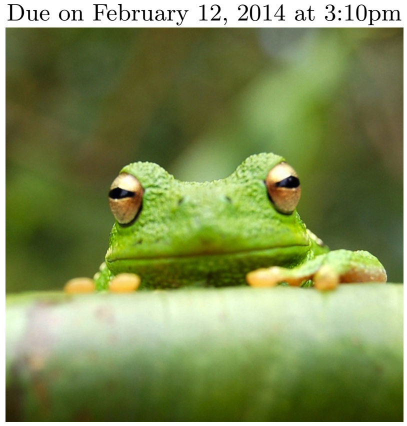
Measure Theory: 5th week hw

Due on February 12, 2014 at 3:10pm



Chapter 3

Problem 1

a. Suppose T is bounded, that means

$$\|T(x+h) - T(x)\|_{\mathfrak{Y}} = \|T(h)\|_{\mathfrak{Y}} \leq c\|h\|_{\mathfrak{X}}$$

so that as $h \to 0$, $\|T(x+h) - T(x)\|_{\mathfrak{Y}} \to 0$ which proves T is continuous.

Now suppose T is continuous, let $\delta > 0$ be such that for all $\|h\|_{\mathfrak{X}} < \delta$

$$\|T(h)\|_{\mathfrak{Y}} < 1$$

and let $0 < c < \delta$, that would mean for all x, $\left\| \frac{cx}{\|x\|_{\mathfrak{X}}} \right\|_{\mathfrak{X}} < \delta$ which means

$$\left\| T\left(\frac{cx}{\|x\|_{\mathfrak{X}}} \right) \right\|_{\mathfrak{Y}} < 1$$

and hence

$$\|T(x)\|_{\mathfrak{Y}} = \frac{\|x\|_{\mathfrak{X}}}{c} \left\| T\left(\frac{cx}{\|x\|_{\mathfrak{X}}} \right) \right\|_{\mathfrak{Y}} \leq \frac{1}{c}\|x\|_{\mathfrak{X}}$$

b. we can define $\|-\|_{\text{op}}$ as

$$\|T\|_{\text{op}} = \inf\{c \,:\, \|T(x)\|_{\mathfrak{Y}} < c\|x\|_{\mathfrak{X}},\, x \in \mathfrak{X}\}$$

which is clearly a norm.

Now, to prove $\mathfrak{X}^*$, suppose $\{T_n\}_0^\infty$ is a cauchy sequence in $\mathfrak{X}^*$, then for all $x \in \mathfrak{X}$ the sequence $\{T_n(x)\}_0^\infty$ is a cuachy sequence since

$$|T_n(x) - T_m(x)| \leq \|T_n - T_m\|_{\text{op}}\|x\|_{\mathfrak{X}}$$

and the RHS goes to 0 as $n, m \to \infty$ and since the $\mathfrak{X}$ is a banach space we can define T such that

$$T(x) = \lim_{n\to\infty} T_n(x)$$

which is clearly a linear operator and it's bounded because

$$|T_n(x)| \leq |T_n(x) - T_N(x)| + |T_N(x)| \leq \|x\|_{\mathfrak{X}} + |T_N(x)| \leq (1 + \|T_N\|_{\text{op}})\|x\|_{\mathfrak{X}}$$

for all $n > N$ for some $N \in \mathbb{N}$ and hence, by letting $n \to \infty$, we find that T is bounded.

Problem 2

a. It's obvious that $M(X)$ is a complex vector space and that $\|-\|_{TV}$ is a norm so it suffices to prove that $M(X)$ is complete under the metric the norm induces.

Let $\{\lambda_n\}_{i=0}^\infty$ is a cauchy sequence in $M(X)$. First, we note that $\{\lambda_n(E)\}_{i=0}^\infty$ is cauchy for all $E \in \mathcal{M}$ since

$$|\lambda_m(E) - \lambda_n(E)| \leq |\lambda_m - \lambda_n|(E) \leq |\lambda_m - \lambda_n|(X) < \varepsilon$$

for some $m > n > N \in \mathbb{N}$ for all $\varepsilon > 0$, and hence we define $\lambda : \mathcal{M} \longrightarrow \mathbb{C}$ as

$$\lambda(E) = \lim_{n\to\infty} \lambda_n(E)$$

and we are going to prove that λ is a complex measure: it's obvious that

$$\lambda(\varnothing) = \lim_{n\to\infty} \lambda_n(\varnothing) = 0$$

and it's obvious that λ is finitely additive. Now let $\varepsilon > 0$ and $\{E_n\}_{n=0}^{\infty}$ is a sequence of disjoint sets of $\mathcal{M}$. Choose $N_1 \in \mathbb{N}$ such that for all $n \geq N_1$

$$|\lambda_n - \lambda_{N_1}|(X) < \varepsilon$$

we know that $|\lambda_n(E)| \to |\lambda(E)|$ for all $E \in \mathcal{M}$ and by using Fatou's lemma on ℓ^1 we can see that

$$\begin{aligned}\sum_{i=0}^{\infty}|\lambda(E_i)| &\leq \liminf_{n\to\infty}\sum_{i=0}^{\infty}|\lambda_n(E_i)| \\ &\leq \liminf_{n\to\infty}\sum_{i=0}^{\infty}|\lambda_n|(E_i) \\ &= \lim_{n\to\infty}|\lambda_n|\left(\bigcup_{i=0}^{\infty}E_i\right) \\ &\leq \lim_{n\to\infty}|\lambda_n|(X)\end{aligned}$$

and the last limit exists since $\{|\lambda_n|(X)\}_{n=0}^{\infty}$ is cauchy and hence $\sum_{i=0}^{\infty}\lambda(E_i)$ converges absolutely. We can then find an $M_1 \in \mathbb{N}$ such that for all $m \geq M_1$ we have

$$\left|\sum_{i=m+1}^{\infty}\lambda(E_i)\right| < \varepsilon \tag{1}$$

and since $\sum_{i=0}^{\infty}\lambda_{N_1}(E_i) = \lambda_{N_1}\left(\bigcup_0^{\infty}E_i\right)$ converges we can find an $M_2 \in \mathbb{N}$ such that for all $m \geq M_2$

$$\left|\sum_{i=m+1}^{\infty}\lambda_{N_1}(E_i)\right| < \varepsilon \tag{2}$$

and put $M = \max(M_1, M_2)$. Now by the finite additivity we can find an $N_2 \in \mathbb{N}$ such that for all $n > N_2$

$$\left|\sum_{i=0}^{M}\lambda(E_i) - \sum_{i=0}^{M}\lambda_n(E_i)\right| < \varepsilon \tag{3}$$

put $N = \max(N_1, N_2)$ now for all $n > N$ we can see that

$$\begin{aligned}\left|\sum_{i=M+1}^{\infty}\lambda_n(E_i)\right| - \left|\sum_{i=M+1}^{\infty}\lambda_{N_1}(E_i)\right| &\leq \left|\sum_{i=M+1}^{\infty}(\lambda_n - \lambda_{N_1})(E_i)\right| \\ &= \left|(\lambda_n - \lambda_{N_1})\left(\bigcup_{i=M+1}^{\infty}E_i\right)\right| \\ &\leq |\lambda_n - \lambda_{N_1}|\left(\bigcup_{i=M+1}^{\infty}E_i\right) \\ &\leq |\lambda_n - \lambda_{N_1}|(X) < \varepsilon\end{aligned}$$

and from (2) we can see that

$$\left|\sum_{i=M+1}^{\infty}\lambda_n(E_i)\right| < \varepsilon + \left|\sum_{i=M+1}^{\infty}\lambda_{N_1}(E_i)\right| < 2\varepsilon \tag{4}$$

and finally, from (1), (3) and (4), we get

$$\begin{aligned}\left|\sum_{i=0}^{\infty}\lambda(E_i)-\lambda_n\left(\bigcup_{i=0}^{\infty}E_i\right)\right| &= \left|\sum_{i=0}^{\infty}\lambda(E_i)-\sum_{i=0}^{\infty}\lambda_n(E_i)\right| \\ &\leq \left|\sum_{i=0}^{M}\lambda(E_i)-\sum_{i=0}^{M}\lambda_n(E_i)\right| + \left|\sum_{i=M+1}^{\infty}\lambda(E_i)\right| + \left|\sum_{i=M+1}^{\infty}\lambda_n(E_i)\right| \\ &< 4\varepsilon\end{aligned}$$

which proves that

$$\sum_{i=0}^{\infty}\lambda(E_i)=\lim_{n\to\infty}\lambda_n\left(\bigcup_{i=0}^{\infty}E_i\right)=\lambda\left(\bigcup_{i=0}^{\infty}E_i\right)$$

and hence λ is a complex measure. And from that it is easy to see that $\lambda_n\to\lambda$ in $\|-\|_{\mathrm{TV}}$ since

$$|\lambda-\lambda_n|(X)=|\mathfrak{Re}(\lambda-\lambda_n)+\mathfrak{Im}(|\lambda-\lambda_n|)|(X)\leq|\mathfrak{Re}(\lambda-\lambda_n)|(X)+|\mathfrak{Im}(|\lambda-\lambda_n|)|(X)$$

and the latter goes to 0 as $n\to\infty$

b. Call this transformation T. It's obvious that

$$\mathrm{d}T(af+bg)=(af+bg)\,\mathrm{d}\mu=a(f\,\mathrm{d}\mu)+b(g\,\mathrm{d}\mu)=a\big(\mathrm{d}T(f)\big)+b\big(\mathrm{d}T(g)\big)$$

for all $f,g\in L^1(\mu)$ and $a,b\in\mathbb{R}$. And since $\mathrm{d}|T(f)|=|f|\,\mathrm{d}\mu$ we can see that

$$\|f\|_1=\int_X|f|\,\mathrm{d}\mu=|T(f)|(X)=\|T(f)\|_{\mathrm{TV}}$$

and, finally, to prove it's an embedding, suppose $T(f)=T(g)$, that means

$$\int_E f\,\mathrm{d}\mu=\int_E g\,\mathrm{d}\mu$$

for all $E\in\mathcal{M}$ which means $f=g$ due to proposition 2.23.b.

Problem 3

a. Due to the concavity of log

$$\log\left(\frac{1}{p}a^p+\frac{1}{q}b^q\right)=\log\left(\frac{1}{p}a^p+\left(1-\frac{1}{p}\right)b^q\right)\geq\frac{1}{p}\log(a^p)+\frac{1}{q}\log(b^q)=\log(ab)$$

the inequality follows immediately.

b. If the inequality holds for f and g then it should hold for af and bg as well and hence we can assume WOLG that $\|f\|_p=\|g\|_q=1$. Now using young's inequality we get

$$|fg|\leq\frac{f^p}{q}+\frac{g^q}{q}$$

from which

$$\int_X|fg|\,\mathrm{d}\mu\leq\frac{(\|f\|_p)^p}{p}+\frac{(\|g\|_q)^q}{q}=1=\|f\|_p\|g\|_q$$

and hence we are done.

Problem 3 continued on next page. . .

To prove minkowski first we need to see that $q = \frac{p}{p-1}$ and that $|f+g|^p \leq (|f|+|g|)|f+g|^{p-1}$ hence

$$\begin{aligned} \|f+g\|_p^p &= \||f+g|^p\|_1 \\ &\leq \left\||f|(f+g)^{p-1}\right\|_1 + \left\||g|(f+g)^{p-1}\right\|_1 \\ &\leq (\|f\|_p + \|g\|_p)\left\|(f+g)^{p-1}\right\|_q \\ &= (\|f\|_p + \|g\|_p)\|f+g\|_p^{p-1} \end{aligned}$$

from which the inequality follows immediately.

c. It's obvious that this mapping is well defined since integrals are invariant under change on zero-measure sets so if $f_1 = f_2$ a.e. then $T(f_1) = T(f_2)$ and $T(f)$ does define a bounded linear operator on $L^p(X, \mathbb{R})$ since

$$(T(f))(ag_1 + bg_2) = \int_X f(ag_1 + bg_2)\,\mathrm{d}\mu = a\int_X fg_1\,\mathrm{d}\mu + b\int_X fg_2\,\mathrm{d}\mu = a\cdot(T(f))(g_1) + b\cdot(T(f))(g_2)$$

and due to Hölder's inequality,

$$\|(T(f))(g)\|_{\mathbb{R}} = \left|\int_X fg\,\mathrm{d}\mu\right| \leq \|f\|_q\|g\|_p = c\|g\|_p$$

where $c = \|f\|_q$ is a constant.

To prove it's an isometric mapping it's suffices to see that $\|T(f)\|_{\mathrm{op}} = \|f\|_q$ due to Hölder's inequality and realizing that the equality holds whenver $f^p = g^q$. That is, the inequality

$$\|T(f)(g)\|_{\mathbb{R}} \leq c\|g\|_p$$

doesn't always hold whenver $c < \|f\|_q$.

Finally, to prove that T is an embedding, suppose that $(T(f_1))(g) = (T(f_2))(g)$ for all $g \in L^p(X, \mathbb{R})$, that means, for all measurable sets E,

$$\int_E f_1\,\mathrm{d}\mu = \int_X f_1\chi_E\,\mathrm{d}\mu = \int_X f_2\chi_E\,\mathrm{d}\mu = \int_E f_2\,\mathrm{d}\mu$$

which proves that $f_1 = f_2$ due to proposition 2.23.b.

d. First we prove that λ is a signed measure: It's obvious that $\lambda(\varnothing) = 0$. Now let $\{A_n\}_0^\infty$ be a sequence of disjoint measurable sets, $A = \bigcup_0^\infty A_n$, and $\varepsilon > 0$

case #1: $\lambda(A_n) \leq \lambda(A) < \infty$: Let $E_n \subseteq A_n$ be a measurable set such that

$$\Lambda(E_n) > \lambda(A_n) - \frac{\varepsilon}{2^n}$$

so that $\bigcup E_n \subset \bigcup A_n$ and

$$\begin{aligned} \lambda(A) &\geq \Lambda\left(\chi_{\bigcup_{n=0}^\infty E_n}\right) \\ &= \sum_{n=0}^\infty \Lambda(\chi_{E_n}) \\ &> \sum_{n=0}^\infty \lambda(A_n) - \varepsilon \end{aligned}$$

from which we can conclude that

$$\lambda(A) \geq \sum_{n=0}^\infty \lambda(A_n) \tag{1}$$

now let $E \subseteq \bigcup_{n=0}^{\infty} A_n$ such that

$$\lambda(A) - \varepsilon < \Lambda(\chi_E)$$

but then we have

$$\begin{aligned}\lambda(A) - \varepsilon &< \Lambda(\chi_E) \\ &= \Lambda(\chi_{\bigcup_0^\infty (E\cap A_n)}) \\ &= \sum_{n=0}^{\infty} \Lambda(\chi_{E\cap A_n}) \\ &\leq \sum_{n=0}^{\infty} \lambda(A_n)\end{aligned}$$

from which

$$\lambda(A) \leq \sum_{n=0}^{\infty} \lambda(A_n) \tag{2}$$

and from (1) and (2) we can see that

$$\lambda(A) = \sum_{n=0}^{\infty} \lambda(A_n)$$

case #2: $\lambda(A_n) = \infty$ for some n: since

$$\{\Lambda(\chi_E) | E \subset A_n, \mu(E) < \infty\} \subset \{\Lambda(\chi_E) | E \subset A, \mu(E) < \infty\}$$

$\lambda(A)$ must be infinite as well.

case #3: $\lambda(A) = \infty$: there must exist a sequence of E_n of measureable sets such that $\Lambda(\chi_{E_i}) \to \infty$ as $i \to \infty$ then for all i we see that

$$\Lambda(\chi_{E_i}) = \sum_{n=0}^{\infty} \Lambda(\chi_{E_i \cap A_n}) \leq \sum_{n=0}^{\infty} \lambda(A_n)$$

and hence λ is a measure.

Now to prove that $\Lambda(f) = \int_X f \,\mathrm{d}\mu$ we'll be satsfied with proving the statement for when f is a simple function and the general case follows from the usual reasoning: First suppose that μ is finite and that $E \subseteq A$ which means

$$\Lambda(\chi_A) = \Lambda(\chi_E) + \Lambda(\chi_{A\setminus E}) \geq \Lambda(E)$$

and hence

$$\int_X \chi_A \,\mathrm{d}\lambda = \lambda(A) = \sup\{\Lambda(\chi_E) | E \subseteq A\} = \Lambda(\chi_A)$$

and to expand this to the case where μ is σ-finite let $\{E_n\}_0^\infty$ is a sequence of finite disjoint measurable sets we then have

$$\int_X \chi_A \,\mathrm{d}\lambda = \sum_{n=0}^{\infty} \int_{E_n} \chi_A \,\mathrm{d}\lambda = \sum_{n=0}^{\infty} \int_{E_n} \chi_{E_n\cap A} \,\mathrm{d}\lambda = \sum_{n=0}^{\infty} \Lambda(\chi_{E_n\cap A}) = \Lambda(\chi_A)$$

Now if $f \in L^p(\mu)$ then $\int_X f \,\mathrm{d}\lambda = \Lambda(f) < \infty$ which means $\int_X |f| \,\mathrm{d}\lambda < \infty$ and hence $L^p(\mu) \subseteq L^1(\lambda)$

e. suppose E is a μ-zero set, that means

$$\Lambda(\chi_E) \leq \|\Lambda\|_{\text{op}} \|\chi_E\|_p = 0$$

hence $\lambda(E) = 0$ therefore $\lambda \ll \mu$ and due to the Radon-Nikodym there exists a g such that $\mathrm{d}\lambda = g \, \mathrm{d}\mu$ and

$$\int_X fg \, \mathrm{d}\mu = \int_X f \, \mathrm{d}\lambda = \Lambda(f)$$

f.