A Model for Community-Acquired Pneumonia in a High-Risk Community

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Background

- Pneumonia is the inflammation of air sacs in the lungs caused by bacteria [6].
- Pneumonia "is the most common cause of sepsis and septic shock" [5].
- Pneumonia has killed an average of $44{,}371$ people per year from 2018-2022 in the United States [3].

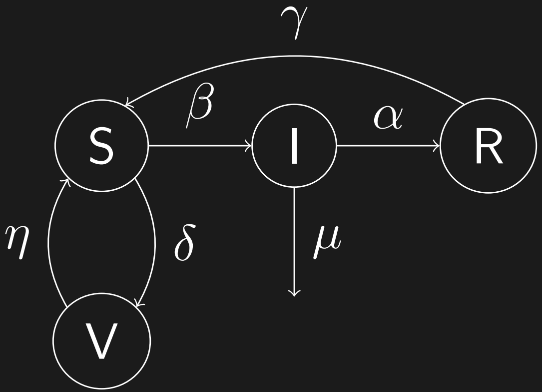
Background

- We will specifically will look at community-acquired pneumonia which can spread bacteria and bacteria-like organisms [6].
- Additionally, our model looks at people who are at higher risk of pneumonia. That is, people who have a chronic lung disease (such as bronchiectasis) or people above the age of 65 [4].

Model - Equations

$$
\begin{aligned}
S' &= -\beta S I - \delta S + \eta V + \gamma R \\
I' &= \beta S I - \alpha I - \mu I \\
R' &= \alpha I - \gamma R \\
V' &= \delta S - \eta V
\end{aligned}
$$

Model - Flow Diagram



Community-acquired pneumonia model flowchart

Analysis - Equilibrium

$$\mathcal{E}_0 = (S^*, 0, 0, \frac{\delta}{\eta} S^*)$$

$$\mathcal{E}^* = (\frac{\alpha + \mu}{\beta}, I^*, \frac{\alpha + \mu}{\gamma} I^*, \frac{\delta(\alpha + \mu)}{\eta\beta})$$

Analysis - The General Jacobian $\mathcal{J}$

$$\mathcal{J} = \begin{pmatrix} -\beta I - \delta & -\beta S & \gamma & \eta \\ \beta I & \beta S - \alpha - \mu & 0 & 0 \\ 0 & \alpha & -\gamma & 0 \\ \delta & 0 & 0 & -\eta \end{pmatrix}$$

Analysis - $\mathcal{J}_{\mathcal{E}_0}$

$$\mathcal{J}_{\mathcal{E}_0} = \begin{pmatrix} -\delta & -\beta S^* & \gamma & \eta \\ 0 & \beta S^* - \alpha - \mu & 0 & 0 \\ 0 & \alpha & -\gamma & 0 \\ \delta & 0 & 0 & -\eta \end{pmatrix}$$

Analysis - Eigenvalues of $\mathcal{J}_{\mathcal{E}_0}$

$$\begin{aligned} 0 &= (-\delta-\lambda)(\beta S^* - \alpha - \mu - \lambda)(-\gamma-\lambda)(-\eta-\lambda) \\ &\quad - \delta(\beta S^* - \alpha - \mu - \lambda)(\eta)(-\gamma-\lambda) \end{aligned}$$

Analysis - $\mathcal{E}_0$ Stability - Next Generation Matrix Method

Let $\mathcal{F} = \beta S I$, and let $\mathcal{V} = (\alpha + \mu) I$.

$$\begin{aligned} F &= \frac{\partial}{\partial I}(\mathcal{F}) \\ &= \frac{\partial}{\partial I}(\beta S I) \\ &= \beta S \end{aligned}$$

Analysis - $\mathcal{E}_0$ Stability - Next Generation Matrix Method

$$
\begin{aligned}
V &= \frac{\partial}{\partial I}(\mathcal{V}) \\
&= \frac{\partial}{\partial I}((\alpha + \mu)I) \\
&= \alpha + \mu
\end{aligned}
$$

Analysis - $\mathcal{E}_0$ Stability - Next Generation Matrix Method

$$
\begin{aligned}
\mathcal{R}_0^{NG} &= FV^{-1} \\
&= \frac{\beta S^*}{\alpha + \mu}
\end{aligned}
$$

Analysis - $\mathcal{E}_0$ Stability Theorem

Conjecture

The equilibrium $\mathcal{E}_0$ is locally asymptotically stable when

$$\frac{\beta S^*}{\alpha + \mu} < 1$$

and unstable if

$$\frac{\beta S^*}{\alpha + \mu} > 1.$$

Analysis - $\mathcal{J}_{\mathcal{E}^*}$

$$\mathcal{J}_{\mathcal{E}^*} = \begin{pmatrix} -\beta I^* - \delta & -\alpha - \mu & \gamma & \eta \\ \beta I^* & 0 & 0 & 0 \\ 0 & \alpha & -\gamma & 0 \\ \delta & 0 & 0 & -\eta \end{pmatrix}$$

Analysis - Eigenvalues of $\mathcal{J}_{\mathcal{E}^*}$ and $\mathcal{E}^*$ Stability

$$
\begin{aligned}
0 = {} & \beta I^*(\lambda+\eta)[(-\alpha-\mu)(-\gamma-\lambda)-\gamma\alpha] \\
& + \lambda(\lambda+\gamma)[(-\beta I^*-\delta-\lambda)(-\eta-\lambda)-\eta\delta]
\end{aligned}
$$

$\mathcal{R}_0^{NG}$ will be used to determine stability

Analysis - $\mathcal{E}^*$ Stability Theorem

Conjecture

The endemic equilibrium $\mathcal{E}^*$ is locally asymptotically stable when

$$\frac{\beta S^*}{\alpha + \mu} > 1$$

and unstable if

$$\frac{\beta S^*}{\alpha + \mu} < 1.$$

Simulation Parameters

Parameter	Simulation 1	Simulation 2
β	0.001	0.9
α	0.9	0.01
γ	0.3	0.6
μ	0.001	0.00000001
δ	0.25	0.2
η	0.05	0.1

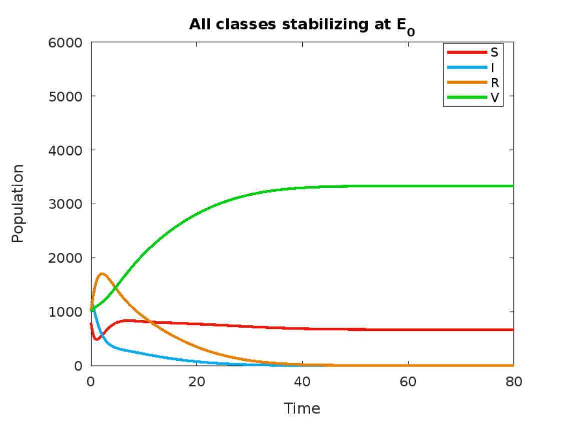
Parameter values for simulations

Simulation Initial Conditions

Initial Condition	Simulation 1	Simulation 2
S_0	800	2000
I_0	1200	2000
R_0	1004	150
V_0	1003	3000

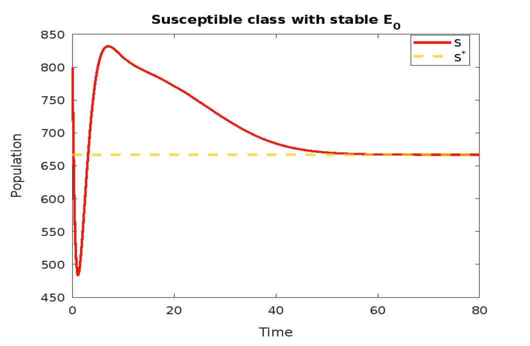
Initial conditions for simulations

Simulation 1

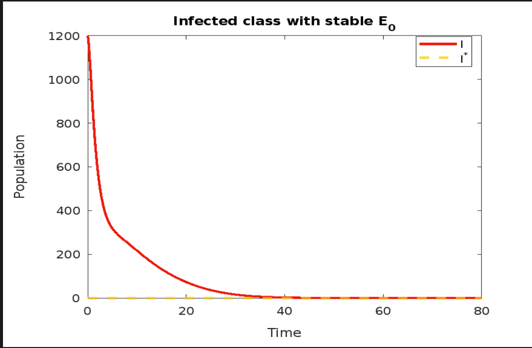


All classes stabilizing at $\mathcal{E}_0$

Simulation 1

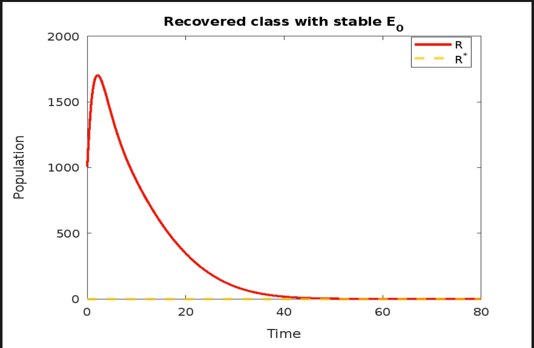


Susceptible class with stable $\mathcal{E}_0$

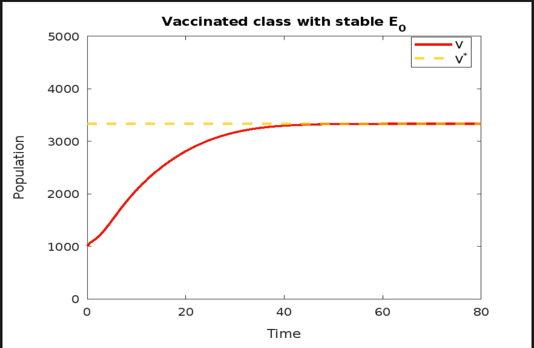


Infected class with stable $\mathcal{E}_0$

Simulation 1

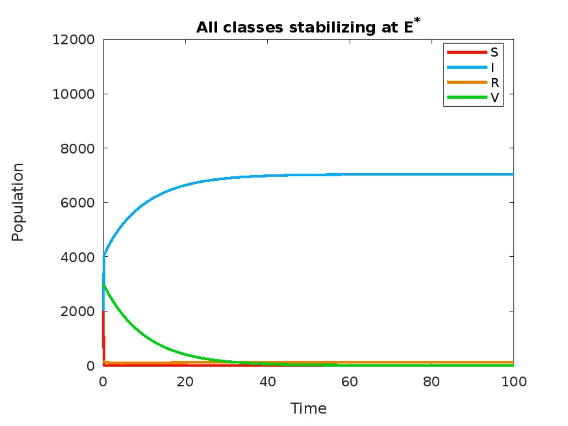


Recovered class with stable $\mathcal{E}_0$



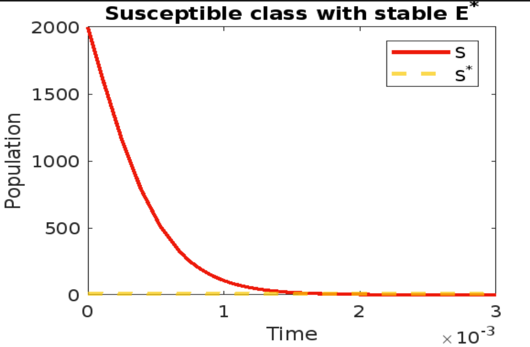
Vaccinated class with stable $\mathcal{E}_0$

Simulation 2

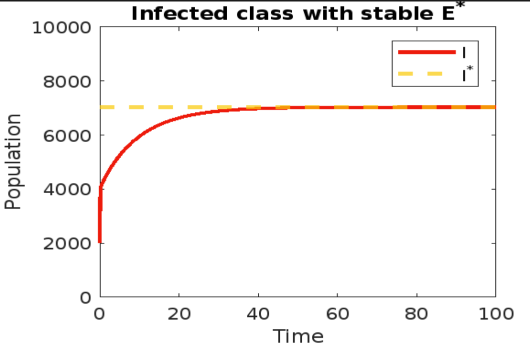


All classes stabilizing at $\mathcal{E}^*$

Simulation 2

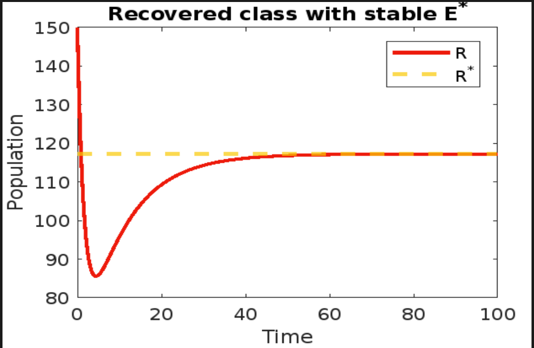


Susceptible class with stable $\mathcal{E}^*$

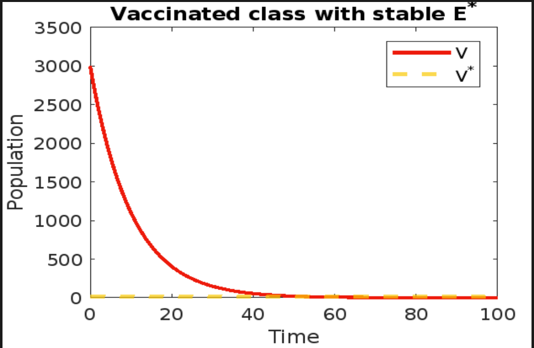


Infected class with stable $\mathcal{E}^*$

Simulation 2



Recovered class with stable $\mathcal{E}^*$



Vaccinated class with stable $\mathcal{E}^*$

References

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[2] Giampiero Foccillo. The infections causing acute respiratory failure in elderly patients. *Ventilatory Support and Oxygen Therapy in Elder, Palliative and End-of-Life Care Patients*, pages 35–45, 2019.

[3] Centers for Disease Control and Prevention. Underlying Cause of Death, 2018-2022, Single Race Results Deaths due to Pneumonia (J12-J18) Deaths occurring through 2022. https://wonder.cdc.gov/controller/datarequest/D158;jsessionid=B6B6391131574F2507C4A5CDE4DB, 2018 - 2022.

[4] Eve Palmer, Nicholas Lane, John Davison, Donna McEvoy, and Anthony De Soyza. Pneumococcal vaccination in bronchiectasis- an area for improvement? *European Respiratory Journal*, 48(suppl 60), 2016.

[5] American Thoracic Society. Top 20 Pneumonia Facts—2019. https://www.thoracic.org/patients/patient-resources/resources/top-pneumonia-facts.pdf, 2019.

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[7] Lewis, Mark Müller, Johannes Vries, Gerda de, Hillen, Thomas and Schönfisch, Birgitt. *A Course in Mathematical Biology*. SIAM, 2006.

Questions?