Measure Theory: 6th hw

Due on February 12, 2014 at 3:10pm

Chapter 7

Problem 7

Let $\mu = \sum_1^n a_i\mu_i$. It suffices to show μ satisfies the definition of a measure

1. $\mu(\varnothing) = \sum_1^n a_i \cdot 0 = 0$
2. let $\{E_i\}_1^\infty \subset \mathcal{M}$ be a collection of disjoint sets. It's evident that $\mu(E_j) \geq 0$ for all j

$$\begin{aligned}\mu\left(\bigcup_1^\infty E_j\right) &= \sum_{i=1}^n \sum_{j=1}^\infty a_i\mu_i(E_j) \\ &= \sum_{j=1}^\infty \sum_{i=1}^n a_i\mu_i(E_j) \qquad (1) \\ &= \sum_{j=1}^\infty \mu(E_j)\end{aligned}$$

(1) was just because all the terms were positive

Problem 8

For all $i \geq n$ for all $N \in \mathbb{N}$ we have

$$\bigcap_{i=1}^n \bigcup_{j=i}^\infty E_j \subseteq E_i$$

which means

$$\begin{aligned}\mu\left(\bigcap_{i=1}^n \bigcup_{j=i}^\infty E_j\right) &\leq \mu(E_i) \\ \mu\left(\bigcap_{i=1}^n \bigcup_{j=i}^\infty E_j\right) &\leq \inf\{\mu(E_i)\}_{i=N}^\infty \\ \lim_{n\to\infty} \mu\left(\bigcap_{i=1}^n \bigcup_{j=i}^\infty E_j\right) &\leq \lim_{n\to\infty} \inf\{\mu(E_i)\}_{i=n}^\infty \\ \mu(\liminf E_i) &\leq \liminf \mu(E_i)\end{aligned}$$

Problem 9

$$\mu(E) + \mu(F) = \mu(E \setminus F) + \mu(F) + \mu(E \cap F) = \mu(E \cup F) + \mu(E \cap F)$$

Problem 10

It suffices to confirm that μ_A satisfies the definition of a measure:

1. $\mu_E(\varnothing) = \mu(E \cap \varnothing) = \mu(\varnothing) = 0$

Problem 10 continued on next page...

2. let $\{E_i\}_1^\infty \subseteq \mathcal{M}$ be a collection of disjoint sets

$$\begin{aligned}\mu_E\left(\bigcup_{i=1}^{\infty} E_i\right) &= \mu\left(E \cap \bigcup_{i=1}^{\infty} E_i\right) \\ &= \mu\left(\bigcup_{i=1}^{\infty} E \cap E_i\right) \\ &= \sum_{i=1}^{\infty} \mu(E \cap E_i) \\ &= \sum_{i=1}^{\infty} \mu_E(E_i)\end{aligned}$$

Problem 11

it suffices to prove the ($\Leftarrow$) direction for both questions. Let $\{E_i\}_1^\infty \subseteq \mathcal{M}$ be a collection of disjoint sets

1. Suppose μ is continuous from below: take $F_n = \bigcup_{i=1}^n E_i$ that means $\{F_i\}_1^\infty$ is an increasing sequences of sets. We then have

$$\begin{aligned}\mu\left(\bigcup_{i=1}^{\infty} E_i\right) &= \mu\left(\bigcup_{i=1}^{\infty} F_i\right) \\ &= \lim_{n\to\infty} \mu(F_n) \\ &= \lim_{n\to\infty} \mu\left(\bigcup_{i=1}^{n} E_i\right) \\ &= \lim_{n\to\infty} \sum_{i=1}^{n} \mu(E_i) \\ &= \sum_{i=1}^{\infty} \mu(E_i)\end{aligned}$$

2. Suppose μ is continuous from above: take $F_n = \bigcap_{i=1}^n E_i^c$ that means $\{F_i\}_1^\infty$ is a decreasing sequences of sets. We then have

$$\begin{aligned}\mu\left(\bigcup_{i=1}^{\infty} E_i\right) &= \mu(X) - \mu\left(\bigcap_{i=1}^{\infty} E_i\right) \\ &= \mu(X) - \mu\left(\bigcap_{i=1}^{\infty} F_i\right) \\ &= \mu(X) - \lim_{n\to\infty} \mu(F_i) \\ &= \lim_{n\to\infty} \mu(X) - \mu\left(\bigcap_{i=1}^{n} E_i^c\right)\end{aligned}$$

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$$
\begin{aligned}
&= \lim_{n\to\infty} \mu\left(\bigcup_{i=1}^{n} E_i\right) \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \mu(E_i) \\
&= \sum_{i=1}^{\infty} \mu(E_i)
\end{aligned}
$$

Problem 12

a. suppose without loss of generality that $\mu(E) > \mu(F)$. we then have

$$\mu(E \cup F) \geq \mu(E)$$

and

$$\mu(E \cap F) \leq \mu(F)$$

but we have

$$
\begin{aligned}
\mu(E \triangle F) &= \mu(E \cup F) - \mu(E \cap F) \\
&\geq \mu(E) - \mu(F) \\
&> 0
\end{aligned}
$$

b. we know that $E \triangle G \subseteq E \triangle F \cup F \triangle G$ so we have

$$0 \leq \mu(E \triangle G) \leq \mu(E \triangle F \cup F \triangle G) \leq \mu(E \triangle F) + \mu(F \triangle G) \leq 0$$

which means $\sim$ is transitive and it's trivially symmetric and reflexive which makes it an equivalence relation.

c. same as before, we know that $E \triangle G \subseteq E \triangle F \cup F \triangle G$ so that means

$$\rho(E, G) = \mu(E \triangle F) \leq \mu(E \triangle F) + \mu(F \triangle G) = \rho(E, F) + \rho(F, G)$$