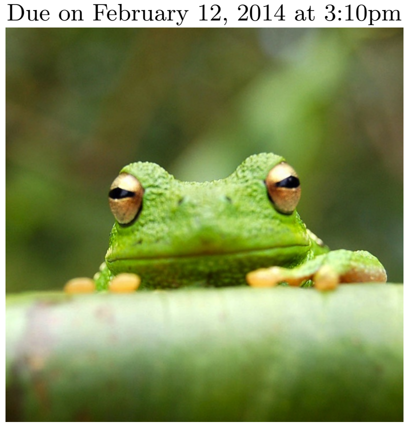
Measure Theory: 5th week hw

Due on February 12, 2014 at 3:10pm



Chapter 2

Problem 1

a. (a) $\nu(\varnothing) = \mu(T^{-1}(\varnothing)) = \mu(\varnothing) = 0$

(b) let $\{E_i\}_{i=0}^{\infty}$ be a sequence of disjoint sets in $\mathcal{M}_Y$

$$\nu\left(\bigcup_{i=0}^{\infty} E_i\right) = \mu\left(T^{-1}\left(\bigcup_{i=0}^{\infty} E_i\right)\right) \tag{1}$$

$$= \mu\left(\bigcup_{i=0}^{\infty} T^{-1}(E_i)\right) \tag{2}$$

$$= \sum_{i=0}^{\infty} \mu\big(T^{-1}(E_i)\big) \tag{3}$$

$$= \sum_{i=0}^{\infty} \nu(E_i) \tag{4}$$

(1) and (4) follow from definition of ν, (2) follows from properties of preimages and (3) is justified coz the preimage of disjoint sets is disjoint.

b. First note that f is measurable iff $f \circ T$ is measurable. We then prove the statement for characteristic functions first: suppose $f = \chi_E$ where $E \in \mathcal{M}_Y$ but note that $f \circ T$ is also a characteristic function and

$$(f \circ T)^{-1}(1) = T^{-1}\big(f^{-1}(1)\big) = T^{-1}(E)$$

which means $f \circ T = \chi_{T^{-1}(E)}$ and hence

$$\int f \circ T \,\mathrm{d}\mu = \int \chi_{T^{-1}(E)} \,\mathrm{d}\mu = \mu(T^{-1}(E)) = \nu(E) = \int f \,\mathrm{d}\nu$$

and hence the statement is true for characteristic functions and also simple functions by linearity. Now for the general case let $\{\phi_i\}$ be an increasing sequence of simple functions that converge to f a.e. so by applying the MCT we get

$$\int f \,\mathrm{d}\nu = \lim_{i\to\infty} \int \phi_i \,\mathrm{d}\nu = \lim_{i\to\infty} \int \phi_i \circ T \,\mathrm{d}\mu = \int f \circ T \,\mathrm{d}\mu$$

Problem 33

suppose $\int f \geq \liminf \int f_n$, that means there exists a subsequence $\{f_{n_k}\}_{k=0}^{\infty}$ such that $\liminf \int f_n = \lim \int f_{n_k}$ but the subsequence obviously still converges to f in measure so it must contain a subsubsequence $\{f_{n_{k_j}}\}_{j=0}^{\infty}$ that converges to f pointwise a.e. but using Fatou's lemma we see that

$$\int f \leq \liminf_{j\to\infty} \int f_{n_{k_j}} = \lim_{k\to\infty} \int f_{n_k} = \liminf_{n\to\infty} \int f_n$$

which would mean

$$\int f = \liminf_{n\to\infty} \int f_n$$

Problem 34

a. f is measurable since it's the limit of some subsequence of $\{f_n\}$ which are all measurable functions and it's integrable since $f = \lim f_n \leq g$. Now we note that $f_n + g \geq 0$ and $g - f_n \geq 0$ and both sequences converge in measure to $f + g$ and $g - f$ respectively so we can use the previous problem to conclude that

$$\int g + \int f \leq \liminf \int (g + f_n) = \int g + \liminf \int f_n$$
$$\int g - \int f \leq \liminf \int (g - f_n) = \int g - \limsup \int f_n$$

from which $\limsup \int f_n \leq \int f \leq \liminf \int f_n$ and hence we're done

b. we have $|f - f_n| \leq 2g$ and it's fairly evident that $|f - f_n| \to 0$ in measure and hence, by using (a), we see that

$$\lim_{n\to\infty} \int |f - f_n| = 0$$

Problem 40

It suffices to prove that $E_n(k)$, as defined in the orginal proof, are eventually finite. Indeed, according to the DCT, $f_n \to f$ in L^1 but if $E_n(k)$ *aren't* eventually finite then

$$\int |f_n - f| \geq \int_{E_n(k)} |f_n - f| \geq \int_{E_n(k)} \frac{1}{k} = \infty$$

which is a contradiction.

Problem 44

Let $\varepsilon > 0$. We're going to prove the result for postive real functions first. Notice that

$$\sum_{i=0}^{\infty} \mu\big(f^{-1}\big([i, i+1)\big)\big) = \mu([a, b])$$

is a convergent sequence which means there exists an $N \in \mathbb{N}$ such that

$$\sum_{i=N}^{\infty} \mu\big(f^{-1}\big([i, i+1)\big)\big) = \mu\big(f^{-1}([N, +\infty))\big) < \varepsilon$$

that means that, if $E_1 = f^{-1}([0, N))$, then $f_{E_1} \in L^1$ and $\mu(E_1^c) < \varepsilon$.

Now using problem 3 from the last hw, we know that there must exist a sequence of continuous functions $\{f_n\}$ that converges a.e. to f in E_1 and using egoroff's there must exist a set E_2 such that $\mu(E_2^c) < \varepsilon$ and $\{f_n\}$ converges uniformly on E_2. And using inner regularity we can find a compact set E_2' such that $\mu(E_2 \setminus E_2') < \varepsilon$ which means $\mu(E_2'^c) < 2\varepsilon$ and $\{f_n\}$ is still converges uniformly on E_2' which means the restriciton of its limit to E_2' is continuous. Now just put $E = E_1 \cap E_2'$ and we can see that $f_| E$ is continuous and it's easy to check that $\mu(E) < 3\varepsilon$.

Now for the general case, we can decompose f as $f = (g^+ - g^-) + i(h^+ - h^-)$ where g^+, g^-, h^+, h^- are all positive functions from which we can extract sets E_1, E_2, E_3, E_4 on the intersection of which all our functions (hence f as well) are continuous and

$$\mu\Big(\bigcup_1^4 E_i\Big) < 4 \cdot \frac{\varepsilon}{4} = \varepsilon$$

Problem 52

(for convennience, let's order the elements in $Y = \{y_i\}_{i=0}^{\infty}$)

First we note that it is evident that if $g \in L^+(\nu)$ then

$$\int g \, \mathrm{d}\nu = \sum_{i=0}^{\infty} g(y_i)$$

and if $f \in L^+(\mu \times \nu)$ then for any $y \in Y$

$$\int \chi_{X \times \{y\}} \cdot f \, \mathrm{d}\mu \times \nu = \int f^y \, \mathrm{d}\mu$$

(it's easy to see that this holds for characteristic funcitons and from that, by applying the MCT, you get the general case) now suppose $f \in L^1(\mu \times \nu)$, not that $f = \lim_{n \to \infty} \sum_{i=0}^{n} \chi_{X \times \{y_i\}} f$ and by using the MCT we see that

$$\begin{aligned}
\int f \, \mathrm{d}\mu \times \nu &= \sum_{i=0}^{\infty} \int \chi_{X \times \{y_i\}} f \, \mathrm{d}\mu \times \nu \\
&= \sum_{i=0}^{\infty} \int f^{y_i} \, \mathrm{d}\mu \\
&= \iint f^y \, \mathrm{d}\mu \, \mathrm{d}\nu
\end{aligned}$$

and

$$\begin{aligned}
\int f \, \mathrm{d}\mu \times \nu &= \sum_{i=0}^{\infty} \int \chi_{X \times \{y_i\}} f \, \mathrm{d}\mu \times \nu \\
&= \sum_{i=0}^{\infty} \int f^{y_i}(x) \, \mathrm{d}\mu(x) \\
&= \int \sum_{i=0}^{\infty} f^{y_i}(x) \, \mathrm{d}\mu(x) \\
&= \iint f_x \, \mathrm{d}\nu \, \mathrm{d}\mu
\end{aligned}$$

This establishes Tonelli 's theorem and also shows that if $f \in L^+(\mu \times \nu)$ and $\int f < \infty$, then $\int f_x \, \mathrm{d}\nu < \infty$ a.e. and $\int f^y \, \mathrm{d}\mu < \infty$ a.e., that is, $f_x \in L^1(\nu)$ for a.e. x and $f^y \in L^1(\mu)$ for a.e. y. If $f \in L^1(\mu \times \nu)$, then, the conclusion of Fubini's theorem follows by applying these results to the positive and negative parts of the real and imaginary parts of f.