$$
\begin{aligned}
P(X \text{ is rational}) &= \int_{\mathbb{Q}\cap[0,1]} 1 \, \mathrm{d}x \\
&= \int_0^1 f(x) \, \mathrm{d}x
\end{aligned}
$$

$$
f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}
$$

$P = \{x_1, x_2, \ldots, x_n\} \subset [0,1]$

$$
\begin{aligned}
&\sup_{x\in[x_i,x_{i+1}]} f(x) = 1 \\
&\inf_{x\in[x_i,x_{i+1}]} f(x) = 0 \\
&\overline{\int_0^1} f(x) \, \mathrm{d}x = 1 \neq 0 = \underline{\int_0^1} f(x) \, \mathrm{d}x \\
&\underline{\int_0^1} f(x) \, \mathrm{d}x = 0 \qquad (1)
\end{aligned}
$$

if $P(X \text{ is rational}) > 0$

then $P(X \text{ is real}) \to \infty$

$\lim_{n\to\infty} \int f_n(x) \, \mathrm{d}\mu = \int \lim_{n\to\infty} f_n(x) \, \mathrm{d}\mu$

$\mu \colon \mathcal{P}(\mathbb{R}^n) \longrightarrow [0, +\infty]$

$\mu(E) =$ the ”volume” of E

1. for a countable collection of disjoint sets $\{E_\alpha\}_{\alpha\in A}$

$$
\mu\left(\bigcup_{\alpha\in A} E_\alpha\right) = \sum_{\alpha\in A} \mu(E_\alpha)
$$

2. congruent sets have the same measure
3. $\mu([0,1]^n) = 1$

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$\mu(E_1 \cup E_2 \cup E_3) = \mu(E_1) + \mu(E_2) + \mu(E_3)$

$\mu(E) = \mu(F)$

$\mu([0,1]^n) = 1$