Prepared for submission to JHEP

General Relavity Tutorials 2026

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Abstract: A set of class notes for the Supersymmetry and Supergravity course at Oxford in Hilary Term 2026.

Contents

1 From special to general relativity

The centre point of special relativity is that physics in any inertial frames are the same. In particular, we can recall the two postulates in special relativity.

Postulate 1.1 (The first postulate of relativity)**.** The laws of nature are the same in all inertial reference frames.

Of course, we also know that there is fundamental speed central to physics and this speed — namely the speed of light — is constant to all observers. This motivates the second postulate of relativity.

Postulate 1.2 (The second postulate of relativity)**.** The speed of light in vacuum, c, is the same in all inertial reference frames.

The key point is we needed something known as an **inertial reference frame** — this is a **system of reference** which is not acted upon by external forces and proceeds with constant velocity. The theory of special relativity is developed with this principle in mind, and we have studied this in detail in B2 with Caroline.

The next obvious question is what happens to non-inertial frames. To this let us actually think about the effects of *gravitational fields*. Gravitational fields have the property that no matter the mass of the test object, all bodies will move in them in the same manner. This is encoded in the Weak Equivalence Principle, stated normally as follows.

Postulate 1.3 (Weak Equivalence Principle)**.** The trajectory of a freely falling test body depends only on its initial position and velocity, and is independent of its composition.

This creates a big problem — if we wish to measure the effects of a gravitational field, we must use some test object and something 'stationary' to compare the effects, but there are simply no such 'stationary' objects as all objects are affected by the gravitational field in the same manner! These stationary objects can only be forced to exist by introducing non-inertial frames (say for example, imagine you are superman and holding a ball in place next to a black hole). So if we work locally, the properties of the motion in a non-inertial system are the same as those in an inertial system in the presence of a gravitational field. Einstein extended the equivalence principle stated above to the now famous *principle of equivalence*, for which we can state as follows.

Postulate 1.4 (Einstein's Equivalence Principle)**.** The following statements hold true.

(i) The WEP is valid.

(ii) In a local inertial frame, the results of all gravitational experiments will be indistinguishable from the results of the same experiments performed in an inertial frame in Minkowski spacetime.

(iii) The properties of motion in a non-inertial system are **locally** the same as those in an inertial system in the presence of a gravitational field. In particular, a non-inertial reference frame is equivalent to a certain gravitational field.

A constant gravitational field therefore corresponds to a uniformly accelerated reference frame; and a variable field corresponds to a non-uniformly accelerated reference frame. As theoretical physicists, the clear next step is to find the correct mathematical tool to describe this. Let us recall what happens in the Minkowski case. In an inertial reference system the spacetime interval is given by the relation,

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 \ . \tag{1.1}$$

Let us think about what happens when we transform to a non-inertial frame. Suppose we have a non-inertial system describing a rotating frame at uniform angular velocity Ω. Clearly we can take the transformation,

$$t = t' \ , \quad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\Omega t & -\sin\Omega t \\ \sin\Omega t & \cos\Omega t \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix} \ , \quad z = z' \ , \tag{1.2}$$

which now gives the following spacetime interval (which we will, with anticipation, call the **metric**),

$$ds^2 = -(c^2 - \Omega^2(x'^2 + y'^2))dt^2 + dx'^2 + dy'^2 + dz'^2 - 2\Omega y' dx' dt + 2\Omega x' dy' dt \ . \tag{1.3}$$

We can no longer write this as a sum of squares of coordinate differentials. This means the metric appears as a quadratic form of general type in coordinate differentials, i.e.

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \ . \tag{1.4}$$

So perhaps we should do the same for gravitational fields too. This means the correct mathematical object to look at is $g_{\mu\nu}$ — the metric of the spacetime. But we aren't done. Notice the word **locally** in EEP. The equivalence principle only states that in a sufficiently small region of spacetime the gravitational effects are the same as a non-inertial frame. There is, however, a fundamental difference between gravitational effects and non-inertial systems. At infinite distances away from the source of the gravitational effects, our physical intuition tells us that the gravitational field goes to zero. After all, *locality* remains in play, and things that are far far away should not affect the local dynamics. This however, does not affect non-inertial frames — at infinity [1], the effects will be the same.

How does this translate to our mathematical structure, namely, the metric? Since a non-inertial frame has globally the same structure (everywhere), it must be true that we can make a global change of coordinates to get back to an inertial reference frame. This is indeed the case in the uniformly rotating frame we have discussed above — we have made a global change of coordinates to obtain the new metric. In mathematical terms, non-inertial metrics, $h_{\mu\nu}$, are related to the Minkowski metric by a change of coordinates $x^\mu \mapsto y^\mu$,

$$ds^2 = h_{\mu\nu} dy^\mu dy^\nu = g_{\mu\nu} dx^\mu dx^\nu \ . \tag{1.5}$$

This is however not true for gravitational metrics — we cannot make a global coordinate transformation as the gravitational field itself goes to zero at infinity! This suggests that the metric associated to gravitational fields cannot be transformed away to a Minkowski one by a simple coordinate transformation!

This forms the basis of general relativity. We want to understand how gravitational effects come into play, and it turns out that they affect the spacetime interval, or metric, in a very non-trivial way. This is what people mean when they say 'gravity curves spacetime'. But now we have a better understanding, and as theorist we shall now proceed to understand the mathematical structures that describes this phenomenon as well as the physical consequences of this mathematical description.

So we have a wish-list. A wish-list for all the mathematical structures we want for describing gravity in spacetime. Clearly, the spacetime interval ds^2 must come into play, so let me list the ingredients down below.

1. **A space (manifold M).** Clearly, before we even begin, we will need some description of space that describes our world and locally admits some inertial frame (coordinates) so we can label events. Mathematically this is known as a differentiable manifold, and this will be the subject of the first tutorial.

2. **A measure on the space for spacetime intervals (metric g).** As mentioned we will need the metric — this object must be able to intrinsically measure gravitational effects, and we will see how notions of connections and curvatures will come into play and tell us something about gravity affects spacetime.

[1] I am giving a very loose physical definition of infinity here. Think of it as spatial infinity, but we should probably do a proper characterisation of infinities later.

3. **Descriptions of fields (tensor fields).** We will of course also need some notion of vectors and tensor to describe physical fields in this space. This turns out to be in general tensor fields, and we will need to set up properly how they work so we have consistent mathematical descriptions of fields in spacetime.

The fundamental mathematical object in general relativity turns out to be (M, g), a manifold equipped with a (Lorentzian metric). In fact, this is the only mathematical object needed to describe spacetime [2]. This is the second important principle in GR, known as general covariance.

Postulate 1.5 (General Covariance)**.** The only mathematical structure to describe the physical spacetime is (M, g).

The special version of this will be that the only mathematical structure needed to describe inertial frames is Minkowski spacetime, with the metric $ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu$.

So here we are. The rest of our discussion should clearly focus on understanding the mathematical structures, and then discuss the physical consequences of these postulates and set-up. General relativity is actually not too hard — once you wrap your head around the mathematics, everything will be straight-forward, but the mathematics, or differential geometry, is of course quite difficult! In fact, the whole field of differential geometry, one might argue, has been sped up by the development of the theory of relativity. But don't fear — I will be here to join you along the way.

2 Manifolds, Vectors and Tensors

In this section we begin our study of differential geometry.

2.1 Manifolds

Our discussion in differential geometry begins with manifolds. A manifold is a space which locally looks like an $\mathbb{R}^n$.

Definition 2.1. A **n-dimensional smooth manifold** M is a topological space such that there exist pairs (U_α, ϕ_α) called **charts**, where,

(i) $\bigcup_\alpha U_\alpha = M$, so (U_α) forms an open cover over M.

(ii) For each α, there is a homeomorphism $\phi : U_\alpha \to V_\alpha \subset \mathbb{R}^n$, where V_α are open sets in $\mathbb{R}^n$.

(iii) For $U_\alpha \cap U_\beta \neq \emptyset$, then $\phi_\beta \circ \phi_\alpha^{-1} : V_\alpha \to V_\beta$ is called the **transition map** which maps the overlap region to the other chart. The transition map is differentiable (smooth).

We call ϕ_α **charts**, **coordinate systems** or **trivialisations**. The set $\{\phi_\alpha\}$ is called an **atlas** (see Figure 2.1).

[2] Here I mean the physical one, which inconveniently sometimes people call (M, g), the mathematical object, spacetime too.

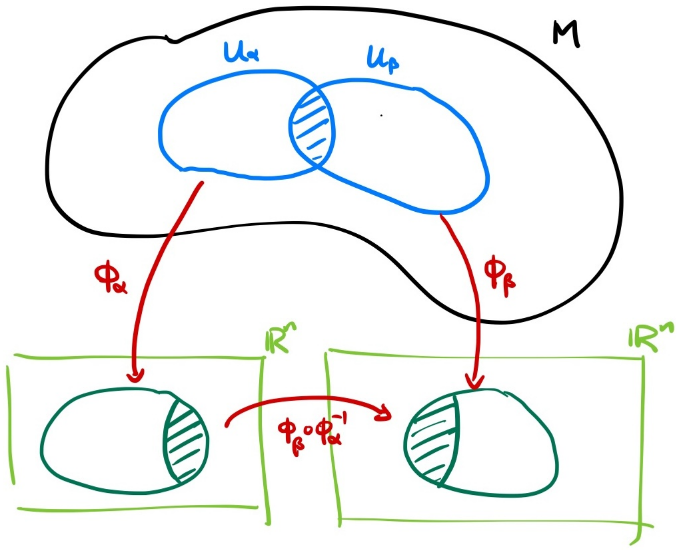

Figure 2.1: A depiction of a smooth manifold.

So a chart is like a local map — it allows you to give local coordinates at a point.

$$\phi_\alpha(p) = (x^1_\alpha(p), x^2_\alpha(p), \ldots, x^n_\alpha(p)) \; . \tag{2.1}$$

Imagine you are walking along Oxford Street in London and you walk by the maps that show the tourist information in the area. You can walk a few blocks along the street and still see a similar map but shifted up a few blocks — but the streets and stations (say, Oxford Circus tube station) that are shown on the two maps must be the same and presented in a similar way (since, clearly, they represent the same buildings)! This means the 'map' going between the two charts must be smooth.

Topological spaces and manifolds

We have gone through a bit fast here. Technically, we should start by defining what is a topological manifold. Let me suppose we don't know any topology.

Definition 2.2. A **topological space** $T = (X, \mathcal{T})$ consists of a non-empty set X together with a fixed family $\mathcal{T}$ of subsets of X satisfying,

(T1) $X, \emptyset \in \mathcal{T}$.

(T2) For $A, B \in \mathcal{T}$, $A \cap B \in \mathcal{T}$.

(T3) For $A_i \in \mathcal{T}$, where $i \in I$ and I can be infinite (I is an indexing set), we have $\bigcup_i A_i \in T$.

The family $\mathcal{T}$ is called a topology of X and the members of $\mathcal{T}$ are called **open sets** of $\mathcal{T}$.

Let us also define what a homeomorphism is.

Definition 2.3. A map of topological spaces $f : (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$ is **continuous** if $U \in \mathcal{T}_Y \Rightarrow f^{-1}(U) \in \mathcal{T}_X$. A **homeomorphism** between topological spaces X and Y is a bijective map $f : X \to Y$ such that both f and its inverse function f^{-1} exist and are both continuous.

With this we can now define a topological manifold.

Definition 2.4. A n**-dimensional topological manifold** M is a topological space that satisfies the following properties.

1. M is a **Hausdorff space**: for every pair of distinct points $p, q \in M$, there are disjoint open subsets $U_p, U_q \subset M$ such that $p \in U_p$ and $q \in U_q$.

2. M is **second-countable**: there exists a countable basis for the topology of M.

3. M is **locally Euclidean of dimension** n: each point of M has a neighbourhood, $p \in U_\alpha$ that is homeomorphic to an open subset $V_\alpha \subset \mathbb{R}^n$, i.e. $\phi_\alpha : U_\alpha \to V_\alpha$ is a homeomorphism.

With a topological manifold, one can then introduce transition maps as before. We can then define smooth structures.

Definition 2.5. Two charts (U_α, ϕ_α) and (U_β, ϕ_β) are said to be **smoothly compatible** if either $U_\alpha \cap U_\beta = 0$ or the transition map $\phi_\beta \circ \phi_\alpha^{-1}$ is a diffeomorphism.

An atlas $\mathcal{A}$ for M is a **smooth atlas** if any two charts are smoothly compatible with each other. We define a **smooth structure** as an equivalence class of smooth atlases which is **maximal**, i.e. it is not properly contained in any larger smooth atlas. A **smooth manifold** is a pair $(M, \mathcal{A})$ where M is a topological manifold and $\mathcal{A}$ is a smooth structure on M.

A bit of a roundabout way to define everything properly, but here i is.

One could perhaps ask why we need to map open sets to open sets. You should really think of open sets as the basic units in topology — they are the members of topology. To properly define homeomorphisms, we will need to find open sets in both M and $\mathbb{R}^n$, and we know that the open sets (the basic units) in $\mathbb{R}^n$ are the 'open sets' in the normal sense — the interval sets that are unbounded.

Let us have a look at a few examples.

Example 2.1 ($\mathbb{R}^n$ as a smooth manifold)**.** The simplest example is $\mathbb{R}^n$ — the open sets are just the open sets in $\mathbb{R}^n$ and the charts are just the identity maps.

Example 2.2 (S^1, the circle)**.** The simplest non-trivial example is the circle S^1. We can define two sets which covers S^1 — $U_N = S^1\backslash\{N\}$ and $U_S = S^1\backslash\{S\}$ where N and S are the north pole and south pole respectively (Figure 2.2). We can construct the charts by using the angular coordinate θ: for U_N we map to $\theta_N \in (0, 2\pi)$ and for U_S we have $\theta_S \in (-\pi, \pi)$. We define the transition function as follows:

- For the left side of the overlap the coordinates match, so $\theta_S = \phi_S \circ \phi_N^{-1}(\theta_N) = \theta_N$.
- For the right side of the overlap the coordinates are off by 2π, so $\theta_S = \phi_S \circ \phi_N^{-1}(\theta_N) = \theta_N - 2\pi$.

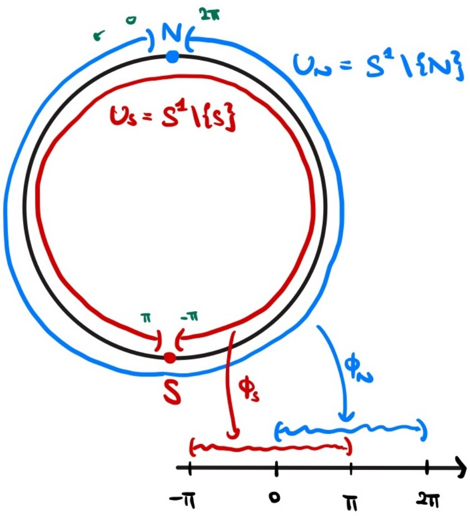

Figure 2.2: The manifold S^1. The blue and red lines illustrate the two open sets U_N and U_S respectively.

2.2 Vectors

The next point is to try and understand objects on the manifold as potential candidates to describe our various fields. To do this we will need some extra notions.

To properly define vectors, we will first need a notion of smooth functions on a smooth manifold. The basic logic in dealing with manifolds is this — we intrinsically only have notions of differentiable when we talk about maps from $\mathbb{R}^n$ to $\mathbb{R}^m$. Therefore, we must define properties of functions with respect to R^n, and manifolds naturally provide us with charts that maps a local region of M to an open set in $\mathbb{R}^n$.

Definition 2.6. A function $f : M \to \mathbb{R}$ is **smooth** iff for any chart ϕ, $f \circ \phi^{-1} : V \to \mathbb{R}$ is a smooth function for $V \subset \mathbb{R}^n$.

Note that this gives us a notion of a scalar field!

To define vectors, it is clear that we can consider some notion of a tangent space at a point. But clearly tangent spaces at different points may not align (think of trying to put a flat

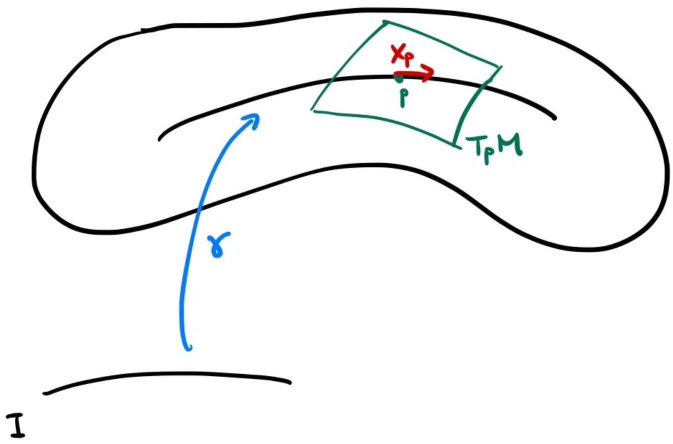

Figure 2.3: Construction of vectors using the directional derivative of a curve at a point.

surface on various points on the surface of a sphere — these flat surfaces do not align). To do this, we will need to first define what curves are on manifolds.

Definition 2.7. A **smooth curve** in a smooth manifold M is a smooth function $\gamma : I \to M$ where I is an open interval in $\mathbb{R}$, i.e. $\phi_\alpha \circ \gamma : I \to \mathbb{R}^n$ is a smooth map for all charts ϕ_α.

Why do we need this notion? Let us think about how we would get tangent vectors in the first place. Suppose we have a function $f : \mathbb{R} \to \mathbb{R}$. It is clear that a notion of a vector is naturally given by the rate of change of f along $\mathbb{R}$ — this is just the local derivative at $p \in \mathbb{R}$ of the domain. In our current case, we fortunately can construct a function $f \circ \gamma : I \to \mathbb{R}$ which has some notion of a derivative. This means we can extend the notion of tangent vectors to a curve — we can look at the rate of change of f along a curves in $\mathbb{R}^n$ as the directional derivative $X_p \cdot (\nabla f)_p$ (see Figure 2.3).

Definition 2.8. Let $\gamma : I \to M$ be a smooth curve without self-intersections and $\lambda(0) = p$ without loss of generality. The **tangent vector to** γ **at** p is the linear map X_p from the space of smooth functions on M to $\mathbb{R}$ given by

$$X_p(f) = \left\{ \frac{d}{dt} f(\gamma(t)) \right\}_{t=t_0} . \tag{2.2}$$

Note that this satisfies two important properties:

(i) Linearity: $X_p(\alpha f + \beta g) = \alpha X_p(f) + \beta X_p(g)$ with α, β constants.

(ii) Leibniz rule: $X_p(fg) = X_p(f)g(p) + f(p)X_p(g)$.

However, we have yet to make any statements about X_p furnishing a vector space. At this point, we have only defined individual vectors using the directional derivative of a curve embedded in the manifold. I would like to take a pause here. Let us jump a bit ahead

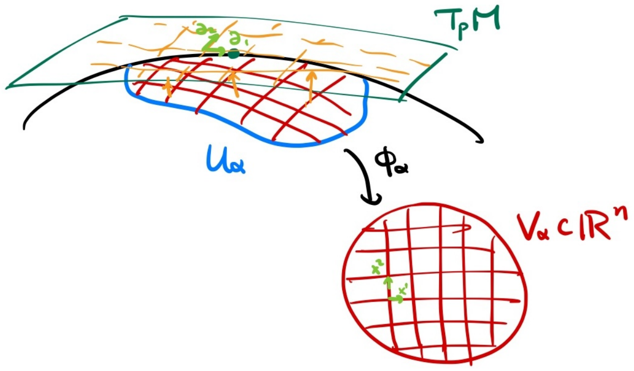

Figure 2.4: Introducing coordinates to the tangent space T_pM.

and assumed the vectors do span a vector space. How do we assign coordinates on this vector space? Clearly, a natural basis is given by a chart at p, ϕ_p, and we can project the coordinates of the chart onto the tangent space (see Figure 2.4 for an illustration). To do this, we introduce the chart $\phi = (x^1, \ldots, x^n)$ in the neighbourhood of p. Then we can write,

$$f \circ \gamma = f \circ \phi^{-1} \circ \phi \circ \gamma \,, \tag{2.3}$$

by inserting an identity map $\text{id} = \phi^{-1} \circ \phi : M \to M$. Then we have the maps $f \circ \phi^{-1} = \mathbb{R}^n \to \mathbb{R}$ and $\phi \circ \gamma : I \to \mathbb{R}^n$ and we write,

$$X_p(f) = \left(\frac{\partial (f \circ \phi^{-1})(x)}{\partial x^\mu}\right)_{\phi(p)} \left(\frac{dx^\mu(\gamma(t))}{dt}\right)_{t=0} = \left(\frac{\partial}{\partial x^\mu}\right)_{\phi(p)} (f) \left(\frac{dx^\mu(\gamma(t))}{dt}\right)_{t=0} \,. \tag{2.4}$$

For the benefit of our sanity let us drop the action on the function f (assume we are looking at elements in a vector space acting on the space of linear functions), and we have [3],

$$X_p = \left(\frac{dx^\mu(\gamma(t))}{dt}\right)_{t=0} \left(\frac{\partial}{\partial x^\mu}\right)_p \,. \tag{2.5}$$

This immediately reminds us of the coordinate basis in special relativity — $\frac{\partial}{\partial x^\mu}$ are just the coordinate basis and clearly in this basis we have,

$$X_p^\mu = \left(\frac{dx^\mu(\gamma(t))}{dt}\right)_{t=0} \,, \tag{2.6}$$

as the components of X_p in this coordinate basis!

We now go back to the question why these tangent vectors span a vector space.

[3] I have dropped $\phi(p)$ for p for notational convenience, so we intuitively treat those things as the same thing when we see them together.

Proposition 2.1. *The set of all tangent vectors at p form an n-dimensional vector space $T_p(M)$ known as the* ***tangent space at*** *p.*

Proof. There are two important properties in showing that some algebraic structure is a vector space. Firstly, linearity. To do this we consider two curves γ and λ through p, and we can define addition and multiplication by constant of tangent vectors by defining $\alpha X_p + \beta Y_p$ to be the linear map acting on the space of functions $f \mapsto \alpha X_p(f) + \beta Y_p(f)$. This is indeed a tangent vector to a curve through p, as one can construct a curve using a local chart ϕ as,

$$\nu(t) = \phi^{-1}\left[\alpha(\phi(\gamma(t)) - \phi(p)) + \beta(\phi(\lambda(t)) - \phi(p)) + \phi(p)\right] \ , \tag{2.7}$$

and you can check that the tangent vector of $\nu(t)$ is $Z_p = \alpha X_p + \beta Y_p$. The second property is a bit more difficult. We first show there is an n-dimensional basis — this is already explained above, but in the chart ϕ we cna pick the curve that only moves in the i-th coordinate, i.e.

$$\lambda_\mu(t) = \phi^{-1}(x^1(p), \ldots, x^{\mu-1}(p), x^\mu(p) + t, x^{\mu+1}(p), \ldots, x^n(p)) \ , \tag{2.8}$$

so the curve only traverses along a coordinate direction in the chart. This defines the tangent vector to this curve as $\left(\frac{\partial}{\partial x^\mu}\right)_p$. The n tangent vectors are linearly independent as the coordinates as linearly independent, and the tangent vectors span the vector space as we can write,

$$X_p = \left(\frac{dx^\mu(\gamma(t))}{dt}\right)_{t=0} \left(\frac{\partial}{\partial x^\mu}\right)_p \ . \tag{2.5}$$

as we described before. $\square$

We can now ask what happens when we change the coordinate basis — i.e. switch to a different chart. To do this consider ϕ and ϕ' being different coordinate basis $\{x^\mu\}$ and $\{x'^\mu\}$ respectively. We look at how the basis vector changes. For any smooth function f, we have,

$$\begin{aligned}\left(\frac{\partial}{\partial x^\mu}\right)_p (f) &= \left(\frac{\partial}{\partial x^\mu}(f \circ \phi^{-1})\right)_{\phi(p)} \\ &= \left(\frac{\partial}{\partial x^\mu}\left[(f \circ \phi'^{-1}) \circ (\phi' \circ \phi^{-1})\right]\right)_{\phi(p)}\end{aligned}$$

Note that $f \circ \phi'^{-1}$ is just the function in coordinates x' and $\phi' \circ \phi^{-1}$ is just the functions $x'^\mu(x)$. So using the chain rule, we can write,

$$\begin{aligned}\left(\frac{\partial}{\partial x^\mu}\right)_p (f) &= \left(\frac{\partial}{\partial x^\mu}\left[(f \circ \phi'^{-1})(x'(x))\right]\right)_{\phi(p)} \\ &= \left(\frac{\partial x'^\nu}{\partial x^\mu}\right)\left(\frac{\partial}{\partial x'^\nu}\right)_p (f) \ .\end{aligned}$$

So now we can compare the coordinate basis to get,

$$\left(\frac{\partial}{\partial x^\mu}\right)_p = \left(\frac{\partial x'^\nu}{\partial x^\mu}\right)\left(\frac{\partial}{\partial x'^\nu}\right)_p \tag{2.9}$$

We like transforming the components of the vectors themselves instead. So let us write $X = X^\mu \partial_\mu = X'^\nu \partial'_\nu$, which gives,

$$X = X^\nu \left(\frac{\partial}{\partial x^\nu}\right)_p = X^\nu \left(\frac{\partial x'^\mu}{\partial x^\nu}\right)\left(\frac{\partial}{\partial x'^\mu}\right)_p \ , \tag{2.10}$$

and hence we have,

$$X'^\mu = \left(\frac{\partial x'^\mu}{\partial x^\nu}\right)_{\phi(p)} X^\nu \ . \tag{2.11}$$

This is exactly the transformation rule for the contravariant vectors you have seen in B2! So it is evident that we have found the correct objects — the vectors we have just described are exactly the vectors of contravariant form in the Minkowski sense. Cool!

2.3 Covectors and tensors

We have now found a natural vector space for our manifolds, the tangent space at p, T_pM. Now we can use what we have learnt in our linear algebra courses to construct new vector spaces.

1. **Dual spaces.** Recall that for V a real vector space, we can define the **dual space of** V, V^* as the vector space of linear maps from V to $\mathbb{R}$. To this basis we can define a dual basis f^μ which acts on the basis e_μ of V as $f^\nu(e_\mu) = \delta^\nu_\mu$. The dual of dual is isomorphic to the original vector space V.

2. **Direct products.** Recall that if V_i, $i = 1, \ldots, n$ are vector spaces, the **direct product** is the vector space which the underlying set is the Cartesian product $V_1 \times \cdots \times V_n$. In particular, if $(v^i_1, \ldots, v^i_n)$, $(w^i_1, \ldots, w^i_n) \in V_1 \times \cdots \times V_n$, we will have,

$$(v^i_1, \ldots, v^i_n) + (w^i_1, \ldots, w^i_n) = (v^i_1 + w^i_1, \ldots, v^i_n + w^i_n) \ , \tag{2.12}$$

$$\lambda \cdot (v^i_1, \ldots, v^i_n) = (\lambda v^i_1, \ldots, \lambda v^i_n) \ . \tag{2.13}$$

These elementary facts allow us to construct new objects using T_pM. Before we continue however, let us have a look at some multilinear algebra which will be important in our discussion.

Definition 2.9. Let $V_1, \ldots, V_n$ and W be vector spaces. A map $f : V_1 \times \ldots V_n \to W$ is **multilinear** if it is linear as a function in each variable separately when the others are held fixed. So for each i,

$$f(v_1, \ldots, \alpha v_k + \beta w_k, \ldots, v_n) = \alpha f(v_1, \ldots, v_k, \ldots, v_n) + \beta f(v_1, \ldots, w_k, \ldots, v_n) \ . \tag{2.14}$$

To construct tensor spaces, we need to think about the vector space of multilinear functions $\text{Maps}(V_1, \ldots, V_k; W)$. This space should really be viewed as the set of all linear combination of objects of the form $(w_1, \ldots, w_n)$ where $w_1 : V_1 \to W$ is the dual vector for V_1. Note that this is different from direct products! The elements of direct products are just the elements of the form $(v_1, \ldots, v_n)$, whereas for tensor products we will now need to define as follows.

Definition 2.10. The **tensor product** $U \otimes V$ is the $\mathbb{K}$-vector space generated by the symbols $u \otimes v$ with $u \in U$ and $v \in V$, modulo the relations,

$$(\lambda_1 u_1 + \lambda_2 u_2) \otimes v = \lambda_1(u_1 \otimes v) + \lambda_2(u_2 \otimes v) \ , \tag{2.15}$$

$$u \otimes (\lambda_1 v_1 + \lambda_2 v_2) = \lambda_1(u \otimes v_1) + \lambda_2(u \otimes v_2) \ . \tag{2.16}$$

Alternatively, tensor products can be defined as the formal linear combinations of vector spaces, with a general element being in the form,

$$\sum c_i(v_1^i, \ldots, v_n^i) \in V_1 \otimes \cdots \otimes V_n \ . \tag{2.17}$$

This abstract way of defining tensor products is perhaps a bit confusing, but in fact there is a canonical isomorphism between the spaces, given by the following proposition,

Proposition 2.2. *If $V_1, \ldots, V_n$ are finite-dimensional vector spaces, there is a canonical isomorphism,*

$$V_1^* \otimes \cdots \otimes V_n^* \cong \mathit{Maps}(V_1, \ldots, V_k; \mathbb{R}) \ . \tag{2.18}$$

Proof. Key is to use a basis to show the isomorphism, see Proposition 12.10 in [1] for details. □

Abstract tensor spaces

We should really, really define tensor spaces using formal linear combinations, given by the following definition.

Definition 2.11. Let S be a set. A **formal linear combination of elements of** S is a function $f : S \to \mathbb{R}$ such that $f(s) = 0$ for all but finitely many $s \in S$. The **free vector space on** S, denoted as $\mathcal{F}(S)$ is the set of all formal linear combination of elements of S. $\mathcal{F}(S)$ is then a vector space over $\mathbb{R}$ under pointwise addition and scalar multiplication.

To construct tensor products, we start with $V_1, \ldots, V_n$ real vector spaces and form the free vector space $\mathcal{F}(V_1 \times \cdots \times V_n)$ given by the set of all finite formal linear combinations of n-tuples $(v_1, \ldots, v_n)$. Then,

Definition 2.12. The **tensor product space** $V_1 \otimes \cdots \otimes V_n$ is the quotient vector space,

$$V_1 \otimes \cdots \otimes V_n = \mathcal{F}(V_1 \times \cdots \times V_n)/\mathcal{R} \tag{2.19}$$

with the subspace $\mathcal{R}$ generated by elements of the forms,

$$(v_1, \ldots, av_k, \ldots, v_n) - a(v_1, \ldots, v_k, \ldots, v_n) \ , \tag{2.20}$$

and

$$(v_1, \dots, v_k + w_k, \dots, v_n) - (v_1, \dots, v_k, \dots, v_n) - (v_1, \dots, w_k, \dots, v_n) \, . \tag{2.21}$$

This is just a formal way of defining tensor spaces, but you will see the mathematical construction $V = A/\sim$ occur very commonly in mathematics and physics — this is just a normal way of constructing a space where we have some additional identifications (here being the 'normal linearity rules' of the tensor product).

Enough of this mathematical spill. What can we now do for T_pM?

Let us first consider dual spaces. Clearly we can construct some dual space of T_pM.

Definition 2.13. The dual space of T_pM is denoted T_p^*M and is called the **cotangent space at** p. An element of this space is called a **covector at** p.

We can similarly construct a coordinate basis for the covector space, but we will first need a notion.

Definition 2.14. Let $f : M \to \mathbb{R}$ be a smooth function. Define a covector $(df)_p$ called **gradient of** f **at** p as,

$$(df)_p(X) = X(f) \tag{2.22}$$

for any vector $X \in T_p(M)$.

In a local chart, we can now pick $f = x^\mu$. This means that we can have,

$$(dx^\mu)_p \left(\left(\frac{\partial}{\partial x^\nu} \right)_p \right) = \delta^\mu_\nu \, , \tag{2.23}$$

and $\{(dx^\mu)_p\}$ is the dual basis of $\left\{ \left(\frac{\partial}{\partial x^\mu} \right)_p \right\}$. You can also check that the gradient of function f now has coordinate basis,

$$[(df)_p]_\mu = \left(\frac{\partial (f \circ \phi^{-1})}{\partial x^\mu} \right)_{\phi(p)} \, , \tag{2.24}$$

and by changing the coordinate basis you will now get the usual covector transformation rule,

$$(dx^\mu)_p = \left(\frac{\partial x^\mu}{\partial x'^\nu} \right)_{\phi'(p)} (dx'^\nu)_p \, , \tag{2.25}$$

$$\omega'_\mu = \left(\frac{\partial x^\nu}{\partial x'^\mu} \right)_{\phi'(p)} \omega_\nu \, . \tag{2.26}$$

Yay! Now let us move on to tensors. Following our discussion of tensor spaces, we can now define tensors as a multilinear map as follows.

Definition 2.15. A tensor of type (p, q) at p is the multilinear map,

$$T : T_p^*M \times \cdots \times T_p^*M \times T_pM \times \cdots \times T_pM \to \mathbb{R} \, , \tag{2.27}$$

with p factors of T_p^*M and q factors of T_pM.

Note that the indices are swapped — for every upper (contravariant) index we want a map $T_p^*M \to \mathbb{R}$ as the dual space of the dual is isomorphic to itself. Similarly for every lower (covariant) index we want a map $T_pM \to \mathbb{R}$. By introducing a basis $\{e_\mu\}$ for T_pM and dual basis $\{f^\mu\}$ for T_p^*M we will have the components of T as,

$$T^{\mu_1 \dots \mu_p}{}_{\nu_1 \dots \nu_q} = T(f^{\mu_1}, \dots, f^{\mu_p}, e_{\nu_1}, \dots, e_{\nu_q}) \ . \tag{2.28}$$

Let me introduce briefly here what **abstract index notation** is. So far we have used Greek letters $\mu, \nu, \dots$ to denote the components of a tensor (vectors and covectors are just $(1,0)$- and $(0,1)$-tensors respectively). The abstract index notation replaces the Greek letters with Latin letters — it simply tells you the object is a tensor of a certain type. This avoids the fact that sometimes we might have $X^1 = 0$ in some basis but $X'^1 \neq 0$ in a different basis. So from now on when you see T^a you should just treat this as, 'ah, this is a vector!'

2.4 Tensor fields

Well, we have really only defined tensors at a point haven't we. But in physics we really want to understand how these things vary in spacetime — so we will need to vary them across the manifold. So we will need some notion for that. Let's start with a vector field.

Definition 2.16. A **vector field** is a map X which maps any point $p \in M$ to a vector $X_p \in T_pM$. This allows us to define a new function $X(f) : M \to \mathbb{R}$ with $X(f) : p \mapsto X_p(f)$. The vector field is **smooth** if $X(f)$ is a smooth function for any smooth f.

This can be easily generalised to covectors and tensors.

Definition 2.17. A **covector field** is a map ω which maps any point $p \in M$ to a covector $\omega_p \in T_p^*M$. Similar to the vector case, with a vector field and covector field we can define the function $\omega(X) : M \to \mathbb{R}$ by $\omega(X) : p \mapsto \omega_p(X_p)$. The covector field is **smooth** if $\omega(X)$ is a smooth function for any smooth X.

The principle is thus — to translate the idea of smoothness we start with some smooth function f (which we can use local coordinates to analyse the properties), and then define new functions by composing $X(f)$ for vectors and then $\omega(X)$ for covectors. Neat. This then gives the definition of tensor fields.

Definition 2.18. A **tensor field** is a map T which maps any point $p \in M$ to a tensor T_p at p. Again we define a function $T(\eta_1, \dots, \eta_p, X_1, \dots X_q) : M \to \mathbb{R}$. The tensor field is **smooth** if $T(X)$ is a smooth function for any collections of smooth η_i and X^i.

Bundles

The neat notion for defining tensor fields is actually to use the idea of bundles. Let me give you a short introduction here, and only illustrate for vector bundles.

Definition 2.19. A **diffeomorphism** between two manifolds $f : M \to N$ is a smooth map with a smooth two-sided inverse.

Definition 2.20. A **vector bundle of rank k over manifold X**, $E \to X$ is a manifold equipped with the following data.

(i) Call E is the **total space**, X is the **base space**.

(ii) A smooth surjection map $\pi : E \to X$ called a **projection**.

(iii) An open cover $\{U_\alpha\}$ of X such that for each α there is a diffeomorphism called **local trivialisations**,

$$\Phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^k \ , \tag{2.29}$$

such that $\mathrm{pr}_1 \circ \Phi_\alpha = \pi$ (pr_1 means projecting to the first item in the map), and that for all α and β the map $\Phi_\beta \circ \Phi_\alpha^{-1}$ has the form,

$$(U_\alpha \cap U_\beta) \times \mathbb{R}^k \to (U_\alpha \cap U_\beta) \times \mathbb{R}^k \tag{2.30}$$

with $x \in \mathbb{R}^k$,

$$(b, x) \mapsto (b, g_{\beta\alpha}(b)(x)) \ , \tag{2.31}$$

for some smooth map $g_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(k, \mathbb{R})$ called **transition functions**.

Effectively, a vector bundle is just a way of organising vector data on a manifold (see Figure 2.5). The transition functions here are very similar to the transition functions we have seen for patches (or charts) — they basically just tells us how to go from one description of vector space to a different description of a vector space. In particular, this will allow some twisting of the vector space — the vector space is no longer necessarily just a product space on top of the manifold $E = X \times \mathbb{R}^k$, but we can twist the $\mathbb{R}^k$ in some weird way that admits this smooth patching using the transition functions.

With this we can define the vector fields as follows.

Definition 2.21. A **local section** of a vector bundle $E \to X$ is the map $s_\alpha : U_\alpha \to E$ such that $\pi \circ s_\alpha = \mathrm{id}_X$. A **gloval section** of a vector bundle $E \to X$ is the map $s : X \to E$ such that $\pi \circ s = \mathrm{id}_X$.

Definition 2.22. A **vector field on X** is a global section on the vector bundle $E \to X$.

Crazy how we can describe objects with geometrical notions! We will revisit this when we talk about connections.

3 Metrics

We have discussed a lot about how important metrics are in §1. So we should really define what they are, as a mathematical object.

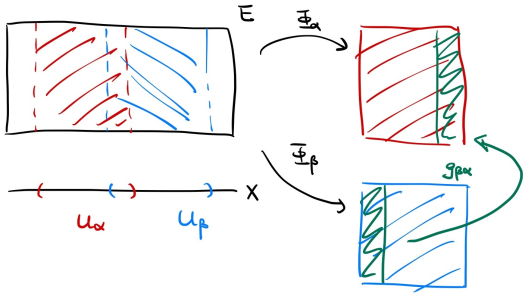

Figure 2.5: An illustration for a vector bundle.

3.1 A first look at metrics

I will define metrics first.

Definition 3.1. A **metric tensor** at $p \in M$ is a $(0, 2)$-tensor g with the following properties:

1. Symmetry: $g(X, Y) = g(Y, X)$ for all $X, Y \in T_p(M)$.
2. Non-degeneracy $g(X, Y) = 0$ for all $Y \in T_p(M)$ iff $X = 0$.

I like to think of the reason why a metric tensor needs to vectors as follows. Imagine you are at some point on the manifold and wants to measure the distance between two points far away. You will, at first instance, need to find a way to go to those two points (since things can only be measured locally, as discussed) so we will need to find the curves that will allow us to traverse to those two points. Hence we will get two vectors! Of course, it does not matter which point we go first (hence the symmetry requirement); nor should this intrinsic notion of distance be defined when I cannot reach there via a path (i.e. when $X = 0$, which justifies the non-degeneracy requirement).

We will have a bit more to say about this next time. In particular, we still need to define some signature on this tensor!

3.2 Embedding

We have looked at embeddings in the first example sheet. Let me define a bit more clear what that means.

Under construction

Lucas is tired and still working on this.

4 Geodesics

5 Connections

6 Curvature

References

[1] J. Lee, *Introduction to Smooth Manifolds.* Graduate Texts in Mathematics, Springer, 2003.