctan(t) \; - \int \frac{t}{1+t^2}\, dt.$$

$$\int \frac{t}{1+t^2}\, dt = \frac{1}{2}\ln(1+t^2).$$

Thus

$$\int \arctan(t)\, dt = t\, \arctan(t) \; - \; \frac{1}{2}\ln(1+t^2).$$

So

$$x(t) = t\, \arctan(t) \; - \; \frac{1}{2}\ln\bigl(1+t^2\bigr) + C.$$

Step 2: Use $x(0) = 0$.

$$x(0) = 0 = 0 - \frac{1}{2}\ln(1+0) + C = 0 - 0 + C \quad \Longrightarrow \quad C = 0.$$

Hence

$$\boxed{x(t) \; = \; t\, \arctan(t) \; - \; \tfrac{1}{2}\ln\bigl(1+t^2\bigr).}$$

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Problem 2(a)

Problem Statement: A projectile is fired directly upward from ground level at a velocity of 1000 meters per second. Assume there is no wind resistance, and that the acceleration due to gravity is 10 meters per second per second
(a) Formulate an initial value problem that determines the height of the projectile as a function of time.

Solution Steps:
1. Define variables.
Let $y(t)$ be the height (in meters) of the projectile at time t (in seconds). Gravity is $g = 10$ m/s^2, acting downward.
2. Write the ODE for acceleration.
Since upward is positive, the acceleration is -10. Hence,

$$\frac{d^2y}{dt^2} = -10.$$

3. Initial conditions.
At $t = 0$, the projectile is at ground level: $y(0) = 0$. It is fired upward at 1000 m/s, so $y'(0) = 1000$.
4. The IVP is:

$$\boxed{\begin{cases} \dfrac{d^2y}{dt^2} = -10, \\ y(0) = 0, \\ y'(0) = 1000. \end{cases}}$$

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Problem 2(b)

(b) What is the maximum altitude attained by the projectile, and at what time does it get there?

Solution Steps:

1. Integrate to get velocity.

$$\frac{d^2y}{dt^2} = -10 \implies y'(t) = \int -10\,dt = -10\,t + C_1.$$

Use $y'(0) = 1000$:

$$y'(0) = -10 \cdot 0 + C_1 = 1000 \implies C_1 = 1000.$$

Thus $y'(t) = 1000 - 10\,t$.

2. Integrate again to get position.

$$y(t) = \int \big(1000 - 10\,t\big)\,dt = 1000\,t - 5\,t^2 + C_2.$$

Use $y(0) = 0$:

$$y(0) = 1000 \cdot 0 - 5 \cdot 0^2 + C_2 = 0 \implies C_2 = 0.$$

So $y(t) = 1000\,t - 5\,t^2$.

3. Find time of maximum altitude.

The maximum occurs when the velocity $y'(t)$ is zero:

$$y'(t) = 1000 - 10\,t = 0 \implies t = 100 \text{ seconds}.$$

4. Evaluate the maximum altitude.

$$y(100) = 1000 \cdot 100 - 5 \cdot (100)^2 = 100{,}000 - 5 \times 10{,}000 = 100{,}000 - 50{,}000 = 50{,}000 \text{ meters}.$$

Answer:

$$\boxed{\text{Maximum height} = 50{,}000 \text{ m at } t = 100 \text{ s.}}$$

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Problem 3(a)

Problem Statement: Find the solution to the initial value problem

$$y' = 2xy, \quad y(0) = 2.$$

Solution Steps:

1. Rewrite the ODE.

$$\frac{dy}{dx} \;=\; 2\,x\,y.$$

2. Separate variables.

$$\frac{1}{y}\,\frac{dy}{dx} \;=\; 2\,x.$$

I'm going to write it like this

$$\int \frac{1}{y}\,dy \;=\; \int 2\,x\,dx.$$

3. Integrate both sides.

$$\int \frac{1}{y}\,dy \;=\; \ln|y|, \quad \int 2\,x\,dx \;=\; x^2 + C.$$

So

$$\ln|y| \;=\; x^2 + C.$$

4. Exponentiate.

$$|y| \;=\; e^{x^2+C} \;=\; e^C\,e^{x^2}.$$

Let $A = e^C > 0$. Then $y = \pm A\,e^{x^2}$. We can absorb the $\pm$ into A, so

$$y(x) = K\,e^{x^2}$$

for some constant $K \in \mathbb{R}$.

5. Use the initial condition $y(0) = 2$.

$$y(0) \;=\; K\,e^{0^2} \;=\; K \;=\; 2.$$

Therefore $K = 2$.

Final Answer:

$$\boxed{y(x) \;=\; 2\,e^{x^2}.}$$

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Problem 3(b)

$$\frac{dz}{dt} \;=\; \frac{t}{z^2}, \quad z(1) = 0.$$

Solution Steps:

1. Write the ODE.

$$z'(t) \;=\; \frac{t}{z^2(t)}.$$

We note that the right-hand side is not defined at $z = 0$, so we must be cautious.

2. Separate variables.

$$\frac{dz}{dt} = \frac{t}{z^2} \quad \Longrightarrow \quad z^2\,dz = t\,dt.$$

3. Integrate both sides.

$$\int z^2\,dz = \int t\,dt.$$

$$\frac{z^3}{3} = \frac{t^2}{2} + C.$$

Multiply through by 3:

$$z^3 = \tfrac{3}{2}\,t^2 + 3C.$$

Let $K = 3C$. Then

$$z^3 = \frac{3}{2}\,t^2 + K.$$

Hence

$$z(t) = \sqrt[3]{\frac{3}{2}\,t^2 + K}\,.$$

4. Use the initial condition $z(1) = 0$.

$$z(1) = 0 \implies 0^3 = \frac{3}{2}\,(1)^2 + K \implies 0 = \frac{3}{2} + K \implies K = -\frac{3}{2}.$$

Thus

$$z^3 = \frac{3}{2}\,t^2 \;-\; \frac{3}{2} = \frac{3}{2}\,\bigl(t^2 - 1\bigr).$$

Therefore

$$z(t) = \sqrt[3]{\frac{3}{2}\,\bigl(t^2 - 1\bigr)}.$$

5. Domain considerations. Because $z'(t) = t/z^2$ is singular at $z = 0$, passing through $z = 0$ at $t = 1$ is delicate.

Final Answer:

$$\boxed{z(t) \;=\; \sqrt[3]{\frac{3}{2}\,\bigl(t^2 - 1\bigr)}, \quad \text{with} \quad z(1) = 0.}$$

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Problem 4(a)

Problem Statement: Suppose that a 400 liter tank contains 10 kilograms of salt in solution at time t = 0, and that pure water is being added to it at a rate of 2 liters per minute, and the resulting mixture (which we assume is always fully homogeneous) is being drawn off at a rate of 2 liters per minute.

Solution Steps:
1. Our variables. $s(t)$ (in kg) is the amount of salt in the tank at time t (in minutes). The tank volume is constant, 400 liters.
2. Inflow of salt. Pure water enters, so no salt is entering. The salt inflow rate is 0 kg/min.
3. Outflow of salt. Well-mixed solution leaves at 2 L/min. Concentration of salt in the tank is $\frac{s(t)}{400}$ kg/L. So the outflow of salt is $\frac{s(t)}{400} \times 2 = \frac{s(t)}{200}$ kg/min.
4. Balance equation $\frac{ds}{dt}$.

$$\frac{ds}{dt} \;=\; (\text{salt in}) \;-\; (\text{salt out}) \;=\; 0 - \frac{s(t)}{200}.$$

So the ODE is

$$\frac{ds}{dt} = -\frac{s}{200}.$$

5. Initial condition. At $t = 0$, it is $s(0) = 10$ kg.

Final IVP:

$$\boxed{\begin{cases} \dfrac{ds}{dt} = -\dfrac{s}{200}, \\ s(0) = 10. \end{cases}}$$

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Problem 4(b) and (c)

(b) Is the amount of salt increasing or decreasing?

Answer: The ODE $\dfrac{ds}{dt} = -\dfrac{s}{200}$ has a negative right-hand side whenever $s > 0$. That means $\frac{ds}{dt} < 0$ for $s > 0$. Therefore, $s(t)$ is *decreasing*.

(c) Long-term behavior: $\lim_{t\to\infty} s(t)$**?**

$$\frac{ds}{dt} = -\frac{s}{200} \implies s(t) = s(0)\, e^{-t/200} = 10\, e^{-t/200}.$$

As $t \to \infty$, $e^{-t/200} \to 0$. Then you get

$$\lim_{t\to\infty} s(t) = 0.$$

Answer: $\boxed{0\,\text{kg.}}$

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Problem 4(d)

New Scenario:

- The tank volume remains 400 liters at all times.
- Inflow: 2 L/min, containing 0.05 kg of salt per minute.
- Outflow: 1 L/min of well-mixed solution (which does carry salt) and 1 L/min of steam (carrying no salt).
- Initially ($t = 0$), there are 10 kg of salt in the tank.

(a) IVP for $s(t)$. Let $s(t)$ = amount of salt (kg) at time t (min).

$$\text{Inflow of salt} = 0.05 \text{ (kg/min)}. \quad \text{Outflow of salt} = \left[\tfrac{s(t)}{400}\right] \times 1 = \frac{s(t)}{400}.$$

Hence

$$\frac{ds}{dt} = 0.05 - \frac{s(t)}{400}, \quad s(0) = 10.$$

$$\boxed{\begin{cases} \dfrac{ds}{dt} = 0.05 - \dfrac{s}{400}, \\ s(0) = 10. \end{cases}}$$

(b) Increasing or decreasing? Since $\frac{ds}{dt} = 0.05 - \frac{s}{400}$, we see

$$\tfrac{ds}{dt} > 0 \iff s < 20, \quad \tfrac{ds}{dt} < 0 \iff s > 20.$$

Given $s(0) = 10 < 20$, $s(t)$ *increases* until it reaches 20.
(c) Long-term behavior: At steady state, $\frac{ds}{dt} = 0$, so

$$0.05 - \frac{s}{400} = 0 \implies s = 20.$$

Thus $\lim_{t\to\infty} s(t) = 20$ kg.