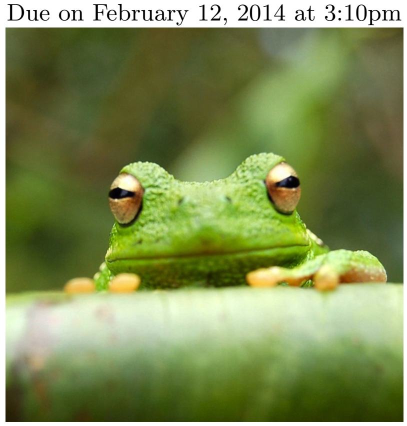
Folland Reading Group: 2nd week hw

Due on February 12, 2014 at 3:10pm



Chapter 1

Problem 1

a. let $\mu^*(E) = \inf\left\{ \sum_{i=1}^{\infty} \mu_{|\mathcal{A}}(A_i) \middle| E \subseteq \bigcup A_i \right\}$ be the outer measure induced by $\mathcal{A}$ and the premeasure $\mu_{|\mathcal{A}}$. Then due to proposition 1.14, $\mu^*_{|\mathcal{M}} = \mu$. We could then find for each $M \in \mathcal{M}$ and $\varepsilon > 0$ a sequence $\{A_i\}_{i=1}^{\infty} \subset \mathcal{A}$ such that $A = \bigcup_{i=1}^{\infty} A_i \supset M$ and

$$\mu_{|\mathcal{A}}(A) < \mu^*(M) + \varepsilon = \mu(M) + \varepsilon$$

and since M is a subset of A, $\mu(M \triangle A) = \mu(A) - \mu(M) < \varepsilon$ and by the finiteness of μ, the series $\sum_{i=1}^{\infty} \mu(A_i)$ is convergent which implies there exists an $N \in \mathbb{N}$ such that

$$\sum_{i=N+1}^{\infty} A_i < \varepsilon$$

which means

$$\mu\left(A \triangle \bigcup_{i=1}^{N} A_i\right) = \mu(A) - \mu\left(\bigcup_{i=1}^{N} A_i\right) \leq \sum_{i=N+1}^{\infty} \mu(A_i) < \varepsilon$$

and now we have

$$\mu\left(M \triangle \bigcup_{i=1}^{N} A_i\right) \leq \mu(M \triangle A) + \mu\left(A \triangle \bigcup_{i=1}^{N} A_i\right) < 2\varepsilon$$

moreover, $\bigcup_{i=1}^{N} A_i \in \mathcal{A}$ which gives us what we want.

Corollary 1.1. if $(X, \mathcal{M}, \mu)$ is a σ-finite measure space, $\mathcal{A}$ is an algebra that generates $\mathcal{M}$ and E is a subset with finite measure then for each measurable subset F of E and $\varepsilon > 0$ there exists a set $A \in \mathcal{A}$ such that $\mu(F \triangle A) < \varepsilon$

Trivial Proof. consider $(E, \mathcal{M}_E, \mu_E)$, where $\mathcal{M}_E$ is the restriction of $\mathcal{M}$ to E and $\mu_E = \mu_{M_E}$, which is a finite measure space and $\mathcal{P}(E) \cap \mathcal{A}$ obviously generates $\mathcal{M}_E$.

enuf said :)))) □

b. consider the algebra $\mathcal{A} \subset \mathcal{P}(\mathbb{R})$ of bounded and co-bounded sets generated by bounded open sets. Since every open set in $\mathbb{R}$ is countable union of bounded open sets:

$$O = O \cap \mathbb{R} = O \cap \Big(\bigcup_{i=0}^{\infty} (-i-1, -i+1) \cup (i-1, i+1) \Big) = \bigcup_{i=0}^{\infty} \Big(O \cap \big((-i-1, -i+1) \cup (i-1, i+1) \big) \Big)$$

that means $\mathcal{A}$ generates $\mathcal{B}_{\mathbb{R}}$ but any member of $\mathcal{A}$ obviously can't approximate, say, $[0, \infty)$ so we're going to restrict ourselves to sets with finite measure.

Now suppose $(X, \mathcal{M}, \mu)$ is a σ-finite measure space, $\mathcal{A}$ is an algebra that generates $\mathcal{M}$ and E is a subset with finite measure and fix $\varepsilon > 0$. There must exist a disjoint sequence $\{E_i\}_{i=1}^{\infty}$ where $\bigcup E_i = X$ and $\mu(E_i)$ is finite for all i. That means

$$\mu(E) = \mu\Big(E \cap \bigcup_{i=1}^{\infty} E_i\Big) = \sum_{i=1}^{\infty} \mu(E \cap E_i)$$

Problem 1 continued on next page...

and since $\mu(E)$ is finite there must exist some $N \in \mathbb{N}$ such that

$$\sum_{i=N+1}^{\infty} \mu(E \cap E_i) < \varepsilon$$

and for each E_i where $1 \leq i \leq N$ we can find an $A_i \in \mathcal{A}$ such that $\mu\big(A_i \triangle (E \cap E_i)\big) < 2^{-i}\varepsilon$

$$\begin{aligned}
\mu\left(\bigcup_{i=0}^{N} A_i \triangle E\right) &= \mu\left(\bigcup_{i=0}^{N} A_i \triangle \bigcup_{i=0}^{\infty}(E \cap E_i)\right) \\
&\leq \mu\left(\bigcup_{i=0}^{N} A_i \triangle \bigcup_{i=0}^{N}(E \cap E_i)\right) + \varepsilon \\
&\leq \sum_{i=0}^{N} \mu(A_i \triangle (E \cap E_i)) + \varepsilon \\
&\leq 2\varepsilon
\end{aligned}$$

c. Suppose $E \in \mathcal{L}$ and $m(E) < \infty$ and let $\varepsilon > 0$, because of outer regularity we can find an open set $O \supset E$ such that $m(O) < m(E) + \varepsilon$. from that it follows

$$\begin{aligned}
m(O \triangle E) &= m(O \cup E) - m(O \cap E) \\
&= m(O) - m(E) \\
&< \varepsilon
\end{aligned}$$

d. suppose $\mu(E) < \infty$ and $\mu(E \cap I) \leq \alpha\mu(I)$ for all intervals I. Put $\varepsilon = \dfrac{(1-\alpha)\mu(E)}{\alpha}$. There exists a sequence of a open intervals $\{I_i\}_{i=0}^{\infty}$ such that $E \subseteq \bigcup I_i$ and

$$\mu\Big(\bigcup I_i\Big) < \mu(E) + \varepsilon$$

but we have

$$\begin{aligned}
\mu(E) &= \mu\Big(\bigcup (E \cap I_i)\Big) \\
&\leq \sum_{i=0}^{\infty} \mu(E \cap I_i) \\
&\leq \alpha \sum_{i=0}^{\infty} \mu(I_i) \\
&< \alpha\Big(\mu(E) + \varepsilon\Big)
\end{aligned}$$

which is a contradiction. Now, if $\mu(E) = \infty$, since $\mathbb{R}$ is σ-finite, it must have a measurable subset E' such that $0 < \mu(E') < \infty$ for which for all $\alpha < 1$ we can find an interval I such that $\mu(E' \cap I) > \alpha \cdot \mu(I)$ but $\mu(E \cap I) \geq \mu(E' \cap I)$ so we're done.

Problem 2

a. follows immediately from caratheodry (see 1.11, Folland).

b. first we note the following lemma

Problem 2 continued on next page...

Lemma 2.1. $|\mathcal{B}_{\mathbb{R}}| = |\mathbb{R}|$

Trivial Proof. Gomez said it's true so it must be. □

and since $|\mathcal{B}_{\mathbb{R}}| = |\mathbb{R}| \prec |\mathcal{P}(\mathbb{R})| = |\mathcal{P}(C)|$ (where C is the cantor set, which is uncountable), there must exist subsets of C that aren't in $\mathcal{B}_{\mathbb{R}}$ and since $\mu(C) = 0$ that means μ isn't complete.

c. It should be obvious that the completion of $(X, \mathcal{B}_{\mathbb{R}}, \mu)$ is the smallest complete space containing it. So it suffices to prove $\mathcal{L} \subseteq \mathcal{B}_{\mathbb{R}}$.

Lemma 2.2. lebesgue measurable sets with 0 measure are subsets of borel sets with 0 measure.

Trivial Proof. Let $E \in \mathcal{L}$ and $m(E) = 0$. Because of outer regularity, we can find an open set $O_i \supset E$ such that $m(O_i) < m(E) + 2^{-i} = 2^{-i}$ for all $i \in \mathbb{N}$. Which means $E \subseteq \bigcap_{i=0}^{\infty} O_i \in \mathcal{B}_{\mathbb{R}}$ and $m\left(\bigcap_{i=0}^{\infty} O_i\right) = 0$. □

now any set $L \in \mathcal{L}$ can be decomposed as $L = H \cup N$ where $H \in F_\sigma \subset \mathcal{B}_{\mathbb{R}}$ and $m(N) = 0$ (see 1.19, Folland) and, because of lemma 2.2, we're done.

Problem 3

a. first we construct a μ for 2 spaces and then use induction.

Lemma 3.1. the set $\mathcal{E} = \{E_1 \times E_2 | E_1 \in \mathcal{M}_1, E_2 \in \mathcal{M}_2\}$ is an elementary family

Trivial Proof. it's obvious that $\varnothing \in \mathcal{E}$ and if $E, F \in \mathcal{E}$ then $E \cap F \in \mathcal{E}$ and $(E_1 \times E_2)^c = (E_1^c \times X_2) \cup (E_1 \times E_2^c)$ □

now define the premeasure $\mu_0\big((E_1 \times E_2)\big) = \mu_1(E_1) \cdot \mu_2(E_2)$ using that we define the outer measure

$$\mu^*(E) = \inf\left\{\sum_{1}^{\infty} \mu_0(E_i) | E_i \in \mathcal{E} \text{ and } E \subseteq \bigcup E_i\right\}$$

and the restriction of that to $\mathcal{M}$ is a measure.