Measure Theory: 3rd week hw

Due on February 12, 2014 at 3:10pm

Chapter 2

Problem 1

a.
- if f is infinite on a set E with positive measure then

$$\int f \mathrm{d}\mu \geq \int N\chi_E \mathrm{d}\mu = N \cdot \mu(E)$$

for any integer N and simply letting $N \to +\infty$ gives us what we want.

- if f is simple then the theorem is trivial. For the general, due to p case there must exist an increasing sequence of simple functions $\{\phi_n\}_0^\infty$ that converge to f pointwise. Let $\mathcal{E}_n$ be the colleciton of sets where ϕ_n is positive, note that $\mathcal{E}_n$ is a finite collection of sets with finite measure. And it's obvious that f is positive on $\bigcup_{i=0}^\infty \bigcup_{E\in\mathcal{E}_n} E$ and hence we're done.

b. it's literally the same proof):<. f is measurable by proposition 2.11 and 2.12, and since for each x, $|f_n(x)| \to |f(x)|$ and $g_n(x) \to g(x)$ and since $|f_n(x)| \leq g_n(x)$ then $|f(x)| \leq g(x)$ then we have $|f| \leq g$ which means $f \in L^1$. By taking real and immaginary parts it suffices to assume f_n and f are real valued. And since $f_n - g_n \geq 0$ and $f_n + g_n \geq 0$ we can use Fatou's lemma:

$$\int g + \int f \leq \liminf \int (g_n - f_n) \leq \liminf \int g_n + \liminf \int f_n$$
$$\int g - \int f \leq \liminf \int (g_n - f_n) \leq \liminf \int g_n - \limsup \int f_n$$

Therefore, $\limsup \int f_n \leq \int f \leq \liminf \int f_n$ and the result follows.

c.
- as $n \to \infty$ we see that $\frac{\sin(\frac{x}{n})}{(1+\frac{x}{n})^n} \to 0$, $\frac{1}{(1+\frac{x}{n})^n} \to e^{-x}$ and we have

$$\int_0^\infty \frac{1}{(1+\frac{x}{n})^n}\mathrm{d}x = \frac{n}{n-1} \to 1 = \int_0^\infty e^{-x}\mathrm{d}x$$

and since $\left|\frac{\sin(\frac{x}{n})}{(1+\frac{x}{n})^n}\right| \leq \frac{1}{(1+\frac{x}{n})^n}$ we can use the generalized DCT to conclude that

$$\lim_{n\to\infty} \int_0^\infty \frac{\sin(\frac{x}{n})}{(1+\frac{x}{n})^n}\mathrm{d}x = \int_0^\infty 0\mathrm{d}x = 0$$

- due to bernouli's inequality, $\frac{1+nx^2}{(1+x^2)^n} < 1$ and we also have

$$\lim_{n\to\infty} \frac{1+nx^2}{(1+x^2)^n} = \lim_{n\to\infty} \frac{n}{(1+x^2)^{n-1}} - \frac{n-1}{(1+x^2)^n} = 0$$

(see baby rudin theorem 3.20) and hence we can use the DCT to get

$$\lim_{n\to\infty} \int_0^1 \frac{1+nx^2}{(1+x^2)^n}\mathrm{d}x = \int_0^1 0\mathrm{d}x = 0$$

- since $\sin(x) < x$ we see that $\left|\frac{\sin(\frac{x}{n})}{\frac{x}{n}} \cdot \frac{1}{1+x^2}\right| < \frac{1}{1+x^2}$ and we know that

$$\int_0^\infty \frac{1}{1+x^2}\mathrm{d}x = [\arctan(x)]_0^\infty = \frac{\pi}{2}$$

and as $n \to \infty$ we see that $\frac{\sin(\frac{x}{n})}{\frac{x}{n}} \cdot \frac{1}{1+x^2} \to \frac{1}{1+x^2}$ so by using DCT we see that

$$\lim_{n\to\infty} \int_0^\infty \frac{\sin(\frac{x}{n})}{\frac{x}{n}} \cdot \frac{1}{1+x^2}\mathrm{d}x = \int_0^\infty \frac{1}{1+x^2}\mathrm{d}x = \frac{\pi}{2}$$

Problem 1 continued on next page...

- a simple $u = nx$ substitution would show us that

$$\lim_{n\to\infty}\int_a^\infty \frac{n}{1+n^2x^2}\mathrm{d}x = \lim_{n\to\infty}\frac{\pi}{2} - \arctan(na)$$

  which is equal to

  - 0 if $a > 0$
  - $\frac{\pi}{2}$ if $a = 0$
  - π if $a < 0$

Problem 2

a. Let $E_{n,k} = \{x \mid |f(x) - f_n(x)| \geq \frac{1}{k}\}$. First, if you fix k and let $n \to \infty$ note that $\mu(E_{n,k}) \to 0$ since $\int |f - f_n| \to 0$.

Choose m_k such that $\mu(E_{N,k}) < 2^{-k}$ for all $N \geq m_k$ and $m_{k-1} < m_k$; we claim $\{f_{m_k}\}_{k=0}^\infty$ converges to f a.e. Indeed, suppose the sequence *doesn't* converge pointwise at x, that means there exists an $\varepsilon > 0$ such that $|f(x) - f_n(x)| \geq \varepsilon$ for infinitely many $n \in \mathbb{N}$ but that would mean $x \in \limsup E_{m_k}$. Now we'll be done if we prove $\mu(\limsup E_{m_k}) = 0$:

$$\begin{aligned}
\mu(\limsup E_{m_k}) &= \mu\left(\bigcap_{i=0}^\infty \bigcup_{k=i}^\infty E_{m_k}\right) \\
&= \lim_{k\to\infty} \mu\left(\bigcup_{k=i}^\infty E_{m_k}\right) \\
&\leq \lim_{k\to\infty} \sum_{i=k}^\infty \mu(E_{m_k}) \\
&< \lim_{k\to\infty} \sum_{i=k}^{i} \frac{1}{2^i} \\
&< \lim_{k\to\infty} \frac{1}{2^{k-1}} = 0
\end{aligned}$$

bonus: the typewriter sequence (I coincidentally came up with this on my own and I don't know if I should type it out ;-;)

b. We're gonna prove b, c and d in one go. Pick m_k such that $\int |f_n - f_m| < 2^{-k}$ for all $n > m \geq m_k$ and $m_{k-1} < m_k$. Put $f = f_{m_1} + \sum_{i=1}^\infty f_{m_{k+1}} - f_{m_k}$, since $\sum_{i=1}^\infty \int \left|f_{m_{k+1}} - f_{m_k}\right| < \sum_{i=1}^\infty 2^{-k} < \infty$, due to the DCT, $f \in L^1(X)$ and

$$\begin{aligned}
f &= f_{m_1} + \lim_{j\to\infty}\sum_{i=1}^{j}(f_{m_{k+1}} - f_{m_k}) \\
&= f_{m_1} + \lim_{j\to\infty} f_{m_{j+1}} - f_{m_1} \\
&= \lim_{j\to\infty} f_{m_{j+1}}
\end{aligned}$$

now choose a N such that $\varepsilon > 2^{-N}$. So for all $m > n > m_N$ we have

$$\int |f_m - f_n| < \varepsilon$$

and

$$|f - f_n| = \lim_{j\to\infty} \left|f_{m_{j+1}} - f_n\right|$$

and by using Fatou's lemma we have

$$\int |f - f_n| \leq \liminf_{j\to\infty} \int \left|f_{m_{j+1}} - f_n\right| < \varepsilon$$

which would prove the sequence converges to f in $L^1(X)$ which would give us d, and by using a that would give us both b and c.

Problem 3

Since simple functions are dense in $L^1(X)$ it suffices to prove that for all $\varepsilon > 0$ and all simple functions $\phi \in L^1(X)$ there exists a function $f \in C_c(X)$ such that $\int |f - \phi| < \varepsilon$.

Let $\sum z_j \chi_{E_j}$ be the standard representation of ϕ. For each j, we can find sets $K_j \subset E_j \subset O_j$ where $\mu(O_j \setminus E_j) < \frac{\varepsilon}{3\cdot 2^j |z_j|}$ and $\mu(E_j \setminus K_j) < \frac{\varepsilon}{3\cdot 2^j |z_j|}$ and using urysohn's lemma there must exist a continuous function f_j such that $\chi_{K_j} < f < \chi_{O_j}$ and hence we have

$$\left|f_j - \chi_{E_j}\right| \leq \left|\chi_{O_j} - f_j\right| + \left|\chi_{O_j} - \chi_{E_j}\right|$$
$$\left|f_j - \chi_{E_j}\right| < (\chi_{O_j} - \chi_{K_j}) + (\chi_{O_j} - \chi_{E_j})$$
$$\left|f_j - \chi_{E_j}\right| < (\chi_{O_j} - \chi_{E_j}) + (\chi_{E_j} - \chi_{K_j}) + (\chi_{O_j} - \chi_{E_j})$$
$$\int \left|z_j f_j - z_j \chi_{E_j}\right| < |z_j| \int (\chi_{O_j} - \chi_{E_j}) + |z_j| \int (\chi_{E_j} - \chi_{K_j}) + |z_j| \int (\chi_{O_j} - \chi_{E_j})$$
$$\int \left|z_j f_j - z_j \chi_{E_j}\right| < \frac{\varepsilon}{2^j}$$

and now just put $f = \sum z_j f_j$ and hence we have

$$\begin{aligned} \int |f - \phi| &\leq \sum \int \left|z_j f_j - z_j \chi_{E_j}\right| \\ &< \sum \frac{\varepsilon}{2^j} \\ &< \varepsilon \end{aligned}$$

and we are done.