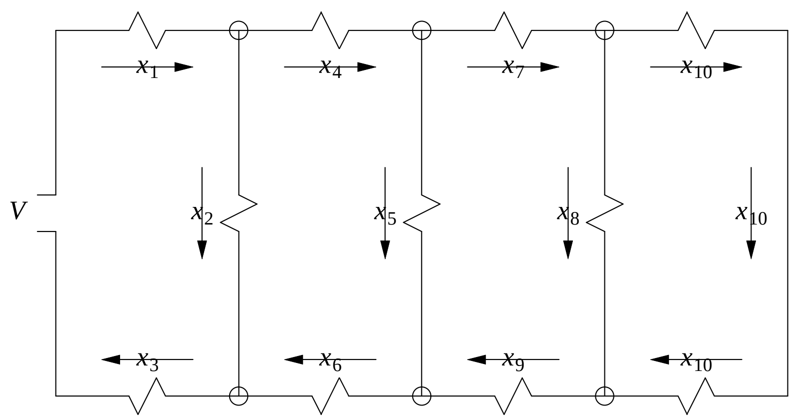
Please write your family and given names and **underline** your family name on the front page of your paper.

1. Consider an electrical circuit with n loops, and $2(n-1)$ nodes, positioned as in the picture below. (The picture is an example, for $n = 4$.) Note that the leftmost loop has 3 resistances and a voltage source, while the rightmost loop has only 3 resistances (with two of them corresponding to a common intensity). Every other loop in between has 4 resistances. The nodes are denoted by (small) circles. Each resistance is equal to 1Ω, while the voltage source is 100 Volts. The intensities (currents) at each wire are unknown. (Intensities are often denoted by i_*, but, here, we denote them by x_*.) Kirchhoff's current law states that the sum of intensities at each node (taking proper signs) must be zero. Kirchhoff's voltage law states that the sum of voltages (recall: for each wire, voltage = intensity · resistance) along each loop must be zero. Writing the equations arising from Kirchhoff's laws for all loops and nodes, we get a linear system of $N = 3n-2$ equations with respect to the $3n-2$ unknown intensities.



For the example figure, the equations are:
Left loop: $x_1 + x_2 + x_3 = V$
Top left node: $x_1 - x_2 - x_4 = 0$
Bot left node: $x_2 - x_3 + x_6 = 0$
Mid-left loop: $-x_2 + x_4 + x_5 + x_6 = 0$
Top mid node: $x_4 - x_5 - x_7 = 0$
Bot mid node: $x_5 - x_6 + x_9 = 0$
Mid-right loop: $-x_5 + x_7 + x_8 + x_9 = 0$
Top right node: $x_7 - x_8 - x_{10} = 0$
Bot right node: $x_8 - x_9 + x_{10} = 0$
Right loop: $-x_8 + 2x_{10} = 0$

Generalizing the above for n loops, the general loop equation is of the form $-x_{i-2} + x_i + x_{i+1} + x_{i+2} = 0$, $i = 4, 7, 10, \cdots, 3n-3$, the general top node equation is of the form $x_i - x_{i+1} - x_{i+3} = 0$, $i = 1, 4, 7, 10, \cdots, 3n-5$, the general bottom node equation is of the form $x_{i+1} - x_{i+2} + x_{i+5} = 0$, $i = 1, 4, 7, 10, \cdots, 3n-5$. Note that the leftmost and rightmost loops, as well as the rightmost bottom node do not follow exactly the general equations. Also note that the matrix of the system is very sparse; it has at most 4 non-zero entries per row, independently of the size of n.

(a) [*20 points*] Write a MATLAB script which, for $n = 4, 8, 16, 32$, generates the matrix and right-hand side vector of the linear system, then solves the linear system (using backslash). For each n, the script also calculates and outputs the maximum and minimum intensities, as well as the condition number of the matrix. After the loop of n, the script plots the top line intensities (x_i, $i = 1, 4, 7, \cdots, 3n-2$) versus their normalized (by the respective n) index, in one plot (four lines plotted).
Because the matrix A is sparse, we use sparse matrix techniques to generate it and store it. E.g `N = 3*n-2;`
`A = speye(N, N); A(1, 1:3) = [1  1 1]; A(2, 1:4) = [1 -1 0 -1];`
Typical loop equ: `A(k, k-2:k+2) = [-1 0 1  1 1];`
Typical top node equ: `A(k+1, k:k+3) = [ 1 -1 0 -1];`
Typical bot node equ: `A(k-1, k-2:k+2) = [ 1 -1 0 0 1];` (incl. left bottom) etc.
(In the above, you have find what values `k` takes.)
Note that you can visualize the sparsity pattern of a sparse matrix A by `spy(A)`.
To get (an estimate of) the condition number of a sparse matrix A, use `condest`.
If you have four vectors of n_i, $i = 1, \cdots, 4$, components respectively, stored as columns of a matrix `t`, to plot their components versus their normalized index use

```
plot([1:ni(1)]/ni(1), t(1:ni(1), 1), 'r-', ...
     [1:ni(2)]/ni(2), t(1:ni(2), 2), 'g--', ...
     [1:ni(3)]/ni(3), t(1:ni(3), 3), 'b-.', ...
     [1:ni(4)]/ni(4), t(1:ni(4), 4), 'k.');
```

For the case $n = 8$, also calculate the LU factorization (using the `lu` function in matlab) of A, and plot (using `spy`), the sparsity patterns of A, L and U. *You must keep an ordering of the equations and unknowns as in the example*, otherwise, the sparsity patterns will not be easy to study. Starting from the left and proceeding to the right, for the equations go: loop, top node, bottom node, loop, top node, bottom node, etc; for unknowns, for each loop, go clockwise.

(b) [*20 points*] What are the lower, upper and total bandwidths of A for any n? Explain. What is the (exact) number of nonzero entries in A in terms of n? Explain.
What can you say about the permutation matrix P (for any n) arising from the LU factorization with partial pivoting? How can you computationally verify the form of P? (You are welcome to use `normest` and `speye` in matlab.)

What are the lower, upper and total bandwidths of L and what of U for any n? Explain. What is the (exact) number of nonzero entries in L in terms of n, and what in U? Explain. (Count the main diagonal entries in both L and U.)
Based on the numerical results, how does the condition number of A behave approximately in terms of n?
Based on the numerical results, how do the values of the intensities of the top line behave as we go from left to right? Do the maximum and minimum values depend on n?

Notes: `speye(A)`, `normest(A)` and `condest(A)` are the sparse matrix equivalent MATLAB functions to `eye(A)`, `norm(A)` and `cond(A)`, respectively.

2. [*20 points*] Consider the linear system

$$0.03x_1 + 58.9x_2 = 59.2$$

$$5.31x_1 - 6.1x_2 = 47.0$$

Solve the system using Gauss elimination and applying 3-decimal-digits floating-point arithmetic with rounding. The results of *each* operation (addition, multiplication, division) of GE must be stored using 3-decimal-digits floating-point representation, before proceeding to the next operation. (Thus, the elimination is carried out as $a_{ij} = fl(a_{ij} - fl(fl(\frac{a_{ik}}{a_{kk}})a_{kj}))$, assuming all past entries are already stored in 3-decimal-digits floating-point representation.) Do this three times: (a) without pivoting, (b) with partial pivoting, (c) with complete pivoting.
What is the relative error for x in the infinity norm in each of the cases? Exact solution is $(10, 1)^T$.

3. Let A be a square matrix, with $||A|| < 1$ for some matrix norm $|| \cdot ||$. Show that

(a) [*5 points*] $(\mathbb{I} - A)^{-1}$ exists (where $\mathbb{I}$ is the identity matrix of the same size as A)

(b) [*10 points*] $||(\mathbb{I} - A)^{-1}|| \leq \dfrac{1}{1 - ||A||}$

(c) [*10 points*] $||(\mathbb{I} - A)^{-1} - (\mathbb{I} + A)|| \leq \dfrac{||A||^2}{1 - ||A||}$

(Note: (c) indicates that $(\mathbb{I} - A)^{-1}$ can be approximated by $\mathbb{I} + A$, with accuracy $O(||A||^2)$.)

4.

(a) [*7 points*] Let $|| \cdot ||_p$ denote a p-norm (or induced or Hölder norm) of a matrix, and let $\kappa_p(\cdot)$ be the respective condition number of the matrix. Show that, for any two invertible $n \times n$ matrices A and B, $\kappa_p(AB) \leq \kappa_p(A)\kappa_p(B)$.

(b) [*8 points*] Show that, if A is an $n \times n$ (row) diagonally dominant matrix, then $||A||_\infty \leq 2 \max_{i=1,\cdots,n} \{|a_{ii}|\}$, where $|| \cdot ||_\infty$ is the infinity norm (or max-norm or row-norm) of a matrix.

Help: An $n \times n$ matrix A is called (row) **diagonally dominant** if $|a_{ii}| \geq \sum_{j=1\ j \neq i}^{n} |a_{ij}|$ for all $i = 1, \cdots, n$.