HAMILTONIAN MECHANICS

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1. Hamilton's Equations

Hamiltonian mechanics is related to Lagrangian mechanics via a Legendre transform [1–3]. Recall that the i^{th} component of generalised (canonical momentum is defined as

$$p_i = \frac{\partial L}{\partial \dot{q}_i} \tag{1.1}$$

We want to replace the $\dot{q}_i$ in the Lagrangian for p_i, the conjugate momenta. This will allow us to have a function that treats q_i and p_i on equal footing. To do this we need to perform the Legendre Transform.

Definition 1.1. *A Legendre transform of a convex function $f(x)$ is*

$$g(y) = xy - f \tag{1.2}$$

where $y = \frac{df}{dx}$. To further see how this works, we have that

$$\frac{dg}{dy} = x \tag{1.3}$$

so $y = \frac{df}{dx}$ and $x = \frac{dg}{dy}$. The pairs $(y, f(x))$ and $(x, g(y))$ are conjugate.

Definition 1.2. *The Hamiltonian $H(p_i, q_i)$ is defined as the Legendre transform of the Lagrangian $L(q_i, \dot{q}_i)$.*

$$H(p_i, q_i) = p_i\dot{q}_i - L(q_i, \dot{q}_i) \tag{1.4}$$

Then we have

$$\frac{\partial H}{\partial p_i} = \dot{q}_i \tag{1.5}$$

$$\frac{\partial H}{\partial q_i} = -\frac{\partial L}{\partial q_i} = -\dot{p}_i \tag{1.6}$$

Equations 1.5 *and* 1.6 *are called Hamilton's equations.*

Example 1.1. *Consider an SHM system. $L = \frac{1}{2}(\dot{x}^2 - x^2)$ compute H. $H = \frac{1}{2}(\dot{x}^2 + x^2)$*

We want to show that H can be usually interpreted as the total energy in the system. We recall that the time-translation symmetry is related to the conservation of energy. We can therefore consider the following proposition.

Proposition 1.1. *If H is independent of t, then H is conserved.*

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Proof.

$$\frac{dH}{dt}(x,p,t) = \frac{\partial H}{\partial p}\dot{p} + \frac{\partial H}{\partial x}\dot{x} + \frac{\partial H}{\partial t} = -\frac{\partial H}{\partial p}\frac{\partial H}{\partial x} + \frac{\partial H}{\partial p}\frac{\partial H}{\partial x} + 0 = 0$$

□

2. Poisson Brackets

Definition 2.1. *A **Poisson bracket** is a bilinear form* $\{\quad,\quad\} : (X, X) \to \mathbb{F}$*, where* X *is the functional space and* $\mathbb{F}$ *is a field.*

Let us look at the total time derivative of a function $f(q_i, p_i, t)$.

$$\begin{aligned} \frac{df}{dt} &= \frac{\partial f}{\partial q_i}\dot{q}_i + \frac{\partial f}{\partial p}\dot{p}_i + \frac{\partial f}{\partial t} \\ &= \frac{\partial f}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial H}{\partial q_i} + \frac{\partial f}{\partial t} \\ &= \{f, H\} + \frac{\partial f}{\partial t} \end{aligned} \tag{2.1}$$

Let us suppose that f has no explicit dependence on t. Then $\frac{\partial f}{\partial t}$. We then must have

$$\frac{df}{dt} = \{f, H\} \tag{2.2}$$

The time derivative of a variable is therefore the Poisson bracket of it with the Hamiltonian of the system.

In Hamiltonian mechanics we can explicitly define the Poisson bracket in terms of the coordinates in phase space.

Definition 2.2. *In Hamiltonian Mechanics, the Poisson Bracket is a map* $\{\quad,\quad\} : (X, X) \to \mathbb{F}$*, where* $X = C^\infty(M; \mathbb{R})$ *and* $\mathbb{F}$ *is a field. It is defined as:*

$$\{f, g\} = \sum_i \left(\frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i} \right) \tag{2.3}$$

Let us take a look at some examples.

Example 2.1. *In 1D,*

$$\{q, p\} = 1 - 0 = 1 \tag{2.4}$$

Example 2.2. *Now operate in* n *dimensions.*

$$\begin{aligned} \{q_i, q_j\} &= \frac{\partial q_i}{\partial q_k}\frac{\partial q_j}{\partial p_k} - \frac{\partial q_i}{\partial p_k}\frac{\partial q_j}{\partial q_k} \\ &= \delta_{ik} \cdot 0 + \delta_{jk} \cdot 0 = 0 \end{aligned} \tag{2.5}$$

$$\begin{aligned} \{q_i, p_j\} &= \frac{\partial q_i}{\partial q_k}\frac{\partial p_j}{\partial p_k} - \frac{\partial q_i}{\partial p_k}\frac{\partial p_j}{\partial q_k} \\ &= \delta_{ik}\delta_{jk} = \delta_{ij} \end{aligned} \tag{2.6}$$

Remark 2.1. *Why have we bothered to define Poisson brackets in the first place? It turns out it is one of the ways of quantise a system (and hence is start of the Hamiltonian formulation of quantum mechanics!). We can relate the following equation*

$$\frac{df}{dt} = \{f, H\} + \frac{\partial f}{\partial t} \tag{2.7}$$

with the Ehrenfest Theorem in quantum mechanics:

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle O\rangle = \frac{1}{i\hbar}\langle [O, H]\rangle + \left\langle \frac{\partial O}{\partial t}\right\rangle \tag{2.8}$$

where O and H are operators (and H is the Hamiltonian operator). This suggests the following map

$$\{\quad,\quad\} \leftrightarrow \frac{1}{i\hbar}[\quad,\quad] \tag{2.9}$$

between Poisson brackets and commutators. This is in fact the start of canonical quantisation in quantum field theory.

Remark 2.2. *Let us suppose we construct a vector in phase space using the basis (q_i, p_i) in the following form*

$$\begin{pmatrix} q_i \\ p_i \end{pmatrix} \tag{2.10}$$

Using the Poisson bracket, we can in fact find a matrix representation of the bilinear form as described. This turns out to be:

$$\begin{pmatrix} 0 & \mathbb{1}_n \\ -\mathbb{1}_n & 0 \end{pmatrix} \tag{2.11}$$

where $\mathbb{1}_n$ is the $n \times n$ identity matrix. This is the canonical symplectic basis of the system - and the Hamiltonian system is known to exist in a symplectic manifold. Hamiltonian mechanics has inspired to study of symplectic geometry, a subfield of geometry which has slowly evolved into a defining field in the late 20th century. This is further explored in [3].

References

[1] T. W. B. Kibble and F. H. Berkshire, *Classical mechanics*. Imperial College Press ; Distributed by World Scientific Pub., 0.

[2] H. Goldstein, J. Safko, and C. P, *Classical mechanics*. Pearson, 2014.

[3] V. I. Arnold, K. Vogtmann, and A. Weinstein, *Mathematical methods of classical mechanics*. Springer, 1989.