Below are some Theorems used to assist in the homework:

Theorem 5.1 Σ can be represented as a regular language.

Proof. We prove Σ can be represented as a regular language by constructing a DFA that accepts any symbol in Σ and denies strings with length greater than 1. Let $M = (Q, \Sigma, \delta, q_0, F)$. Let $Q = \{0, 1, 2\}$, $q_0 = 0$, and $F = \{1\}$. Next, let $\delta(q_0, a) = 1$, $\delta(1, a) = 2$, and $\delta(2, a) = 2$ for any $a \in \Sigma$. Then, our DFA, M, is constructed, and $L(M)$ is regular. Therefore Σ can represented as a regular language. □

Theorem 5.2 (Question 19 on page 119) The *left quotient* of a language L_1 with respect to L_2, defined as

$$L_2 \setminus L_1 = \{y : x \in L_2, xy : L_1\},$$

is closed under the operation.

Proof. Let $L_1 = L(M)$, where $M = (Q, \Sigma, \delta, q_0, F)$ is a DFA. We construct another DFA $\hat{M} = (Q, \Sigma, \delta, \hat{q_0}, F)$. Next, let $xy \in L_1$ and $x \in L_2$. Then, we will pick $\hat{q_0}$ to be $\delta^*(q_0, x)$ from the DFA M, such that $\delta^*(\hat{q_0}, y) = q_f \in F$. We know $\hat{q_0}$ exists since $xy \in L_1$ and $x \in L_2$, so $\delta^*(q_0, x)$ must be the beginning of y. Thus, with $\hat{q_0}$, we have constructed $\hat{M}$, so $L(\hat{M})$ is regular. Next, To show $L(\hat{M}) = L_2 \setminus L_1$, let $y \in L_2 \setminus L_1$, then (from $\hat{M}$), $\delta^*(\hat{q_0}, y) = q_f \in F$ as defined. Therefore, The *left quotient* of a language L_1 with respect to L_2 is regular, so the operation is closed. □

The Homework:

A) Complete problems 10 and 11 on page 118.

10. The *symmetric difference* of two sets S_1 and S_2 is defined as

$$S_1 \ominus S_2 = \{x : x \in S_1 \text{ or } x \in S_2, \text{ but } x \text{ is not in both } S_1 \text{ and } S_2\}.$$

Show that the family of regular languages is closed under symmetric difference.

Answer. To begin, $S_1 \ominus S_2 = (S_1 \cup S_2) - (S_1 \cap S_2)$. First, by Theorem 4.1 we have that $(S_1 \cup S_2)$ and $(S_1 \cap S_2)$ are both closed operations and produce regular langauges. Next, the operation for set difference $(-)$ was shown to be regular in Example 4.1 (in the textbook). Thus, since $(-)$ is closed and $(S_1 \cup S_2), (S_1 \cap S_2)$ are regular, then the operation $(S_1 \cup S_2) - (S_1 \cap S_2)$ is closed. Therefore, $S_1 \ominus S_2$ is closed under the operation.

11. The *nor* of two languages is

$$nor(L_1, L_2) = \{w : w \notin L_1 \text{ and } w : w \notin L_2\}.$$

Show that the family of regular languages is closed under the *nor* operation.

Answer. To begin, $nor(L_1, L_2) = \overline{L_1} \cap \overline{L_2}$. Then, by Theorem 4.1, we have that the complement of a language is regular and so is the intersection, so it follows that $\overline{L_1}$ is regular, $\overline{L_2}$ is regular, and thus $\overline{L_1} \cap \overline{L_2}$ is regular. Therefore, $nor(L_1, L_2)$ is closed under the operation.

B) Complete problems 21 and 22 on page 119

21. The *tail* of a language is defined as the set of all suffixes of its strings, that is,

$$tail(L) = \{y : xy \in L \text{ for some } x \in \Sigma^*\}.$$

Show that if L is regular, so is $tail(L)$.

Answer. To begin, $tail(L) = \Sigma^* \setminus L$ (the *left quotient* of L with respect to Σ^*). Then, by Theorem 5.1 (from above) and 4.1 (from the textbook), Σ^* is regular. Next, since L is regular and Σ^* is regular, by Theorem 5.2 (from above), we have that $\Sigma^* \setminus L$ is regular. Therefore, $tail(L)$ is regular.

22. The head of a language is the set of all prefixes of its strings, that is,

$$head(L) = \{x : xy \in L \text{ for some } y \in \Sigma^*\}.$$

Show that the family of regular languages is closed under this operation.

Answer. To begin, $head(L) = L \setminus \Sigma^*$ (the *right quotient* of L with respect to Σ^*). Then, by Theorem 5.1 (from above) and 4.1 (from the textbook), Σ^* is regular. Next, since L is regular and Σ^* is regular, by Theorem 4.4 (from the textbook), we have that $L \setminus \Sigma^*$ is regular. Therefore, $head(L)$ is closed under the operation.