

Groups and Representations Tutorial Class

Supplementary Notes

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ABSTRACT: This set of notes is my attempt on summarising and expanding on the topics and discussion points made during the tutorial classes. These only act as a set of supplementary notes and are not designed to replace the course notes or as solutions to the problem sheets. You should not share these notes without permission.

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1 General References, Articles and Books

The canonical reference for the course is Fulton and Harris' *Representation Theory* [1]. This is a really good book for both mathematicians and theoretical physicists and it provides a rigorous treatment to all the topics covered in the course. It might be, however, useful to look elsewhere if this is a bit too mathematically rigorous for you.

Here are some books mainly catering to mathematicians. For general algebra Dummit and Foote is a standard [2]. For finite groups it might be helpful to look at James and Liebeck's "Representations and Characters of Groups" [3]. For Lie algebras and Lie groups, have a look at Humphrey, Erdmann or Hall [4–6]. For some differential geometry background, Lee's "Introduction to Smooth Manifolds" [7] is a very good introduction and reference.

For physicist-oriented books, the best book is Georgi's "Lie Algebras in Particle Physics" [8]. I didn't like it when I was doing the course¹, but I later found out it is one of the best books on the subject! Fuchs and Schweigert's "Symmetries, Lie algebras and Representations" [9] offers a comprehensive rigorous view on the subject which also has a lot of interesting topics in it. Finally Costa and Fogli's book is quite good as well [10] - I did use it for quite a bit when I was learning the subject.

Throughout the notes I will cite the relevant references and articles. You should go look at them if you want a bit more information on the topics I have selected here. My presentation is nowhere coherent or formal enough - it is just the kind of style I like². I apologise if there are any typos/ mistakes in the notes but do email me if you spot them!

¹Well, okay, the Symmetries, Particles and Fields course in Cambridge.

²It's quite formal for most theoretical physicists. Sorry. I just like my weird sense of mathematical and physical intuition so you will find comments dotted around...

2 Basic Group Theory

In the first class we mainly discussed finite groups and their representations. The main references here are [1, 2]. In this section, I would like to highlight some important topics that you might not have encountered as part of your undergraduate course, namely the subgroup test and some facts about permutation groups.

2.1 Subgroup Test

When marking I noticed a lot of you tried to show $H \leq G$ (H is a subgroup of G) by showing all the axioms, namely: Closure, Associativity, Identity and Inverse³. This is generally not too helpful when you are trying to show a number of groups as subgroups of a larger group - you will be doing the same tedious arguments again and again! So I want to introduce some test here that will shorten the argument by a bit [11].

Theorem 2.1 (Subgroup Test). *Let H be a non-empty subset of a group G . Then H is a subgroup if and only if $\forall h_1, h_2 \in H, h_1 h_2^{-1} \in H$.*

Proof. The forward direction is obvious (from group axioms, check!). So let us show the converse. Assume $\forall h_1, h_2 \in H, h_1 h_2^{-1} \in H$. We see that $e \in H$, since $h h^{-1} \in H$. This also shows that $h^{-1} \in H$. Now take $h_2^{-1} \in H$, so $h_1 (h_2^{-1})^{-1} \in H, \forall h_1, h_2 \in H$. This means H is closed under the multiplication inherited from G . Associativity clearly descends from G , so all axioms are satisfied. \square

In the exam I would clearly state I am using the **subgroup test** if I am using this theorem - this is a trivial group theoretical result that you probably do not need to provide a proof for.

2.2 Permutation Groups

In Q3 of the first problem sheet we were asked about the conjugacy classes of S_3 . However, a lot of you simply explicitly calculated that these are in fact the only conjugacy classes of the group - this is doable for small finite groups, but in general not a feasible method. Some people stated the fact that the conjugacy classes are all defined by the cycle type, but without proof. Here I want to show that this is true in general for S_n . First with the following Lemma.

Lemma 2.1. *Let $\alpha, \beta \in S_n$, α a k th cycle $(a_1 \dots a_k)$. Then*

$$\beta \alpha \beta^{-1} = (\beta(a_1) \dots \beta(a_k)) , \quad (2.1)$$

i.e. conjugation does not change the cycle type.

Proof. Consider $\beta(a_i)$, then $\beta^{-1} \beta(a_i) = a_i$, $\alpha(a_i) = a_{i+1 \bmod k}$, and

$$\beta \alpha \beta^{-1}(\beta(a_i)) = (\beta(a_{i+1 \bmod k})) . \quad (2.2)$$

³Kit: By the way, it will be a great idea if you name these during an exam!

Take $b_j \neq a_i, \forall i$. Then $\alpha(b_j) = b_j$ and

$$\beta\alpha\beta^{-1}(\beta(b_j)) = \beta(b_j) . \quad (2.3)$$

So $\beta\alpha\beta^{-1}$ fixes any number not in the form $\beta(a_i)$ for some i . Therefore, we deduce that

$$\beta\alpha\beta^{-1} = (\beta(a_1)\dots\beta(a_k)) . \quad (2.4)$$

□

This lemma tells us that the conjugacy classes are determined by cycle type. In fact, the cycle type uniquely determines the conjugacy classes. To see this we need:

Theorem 2.2. *Every permutation is uniquely expressible as a product of disjoint cycles.*

Proof. Given $g \in S_n$, the cyclic subgroup $\langle g \rangle \leq S_n$ generated by g acts on some set $X = \{1, \dots, n\}$ decomposes X into distinct orbits:

$$O_x = \{g^i x \mid i \in \mathbb{Z}\} \quad (2.5)$$

for choices of orbit representations of $x \in X$. For each x , let N_x be the order of g when restricted to the orbit $\langle g \rangle \cdot x$ and define the cycle

$$C_x = (x, gx, \dots, g^{N_x-1}x) . \quad (2.6)$$

Since distinct orbits are disjoint, the cycles are disjoint. Given $y \in X$, we choose an orbit representation x such that $y \in \langle g \rangle \cdot x$. Then $g \cdot y = C_x y$ and g is the product of the cycles C_x over the orbit representations x . □

So now we have the theorem:

Theorem 2.3. *The conjugacy classes of any S_n are uniquely determined by cycle type.*

Proof. This follows immediately from Theorem 2.2 but now we are considering a product of cycles (α in Theorem 2.2 is a single cycle)⁴. □

We can think of this as follows - the conjugation simply relabels the numbers inside the cycle when the element $\sigma \in S_n$ is written in cycle notation. So the cycle type is just the conjugacy class!

⁴If you want it more fleshed out here are the details: Write $\sigma = \alpha_1 \dots \alpha_l$ for $\sigma \in S_n$. Then for $\tau \in S_n$, we have

$$\tau\sigma\tau^{-1} = (\tau\alpha_1\tau^{-1})\dots(\tau\alpha_l\tau^{-1}) \quad (2.7)$$

From Lemma 2.1, we have that $\tau\alpha_i\tau^{-1}$ also a k_i -cycle. Now for $i \neq j$, α_i and α_j are disjoint and $\tau\alpha_i\tau^{-1}$ and $\tau\alpha_j\tau^{-1}$ are also disjoint since τ is injective. Therefore, $\tau\sigma\tau^{-1}$ are also separated into disjoint cycles. To show uniqueness, we note that Theorem 2.2 states that every permutation can be uniquely described as a product of disjoint cycles. Since this cycle type is unchanged under conjugation, the conjugacy classes therefore uniquely correspond to all possible cycle types constructible in S_n .

3 Representation Theory - Character Tables

Most of you did very well when tackling character theory questions. I just want to summarise some facts here and give you a general direction as to how to construct character tables for finite groups (you can probably generalise this to Lie groups - try it)!

To construct the character table, we use the steps:

1. Construct/Determine all the conjugacy classes of the group. You can probably use some smart argument with the symmetric group S_n - using Cayley's Theorem [2] that all groups are isomorphic to subgroups of the symmetric group.
2. Now use Theorem 2.8 of notes - the number of irreducible representations of G is equal to the number of conjugacy classes.
3. Use Theorem 2.3 of notes:

$$\sum_i (\dim R_i)^2 = |G| \quad (3.1)$$

to find the dimensions of the irreps.

4. Use the fact that there is always a trivial representation to construct one dim-1 representation.
5. Use row and column orthogonality of characters - it is the easiest!
6. If you don't want/cannot to use orthogonality - find some clever ways to construct other dimension-1 characters, e.g. taking determinants of an existing representation. Or see if you can get anything from the regular representation. Or... see if you can argue that the matrix representation given is in fact an irrep!

You already know what *row orthogonality* is: this is just Theorem 2.1 of notes:

$$(\chi, \tilde{\chi}) = \frac{1}{|G|} \sum_{g \in G} \overline{\chi(g)} \tilde{\chi}(g) \quad (3.2)$$

or

$$(\chi_i, \chi_j) = \delta_{ij} \quad (3.3)$$

for irreps R_i and R_j . What is *column orthogonality*? It is in fact a corollary of Theorem 2.1 of the notes:

Corollary 3.1. *For irreps R_i , and conjugacy class c_I , we have*

$$[\chi_I, \tilde{\chi}_J] = \frac{|c_I|}{|G|} \sum_i \overline{\chi_i(g_I)} \tilde{\chi}_i(g_J) = \delta_{IJ} \quad (3.4)$$

with A, B indicating the different conjugacy classes and $|c_I|$ the number of elements in the conjugacy class labelled I .

Proof. From simple linear algebra. If we rescale the original character table by $\sqrt{\frac{\dim(R_i)}{|G|}}$ for each i (i.e. each row), we obtain a unitary matrix. The transpose of a unitary matrix is unitary, so the column vectors are orthonormal to each other. The correct scaling is deduced by the redefinition of the sum. An alternative proof is as follows. We define class functions ψ_s for $1 \leq s \leq k$ by

$$\psi_s(g_r) = \delta_{rs} , \quad (3.5)$$

where k is the number of conjugacy classes. As characters form a basis of the space of class functions⁵, ψ_s is a linear combination of χ_i ,

$$\psi_s = \sum_{i=1}^k \lambda_i \chi_i . \quad (3.6)$$

We know that for irreps $(\chi_i, \chi_j) = \delta_{ij}$. So we must have

$$\lambda_i = (\chi_i, \psi_s) = \frac{1}{|G|} \sum_{g \in G} \overline{\chi_i(g)} \psi_s(g) . \quad (3.7)$$

By definition of ψ_s , it is 1 if g is conjugate to g_I and 0 otherwise. So this gives:

$$\lambda_i = \frac{\chi_i(g_I) |g_I|}{|G|} , \quad (3.8)$$

where $|g_I|$ is the number of elements in the conjugacy class represented by g_I . So substituting into the definition of ψ_s gives

$$\delta_{IJ} = \frac{|g_I|}{|G|} \sum_i \overline{\chi_i(g_I)} \chi_i(g_J) , \quad (3.9)$$

as required. □

Remark 3.1. Column orthogonality strictly speaking gives no new information - but it might speed up your calculation when you are trying to deduce the entire character table!

⁵See, for example, [1].

4 More on Group Algebras

In the lectures **group algebras** are defined in a really concise manner - it might be difficult to really understand what it is though! Here I would like to highlight some facts about group algebras and how they are constructed. This section is mainly for your reference and is entirely out-of-syllabus!

4.1 Groups algebras - Definitions

First recall the definition in the lecture.

Definition 4.1. The **group algebra** A_G of group G is defined as the set of formal linear combinations:

$$v = \sum_{g \in G} v(g)g \quad (4.1)$$

where $v(g) \in \mathbb{C}$ all $g \in G$ are considered to be linearly independent.

This definition might appear to be a bit confusing - but if you look at it again this is actually not too bad to work with. The group algebra is effectively like “a vector space” with the group elements $\{e_g\}$ being the basis for this vector space. Let us make this notion a bit more precise by recalling some definitions from elementary algebra⁶:

Definition 4.2. A **ring** R is a set with two binary operations (addition and multiplication) such that

1. R is an abelian group with respect to addition.
2. Multiplication is associative and distributive over addition. i.e.

$$(xy)z = x(yz) \quad (4.2)$$

$$x(y+z) = xy + xz, (y+z)x = yx + zx \quad (4.3)$$

3. $xy = yx, \quad \forall x, y \in R$
4. There is an identity element $\exists 1 \in R$ such that $x1 = 1x = x, \quad \forall x \in R$. This identity element is then unique.

Definition 4.3. Let R be a ring. An **R -module** is an abelian group M on which R acts linearly. More precisely, it is the pair (M, μ) where M is an abelian group and $\mu : R \times M \rightarrow M$ is a mapping such that (writing $\mu(r, m)$ as rm for $r \in R, m \in M$) the following axioms are satisfied:

1. $r(x+y) = rx + ry$
2. $(r+s)x = rx + sx$

⁶I say elementary - this might not be if you haven't seen these notions before! One can consult a basic algebra book for details, say [2, 11].

$$3. (rs)x = r(sx)$$

$$4. 1x = x$$

A module is kind of like a vector space using rings as the “scalar”. In fact, if R is a field k , then an R -module is just a k -vector space. Saying modules is kind of like vector spaces is a very imprecise way of putting it at that - but it’s close enough for what we need to know. We note the following equivalent definition of a module (which will become important later).

Definition 4.4. An R -**module** M is an abelian group together with a ring homomorphism $R \rightarrow E(M)$ where $E(M)$ is the ring of endomorphisms of the abelian group M .

So why do we need to know something about modules? It turns out that group algebra, or FG -**modules**, turns out to be representations of G . This requires a bit more explanation. Let’s start with a definition of FG -modules.

Definition 4.5. Let R be a commutative⁷ ring. An **associative R -algebra** (or an R -algebra simply) is a ring S that is also an R -module such that the two additions (ring and module) are the same operation and scalar multiplication satisfies:

$$r \cdot (xy) = (r \cdot x)y = x(r \cdot y) . \quad (4.4)$$

Equivalently, let $f : R \rightarrow S$ be a ring homomorphism. Define a product

$$r \cdot s \equiv f(r)s , \quad (4.5)$$

with $r \in R$ and $s \in S$. This definition of scalar multiplication turns S into an R -module. The ring S equipped with this R -module structure is then an R -**algebra**.

We also define the following notion.

Definition 4.6. A **free R -module** is one which is isomorphic to an R -module of the form

$$\bigoplus_{i \in I} M_i , \quad (4.6)$$

where each $M_i \cong R$ behaving as an R -module.

Again, looks quite complicated but the idea is simple. A free R -module is one that has a basis - which is a subset of M being the generating set of M and also having linearly-independent elements. This means that for some element in the free R -module x , there exists unique non-zero elements $r_i \in R$ and $m_i \in M$ such that

$$x = \sum_i r_i m_i . \quad (4.7)$$

This is quite important - it means that you can effectively disentangle your complicated module into simple R -modules. Now with all of these defined, let us now look at what a group algebra is.

⁷Recall this means multiplication defined above is commutative.

Definition 4.7. The **group algebra** $R[G]$ of a group G over a ring R is the associative algebra whose elements underlying R -module is the free module over R on the underlying set of G and whose multiplication is given on these basis elements by group operation in G .

Let us unpack this definition - it looks a bit complicated. Firstly, there is an underlying R -module. This means that we are kind of working with a vector space⁸ of some-sort. This underlying module is a free module - this means that any element x in this group algebra can be uniquely written in the form

$$x = \sum_{i \in I} r_i g_i \quad (4.8)$$

This exactly matches up with what we have defined at the very start of the section (i.e. in Andre's notes). The group algebra is the "formal" linear combination of group elements. Now the second point - multiplication of these elements are given by the group operation in G . Let us, for clarity, write the basis elements of the module as $\{e_g\}$ for the moment. This warrants the following algebraic structure:

$$e_g \cdot e_h = e_{gh} \quad (4.9)$$

So it is not actually that hard - RG is kind of like a vector space with the following structure. Writing

$$x = \sum_{g \in G} r_g e_g \quad (4.10)$$

$$y = \sum_{h \in G} s_h e_h \quad (4.11)$$

Then in the group algebra we can write

$$\begin{aligned} x \cdot y &= \sum_{g, h \in G} r_g s_h e_g e_h \\ &= \sum_{g \in G} \sum_{h \in G} r_g s_h e_{gh} \\ &= \sum_{g \in G} \sum_{h \in G} r_h s_{h^{-1}g} e_g \end{aligned} \quad (4.12)$$

So the two definitions align. Phew!

4.2 FG-modules and Representations

Recall in the lectures we defined a representation as follows (for finite groups):

Definition 4.8. A **representation** of a finite group G on a finite-dimensional complex vector space V is a homomorphism $\rho : G \rightarrow GL(V)$ of G to the group of automorphisms of V .

⁸Recall that a vector space is a specific kind of modules.

It's quite clear what this means - a representation "represents" how the group elements "acts" on your vector space of choice V . Since any map $g' : V \rightarrow V$ will be an automorphism - it is quite clear that this definition aligns with the idea of what a group element should do. We want to rephrase this using group algebras. How do we do that? Let us first make a definition.

Definition 4.9. An FG -module is a vector space over field F with left multiplication by elements of the group G . The following axioms must then be satisfied $\forall u, v \in V, \lambda \in F, g, h \in G$:

1. $gv \in V$
2. $(gh)v = g(hv)$
3. $1v = v$
4. $\lambda gv = g(\lambda v)$
5. $g(u + v) = gu + gv$

Technically you can generalise this notion to RG -modules - an algebra is a ring so the module in general will not be a vector space. But with this specific algebra this is indeed the case. Now let us make the following definition.

Definition 4.10. A representation of the group algebra on a vector space V is an algebra homomorphism $\rho : FG \rightarrow \text{End}(V)$.

In the following we will specifically consider the case where the field F is chosen to be the field of complex numbers \mathbb{C} .

Proposition 4.1. A representation on V of $\mathbb{C}G$ is a left $\mathbb{C}G$ -module.

Proof. This is obvious. Check the definitions. \square

Now we can draw the following conclusion. A representation $\rho : G \rightarrow \text{Aut}(V)$ will extend by linearity to a map $\tilde{\rho} : \mathbb{C}G \rightarrow \text{End}(V)$. This means that representations of $\mathbb{C}G$ will correspond exactly to representations of G . In particular, we have the following special case:

Definition 4.11. The **regular representation** R_G corresponds to the left action of G on itself. Equivalently, R_G is the space of complex-valued functions on G where an element $g \in G$ acts on a function α by

$$(g \cdot \alpha)(h) = \alpha(g^{-1}h) \tag{4.13}$$

Proposition 4.2. The left $\mathbb{C}G$ -module given by $\mathbb{C}G$ itself corresponds to the regular representation.

Proof. This is a simple proof - the action of an element on $\mathbb{C}G$ on itself is expressed in Equation 4.12. This is exactly the regular representation where this action is simply the lift by extending the definition of the regular representation above by linearity. \square

Remark 4.1. It is now trivial to check any definition of regular representations in your undergraduate course is exactly the same as the definition above. Simply set a basis for V , the representation space, and it should be obvious how Definition [4.11](#) matches up with the stated definition.

5 Young Tableaux and Projectors

In this section I would like to discuss about Young Tableaux and representation projectors. Young Tableaux is a vast subject - they are extremely useful in describing the representations of S_n . As a result, they are used in a variety of contexts such as describing tensors and projectors. Meanwhile, projectors are extremely important in physics as they are effectively the operators that allow take you to the relevant space of the irreducible representation. Here I aim to provide a summary of results regarding Young Tableaux, particularly highlighting the use of Young Tableaux in describing projectors. The main references for this section are [1, 12].

5.1 Young Diagrams and Tableaux

We first recall that the number of irreducible representations of S_n is the number of conjugacy classes, i.e. the number of partitions $p(n)$ of $n = \lambda_1 + \dots + \lambda_k$. To each partition we will associate a *Young diagram*⁹:

Definition 5.1. A **Young diagram** is a collection of boxes, or cells, arranged in left-justified rows, with a weakly decreasing number of boxes in each row. Listing the number of boxes in each row gives a partition $p(n)$ of integer n , where it is also the total number of boxes. Conversely, as stated above, each partition correspond to a Young diagram.

Example 5.1. Consider the partition $11 = 6 + 4 + 1$, or $(6,4,1)$. Then the corresponding Young diagram is

$$\begin{array}{|c|c|c|c|c|c|} \hline & & & & & \\ \hline & & & & & \\ \hline & & & & & \\ \hline & & & & & \\ \hline & & & & & \\ \hline \end{array} \quad (5.1)$$

Definition 5.2. A **Young tableau** is a filling (i.e. entering numbers into boxes) of a Young diagram such that the numbers are

1. weakly increasing across each row
2. strictly increasing down each column

A **standard Young tableau** is a tableau in which entries are numbers from 1 to n , n being the total number of boxes.

Example 5.2. Consider the partition $(5,2,1)$. Then one of the possible corresponding Young tableau is:

$$\begin{array}{|c|c|c|c|c|} \hline 1 & 2 & 5 & 6 & 8 \\ \hline 3 & 4 & & & \\ \hline 7 & & & & \\ \hline \end{array} \quad (5.2)$$

⁹This is also called a Ferrers diagram or a Young frame according to [1]. Haven't really seen these names before to be honest.

5.2 Projection Formulae

Before we continue, let us revisit the topic of projection formulae. Firstly, recall that we have the trivial projection operator that projects onto the direct sum of trivial factors in the decomposition. In the lectures, we have used this to obtain several results in character theory, namely to derive character orthogonality. For completeness I will recap this here.

Definition 5.3. We define the direct sum of trivial factors in a representation¹⁰ V of a group G to be

$$V^G = \{v \in V \mid gv = v \quad \forall g \in G\} . \quad (5.3)$$

We want a way to find V^G explicitly - to do this we will need a G -module homomorphism¹¹ so we take the average of all these endomorphisms and set

$$\varphi = \frac{1}{|G|} \sum_{g \in G} g \in \text{End}(V) , \quad (5.4)$$

which this will be G -linear as $\sum g = \sum hgh^{-1}$. We then have the following proposition.

Proposition 5.1. *The map φ is a projection of V onto V^G .*

Proof. First, suppose $v = \varphi(w) = \frac{1}{|G|} \sum gw$. Then for any $h \in G$ we have

$$hv = \frac{1}{|G|} \sum hgw = \frac{1}{|G|} \sum gw , \quad (5.5)$$

such that the image of φ is contained in V^G . Conversely, if $v \in V^G$, then $\varphi(v) = \frac{1}{|G|} \sum v = v$, so $V^G \subset \text{Im}(\varphi)$ and $\varphi \circ \varphi = \varphi$. \square

Remark 5.1. In particular, if we wish to know the number m of the copies of trivial representation appearing in the decomposition of V then we can just compute the trace of φ .

Now let us move on to derive a more general projection formula. Let us first have the following Proposition.

Proposition 5.2. *Let $\alpha : G \rightarrow \mathbb{C}$ be any function on the group G , and for any representation V of G set*

$$\varphi_{\alpha,V} = \sum \alpha(g) \cdot g : V \rightarrow V . \quad (5.6)$$

Then $\varphi_{\alpha,V}$ is a homomorphism of G -modules for all V if and only if α is a class function.

¹⁰In this section a representation $R : G \rightarrow \text{End}(V)$ will be indicated by its representation space V . This is just a notation.

¹¹Observe in general that if V is the representation of a finite group G then each G will give a map $\rho(g) : V \rightarrow V$ but this is in general not a G -module homomorphism as for general $h \in G$ we have $g(h(v)) \neq h(g(v))$.

Proof. We have

$$\begin{aligned}
\varphi_{\alpha,V}(hv) &= \sum \alpha(g) \cdot g(hv) \\
&= \sum \alpha(hgh^{-1}) \cdot hgh^{-1}(hv) \\
&= h \left(\sum \alpha(hgh^{-1}) \cdot g(v) \right) \\
&= h \left(\sum \alpha(g) \cdot g(v) \right) \\
&= h(\varphi_{\alpha,V}(v)) ,
\end{aligned} \tag{5.7}$$

where the second line follows from substitution and the fourth follows from α being a class function. This is just the condition that $\varphi_{\alpha,V}$ is G -linear. \square

We can now use this to construct a projection formula. Let us suppose W is a fixed irreducible representation. Then for representation V , we can look at the weighted sum:

$$\psi = \frac{1}{|G|} \sum_{g \in G} \overline{\chi_W(g)} \cdot g \in \text{End}(V) . \tag{5.8}$$

From Proposition 5.2, it is clear that ψ is a G -module homomorphism. Therefore, if V is an irreducible representation, we will then have $\psi = \lambda \text{id}$ (id being the identity map) and therefore

$$\begin{aligned}
\lambda &= \frac{1}{\dim V} \text{Tr } \psi \\
&= \frac{1}{|G| \dim V} \sum \overline{\chi_W(g)} \cdot \chi_V(g) \\
&= \begin{cases} \frac{1}{\dim V} & \text{if } V = W \\ 0 & \text{if } V \neq W \end{cases} .
\end{aligned} \tag{5.9}$$

Therefore, what we have is for arbitrary V , the projector

$$\psi_V = \frac{\dim W}{|G|} \sum_{g \in G} \overline{\chi_W(g)} \cdot g : V \rightarrow V \tag{5.10}$$

is the projection of V onto the factor consisting of the sum of all copies of W (compare with the case with the trivial factors, the projectors act on the sum of the spaces). Therefore, we have that

Definition 5.4. Let the decomposition for a general representation V as

$$V = \bigoplus V_i^{\oplus a_i} , \tag{5.11}$$

where V_i are the inequivalent irreps, the **projector** of V onto $V_i^{\oplus a_i}$ is

$$\pi_i = \frac{\dim V_i}{|G|} \sum_{g \in G} \overline{\chi_{V_i}(g)} \cdot g . \tag{5.12}$$

This means that the projector $\pi^{(i)} : R_{\text{reg}} \rightarrow R_i^{\oplus \Delta_i}$ of the regular representation onto the irreducible subspaces $R_i^{\oplus \Delta_i}$, where $\Delta_i = \dim R_i$ is the dimension of the irrep R_i , is then

$$\pi^{(i)} = \frac{\dim(R_i)}{|G|} \sum_{g \in G} \overline{\chi_i(g)} \cdot g, \quad (5.13)$$

where $|G|$ is the order of the group and $\chi_i(g)$ is the character of the irrep R_i and element g .

5.3 Young Tableaux as Projectors

A standard Young tableau is what is used to describe projection operators for the regular representation onto irreducible representations of S_n . To show this, let us associate

$$P = P_\lambda = \{g \in S_n \mid g \text{ preserves each row}\}, \quad (5.14)$$

together with

$$Q = Q_\lambda = \{g \in S_n \mid g \text{ preserves each column}\}, \quad (5.15)$$

where e_g is the corresponding element of g in the group algebra $\mathbb{C}S_n$. In the same space we can introduce the following terms:

$$a_\lambda = \sum e_g, \quad g \in P, \quad (5.16)$$

and

$$b_\lambda = \sum \text{sgn}(g) e_g, \quad g \in Q, \quad (5.17)$$

λ here is a free index labelling the invariant subspaces of the regular representation. We can now define the following.

Definition 5.5. The **Young symmetriser** is defined to be

$$c_\lambda = a_\lambda \cdot b_\lambda \in \mathbb{C}S_n. \quad (5.18)$$

This gives the following Theorem.

Theorem 5.1. *Some scalar multiple of c_λ is idempotent, i.e. $c_\lambda^2 = n_\lambda c_\lambda$. $\text{im}(c_\lambda)$ is the irreducible invariant subspace V_λ of S_n . Every irreducible representation of S_n can be obtained in this manner for unique partition (i.e. for a valid standard Young tableau).*

Proof. See §4.2 and Theorem 4.3 of [1]. □

I also state two very useful formula for computing the projection operators.

Lemma 5.1. (*Hook Length Formula*) *We have*

$$\dim V_\lambda = \frac{d!}{\prod (\text{Hook lengths})}, \quad (5.19)$$

where the hook length of a box in a Young diagram is the number of squares directly below and to the right of the box, including the box itself once.

Proof. Again see §4, Hook Length Formula 4.12 of [1]. □

In fact we can then write

$$n_\lambda = \frac{d!}{\dim V_\lambda} = \prod (\text{Hook lengths}) . \quad (5.20)$$

This is the inverse of the coefficient c you typically need to find when you write down the projector^{12 13}:

$$\pi_\lambda = c \left(\sum_{\sigma \in R_\lambda} \sigma \right) \left(\sum_{\sigma \in C_\lambda} \text{sgn}(\sigma) \sigma \right) = cc_\lambda \quad (5.22)$$

How does this relate to the projector stated in the previous section, in Eq. (5.13)?

$$\pi^{(i)} = \frac{\dim(R_i)}{|G|} \sum_{g \in G} \overline{\chi_i(g)} \cdot g \quad (5.13)$$

The projector π_i projects onto the sum of irreducible representations R_i of the same dimensions, i.e. onto $R_i^{\oplus \Delta_i}$ where $\Delta_i = \dim R_i$; whereas π_λ in S_n projects onto the individual irreducible representations R_λ of the regular representation, the individual invariant subspaces. Labelling the individual irreps as $R_\lambda^{(i)}$, where λ is the index denoting the irrep within the irreps of the same dimensions and i labelling the irreps of different dimensions, we then have the following **completeness relations**:

$$\sum_i \pi^{(i)} = \text{id} , \quad (5.23)$$

together with

$$\sum_\lambda \pi_\lambda^{(i)} = \pi^{(i)} \quad (5.24)$$

We can check this explicitly in Q3 of the first problem sheet for the two two-dimensional irreducible representations of the symmetric group S_3 . This gives us the following example.

Distributed version:

Exercise 5.1. Try and find the projectors of the two-dimensional irreps of S_3 in Q3 of Sheet 1 again. You should find that the projector π_{2D} in Eq. (5.13) is the sum of the projector into the 2 irreps (invariant subspaces) of the group algebra $\mathbb{C}G$ in Eq. (5.22), or,

$$\pi_{2D} = \pi_{2D,1} + \pi_{2D,2} \quad (5.25)$$

¹²See p.32 of notes.

¹³Here, to calculate coefficient c , a quick way is to use idempotency, i.e.

$$\pi_\lambda^2 = \pi_\lambda . \quad (5.21)$$

6 Spin Groups and Lie Algebra Isomorphisms

In Question 1 of the second sheet we encountered the following Lie algebra isomorphisms:

$$\mathfrak{so}(4) \cong \mathfrak{su}(2) \oplus \mathfrak{su}(2) \quad (6.1)$$

$$\mathfrak{so}(6) \cong \mathfrak{su}(4) \quad (6.2)$$

Of course, in the question you are asked to construct an explicit basis to show that the Lie algebras are isomorphic¹⁴. But now we have the questions:

1. Are there any more Lie algebra isomorphisms like this? Is there an exhaustive list?
2. Is there an alternative proof to this, perhaps without using an explicit basis?

The answer to the first question is actually really straightforward - these are known as real form isomorphisms. Here is a list of these ‘accidental’ isomorphisms of complex classical Lie algebras:

$$\mathfrak{u}(1) \cong \mathfrak{so}(2) \quad (6.3)$$

$$\mathfrak{su}(2) \cong \mathfrak{so}(3) \quad (6.4)$$

$$\mathfrak{su}(2) \oplus \mathfrak{su}(2) \cong \mathfrak{so}(4) \quad (6.1)$$

$$\mathfrak{sp}(2) \cong \mathfrak{so}(5) \quad (6.5)$$

$$\mathfrak{su}(4) \cong \mathfrak{so}(6) \quad (6.2)$$

$$\mathfrak{so}(8) \cong \mathfrak{so}(8) \quad (6.6)$$

The last one (Equation 6.6) is in fact a nontrivial “triality” automorphism of $\mathfrak{so}(8)$ and it is somehow closely related to the exceptional Lie group G_2 .

You can definitely see a pattern here - most of the nontrivial ones are of the form $\mathfrak{so}(n)$ where $n \in \{3, 4, 5, 6\}$. This allows me to answer the second question - of course, one can always check this by using some clever way of finding a basis and checking that these is an explicit Lie algebra isomorphism between them. Another way, of course, is to use the Dynkin formalism to show that the Lie algebras have the same Dynkin diagram (hence they must be isomorphic). These two methods however are not basis-independent - the former explicitly involves a choice of basis, but the latter also implicitly uses the Cartan-Weyl basis. The trick to find a basis-independent method is to notice that this isomorphism is related to the theory of Clifford algebras and spin groups¹⁵. Understanding this is of course way beyond our course, but it is in fact illuminating - you can establish the isomorphism of $\mathfrak{su}(2)$ and $\mathfrak{so}(3)$ by first finding an appropriate cover of a Lie group and translating it back to the Lie algebra level by taking the tangent map. This covering business is intrinsically related to how spins arise and in fact are the reason why the isomorphism arises from spin structures in the first place. To fully understand what is going on we will need to first establish some relevant concepts.

¹⁴I am not allowed to reproduce the answer here sorry - internal policy!

¹⁵I might add a section in the Appendix to explain the concepts in the future, but this is in fact really out of the course and I suggest you actually spend more time on the course material (which is already a lot!). But for enthusiasts (like me), a good place to start is §20 of [1]. The book [13] also has a section on this stuff §6 so feel free to check that out as well.

6.1 Universal Covers and Projective Representations

In the lectures we are introduced to the idea that $SU(2)$ is a *two-fold cover* of the manifold $SO(3)$. This is in fact the exact reason why we have spins. To explain this, we need to first understand the relationships between covers and projective representations.

6.1.1 Universal Coverings

We begin by defining what a covering map is [7].

Definition 6.1. A subset $U \subset X$ is **evenly covered** by π if U is connected and open, and each component $\pi^{-1}(U)$ is an open set that is mapped homeomorphically onto U by π .

Definition 6.2. A **covering map** is a continuous surjective map $\pi : \tilde{X} \rightarrow X$ such that \tilde{X} is path-connected and locally path-connected¹⁶, and every point $p \in X$ has an evenly covered neighbourhood. We call \tilde{X} the covering space of X and X the base of the covering.

Of course, everything so far is in the topology context. To specialise this to smooth manifolds (which is what we want), we will need to restrict the definition to a very specific type of covering map¹⁷.

Definition 6.3. Take E and M connected smooth manifolds with or without boundary. A map $\pi : E \rightarrow M$ is called a **smooth covering map** if π is smooth and surjective, and each point in M has a neighbourhood U such that each component of $\pi^{-1}(U)$ is mapped diffeomorphically onto U by π . We say U is evenly covered. We call M to be the **base manifold**, and E a **covering manifold of M** . If E is simply-connected, it is called the **universal covering manifold of M** .

Here *simply-connected* means every loop is path-isomorphic to a constant path¹⁸. We want to show that this universal covering exists and is in fact unique. I will here quote a few lemmas and theorems without detailed proof - the details can be found in the references [7, 15].

Theorem 6.1. *Suppose M is a connected smooth- n -manifold, and $\pi : E \rightarrow M$ is a topological covering map. Then E is a topological n -manifold, and has a unique smooth structure such that π is a smooth covering map.*

Proof. See Proposition 4.40 of [7]. □

¹⁶It might be surprising to see how path-connectedness does not generally imply locally path-connectedness. A counterexample is the topologist's sine curve, $y = \sin(\frac{1}{x})$ for $x \in (0, \pi)$ together with closed arc connecting $(0, 0)$ and $(\pi, 0)$ where the space is path-connected but not locally path-connected at $(0, 0)$.

¹⁷By the way, if you are completely baffled by the definitions I just made, these are just mathematical details that you can skip (if you want, but I am weird so I will babble on). Alternatively you should pick up some topology books and start learning what topology is.

¹⁸Phrased in the language of the fundamental group at X , simply-connectedness simply means that the fundamental group of a manifold at every point $q \in M$ is the trivial group [7, 14].

Corollary 6.1. *If M is a connected smooth manifold, there exists a simply connected manifold \tilde{M} - the **universal covering manifold of M** , and a smooth covering map $\pi : \tilde{M} \rightarrow M$. The universal covering manifold is unique such that for any other universal covering manifold \tilde{M}' with projection map π' , then there exists a diffeomorphism $\Phi : \tilde{M} \rightarrow \tilde{M}'$ such that $\pi' \circ \Phi = \pi$.*

Proof. This is Corollary 4.43 of [7]. Since a proof is not given there I will give a sketch of the proof. The first step is show that any connected and locally simply connected space admits a unique universal cover. You will need some sort of path-connectedness arguments (see Theorem 12.8 of [15]) - to show the path classes are lifted to the upper space, then check several topological requirements (path-connectedness, topologies, covering maps). Now by Theorem 6.1 you have the existence of a smooth covering manifold of M that is simply-connected. To show uniqueness, we need to find ϕ between any two universal covers that is a diffeomorphism - to show this find open sets such that you can find a surjective smooth submersion of $\pi|_{V^{-1}}$ must give you a smooth ϕ and ϕ^{-1} in both directions. \square

Now it is straightforward to generalise this to Lie groups.

Theorem 6.2. *Let G be a connected Lie group. There exists a simply connected Lie group \tilde{G} , called the universal covering group of G , that admits a smooth covering map $\pi : \tilde{G} \rightarrow G$ that is also a Lie group homomorphism.*

Proof. See Theorem 7.7 of [7]. Essentially the idea is you now need to also do group axiom checks on the universal covering group. \square

Theorem 6.3. *For any connected Lie group G , the universal covering group is unique in the following sense: if \tilde{G} and \tilde{G}' are connected Lie groups with corresponding smooth covering maps π and π' , then there exists a Lie group homomorphism $\Phi : \tilde{G} \rightarrow \tilde{G}'$ such that $\pi' \circ \Phi = \pi$.*

Proof. Again - this is similar to the proofs done above. See Theorem 7.9 of [7]. \square

What is the point of establishing all this notation? Turns out establishing these universal covering groups on the Lie groups we know and love, say $SO(3)$, are extremely important when it comes to constructing so-called spins!

6.1.2 Projective Representations

Let us first define what a projective representation is [16, 17].

Definition 6.4. Let G be a group and V a finite-dimensional vector space over a field F . A map $\rho : G \rightarrow GL(V)$ is a **projective representation** of G over F if there exists a mapping $\alpha : G \times G \rightarrow F^*$ such that the following two properties hold:

1. $\rho(x)\rho(y) = \alpha(x, y)\rho(xy) \quad \forall x, y \in G$
2. $\rho(1) = \text{id}_V$

The two conditions imply that α satisfies the following properties:

- $\alpha(x, y)\alpha(xy, z) = \alpha(y, z)\alpha(x, yz) \quad \forall x, y, z \in G$
- $\alpha(x, 1) = \alpha(1, x) = 1 \quad \forall x \in G$

We provide an alternative definition most useful to this discussion. We note that for quantum mechanical objects, it is most useful to look at unitary representations¹⁹. First recall the following:

Definition 6.5. Let $U(V)$ be the group of invertible linear transformations of a finite-dimensional Hilbert space V over \mathbb{C} that preserve the inner product. A finite-dimensional **unitary representation** of a matrix Lie group G is a continuous homomorphism of $\Pi : G \rightarrow U(V)$ for some finite-dimensional Hilbert space V .

Definition 6.6. Let V be a finite-dimensional Hilbert space over \mathbb{C} . The **projective unitary group** over V , denoted $PU(V)$ is then the quotient group

$$PU(V) = U(V)/e^{i\theta}I \quad (6.7)$$

where $e^{i\theta}I$ denotes the group of matrices in $U(1)I$, I being the identity matrix here.

Now let $Q : U(V) \rightarrow PU(V)$ be the quotient homomorphism and let $q : \mathfrak{u}(V) \rightarrow \mathfrak{pu}(V)$ be the associated Lie algebra isomorphism. We note that given an ordinary unitary representation $\Sigma : G \rightarrow U(V)$, we can always form a projective representation $\Pi : G \rightarrow PU(V)$ by setting $\Pi = Q \circ \Sigma$. This is equivalent to saying the following diagram commutes:

$$\begin{array}{ccc} & G & \\ \Sigma \swarrow & & \searrow \Pi \\ U(V) & \xrightarrow{Q} & PU(V) \end{array}$$

Note that not all projective representations arise in this fashion. I will state the following propositions without detailed proof.

Proposition 6.1. *If V is a finite-dimensional Hilbert space over \mathbb{C} , then $PU(V)$ is isomorphic to a matrix Lie group. The associated Lie algebra homomorphism q defined above has the kernel $\{iaI\}$, so $\mathfrak{pu}(V)$ is isomorphic to $\mathfrak{u}(V)/\{iaI\}$.*

Proof. Consider the homomorphism $\Gamma : U(V) \rightarrow GL(\mathfrak{gl}(V))$, such that for given $U \in U(V)$, $\Gamma : U \mapsto C_U(X) = UXU^{-1}$. Then one can show that $\ker \Gamma = \{U(1)I\}$, so the image under this homomorphism is isomorphic to the quotient group $U(V)/\{e^{i\theta}I\}$, compact, and closed, i.e. a matrix Lie group isomorphic to $PU(V)$. To find the related Lie algebra homomorphism, we note that $c_X(Y) = [X, Y]$, with the kernel of c being the scalar multiples of I in $\mathfrak{u}(V)$ - the group $\{iaI\}$. c therefore must map onto $\mathfrak{pu}(V)$, giving the required isomorphism. \square

¹⁹Recall this is to make sure that the states are positive-definite so we can define a notion of probability on the Hilbert space.

Every finite-dimensional projective representation can be “de-projectivised” at the Lie-algebra level. To state this we have the following proposition.

Proposition 6.2. *Let $\Pi : G \rightarrow \text{PU}(V)$ be a finite-dimensional projective unitary representation of a matrix Lie group G , and $\pi : \mathfrak{g} \rightarrow \mathfrak{pu}(V)$ be the associated Lie algebra homomorphism. Then there exists a Lie algebra homomorphism $\sigma : \mathfrak{g} \rightarrow \mathfrak{u}(V)$ such that $\pi(X) = q(\sigma(X)) \quad \forall X \in \mathfrak{g}$. So the following diagram commutes:*

$$\begin{array}{ccc} & \mathfrak{g} & \\ \sigma \swarrow & & \searrow \pi \\ \mathfrak{u}(V) & \xrightarrow{q} & \mathfrak{pu}(V) \end{array}$$

This σ is unique upon fixing that $\text{tr } \sigma(X) = 0 \quad \forall X \in \mathfrak{g}$.

Proof. This proposition boils down to the fact that you can always fix $\sigma(X)$ to have trace zero by choosing for $Y \in \mathfrak{u}(1)$, pick $\sigma(X) = Y + cI$ where c is an appropriate pure-imaginary constant. Such σ therefore always exist. (See Proposition 16.46 of [17] for more details.) \square

Now we can say the most important theorem in this subsection:

Theorem 6.4. *Suppose G is a matrix Lie group and \tilde{G} is a universal cover of G with the covering map Φ . Then the following hold:*

1. *Let $\Pi : G \rightarrow \text{PU}(V)$ be a finite-dimensional projective unitary representation of G . Then there is an ordinary unitary representation $\Sigma : \tilde{G} \rightarrow \text{U}(V)$ of \tilde{G} such that $\Pi \circ \Phi = Q \circ \Sigma$. Any such Σ is irreducible if and only if Π is irreducible. Σ is unique if we choose $\det(\Sigma(A)) = 1, A \in \tilde{G}$.*
2. *Let Σ be a finite-dimensional irreducible unitary representation of \tilde{G} . Then the kernel of the associated projective unitary representation $Q \circ \Sigma$ contains the kernel of the covering map Φ . Therefore $Q \circ \Sigma$ factors through G and gives rise to a projective unitary representation of G .*

Point 1 is equivalent to saying that the following box diagram commutes:

$$\begin{array}{ccc} \tilde{G} & \xrightarrow{\Pi} & \text{U}(V) \\ \Phi \downarrow & & \downarrow Q \\ G & \xrightarrow{\Sigma} & \text{PU}(V) \end{array}$$

Proof. See Theorem 16.47 of [17]. The idea is actually really simple - we make use of Proposition 6.2 to find an ordinary representation of \mathfrak{g} at the base level, and then simply lift it up and apply Lie’s Theorem at the cover level. The second half the theorem rests on the fact that $\ker \Phi$ is a discrete normal subgroup \tilde{G} and is therefore central. We can then

show that $\Sigma(A)$, where $A \in \ker \Phi$ under Schur's lemma gives $\Sigma(A) = cI$ as it intertwines V to itself. The A is in the kernel of the associated projective representation $Q \circ \Sigma$. \square

Of course, we should be dealing with the infinite-dimensional case. Here of course the unitary representation needs to be defined slightly different²⁰ The main thing to note here is that we can no longer do the de-projectivisation by passing to the Lie algebra since there is no unique member we can choose - the notion of trace doesn't work for unbounded operators on the Hilbert space. Point 1 in Theorem 6.4 no longer works. However, if G is connected and "semi-simple", every projective unitary representation of G can be de-projectivised after passing to the universal cover. This is in fact the crucial reason why we need to study the universal covering manifolds! The spins intrinsically comes from this universal cover, and it is precisely since we are looking at the de-projectivised version of the representation that brought us there in the first place!

6.2 Spins in Lorentz group

We state without proof the following relation related to the proper orthochronous Lorentz group $SO^+(1, 3)$:

$$SO^+(1, 3) = \frac{SL(2, \mathbb{C})}{\mathbb{Z}_2} \quad (6.8)$$

In the lecture notes you have precisely looked at the representations of $SL(2, \mathbb{C})$ and noted that we can generate irreducible representations classified by two spins $(j_+, j_-) \in \frac{\mathbb{Z}}{2} \times \frac{\mathbb{Z}}{2}$. The reason for that, by the argument in the previous section, is the fact that the representations that we actually really want to look at are states in our quantum field theory, which are intrinsically projective in nature. By looking at projective unitary representations on $SO^+(1, 3)$, we can therefore look at the universal cover $SL(2, \mathbb{C})$ and analyse its unitary (de-projectivised) representations. This is why we have half-spins, i.e. our lovely fermions in our theory!

6.3 Clifford Algebras and Gamma Matrices

With the intuition sketched out in the previous section, let us try and make things concrete. The following discussion mainly follows [1].

We follow from the discussion of the representations of $\mathfrak{so}_n \mathbb{C}$. Recall that tensor powers of the fundamental and anti-fundamental representations of $\mathfrak{su}_n \mathbb{C}$ completely characterises all the representations of $\mathfrak{su}_n \mathbb{C}$, but this is not the case for the representations of $\mathfrak{so}_n \mathbb{C}$. As a matter of fact, only half of the representations of $\mathfrak{so}_n \mathbb{C}$ arises in this way. As we have seen above, this could be attributed to the different underlying topology of the two groups - the missing representations are exactly the $\text{Spin}_m \mathbb{C}$ reps where $\text{Spin}_m \mathbb{C} \rightarrow SO_m \mathbb{C}$ is a double cover.

How do we generate this double cover then? The theory of projective representations, as described above, seems to be hard to access from a construction point-of-view - where do we even start? It turns out to be helpful to define something called *Clifford Algebras*. Each

²⁰You will need some sort of a strong continuity homomorphism $\Phi : G \rightarrow U(\mathbf{H})$. You can read more about this in [17].

complex Clifford algebra contains an orthogonal Lie algebra as a subalgebra - it is isomorphic either to a matrix algebra or a sum of two. This determines one or two representations of the orthogonal Lie algebras, as we will see, which are exactly the representations that we are missing from the tensor product constructions.

So, first a definition. In this section we fix V to be a vector space and $B : V \times V \rightarrow \mathbb{K}$ to be a symmetric bilinear form on V (here \mathbb{K} is a field). To construct a corresponding quadratic form Q , we can define

$$Q(x) = B(x, x) \quad (6.9)$$

such that (V, Q) is a **quadratic vector space over \mathbb{K}** . One can conversely reconstruct the symmetric bilinear form B from Q by polarisation, i.e.

$$B(x, y) = \frac{1}{2} (Q(x + y) - Q(x) - Q(y)) \quad (6.10)$$

so we hereby denote Q and B interchangeably, writing $Q(x) = Q(x, x)$.

Now we have the following definition.

Definition 6.7. Let A be an associative \mathbb{K} -algebra and (V, Q) be a quadratic vector space. A \mathbb{K} -linear map $\phi : V \rightarrow A$ is **Clifford** if $\forall x \in V$,

$$\phi(x)^2 = Q(x)1_A \quad (6.11)$$

where 1_A is the unit of A .

Definition 6.8. The **Clifford algebra** $C = C(Q) = \text{Cliff}(V, Q)$ is an associative algebra with unit 1 and is generated by V , with $\forall v \in V$

$$v \cdot v = Q(v, v) \cdot 1 \quad (6.12)$$

Equivalently (if the characteristic of k is not 2), we have $\forall v, w \in V$,

$$v \cdot w + w \cdot v = 2Q(v, w) \quad (6.13)$$

A note about construction. The Clifford algebra can be constructed quickly by taking the tensor algebra

$$T^\bullet(V) = \bigoplus_{n \geq 0} V^{\otimes n} \quad (6.14)$$

and setting

$$C(Q) = \frac{T^\bullet(V)}{I(Q)} \quad (6.15)$$

Here $I(Q)$ is the two-sided ideal generated by all elements of the form $v \otimes v - Q(v, v) \cdot 1$. Clearly $C(Q)$ satisfies the universal property. From this we can see that the dimension of C is 2^m where $m = \dim(V)$ and that the canonical mapping $V \rightarrow C$ is an embedding, with the basis of $C(Q)$ being the products $e_I = e_{i_1} \cdot e_{i_2} \cdot \dots \cdot e_{i_k}$ where e_i are the basis of V . To see this, in particular, we can check the following.

Proposition 6.3. *There is a natural embedding $V \hookrightarrow C(Q)$ which is the image of $V = V^{\otimes 1}$ under the canonical projection*

$$\pi_q : T^\bullet(V) \rightarrow C(Q) , \quad (6.16)$$

and this is an injection.

Proof. Say that an element $\varphi \in T^\bullet(V)$ is of pure degree s if $\varphi \in V^{\otimes s}$. We want to show that any element $\varphi \in T^\bullet(V) \cap V$ is zero. Suppose this is not true. Then we can write $\varphi = \sum_i a_i \otimes (v_i \otimes v_i + Q(v_i)) \otimes b_i$ where we assume that a_i and b_i is of pure degree. Now since $\varphi \in V$ we must have that the expression is equal to zero, with the sum taken over those indices with $\deg a_i + \deg b_i$ maximal. Contracting with Q means $\sum_i a_i Q(v_i) \cdot b_i = 0$. Proceed with induction to show $\varphi = 0$. \square

The Clifford Algebra has a universal property as follows. This also gives a categorical definition of Clifford Algebras.

Proposition 6.4. *The Clifford algebra can be defined to be the universal algebra with the following property: If A is any associative algebra with unit and a linear mapping $j : V \rightarrow A$ is given such that*

$$j(v) \cdot j(v) = Q(v, v) \cdot 1, \quad \forall v \in V , \quad (6.17)$$

or equivalently $\forall v, w \in V$, (given that k has a characteristic not equal to 2,)

$$j(v) \cdot j(w) + j(w) \cdot j(v) = 2Q(v, w) \cdot 1 \quad (6.18)$$

then there should be a unique homomorphism of algebras from $C(Q)$ to A extending j , i.e. j extends uniquely to a \mathbb{K} -algebra homomorphism $\tilde{j} : C(Q) \rightarrow A$, and $C(Q)$ is the unique associative \mathbb{K} -algebra with this property.

Proof. Any linear map $j : V \rightarrow A$ extends to a unique algebra homomorphism $\bar{j} : T^\bullet(V) \rightarrow A$. Now Eq. (6.17) implies that $\bar{j} = 0$ on $I(Q)$ so \bar{j} descends to $C(Q)$. Suppose now B is an associative \mathbb{K} -algebra with unit and that $\iota : V \rightarrow B$ is an embedding with the property that any linear map $j : V \rightarrow A$ with the property in Eq. (6.17) extends uniquely to an algebra homomorphism $\tilde{j} : A \rightarrow B$. Then the isomorphism from $V \subset Q$ to $\iota(V) \subset B$ clearly induces an algebra isomorphism $C(Q) \xrightarrow{\cong} B$. \square

The proposition above effectively states the following. Given an associative algebra with unit A , together with a Clifford map $i : V \rightarrow C(Q)$ such that for every Clifford map $\phi : V \rightarrow A$ there is a unique algebra morphism $\Phi : C(Q) \rightarrow A$ that makes the following triangle commute.

$$\begin{array}{ccc} & V & \\ i \swarrow & & \searrow \phi \\ C(Q) & \xrightarrow{\Phi} & A \end{array}$$

Categorically, the Clifford Algebra is an initial object in the category $\mathbf{Cliff}(V, Q)$, which has Clifford maps $\phi : V \rightarrow \cdot$ from a fixed vector space equipped with a quadratic form Q as objects. The morphism from $V \rightarrow A$ to $V \rightarrow A'$ is given by a commuting triangle

$$\begin{array}{ccc} & V & \\ & \swarrow \quad \searrow & \\ A & \xrightarrow{f} & A' \end{array}$$

with $f : A \rightarrow A'$ as a homomorphism of associative algebras. This initial object is unique up to a unique isomorphism. In other words, the Clifford algebra $C(Q)$ is universal for Clifford maps to associative algebras. The construction via tensor algebra as before implies the following statement. If $\phi : V \rightarrow A$ is a Clifford map and $\tilde{\Phi} : T^\bullet(V) \rightarrow A$ is the unique extension of ϕ to the tensor algebra, then $\tilde{\Phi}$ indeed annihilates the ideal $I(Q)$ and therefore factors through a unique map $\Phi : T^\bullet(V)/I(Q) \rightarrow A$ from the quotient. Therefore, we have a commutative diagram:

$$\begin{array}{ccccc} V & \xrightarrow{\quad} & T^\bullet(V) & & \\ & \searrow i & \downarrow \phi & \nearrow \tilde{\Phi} & \\ & & C(Q) & \xrightarrow{\quad} & A \end{array}$$

Here i is really injective as the ideal only comes into play for $V^{\geq \otimes 2}$.

6.3.1 Constructing Clifford algebras

The way we have been discussing about Clifford algebras is not very suitable for computations. Instead, we will discuss the way that Clifford introduced the algebras. This is the way Clifford algebras are still taught in physics courses, following Dirac.

Traditionally, the discussion of Clifford algebras started with Dirac matrices.

Definition 6.9. Suppose $\{e_i\}$ is a \mathbb{K} -basis for V , where $i = 1, \dots, \dim V$. The vector space V is equipped with the symmetric bilinear form where $B(e_i, e_j) = B_{ij} = B_{ji}$. The **Clifford generators** Γ_i is the image of e_i under the map $i : V \rightarrow C(Q)$, which satisfy the relations

$$\Gamma_i \Gamma_j + \Gamma_j \Gamma_i = 2B_{ij} \mathbb{1} \quad (6.19)$$

where $\mathbb{1}$ is the unit in the Clifford algebra $C(Q)$.

Following this, we can define Clifford algebras by using generators in the following manner.

Definition 6.10 (Clifford algebras - generators). An associative algebra over field \mathbb{K} with unity 1 is the **Clifford algebra** $C(Q)$ of a non-degenerate quadratic form Q on V if it contains V and $\mathbb{K} = \mathbb{K} \cdot 1$ as distinct subspaces such that the following three conditions hold:

- (i) $v^2 = Q(v)$ for any $v \in V$.
- (ii) V generates $C(Q)$ as an algebra over \mathbb{K} .
- (iii) $C(Q)$ is not generated by any proper subspace of V .

We can immediately how the Dirac matrices furnishes a representation of the Clifford algebra. We then define

$$\Gamma_{ij} = \frac{1}{2} (\Gamma_i \Gamma_j - \Gamma_j \Gamma_i) \quad (6.20)$$

as the product of two generators. More generally, we have

$$\Gamma_{i_1 \dots i_p} = \frac{1}{p!} \sum_{\sigma \in S_p} (-1)^\sigma \Gamma_{i_{\sigma(1)}} \dots \Gamma_{i_{\sigma(p)}} \quad (6.21)$$

where $(-1)^\sigma$ indicates the sign of the permutation in S_p . We then see that since $C(Q)$ is generated by V and the identity it is the linear span of $1, \Gamma_i, \Gamma_{ij}, \dots$ in total there are $1 + n + C_2^n + \dots C_n^n = 2^n$ monomials. So $\dim C(Q) = 2^{\dim V}$. This is the same dimension as the exterior algebra $\bigwedge^\bullet V$ so we can establish a vector space isomorphism between the two.

In particular, if we use an orthonormal basis to generate $C(Q)$, then the first condition in the above Definition 6.10 then becomes

$$\Gamma_i^2 = 1, \quad 1 \leq i \leq p, \quad (6.22)$$

$$\Gamma_i^2 = -1, \quad p < i \leq p, \quad (6.23)$$

$$\Gamma_i \Gamma_j = -\Gamma_j \Gamma_i, \quad i < j. \quad (6.24)$$

whilst condition (iii) becomes

$$\Gamma_1 \dots \Gamma_n \neq \pm 1 \quad (6.25)$$

This is important in constructing a representation of Clifford algebras in general dimensions. Typically, this is needed in the discussion of supersymmetry and supergravity (and spinors) in various dimensions. The construction typically involves a set of matrices called **Dirac matrices** or **Gamma matrices**, defined as matrix representations of the Clifford algebra in various dimensions. You should have seen **Pauli matrices** in your elementary quantum mechanics courses:

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (6.26)$$

These matrices can generate a basis for Clifford algebras of arbitrary dimensions. Here we follow the discussion in [18–20]. We construct the Euclidean γ -matrices from Gamma matrices which are the basic building block of the matrix representations of the Clifford algebras. We define $(2k+1)$ -matrices by the tensor products of k Pauli matrices to get a

$2^k \times 2^k$ matrix representation as follows:

$$\begin{aligned}
\Gamma_1^{(k)} &= \sigma_1 \otimes \underbrace{\sigma_0 \otimes \dots \otimes \sigma_0}_{k-1}, & \Gamma_2^{(k)} &= \sigma_2 \otimes \underbrace{\sigma_0 \otimes \dots \otimes \sigma_0}_{k-1}, \\
\Gamma_3^{(k)} &= \sigma_3 \otimes \sigma_1 \otimes \underbrace{\sigma_0 \otimes \dots \otimes \sigma_0}_{k-2}, & \Gamma_4^{(k)} &= \sigma_3 \otimes \sigma_2 \otimes \underbrace{\sigma_0 \otimes \dots \otimes \sigma_0}_{k-2}, \\
&\vdots \\
\Gamma_{2k-1}^{(k)} &= \underbrace{\sigma_3 \otimes \dots \otimes \sigma_3}_{k-1} \otimes \sigma_1, & \Gamma_{2k}^{(k)} &= \underbrace{\sigma_3 \otimes \dots \otimes \sigma_3}_{k-1} \otimes \sigma_2, \\
\Gamma_{2k+1}^{(k)} &= \underbrace{\sigma_3 \otimes \dots \otimes \sigma_3}_k
\end{aligned} \tag{6.27}$$

The matrices listed above can be generated using the recurring relations:

$$\Gamma_M^{(k+1)} = \Sigma_M^{(k)} \otimes \sigma_0, \quad M = 1, \dots, 2k \tag{6.28}$$

$$\Gamma_{2k+i}^{(k+1)} = \Sigma_{2k+1}^{(k)} \otimes \sigma_i, \quad i = 1, 2, 3 \tag{6.29}$$

which gives

$$\{\Gamma_M^{(k)}, \Gamma_N^{(k)}\} = 2\delta_{MN} \tag{6.30}$$

So then we have the following definition.

Definition 6.11 (Gamma matrices). The **Gamma** or **Dirac matrices** are matrix representations of the Clifford algebras, i.e. the map: $\Gamma : C(Q) \rightarrow \text{GL}(\mathbb{C}^{2^k})$ where we map the generators $e_M \mapsto \Gamma_M$. The representation is faithful when $d = 2k$ and non-faithful when $d = 2k + 1$ where $\Gamma(\epsilon) = \Gamma_1^{(k)} \dots \Gamma_{2k+1}^{(k)} = i^k$.

We will find Gamma matrices extremely helpful later when we construct spinors in spaces of Euclidean and Lorentzian signatures.

6.3.2 \mathbb{Z}_2 -grading

Let us return to the tensor construction of the Clifford algebras. Since the ideal $I(Q)$ is not homogeneous, $C(Q)$ does not inherit a \mathbb{Z} -grading from $T^\bullet(V)$. However, notice that the ideal $I(Q)$ is generated by elements of an even degree. This means the Clifford algebra does inherit a \mathbb{Z}_2 grading. To study this grading recall the following definitions from elementary algebra.

Definition 6.12. A **graded ring** is a ring that is decomposed into a direct sum of additive groups

$$R = \bigoplus_{n=0}^{\infty} R_n = R_0 \oplus R_1 \oplus R_2 \oplus \dots \tag{6.31}$$

such that

$$R_m R_n \subseteq R_{m+n} \tag{6.32}$$

for all nonnegative integers m and n .

Definition 6.13. An associative algebra A over a ring R is **graded** if it is graded as a ring.

So we can now go back to Clifford Algebras. Consider the automorphism $\alpha : C(Q) \rightarrow C(Q)$ which sends $\alpha(v) = -v$ on V . Since $\alpha^2 = \text{id}$, the ideal $I(Q)$ is generated by elements of an even degree, and hence Clifford algebra inherits a \mathbb{Z}_2 grading:

$$C(Q) = C^0(Q) \oplus C^1(Q) \quad (6.33)$$

where $C^i(Q) = \{\varphi \in C(Q) \mid \alpha(\varphi) = (-1)^i \varphi\}$ are the eigenspaces of α . Since α is a homomorphism, we have

$$C^i(Q) \cdot C^j(Q) \subset C^{i+j}(Q) \quad (6.34)$$

with the indices taken modulo 2. This \mathbb{Z}_2 -grading plays an important role in the analysis and application of Clifford algebras. In particular, $C^0(Q)$ is often called $C^{\text{even}}(Q)$ and is a subalgebra of dimension 2^{m-1} , where as $C^1(Q)$ is often called $C^{\text{odd}}(Q)$.

6.3.3 Grading and exterior algebras

The \mathbb{Z}_2 -gradedness of the Clifford algebra is very different from the graded nature of the tensor algebra which inherently has a \mathbb{Z} -graded structure. To see this, define $\tilde{\mathcal{F}}$ as

$$\tilde{\mathcal{F}}^r = \sum_{s \leq r} V^{\otimes s} . \quad (6.35)$$

This has the property

$$\tilde{\mathcal{F}}^r \otimes \tilde{\mathcal{F}}^s \subset \tilde{\mathcal{F}}^{r+s} . \quad (6.36)$$

The tensor algebra therefore has a natural filtration

$$\tilde{\mathcal{F}}^0 \subset \tilde{\mathcal{F}}^1 \subset \dots \subset T^\bullet(V) , \quad (6.37)$$

which makes the tensor algebra into a **filtered algebra**. Every filtered algebra has an associated graded algebra. For the tensor algebra with the canonical filtration described above, the associated graded algebra is described by

$$\tilde{\mathcal{G}}^p = \tilde{\mathcal{F}}^p / \tilde{\mathcal{F}}^{p-1} \quad (6.38)$$

Then $\tilde{\mathcal{G}}^\bullet$ is a graded algebra where the product map is defined by

$$\tilde{\mathcal{G}}^p \times \tilde{\mathcal{G}}^q \rightarrow \tilde{\mathcal{G}}^{p+q} . \quad (6.39)$$

The canonical filtration of the tensor algebra $T^\bullet(V)$ defines a natural filtration on the Clifford algebra $C(Q)$. Suppose $\pi_q : T^\bullet(V) \rightarrow T^\bullet(V)/I(Q)$ where $I(Q)$ is the ideal that generates the Clifford algebras. Then $\mathcal{F}^i = \pi_q(\tilde{\mathcal{F}}^i)$ naturally has a natural filtration,

$$\mathcal{F}^0 \subset \mathcal{F}^1 \subset \mathcal{F}^2 \subset \dots , \quad (6.40)$$

and naturally the associated graded algebra $\mathcal{G}^r = \mathcal{F}^r / \mathcal{F}^{r-1}$ naturally inherits the filtration. We now have the following proposition.

Proposition 6.5. *For any quadratic form Q , the associated graded algebra of $C(Q)$ is naturally isomorphic to the exterior algebra $\bigwedge^\bullet V$.*

Proof. The map $\bigotimes^r V \xrightarrow{\pi_r} \mathcal{F}^r \rightarrow \mathcal{G}^r = \mathcal{F}^r / \mathcal{F}^{r-1}$ given by $v_{i_1} \otimes \dots \otimes v_{i_r} \mapsto [v_{i_1} \dots v_{i_r}]$ descends to a map $\bigwedge^r V \rightarrow \mathcal{F}^r$ by the property in Eq. (6.18). (Note that when the characteristic of \mathbb{K} is 2 then we will have to use the other condition.) This map is surjective and gives a homomorphism of graded algebras $\bigwedge^\bullet V \rightarrow \mathcal{G}^\bullet$. It remains to check the map is injective. The kernel of $\bigotimes^r V \rightarrow \mathcal{G}^r$ consists of the r -homogeneous pieces of elements $\varphi \in I_q(V)$ of degree less than r . Any such φ can be written as a finite sum $\varphi = \sum a_i \otimes (v_i \otimes v_i + q(v_i)) \otimes b_i$ where $v_i \in V$ and where we may assume that the a_i and b_i are of pure degree with $\deg a_i + \deg b_i \leq r - 2$. The r -homogeneous part of φ is then of the form $\varphi_r = \sum a_i \otimes v_i \otimes v_i \otimes v_i$ where $\deg a_i + \deg b_i = r - 2$ for each i . The image of φ in the exterior algebra is however zero as $v_i \wedge v_i = 0$. So the map $\bigwedge^r V \rightarrow \mathcal{G}^r$ is injective. \square

Note that the proposition above gives a canonical vector space isomorphism that is compatible with the filtrations as follows,

$$\bigwedge^\bullet V \rightarrow C(Q). \quad (6.41)$$

The map in Eq. (6.41) is of course not an isomorphism of algebras unless $q = 0$. However the map is indeed canonical so we can discuss embeddings of the form $\bigwedge^r V \subset C(Q)$ for all $r \geq 0$. To see that the isomorphism is only true when $q = 0$, consider the \mathbb{Z}_2 -grading on the tensor algebra defined with $T^\bullet(V) = T^\bullet(V)_0 + T^\bullet(V)_1$ where

$$T^\bullet(V)_0 = \bigoplus_{k \geq 0} V^{\otimes 2k}, \quad T^\bullet(V)_1 = \bigoplus_{k \geq 0} V^{\otimes 2k+1}. \quad (6.42)$$

where the \mathbb{Z}_2 -grading is the reduction mod-2 of the \mathbb{Z} -grading of the tensor algebra as discussed above. This reduction makes the ideal I_q homogeneous, and hence the projection $T^\bullet(V) \rightarrow C(Q)$ restricts to projections $TV_i \rightarrow C_i$ for $i = 0, 1$. Note however that for $i = 1$ this is only a projection of vector spaces, since neither TV_1 nor C_1 are algebras.

Now the canonical filtration of the tensor algebra $T^\bullet(V)$ defines a filtration on $C(Q)$ as follows. By filtering $T^\bullet(V)_0$ and $T^\bullet(V)_1$ separately, i.e.

$$\mathcal{F}^{2k} T(V)_0 = \bigoplus_{l \leq k} V^{\otimes 2l}, \quad \mathcal{F}^{2k+1} T(V)_1 = \bigoplus_{l \leq k} V^{\otimes 2l+1} \quad (6.43)$$

such that

$$0 \subset \mathcal{F}^0 T(V)_0 \subset \mathcal{F}^2 T(V)_0 \subset \dots \quad (6.44)$$

$$0 \subset \mathcal{F}^1 T(V)_1 \subset \mathcal{F}^3 T(V)_1 \subset \dots \quad (6.45)$$

Now under the projections $TV_0 \rightarrow C_0$ and $TV_1 \rightarrow C_1$, we can similarly identify the filtrations of the Clifford algebra as

$$0 \subset \mathcal{F}^0 C_0 \subset \mathcal{F}^2 C_0 \subset \dots \quad (6.46)$$

$$0 \subset \mathcal{F}^1 C_1 \subset \mathcal{F}^3 C_1 \subset \dots \quad (6.47)$$

We will henceforth use the shorthand $\mathcal{F}^p C$ as $\mathcal{F}^p C_0$ and $\mathcal{F}^p C_1$ if p is even and odd respectively. Now we note that $\mathcal{F}^p C / \mathcal{F}^{p-2} C \cong \bigwedge^p V$ as the corrections in replacing xy by $-yx$ where $x, y \in V$ involve terms of degree 2 less. The corrections are 0 when $q = 0$, so we can identify $Cl(V, 0) \cong \bigwedge^\bullet V$, exactly as mentioned above.

It is possible to understand the relation between the Clifford and exterior algebras in a different way which does not involve filtrations. The bilinear form B defines a map $\flat : V \rightarrow V^*$ where $x \mapsto B(x, \cdot)$. The map \flat is an isomorphism if and only if B is non-degenerate. The inverse is typically defined as \sharp so together with the map \flat they are referred to as the musical isomorphisms induced from the inner product B . We can then define a linear map $\phi : V \rightarrow \text{End } \bigwedge^\bullet V$ by

$$\phi(x)\alpha = x \wedge \iota_x \alpha \quad (6.48)$$

where ι_x is the unique odd derivation defined by $\iota_x 1 = 0$ and $\iota_x y = B(x, y)$ for $y \in V$. So on a monomial we have,

$$\iota_x (y_1 \wedge \dots \wedge y_p) = \sum_{i=1}^p (-1)^{i-1} B(x, y_i) y_1 \wedge \dots \wedge \hat{y}_i \wedge \dots \wedge y_p, \quad (6.49)$$

where the hat denotes omission. Then we can extend this linearly to all of $\bigwedge^\bullet V$ as in the following lemma.

Lemma 6.1. *The map $\phi : V \rightarrow \text{End } V$ in Eq. (6.48) is Clifford.*

Proof. For every $x \in V$ and $\alpha \in \text{End } \bigwedge^\bullet V$, we have

$$\begin{aligned} \phi(x)^2 \alpha &= \phi(x) (x \wedge \alpha - \iota_x \alpha) \\ &= x \wedge x \wedge \alpha - x \wedge \iota_x \alpha - Q(x) \alpha + x \wedge \iota_x \alpha + \iota_x \iota_x \alpha \\ &= -Q(x) \alpha, \end{aligned} \quad (6.50)$$

where $x \wedge x = 0 = \iota_x \iota_x$ and $\iota_x(x \wedge \alpha) = Q(x) \alpha - x \wedge \iota_x \alpha$. \square

By the universality of Clifford algebras we can then extend this to the algebra homomorphism uniquely,

$$\Phi : C(Q) \rightarrow \text{End } \bigwedge^\bullet V \quad (6.51)$$

So composing this with the evaluation at $1 \in \bigwedge^\bullet V$ gives a linear map $\Phi_1 : C(Q) \rightarrow \bigwedge^\bullet V$. This map obeys $\Phi_1(1) = 1$ and if $x \in V$ then $\Phi_1(i(x)) = x$ where $i : V \rightarrow C(Q)$. Since i is injective from the construction of $C(Q)$, $\Phi_1 \circ i$ is also injective. By further computations, we then get

$$\Phi_1(i(x)i(y)) = x \wedge y - B(x, y), \quad (6.52)$$

and

$$\Phi_1(i(x)i(y)i(z)) = x \wedge y \wedge z - B(x, y)z + B(x, z)y - B(y, z)x, \quad (6.53)$$

so Φ_1 surjects onto $\bigwedge^\bullet V$. This is a vector space isomorphism with the inverse map defined by

$$y_1 \wedge \dots \wedge y_p \mapsto \frac{1}{p!} \sum_{\sigma \in S_p} (-1)^\sigma y_{i_{\sigma(1)}} \wedge \dots \wedge y_{i_{\sigma(p)}} \quad (6.54)$$

which gives an explicit quantisation of the exterior algebra.

6.3.4 Clifford algebras as representations of $\mathfrak{so}_n \mathbb{C}$

It is clear from the construction that Clifford algebras are associative algebras. As a results it determines a Lie algebra with the bracket defined by the associative multiplication. How are Clifford algebras related to the representations of $\mathfrak{so}_m \mathbb{C}$? To see this we will first embed the Lie algebra $\mathfrak{so}(Q)$ inside the Lie algebra of the even part of the Clifford algebra and from there identify $C(Q)$ with one or two copies of matrix algebras.

Let us see how this works in practice. We first need to make an explicit isomorphism of $\bigwedge^2 V$ with $\mathfrak{so}(Q)$, which is defined as

$$\mathfrak{so}(Q) = \{X \in \text{End}(V) \mid Q(Xv, w) = -Q(v, Xw) \forall v, w \in V\} . \quad (6.55)$$

Define the map

$$\varphi_{a \wedge b}(v) = 2(Q(b, v)a - Q(a, v)b) , \quad (6.56)$$

which gives the isomorphism $\phi : \bigwedge^2 V \rightarrow \mathfrak{so}(Q) \subset \text{End}(V)$ with $a \wedge b \mapsto \varphi_{a \wedge b}$. One can check that the bracket on $\bigwedge^2 V$ makes this an isomorphism of Lie algebras with the Clifford algebra²¹, allowing the map $\psi : \bigwedge^2 V \rightarrow C(V, Q)$ to be defined by

$$\psi(a \wedge b) = \frac{1}{2}(a \cdot b - b \cdot a) = a \cdot b - Q(a, b) . \quad (6.57)$$

This is an injective embedding, which shows that the following Lemma.

Lemma 6.2. *The mapping $\psi \circ \varphi^{-1} : \mathfrak{so}(Q) \rightarrow C(Q)^{\text{even}}$ embeds $\mathfrak{so}(Q)$ as a Lie subalgebra of $C(Q)^{\text{even}}$.*

Proof. See discussion above. □

The reason why the embedding only goes into the even part is because, simply, $C(Q)^{\text{odd}}$ is indeed not an algebra. You can also see that in Eq. (6.57) the map is defined with elements of even degree. By looking at the basis elements we can then see that ψ is an embedding and the map exactly maps the exterior algebra to the even part of the Clifford algebra.

What remains is to identify the subalgebra of $C(Q)_0$ or $C(Q)^{\text{even}}$, the image of $\mathfrak{so}(Q)$ as matrix algebras. Let us separate this into two cases.

Case 1: $n = \dim V$ is even.

We first decompose V into two n -dimensional isotropic spaces for Q ,

$$V = W \oplus W' . \quad (6.58)$$

Then we have the following lemma.

Lemma 6.3. *The decomposition $V = W \oplus W'$ determines an isomorphism of algebras,*

$$C(Q) \cong \text{End}(\bigwedge^\bullet W) \quad (6.59)$$

where $\bigwedge^\bullet W = \bigwedge^0 W \oplus \dots \oplus \bigwedge^n W$.

²¹This is done by checking the brackets on $[a \wedge b, c \wedge d]$ and $[a \cdot b, c \cdot d]$

Proof. Let us try and construct the map $\varphi : C(Q) \rightarrow E = \text{End}(\bigwedge^\bullet W)$. The map φ is the same as defining a linearly mapping $V \rightarrow E$ with the condition in Eq. (6.18). We must therefore construct maps $l : W \rightarrow E$ and $l' : W' \rightarrow E$ such that

$$l(w)^2 = 0 = l'(w')^2 \quad (6.60)$$

$$l(w) \circ l'(w') + l'(w') \circ l(w) = 2Q(w, w')I, \quad (6.61)$$

for any $w \in W, w' \in W$. For each $w \in W$, let $L_w \in E$ be the left multiplication by w on the exterior algebra $\bigwedge^\bullet W$,

$$L_w(\xi) = w \wedge \xi, \quad \xi \in \bigwedge^\bullet W. \quad (6.62)$$

For any $\vartheta \in W^*$, let $D_\vartheta \in E$ be the derivation of $\bigwedge^\bullet W$ such that,

$$D_\vartheta(1) = 0 \quad (6.63)$$

$$D_\vartheta(w) = \vartheta \in \bigwedge^0 W = \mathbb{C} \quad (6.64)$$

$$D_\vartheta(\zeta \wedge \xi) = D_\vartheta \zeta \wedge \xi + (-1)^{\deg(\zeta)} \zeta \wedge D_\vartheta(\xi), \quad (6.65)$$

where $w \in W = \bigwedge^1 W$. i.e. Explicitly,

$$D_\vartheta(w_1 \wedge \dots \wedge w_r) = \sum_i (-1)^{i-1} \vartheta(w_i) (w_1 \wedge \dots \wedge \hat{w}_i \wedge \dots \wedge w_r). \quad (6.66)$$

Now we can set

$$l(w) = L_w, \quad l'(w') = D_{\vartheta'}, \quad (6.67)$$

where $\vartheta' \in W^*$ is defined by $\vartheta'(w) = 2Q(w, w')$, $\forall w \in W$. It is straightforward to show that the maps defined obeys the requirements, as well as for $\zeta \wedge \xi$ if they obey for ζ and ξ separately. The map is clearly an isomorphism and one can see that by its action of a basis. \square

Now note that there exists a decomposition of the exterior powers into even and odd parts $\bigwedge^\bullet W = \bigwedge^{\text{even}} W \oplus \bigwedge^{\text{odd}} W$ where $C(W)^{\text{even}}$ respects the splitting. From Lemma 6.3, we then have the isomorphism,

$$C(Q)^{\text{even}} \cong \text{End}(\bigwedge^{\text{even}} W) \oplus \text{End}(\bigwedge^{\text{odd}} W). \quad (6.68)$$

Combining this with Lemma 6.2, we then have an embedding of Lie algebras,

$$\mathfrak{so}(Q) \subset C(Q)^{\text{even}} \cong \mathfrak{gl}(\bigwedge^{\text{even}} W) \oplus \mathfrak{gl}(\bigwedge^{\text{odd}} W), \quad (6.69)$$

and we find that there are two representations of $\mathfrak{so}(Q) = \mathfrak{so}_{2n}\mathbb{C}$. We denote the two representations by,

$$S^+ = \bigwedge^{\text{even}} W, \quad S^- = \bigwedge^{\text{odd}} W. \quad (6.70)$$

Proposition 6.6. *The representations S^\pm are the irreps of $\mathfrak{so}_{2n}\mathbb{C}$ with highest weights $\alpha = \frac{1}{2}(L_1 + \dots + L_n)$ and $\beta = \frac{1}{2}(L_1 + \dots + L_{n-1} - L_n)$. More precisely, we have,*

$$S^+ = \Gamma_\alpha, \quad S^- = \Gamma_\beta, \quad \text{if } n \text{ is even}; \quad (6.71)$$

$$S^+ = \Gamma_\beta, \quad S^- = \Gamma_\alpha, \quad \text{if } n \text{ is odd}. \quad (6.72)$$

Proof. We need to show that the natural basis vectors $e_I = e_{i_1} \wedge \dots \wedge e_{i_k}$ for $\bigwedge^\bullet W$ are weight vectors. Tracing through the isomorphisms, we find that $H_i = E_{i,i} - E_{n+i,n+i}$ in $\mathfrak{h} \subset \mathfrak{so}_{2n}\mathbb{C}$ corresponds to $\frac{1}{2}(e_i \wedge e_{n+i})$ in $\bigwedge^2 V$, which corresponds to $\frac{1}{2}(e_i \cdot e_{n+i} - 1)$ in $C(Q)$, and this maps to,

$$\frac{1}{2}(L_{e_i} \circ D_{2e_i^*} - I) = L_{e_i} \circ D_{e_i^*} - \frac{1}{2}I \in \text{End}(\bigwedge^\bullet W). \quad (6.73)$$

We can compute,

$$L_{e_i} \circ D_{e_i^*}(e_I) = \begin{cases} e_I & \text{if } i \in I \\ 0 & \text{if } i \notin I \end{cases} \quad (6.74)$$

So e_I spans a weight space with weight $\frac{1}{2}(\sum_{i \in I} L_i - \sum_{j \notin I} L_j)$. All such weights with given $|I| \bmod 2$ are congruent by the Weyl group (they are equivalent up to transformations of the Weyl group), so $S^+ = \bigwedge^{\text{even}} W$ and $S^- = \bigwedge^{\text{odd}} W$ must be an irreducible representation. The highest weights are then straightforward to read off - the one for $\bigwedge^{\text{even}} W$ is $\alpha = \frac{1}{2} \sum_i L_i$ if n is even and β if n is odd. \square

Definition 6.14. The representations S^\pm are the **half-spin representations** of $\mathfrak{so}_{2n}\mathbb{C}$, and $S = S^+ \oplus S^- = \bigwedge^\bullet W$ is called the **spin representation**. Elements of S are known as **spinors**.

We are going to come back to spinor representations in the next section.

Case 2: $n = \dim V$ is odd.

This time we decompose the space V as follows,

$$V = W \oplus W' \oplus U, \quad (6.75)$$

where W and W' are n -dimensional isotropic spaces and U is a one-dimensional space perpendicular to them. For the standard Q on \mathbb{C}^{2n+1} these are spanned by the first n , second n , and the last basis vector. We then have the following lemma.

Lemma 6.4. *The decomposition $V = W \oplus W' \oplus U$ determines an isomorphism of algebras,*

$$C(Q) \cong \text{End}(\bigwedge^\bullet W) \oplus \text{End}(\bigwedge^\bullet W'). \quad (6.76)$$

Proof. We can proceed exactly as the even case, as in Lemma 6.3. The only difference is with the element u_0 where $Q(u_0, u_0) = 1$. We send u_0 to the endomorphism that is the identity on $\bigwedge^{\text{even}} W$ and minus the identity on $\bigwedge^{\text{odd}} W$. This involution then skew commutes with all L_w and D_ϑ , which means the map $V \rightarrow E = \text{End}(\bigwedge^\bullet W)$ determines an algebra homomorphism from $C(Q) \rightarrow E$. The map for $\text{End}(\bigwedge^\bullet W')$ is similar but with the roles of W and W' reversed. The maps are isomorphic by checking the basis elements. \square

From Lemma 6.4 we see that the subalgebra $C(Q)^{\text{even}} \subset C(Q)$ is mapped isomorphically onto the factors,

$$C(Q)^{\text{even}} \cong \text{End}(\bigwedge^\bullet W) \quad (6.77)$$

which gives a representation $S = \bigwedge^\bullet W$ of Lie algebras,

$$\mathfrak{so}_{2n+1}\mathbb{C} = \mathfrak{so}(Q) \subset C(Q)^{\text{even}} \cong \mathfrak{gl}(\bigwedge^\bullet W) = \mathfrak{gl}(S) . \quad (6.78)$$

So now we have the following proposition.

Proposition 6.7. *The representation $S = \bigwedge^\bullet W$ is the irrep of $\mathfrak{so}_{2n+1}\mathbb{C}$ with the highest weight*

$$\alpha = \frac{1}{2}(L_1 + \dots + L_n) . \quad (6.79)$$

Proof. This is similar to the even case - each e_I is an eigenvector with weight $\frac{1}{2} \left(\sum_{i \in I} L_i - \sum_{j \notin I} L_j \right)$. All the weights are congruent by the Weyl group so it must be an irrep with highest weight α . \square

We have therefore constructed the spin representations of $\mathfrak{so}_n\mathbb{C}$.

6.4 Classification of Clifford algebras

As we have seen above, Clifford algebras are extremely useful. There exists a classification that classifies real and complex Clifford algebras. The details of which could be found in [21] and I will refer to that for the full classification²².

6.5 Spinor Representations and Clifford Algebras

Having discussed the spin representations of $\mathfrak{so}_n\mathbb{C}$, it is prudent to discuss its relation with the spinors in this section.

6.5.1 Pin and Spin Groups

First, let us define something known as pin and spin groups.

Definition 6.15. The **multiplicative group of units** in the Clifford algebra is defined to be the subset

$$C^\times(Q) = \{ \phi \in C(Q) \mid \exists \phi^{-1}, \phi^{-1}\phi = \phi\phi^{-1} = 1 \} . \quad (6.80)$$

This group contains all elements $v \in V$ with $Q(v) \neq 0$.

The group of units always acts naturally as automorphisms of the algebra, i.e. the **adjoint representation**,

$$\text{Ad} : C^\times(Q) \rightarrow \text{Aut}(C(Q)) , \quad (6.81)$$

which is given by,

$$\text{Ad}_\phi(x) = \phi x \phi^{-1} . \quad (6.82)$$

Taking the derivation of this map gives the usual Lie bracket action $\text{ad}_y(x) = [y, x]$. Hitherto we have assumed that the characteristic of the field could be any integer. Let us assume from now that the characteristic of the field $k \neq 2$. Then we have the following important proposition.

²²If I have time in the future I will come back to add more information but this is not relevant to the current discussion.

Proposition 6.8. *Let $v \in V \subset C(Q)$ be an element with $Q(v) \neq 0$. Then $\text{Ad}_v(V) = V$, and $\forall w \in V$, we have,*

$$-\text{Ad}_v(w) = w - 2\frac{Q(v, w)}{Q(v)}v. \quad (6.83)$$

Proof. We have that $v^{-1} = -\frac{v}{Q(v)}$, so

$$-Q(v)\text{Ad}_v(w) = -Q(v)v w v^{-1} = v w v = -v^2 w - 2Q(v, w)v = Q(v)w - 2Q(v, w)v. \quad (6.84)$$

□

Naturally, this lead us to consider the subgroup of elements $\phi \in C^\times(Q)$ such that $\text{Ad}_\phi(V) = V$. From Proposition 6.8 above, we see that the group contains all the elements $v \in V$ with $Q(v) \neq 0$, and when this happens the transformation Ad_v preserves the quadratic form Q ,

$$(\text{Ad}_v^* Q)(w) = Q(\text{Ad}_v(w)) = Q(w), \quad (6.85)$$

for all $w \in V$. We define $P(Q) \subset C(Q)$ to be the subgroup generated by the elements $v \in V$ with $Q(v) \neq 0$. Note that then there is a representation,

$$P(Q) \rightarrow O(V, Q), \quad (6.86)$$

where

$$O(V, Q) = \{\lambda \in GL(V) \mid \lambda^* Q = Q\} \quad (6.87)$$

is the orthogonal group of the form Q .

We are now ready to explore the important subgroups of $P(Q)$.

Definition 6.16. The **Pin group** is the subgroup of $P(Q)$ generated by the elements $v \in V$ with $Q(v) = \pm 1$, i.e.

$$\text{Pin}(V) = \{v_1 \cdot \dots \cdot v_m \in C(Q) \mid v_j \in S^{n-1} \subset \mathbb{F}^n, m \in \mathbb{N}\} \quad (6.88)$$

Definition 6.17. The **Spin group** is similarly defined as,

$$\text{Spin}(V) = \text{Pin}(n) \cap C^0(Q) \quad (6.89)$$

$$= \{v_1 \cdot \dots \cdot v_m \in C(Q) \mid v_j \in S^{n-1} \subset \mathbb{F}^n, m \in 2\mathbb{N}\} \quad (6.90)$$

Note that from the definition of the Pin group, the inverse element to $v_1 \cdot \dots \cdot v_m$ is,

$$(v_1 \cdot \dots \cdot v_m)^{-1} = (-v_m) \dots (-v_1) \in \text{Pin}(V) \quad (6.91)$$

Let us take a deep look at Eq. (6.83). Notice that the right hand side of the equation is basically the reflection $R_v(x)$ of the vector $x \in V$ where v is the vector marking the perpendicular direction of the reflection hyperplane. To remove this sign we therefore consider the following action.

Definition 6.18. The **twisted adjoint representation** is the map $\tilde{\text{Ad}} : C^\times(Q) \rightarrow GL(C(Q))$ where,

$$\tilde{\text{Ad}}_\phi(y) = \alpha(\phi)y\phi^{-1}. \quad (6.92)$$

For even elements ϕ , $\tilde{\text{Ad}}_\phi = \text{Ad}_\phi$. We also have $\tilde{\text{Ad}}_{\phi_1\phi_2} = \tilde{\text{Ad}}_{\phi_1} \circ \tilde{\text{Ad}}_{\phi_2}$. Explicitly,

$$\tilde{\text{Ad}}_v(w) = w - 2\frac{Q(v,w)}{Q(v)}v. \quad (6.93)$$

We state without proof the following result.

Theorem 6.5 (Cartan-Dieudonné). *Every $g \in O(V)$ is the product of a finite number of reflections $g = R_{u_1} \circ \dots \circ R_{u_r}$ along the null lines where $Q(u_i) \neq 0$ and $r \leq \dim V$.*

Proof. See [22]. □

We note that the twisted adjoint action must define a group homomorphism $\tilde{\text{Ad}} : \text{Pin}(V) \rightarrow O(V)$. It follows from the Cartan-Dieudonné Theorem 6.5 that $\tilde{\text{Ad}}$ is surjective. But what is the kernel of $\tilde{\text{Ad}}$?

Proposition 6.9. *Suppose V is finite dimensional and Q is non-degenerate. Then the kernel of the homomorphism $\tilde{\text{Ad}} : \tilde{P}(V, Q) \rightarrow GL(V)$ is the group \mathbb{K}^\times of non-zero multiples of 1. Here the group $\tilde{P}(V, Q)$ is defined as,*

$$\tilde{P}(V, Q) = \left\{ \phi \in C^\times(V, Q) \mid \tilde{\text{Ad}}_\phi(V) = V \right\}, \quad (6.94)$$

where $P(V, Q) \subset \tilde{P}(V, Q)$.

Proof. See [21] for a complete proof. The proof is also outlined in [23]. □

[21] goes into a bit more detail in how you would define the homomorphism from the group $\tilde{P}(V, Q)$ to the orthogonal group $O(V)$ (see Propositions 2.5 and Corollary 2.6 of [21]). It also shows how the images $\tilde{\text{Ad}}(\text{Pin}(V, Q))$ and $\tilde{\text{Ad}}(\text{Spin}(V, Q))$ is a normal subgroup of $O(V)$ (see Proposition 2.8 of [21]). To summarise there are two exact short sequences.

Theorem 6.6. *Let V be a finite-dimensional vector space over a field \mathbb{K} and Q a non-degenerate quadratic form on V . Suppose the field \mathbb{K} of characteristic $\neq 2$ is **spin**, i.e. at least one of the two equations $t^2 = a$ and $t^2 = -a$ can be solved in \mathbb{K} for each non-zero element $a \in \mathbb{K}^\times$. Then there are two short exact sequences.*

$$0 \rightarrow F \rightarrow \text{Spin}(V, Q) \xrightarrow{\tilde{\text{Ad}}} SO(V) \rightarrow 1, \quad (6.95)$$

$$0 \rightarrow F \rightarrow \text{Pin}(V, Q) \xrightarrow{\tilde{\text{Ad}}} O(V) \rightarrow 1, \quad (6.96)$$

where

$$F = \begin{cases} \mathbb{Z}_2 = \{1, -1\} & \text{if } \sqrt{-1} \notin \mathbb{K} \\ \mathbb{Z}_4 = \{\pm 1, \pm \sqrt{-1}\} & \text{otherwise} \end{cases} \quad (6.97)$$

The sequences above hold for general fields provided that $SO(V)$ and $O(V)$ are replaced by appropriate normal subgroups of $O(V)$ (since the map $\tilde{\text{Ad}}$ maps to normal subgroups of $O(V)$ in general and field \mathbb{K} , which is spin, solves the equation $t^2 Q(v) = \pm 1$ so every $v \in V^\times$ can be renormalised to have $Q(v) = 1$).

Proof. See [21] Theorem 2.9. The details of the field being **spin** is not relevant if we restrict to $\mathbb{K} = \mathbb{R}, \mathbb{C}$ as both fields are spin. \square

The real case of the above Theorem 6.6 is reduced to the simple short exact sequences,

$$0 \rightarrow \mathbb{Z}_2 \rightarrow \text{Spin}_{r,s} \rightarrow SO_{r,s} \rightarrow 1, \quad (6.98)$$

$$0 \rightarrow \mathbb{Z}_2 \rightarrow \text{Spin}_{r,s} \rightarrow SO_{r,s} \rightarrow 1, \quad (6.99)$$

for all (r, s) , where the subscripts denote the signature of the quadratic form Q . In particular, for $SO_n = SO_{n,0} = SO_{0,n}$, we have

$$0 \rightarrow \mathbb{Z}_2 \rightarrow \text{Spin}_n \xrightarrow{\tilde{\text{Ad}}} SO_n \rightarrow 1, \quad (6.100)$$

where the map $\tilde{\text{Ad}}$ acts as the universal covering map of SO_n for all $n \geq 3$. From this we see that the Pin and Spin groups are covering groups of the orthogonal group. We will now examine the representations of the Pin and Spin groups, which in term will give representations of the orthonal groups. We will see how this unifies with the picture we have taken in two subsections ago.

6.5.2 Spinor Representations

Before we begin let us recall what representations of an algebra is.

Definition 6.19. Suppose A is an associative algebra and $\mathbb{K} = \mathbb{R}, \mathbb{C}$ or \mathbb{H} . A **\mathbb{K} -representation** of A is a \mathbb{K} -linear homomorphism $\rho : A \rightarrow \text{End}_{\mathbb{K}}(E)$ for some \mathbb{K} -vector space E . Two \mathbb{K} -representations $\rho : A \rightarrow \text{End}_{\mathbb{K}}(E)$ and $\rho' : A \rightarrow \text{End}_{\mathbb{K}}(E')$ are **equivalent** if there exists a \mathbb{K} -linear isomorphism $f : E \rightarrow E'$ such that the following triangle commutes:

$$\begin{array}{ccc} & A & \\ \rho \swarrow & & \searrow \rho' \\ \text{End}_{\mathbb{K}}(E) & \xrightarrow{\text{Ad } f} & \text{End}_{\mathbb{K}}(E') \end{array}$$

where $\text{Ad } f : \text{End}_{\mathbb{K}}(E) \rightarrow \text{End}_{\mathbb{K}}(E')$ is defined as $\phi \mapsto f \circ \phi \circ f^{-1}$, so $f \circ \rho(a) = \rho'(a) \circ f$ for all $a \in A$.

We can now define the following representations.

Definition 6.20. A **pinor representation** of $\text{Pin}(V)$ is the restriction of an irreducible representation of $C(Q)$. Similarly, a **spinor representation** of $\text{Spin}(V)$ is the restriction of an irreducible representation of $C_0(Q)$

We here note that most representations of Clifford algebras (similar to Lie algebras) are reducible. The volume element plays a key role in determining irreducible representations. In particular, since the volume element is used in determining the classification of real and complex Clifford algebras [21], one can use it to determine the properties of irreducible real and complex representations of Clifford algebra. You can read about the details of the

classification and determination of irreducible representations in [21] and [23]. The crux of the discussion there is that one must distinguish between (broadly) even and odd-dimension V cases, similar to our discussion above. In particular, let us suppose Cl_n is the Clifford algebra of \mathbb{R}^n , and its complexification is $\mathbb{C}l_n = Cl_n \otimes_{\mathbb{R}} \mathbb{C}$. For the standard basis e_i of \mathbb{R}^n , define

$$z_j = \frac{1}{2}(e_{2j-1} - ie_{2j}) \in \mathbb{C}l_n \quad (6.101)$$

for $j = 1, \dots, m$, $n = 2m$ and \bar{z}_j be its conjugate. Then the span

$$\Sigma = \{z_{j_1} \cdot \dots \cdot z_{j_k} \cdot \bar{z}_1 \cdot \dots \cdot \bar{z}_m \mid k = 0, \dots, m, \quad 1 \leq j_1 \leq \dots \leq j_k \leq m\} , \quad (6.102)$$

defines a complex vector subspace of $\mathbb{C}l_n$ of dimension 2^m . The space that is known as the **spinor space** with its elements being **spinors**²³. The spinor space is invariant under Clifford multiplications (multiplication by e_j). We further define Σ^{\pm} to be the spans where k is even and odd respectively. The sign of the space is then the **chirality** of the space, and by setting the above notation $E = \Sigma$ in Definition 6.20, we now have reobtained our definitions of spinor representations in terms of action on spinor spaces. We then have the following propositions.

Proposition 6.10. *Suppose the complex spinor representation is $\Delta_n^{\mathbb{C}} : \text{Spin}_n \rightarrow GL_{\mathbb{C}}(\Sigma)$ which is given by restricting an irreducible complex representation $\mathbb{C}l_n \rightarrow \text{Hom}_{\mathbb{C}}(\Sigma, \Sigma)$ to $\text{Spin}_n \subset Cl_n^0 \subset \mathbb{C}l_n$. Then for n odd this definition is independent of which irrep of $\mathbb{C}l_n$ is used, and that the representation $\rho_n^{\mathbb{C}}$ is irreducible. When n is even then there is a decomposition,*

$$\Delta_n^{\mathbb{C}} = \Delta_n^{\mathbb{C}^+} \oplus \Delta_n^{\mathbb{C}^-} , \quad (6.103)$$

i.e. into a direct sum of two inequivalent irreducible complex representations of Spin_n .

Proof. See Proposition 5.15 of [21], which uses the argument for the real case. For more information one can also look at [24]. \square

We now see how this is exactly the same as how we have defined spinor representations to start with. This unifies the two pictures of spinor representations in [1] and in [21] - the subspaces in [1] are exactly the $\text{Spin}(\text{Pin})$ groups that are defined at the start of this subsection.

6.5.3 Weyl, Dirac, Pauli and Majorana Spinors

Physicists like calling spinor with names that confuse mathematicians. In this section I will clarify the relationship between the spin(or) representations we have seen in the previous section with the ones often defined in a hand-waving manner in the physics literature. In the following I will fix V to be a vector space where spinors live.

Case 1: $n = \dim V$ is odd.

²³Note very carefully that the spinor space Σ dimensions has nothing to do with the dimensions of V , the vector space where we have defined the Clifford algebra. This is important for the classification of spinors in the next subsection.

I want to cover the odd case first as this is a bit easier. The spin representations are defined as ($n = p + q$),

$$\Delta_n : \text{Spin}_{p,q} \rightarrow GL_{\mathbb{C}}(\Sigma) , \quad (6.104)$$

where Σ is the spinor space. We note that we can design the map Δ_n to map to a subspace of the spinor space defined above. We then have the following ‘definitions’.

Definition 6.21. For $p + q = \dim V$ odd, we have the following spinors.

- $\dim \Sigma = 1$: This is the **scalar representation**, the complexified spin group acts on the space of complex scalars \mathbb{C} .
- $\dim \Sigma = \dim V$: This is the **vector representation**, so the complexified spin group acts on the space of complexified vectors $\mathbb{C}^{\dim V}$.
- $\dim \Sigma = m$, so the action of the spin group is represented by complex matrices $M \in \text{Mat}(m, \mathbb{C})$. This is called the **spin- $\frac{1}{2}$ representation**, the spinors are known as **Pauli spinors**.

Ah, that is exactly what we called Pauli spinors. The easiest case is to look at the spinor representations of \mathfrak{so}_3 which represents the Clifford algebra as Pauli spinors (and transformations as Pauli matrices).

Case 2: $n = \dim V$ is even.

Now we move on to the case where n is even. From the analysis of the odd case we have for each separate decomposition Σ^{\pm} a scalar representation, a set of spin- $\frac{1}{2}$ representations and a vector representation. Now we need to consider the pair of irreducible representations.

Definition 6.22. For $p + q = \dim V$ even, we have the following spinors.

- $0 \oplus 0 = (0, 0)$: This is the **scalar representation** acting on the space $\mathbb{C} \oplus \mathbb{C}$.
- $\frac{1}{2} \oplus 0 = (\frac{1}{2}, 0)$: This is the **left spin- $\frac{1}{2}$ representation** which acts on the space of **left Weyl spinors**.
- $0 \oplus \frac{1}{2} = (0, \frac{1}{2})$: This is the **right spin- $\frac{1}{2}$ representation** which acts on the space of **right Weyl spinors**.
- $(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$: This is the **Dirac representation** which acts on the space of **Dirac spinors**. This can be viewed as two Weyl spinors stacked on top of each other.
- $(\frac{1}{2}, \frac{1}{2})$: This is called the **vector representation**, where the spin group acts on the space of complexified vectors.

We note that the Weyl spinors are indeed the building blocks of spinor representations in this case. This addresses how the complicated spinor definitions are related to the spinors you have left in Quantum Field Theory courses ²⁴!

²⁴Majorana spinors require the definition of a symplectic structure. I will not cover the details here but please refer to [25].

6.5.4 Spin Structures

To fully address the mathematical structures one needs to also define something known as spin structures and spin bundles. Since this is completely unrelated to our main discussion, I will leave the interested amongst you to have a go at [\[21, 23, 24\]](#). This concludes our excursion into spinors!

7 Correspondence between Lie Groups and Lie Algebras

In the lectures we have established that the following facts.

1. The tangent map e of a group homomorphism is a Lie algebra homomorphism, and vice versa.
2. Lie groups can be recovered from Lie algebras using the exponential map.

Effectively what this says is that there is a natural correspondence between Lie groups and Lie algebras. However, from the Lie algebra isomorphism discussion we have seen that in fact the Lie algebra $\mathfrak{su}(2)$ and $\mathfrak{so}(3)$ are isomorphic to one another. We also happen to know that $SU(2)$ as a real Lie group is a universal (2:1) covering of the Lie group $SO(3)$. Is there a more precise statement of the facts as discussed in the lectures? In particular, what kind of manifold will I get if I exponentiate the $\mathfrak{so}(3)$ and $\mathfrak{su}(2)$ real Lie algebras?

The answer lies again in the global construction of Lie groups. The statement made in the lectures follows Lie's original argument - the construction is *local*. In this section, we will discuss the consequences of the exponential map in the global setting, and fully complete the correspondence discussion set out in the notes.

7.1 Lie Brackets and Flows

The first question to address is why the tangent map e of a group homomorphism is a Lie algebra homomorphism. Additionally there is a side question - what is the need of defining a left-invariant vector field, when we can locally define vector fields at the tangent space of identity e ?

To answer these two questions we must first go back and define what an integral curve, a flow and vector fields are. To do this I will go a bit deeper - is there a more general notion for tangent spaces at a point? Turns out there is mathematical notion for the 'collection of tangent spaces' $T_p M$ of the manifold M , typically known as a tangent bundle. This will be very useful in our discussion (and typically covered in a 'second' course in differential geometry) so here is the definition. Note throughout this section we will fix M to be a smooth n -manifold²⁵.

Definition 7.1. The **tangent bundle of M** , denoted by TM , is *loosely* the disjoint union of the tangent spaces at all points of M ,

$$TM = \coprod_{p \in M} T_p M, \quad (7.1)$$

where $T_p M$ denotes the tangent space at $p \in M$.

More accurately, the **tangent bundle of M** TM is defined as a smooth topological space with the following data.

- There is a natural surjective **projection map** $\pi : TM \rightarrow M$. Denoting the points of TM as the ordered pair (p, v) where $p \in M$ and $v \in T_p M$, then $\pi(p, v) = p$.

²⁵The notion of smoothness is defined in Andre's notes. If it is not clear then have a look at [7].

- Suppose $\{U_i\}_{i \in I}$ is an open cover of M (here I is an indexing set). For each i there exists a diffeomorphism

$$\Phi_i = \pi^{-1}(U_i) \rightarrow U_i \times \mathbb{R}^n, \quad (7.2)$$

such that $\text{pr}_1 \circ \Phi_i = \pi$ and for all i, j the map $\Phi_i \circ \Phi_j^{-1}$ has the form $(U_i \cap U_j) \times \mathbb{R}^n \rightarrow (U_i \cap U_j) \times \mathbb{R}^n$ where

$$(x, v \in \mathbb{R}^n) \mapsto (x, g_{ji}(x)(v)) \quad (7.3)$$

for some smooth map $g_{ji} : U_i \cap U_j \rightarrow GL(n, \mathbb{R})$. This smooth map g_{ij} is precisely the transition map defined between the two charts.

There is of course the notion of a vector bundle, which we now can define below.

Definition 7.2. A **vector bundle of M of rank k** $\pi : E \rightarrow M$ is defined as a smooth topological space with the following data.

- There is a natural surjective **projection map** $\pi : E \rightarrow M$. Denoting the points of TM as the ordered pair (p, v) where $p \in M$ and $v \in E$, then $\pi(p, v) = p$.
- Suppose $\{U_i\}_{i \in I}$ is an open cover of M (here I is an indexing set). For each i there exists a diffeomorphism

$$\Phi_i = \pi^{-1}(U_i) \rightarrow U_i \times \mathbb{R}^k, \quad (7.4)$$

such that $\text{pr}_1 \circ \Phi_i = \pi$ and for all i, j the map $\Phi_i \circ \Phi_j^{-1}$ has the form $(U_i \cap U_j) \times \mathbb{R}^k \rightarrow (U_i \cap U_j) \times \mathbb{R}^k$ where

$$(x, v \in \mathbb{R}^k) \mapsto (x, g_{ji}(x)(v)) \quad (7.5)$$

for some smooth map $g_{ji} : U_i \cap U_j \rightarrow GL(k, \mathbb{R})$.

The only thing that has changed is the dimension of the total space E and the transition maps g_{ji} which are no longer directly related to the coordinate transformations when we defined different charts. We can then define a vector field as follows.

Definition 7.3. A **section** on E is a map $s : M \rightarrow E$ such that $s \circ \pi = \text{id}_M$. A section of TM is a **(rough) vector field on M** . If the restrictions of X to the coordinate chart on M is smooth and the component functions with respect to this chart is also smooth then the vector field is **smooth**.

Now we see how the notion of Lie brackets can be defined naturally on vector fields. To see this first note that (see Proposition 8.14 of [7]) if X is a smooth vector field, then for every $f \in C^\infty(M)$ function, then the function Xf is smooth on M . We can then apply another smooth vector field Y onto this function to get $f \mapsto YXf$ - but this does not in general satisfy the product rule so cannot be a vector field. The way to do this is to define something known as the Lie bracket of X and Y ,

$$[X, Y]f = XYf - YXf \quad (7.6)$$

This operator now is a vector field and gives the map $[X, Y] : C^\infty(M) \rightarrow C^\infty(M)$. It is then a general question whether the set of vector fields $\{X_i\}$ form a Lie algebra (the answer is it does... but we need to wait). To see this let us return to what a flow is.

Definition 7.4. Suppose V is a vector field on M . An integral curve of V is a differentiable curve $\gamma : J \rightarrow M$ such that

$$\gamma'(t) = V_{\gamma(t)} \quad \forall t \in J. \quad (7.7)$$

The point $\gamma(0)$ is called the **starting point** of γ if $0 \in J$.

Integral curves are the curves generated by moving along the vector field - the curve is entirely defined by the vector field V so given a starting point you will know what the curve is. Another way to visualise this is as follows. Suppose for each point $p \in M$, V has a unique integral curve starting at p defined for all $t \in \mathbb{R}$. We can define a map $\theta_t : M \rightarrow M$ by sending $p \in M$ to the point which is obtained after transporting t units along the integral curve defined by V . This motivates the following definition.

Definition 7.5. A **flow domain** is an open neighbourhood U of $0 \times M$ in $\mathbb{R} \times M$ such that for each $p \in M$ the set $U \cap (\mathbb{R} \times \{p\})$ is connected (i.e. it is an open interval around 0)²⁶.

A **local flow** of a vector field V is a smooth map $\theta : U \rightarrow M$ where U is a flow domain, and satisfying the following two conditions.

- $\theta(0, \cdot) = \text{id}_M$.
- $\frac{d}{dt}\theta(t, p) = V(\theta(t, p)), \forall (t, p) \in U$.

It is a **global flow** if $U = \mathbb{R} \times M$. For shorthand write θ^t for $\theta(t, \cdot)$.

Defined in the above manner, the flow θ as a continuous map then automatically satisfies $\theta^t \circ \theta^s = \theta^{t+s}$. In that sense $\theta^{-t} = (\theta^{-1})^t$. One can also define the map as one with the condition above (together with $\theta^0 = \text{id}$ are known as the group laws of flow.). V is then known as the **infinitesimal generator** of θ .

Definition 7.6. A vector field is **complete** if it admits a global flow.

With a vector field V and a local flow θ one can now define something known as the Lie derivative.

Definition 7.7. Given a smooth map between two manifolds X and Y , $F : X \rightarrow Y$, the **derivative of F at p** is the map $D_p F : T_p X \rightarrow T_{F(p)} Y$ given by $[\gamma] \rightarrow [F \circ \gamma]$, where γ is the integral curve for some vector field v , with the bracket indicating the tangent vector represented by curve γ . We denote D_p by F_* the **pushforward** by F on tangent vectors. Similarly, we define the dual map $(D_p F)^\vee : T_{F(p)}^* Y \rightarrow T_p^* X$ to be the **pullback** F^* .

Definition 7.8. The **Lie derivative** of a vector field W , denoted $\mathcal{L}_V W$, is

$$\left. \frac{d}{dt} \right|_{t=0} (\theta^t)^* W \quad (7.8)$$

Here \cdot^* indicates the pullback map.

²⁶This definition is not standard, and is credited to Dr. Jack Smith from whom I learnt most of these my differential geometry.

The Lie derivative is defined as above as we want to pullback the vector transported to compare with the original vector at the origin (and not the other way - otherwise there will be a minus sign). We then note that we can define the Lie bracket of vector fields as follows.

Definition 7.9. The **Lie bracket** of vector fields V and W is defined as

$$[V, W] = \mathcal{L}_V W = -\mathcal{L}_W V . \quad (7.9)$$

You can check that this makes the space of vector fields on M , $\mathfrak{X}(M)$ into a Lie algebra as this definition of Lie bracket is now alternating and satisfies the Jacobi identity. This is surprising - the set of vector fields $\mathfrak{X}(M)$ forms a Lie algebra under the Lie bracket of vector fields. Why do we care so much about Lie groups then, if this structure is already prevalent in a typical smooth manifold X ?

The reason why is because Lie groups have a lot more structure to a general smooth manifold. The notion of left-invariant vector fields are specifically defined only when you have a group action G on the manifold, and since this group action already exists for Lie groups, the notion of left-invariant vector fields naturally exist and hence one can naturally define a Lie algebra on the set of vector fields. For a generic smooth manifold, there may not exist such smooth right action that maps any point to another by a global diffeomorphism ²⁷. Of course, one can also take the viewpoint that the group of all diffeomorphisms of a smooth manifold M as an infinite-dimensional Fréchet Lie group - in that sense its Lie algebra is exactly $\mathfrak{X}(M)$ with its Lie bracket structure [26]. How do we connect all these notions with the ‘local’-sense of how a Lie algebra is defined - as the Lie bracket structure of tangent vector fields at the identity? This point is explored in the lectures but it is worth emphasising once more - the left-invariant property of the vector fields means that once the vector field is determined we will know its value at each point $p \in M$. We then have the vector space isomorphism,

$$\mathcal{L}(G) \rightarrow T_e G, \quad \xi \mapsto \xi_e , \quad (7.10)$$

so the tangent vector at identity is uniquely defined ²⁸. This global to local property is again unique to the Lie group structure. All that means is it is sufficient to understand the local tangent space (germ) of G to understand the properties of the manifold (bar some global properties as we will see soon). Now the remaining question to address is why the tangent map e of a group homomorphism is a Lie algebra homomorphism. This is again sketched out in the lecture note but it is worth emphasising again. There is a natural map $\Psi : G \rightarrow \text{Aut}(G)$ ²⁹, where $\Psi_g(h) = g \cdot h \cdot g^{-1}$. A homomorphism ρ respects the adjoint action of a group G on its tangent space $T_e G$ at the identity and therefore by taking the differential of the characterisation $(d\rho)_e$ respects the adjoint action of the tangent space to

²⁷This point is explored in more detail in §20 of [7] where Lee talks about the homomorphism of $\mathfrak{g} \rightarrow \mathfrak{X}(M)$ if such a smooth action G on M exists.

²⁸The inverse of this map of course is $\xi_e \mapsto g_* \xi_e = \xi_g$.

²⁹Here we look at the automorphisms of G by conjugation as there are no fixed points for multiplication m_g .

G on itself. Of course, with the arguments above, we now see how this map relates to the Lie bracket definitions of left-invariant fields and hence everything is now consistent.

7.2 The exponential map and BCH Formula

We have seen in the previous subsection that the correspondence is a local construction. The Lie group structure naturally endows a Lie bracket on the space of tangent vectors at identity (or the space of left-invariant vector fields by the arguments above) so it naturally has a Lie algebra structure. The local construction we discussed follows two important principles (see §8 of [1]):

1. Let G and H be Lie groups, with G connected. A map $f : G \rightarrow H$ is uniquely determined by its differential $df_e : T_e G \rightarrow T_e H$ at the identity.
2. Let G and H be Lie groups, with G connected and simply-connected. A linear map $T_e G \rightarrow T_e H$ is the differential of a homomorphism $f : G \rightarrow H$ if and only if it preserves the bracket operation:

$$df_e([X, Y]) = [df_e(X), df_e(Y)] \quad (7.11)$$

Alternatively, a linear map between the Lie algebras $\alpha : \mathfrak{g} \rightarrow \mathfrak{h}$ is the differential of a map $f : G \rightarrow H$ of Lie groups if and only if α is a map of Lie algebras.

The natural question to ask now is how to perform the inverse, i.e. could we possibly reconstruct a Lie group given a Lie algebra. The answer is yes, and it is often done with the help of one-parameter subgroups and exponential maps which you have all encountered in the lecture notes. These are also discussed in §3.4 of the notes and also §8.3 of [1]. We note here that the exponential map is the unique map from \mathfrak{g} to G taking 0 to e whose differential at the origin $(\exp_*)_0 : T_0 \mathfrak{g} \rightarrow T_e G$ is the identity and the restrictions to the lines through the origin in \mathfrak{g} are one-parameter subgroups of G . So in this so-called local construction we have shown the following - since the differential of the exponential map at the origin of \mathfrak{g} is an isomorphism, the image of \exp will contain a neighbourhood of the identity in G .

Before we continue let us see this more explicitly by constructing the actual correspondence in detail for matrix Lie groups. Using the standard power series for e^x :

$$\exp(X) = 1 + X + \frac{1}{2}X^2 + \frac{1}{3!}X^3 + \dots \quad (7.12)$$

We want to show that the group structure of G is encoded in the Lie algebra. Explicitly we want to show that for $X, Y \in \mathfrak{g}$ in a sufficiently small neighbourhood near the origin we can write down the product $\exp(X) \cdot \exp(Y)$ as an exponential. By defining,

$$\log(g) = (g - I) - \frac{1}{2}(g - I)^2 + \frac{1}{3}(g - I)^3 + \dots \in \mathfrak{gl}_n \mathbb{R}, \quad (7.13)$$

where $g \in G$, we want to set a new bilinear operation $*$ on $\mathfrak{gl}_n \mathbb{R}$ where

$$X * Y = \log(\exp(X) \cdot \exp(Y)) \quad (7.14)$$

The formula of $X * Y$ in terms of X , Y and brackets is known as the Baker-Campbell-Hausdorff formula and gives the local identification of Lie algebras and Lie groups. To degree three it is

$$X * Y = X + Y + \frac{1}{2}[X, Y] \pm \frac{1}{12}[X, [X, Y]] \pm \frac{1}{12}[Y, [Y, X]] + \dots \quad (7.15)$$

If G is connected the exponential map generates all of G (provided G is simply-connected, as we will see soon in the next subsection) so every finite-dimensional Lie algebra is the Lie algebra of a Lie group, by applying to an embedding of a Lie algebra \mathfrak{g} into \mathfrak{gl}_n .

I want to emphasise that this is a local construction. The exponential map fails to be injective and surjective in different circumstances. We will leave the surjective case to the following subsection, so let us simply focus on the injective case. A simple counter-example is the group $SO(2)$ which is isomorphic to S^1 , its Lie algebra is the space of two-dimensional skew-symmetric matrices with the exponential map given by:

$$\exp : \begin{pmatrix} 0 & t \\ -t & 0 \end{pmatrix} \mapsto \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \quad (7.16)$$

which clearly is not injective. In general, we have the following proposition.

Proposition 7.1. *\exp is injective if G has no closed subgroup isomorphic to either the circle \mathbb{R}/\mathbb{Z} , the universal covering $\widetilde{SL_2(\mathbb{R})}$, E the isometry group of the plane and its central extension \tilde{E} .*

Proof. See [27] for proof. □

7.3 Centres and Covers

What remains to be answered is whether the exponential map indeed extends to the entire manifold G given a Lie algebra \mathfrak{g} , i.e. whether the exponential map is surjective in all cases. The answer is clearly no, but why? Having discussed covering spaces, we have the following propositions.

Definition 7.10. The **centre** $Z(G)$ of a group G is defined as

$$Z(G) = \{z \in G \mid \forall g \in G, zg = gz\} \quad (7.17)$$

Proposition 7.2. *Let G be a Lie group, H be a connected manifold and $\phi : H \rightarrow G$ a covering space map. Let e' be an element lying over the identity e of G . Then there is a unique Lie group structure on H such that e' is the identity and ϕ is a map of Lie groups. The kernel of ϕ is the centre of H .*

Proof. The multiplication on G lifts uniquely to a map $H \times H \rightarrow H$ which takes $(e', e') \mapsto e'$. This product satisfies the group axioms (check). From Proposition 7.3 we can in fact do this for the universal cover \tilde{H} of G and then use the proposition to obtain the appropriate cover H . □

Proposition 7.3. *Let H be a Lie group and $\Gamma \subset Z(H)$ a discrete subgroup of its centre. Then there is a unique Lie group structure on the quotient group $G = H/\Gamma$ such that the quotient map $q : H \rightarrow G$ is a Lie group map.*

Proof. This is straightforward from the axioms. \square

This motivates the following definition.

Definition 7.11. A Lie group map between two Lie groups G and H is an **isogeny** if it is a covering space map of the underlying manifolds. Then we say G and H are **isogenous**.

Isogeny generates an equivalence notion - by identifying Lie groups with their universal covers. This means that starting from a Lie group G , one can generate the universal covering space \tilde{G} and then if the centre of the universal cover is discrete (as for all semisimple groups) $\tilde{G}/Z(\tilde{G})$ can be generated. This identifies

$$G \simeq \tilde{G} \simeq \tilde{G}/Z(\tilde{G}) \quad (7.18)$$

as an equivalence class in the set (category) of Lie groups.

Definition 7.12. Using the notation above, we call \tilde{G} the **simply connected form** of the group G and $\tilde{G}/Z(\tilde{G})$, if it exists, the **adjoint form** of the group G .

We can now state the following theorem.

Theorem 7.1. *Let \mathfrak{g} (over $k = \mathbb{R}, \mathbb{C}$) be a semisimple Lie algebra of finite dimension. Then there is a connected, simply connected Lie group G such that \mathfrak{g} is the Lie algebra of G .*

Proof. This is Lie's Third Theorem and its modern version was first proved by Cartan. A discussion can be found in [28]. The actual proof is extremely complicated - the theorem can be proved by Ado's Theorem (as a Corollary, see Tao's webpage [29]) or topological [30] and cohomological proofs [31]. Also see [32–34]. \square

In fact, one can look at this from a functorial approach [32]. This consists of describing the assignment of the category of Lie groups G to the category of Lie algebras \mathfrak{g} via the contravariant functor Lie:

$$\text{Lie} : G \mapsto \mathfrak{g} \quad (7.19)$$

I shall not delve further into this. If you are interested you can have a look at the discussion in [32].

But the answer to the question of surjectivity is clear. The exponential map in general is not surjective due to this covering property: the local construction, constructed as a smooth connected map, can never map to a point not connected to the identity. The exponential map is only unique when you specify that the Lie groups are simply-connected - otherwise simply using the formula of the exponential map (for example, the matrix exponential) will not allow you to construct the homomorphism as discussed at the start of the section to a different simply-connected Lie group.

Example 7.1 ($SU(2)$ and $SO(3)$ revisited). To illustrate the point above let us revisit the $SU(2)$ and $SO(3)$ homomorphism, $\varphi : SO(3) \rightarrow SU(2)/\mathbb{Z}_2$. Suppose $A \in SU(2)$ and $R \in SO(3)$, where

$$A = \frac{1}{2} \text{tr}(A) \mathbb{1} + \frac{1}{2} \text{tr}(\boldsymbol{\sigma} A) \cdot \boldsymbol{\sigma} . \quad (7.20)$$

where $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ are the Pauli matrices. Then for a 2d complex vector \mathbf{x} we then have,

$$\det(\mathbf{x} \cdot \boldsymbol{\sigma}) = -\mathbf{x}^2 . \quad (7.21)$$

Define a linear transformation $\mathbf{x} \rightarrow \mathbf{x}'$ as

$$\mathbf{x}' \cdot \boldsymbol{\sigma} = A \mathbf{x} \cdot \boldsymbol{\sigma} A^\dagger , \quad (7.22)$$

where $\mathbf{x}^{2'} = \mathbf{x}^2$. We can now use component-wise the $SO(3)$ transformation,

$$x'_i = R_{ij} x_j , \quad (7.23)$$

to get (using also $\text{tr}(\sigma_i \sigma_j) = 2\delta_{ij}$),

$$\sigma_i R_{ij} = A \sigma_j A^\dagger , \quad (7.24)$$

i.e.

$$R_{ij} = \frac{1}{2} \text{tr} \left(\sigma_i A \sigma_j A^\dagger \right) . \quad (7.25)$$

The converse can be constructed from $\sigma_j A^\dagger \sigma_j = 2 \text{tr}(A^\dagger) \mathbb{1} - A^\dagger$ to get

$$R_{jj} = |\text{tr}(A)|^2 - 1, \quad \sigma_i R_{ij} \sigma_j = 2 \text{tr}(A^\dagger) A - \mathbb{1} , \quad (7.26)$$

which gives

$$A = \pm \frac{\mathbb{1} + \sigma_i R_{ij} \sigma_j}{2(1 + R_{jj})^{\frac{1}{2}}} . \quad (7.27)$$

The sign \pm then signifies the correspondence $\pm A \leftrightarrow R_{ij}$ which gives the map φ .

Distributed version

Exercise 7.1. Try repeating the same procedure of Example 7.1 for the map

$$\tilde{\varphi} : SO(3, 1) \xrightarrow{\cong} SL(2, \mathbb{C})/\mathbb{Z}_2 . \quad (7.28)$$

8 Tensor Methods and Young Diagrams

I know a lot of you are extremely confused about Young diagrams and tensor methods. Here I am going to provide a pedagogical account of tensor methods, focussing on how you can utilise Young diagrams to produce irreducible tensorial representations of compact Lie groups. We will only focus on $SU(n)$ -tensors in this short account³⁰, and you can look up on [1] for the other classical Lie groups. For completeness I will reproduce some results presented in the last class, but for efficient presentation of the material I will ignore most proofs (but I will try and point you to resources where you can look it up - they are not pretty in general!).

8.1 Tensor Irreps in $GL(n)$ and $SU(n)$

First let us define tensors and tensor representations.

Definition 8.1. An r -**tensor representation** of a group G is a map $T : G \rightarrow GL(V^{\otimes r})$ such that T is a representation of G . A **rank r G -tensor**³¹ is defined to be an element of the product space $V^{\otimes r}$. For any given basis $\{v_1, \dots, v_n\}$, we can expand a tensor in the form:

$$T = T_{i_1 i_2 \dots i_r} v_{i_1} \otimes \dots \otimes v_{i_r} \quad (8.1)$$

$T_{i_1 i_2 \dots i_r}$ are then called the **components** of the tensor T with respect to the basis $\{v_i\}$.

Similarly, one can define a (p, q) tensor with dual vector spaces.

Definition 8.2. Take V^* to be the dual space of V (the set of linear maps $\varphi : V \rightarrow \mathbb{F}$ where \mathbb{F} is a field). An (p, q) -**tensor representation** of a group G is a map $T : G \rightarrow GL(V^{\otimes p} \otimes V^{*\otimes q})$ such that T is a representation of G . A **rank (p, q) G -tensor** is defined to be an element of the product space $V^{\otimes p} \otimes V^{*\otimes q}$. For any given basis $\{v_1, \dots, v_p\}$ and dual basis $\{w^1, \dots, w^q\}$, we can expand a tensor in the form:

$$T = T_{i_1 \dots i_p}^{j_1 \dots j_q} v_{i_1} \otimes \dots \otimes v_{i_p} \otimes w^{j_1} \otimes \dots \otimes w^{j_q} \quad (8.2)$$

In particular, we want to look at how tensor transforms under a group G . The most general case is the general linear group $GL(n)$. For this, we see that a (p, q) -tensor in $GL(n)$ will transform as

$$T_{i_1 \dots i_p}^{j_1 \dots j_q} \mapsto A_{i_1}^{k_1} \dots A_{i_p}^{k_p} B^{j_1}_{l_1} \dots B^{j_q}_{l_q} T_{k_1 \dots k_p}^{l_1 \dots l_q} \quad (8.3)$$

Here the matrices B are related to $A \in GL(n)$, the fundamental transformation, as

$$B^T = A^{-1} \quad (8.4)$$

³⁰We will in fact start with a discussion of GL_n tensors, but we will see later the invariant tensors of GL_n is exactly the same as $SU(n)$.

³¹We note that the tensors are not immediate representations of the groups themselves. They are in fact the objects that the representations transform on and forms space called the tensor space. The tensor space however is defined with respect to the group G ; given a different group G the law of transformations on the space will be different and hence the space of tensors will be different.

Our central goal is to find irreducible representations of $\text{GL}(n)$. From the above definitions, we can see that the tensors are in fact the objects that the matrices transform on. They therefore give a representation of the invariant subspace that the irreducible representations of $\text{GL}(n)$ (or in fact a matrix subgroup $G \subset \text{GL}(n)$) act on.

Let us look at second-rank tensors to illustrate this point. Recall that the direct product of two vector spaces is a reducible space:

$$V \otimes V = \text{Sym}^2 V \oplus \wedge^2 V \quad (8.5)$$

where $\text{Sym}^2 V$ and $\wedge^2 V$ denotes the symmetric and antisymmetric part of the space. Suppose T_{ij} is a second-rank tensor and \mathbf{p} is a permutation operator that switches the two indices of the second-rank tensor. Since $T_{ij} \in V \otimes V$, we can write the decomposition of Eq. (8.5) as

$$T_{ij} = T_{(ij)} + T_{[ij]} = (\text{id} + \mathbf{p}) T_{ij} \quad (8.6)$$

where id is just the identity map. The $\text{GL}(n)$ -transformation acts on the tensor in the following manner,

$$T_{ij} \mapsto A_i^k A_j^l T_{kl}, \quad (8.7)$$

and we note in particular that the operator \mathbf{p} commutes with this $\text{GL}(n)$ -action. The reason for this is the product of matrices $A_i^k A_j^l$ is **bisymmetric**, the product indeed remains unchanged when the same permutation is applied to both $\text{GL}(n)$ actions:

$$\begin{aligned} \mathbf{p} \cdot T_{ij} &= T_{ji} \\ &= A_j^k A_i^l T_{kl} \\ &= A_i^k A_j^l (\mathbf{p} \cdot T)_{kl} \end{aligned} \quad (8.8)$$

A similar calculation will show that the permutation operator \mathbf{p} of any two indices indeed commute with the $\text{GL}(n)$ -action acting on the tensors since tensor transformation is bisymmetric. We conclude that the whole space of rank- r tensors is reducible into subspaces consisting of tensors of different symmetry properties. The rank- r tensors of a given symmetry forms the basis of an irreducible representation of $\text{GL}(n)$ - this corresponds to the irreducible sub-block of the matrix product $A_i^k \dots$ in the $\text{GL}(n)$ transformations of tensors.

Definition 8.3. The rank- r tensors of a particular symmetry type forms a basis of irreps of $\text{GL}(n)$; they are known as **irreducible tensors with respect to $\text{GL}(n)$** .

We will henceforth discuss the irreps of a subgroup G of $\text{GL}(n)$ by its representation as the invariant space in the tensor space, i.e. we will associate the irrep of G with the tensor itself.

The question is, how do we specialise this to the $\text{SU}(n)$ group? To do this we need to work out how irreducible representations of $\text{GL}(n)$ can remain irreducible when we restrict to subgroups of $\text{GL}(n)$. Firstly, the elements of the matrices of a $\text{GL}(n)$ irrep in terms of the rank- r tensors are homogeneous polynomials of degree r in the elements A_{ij} of

the transformation matrices ³². Suppose the matrices are reducible when restricted to the subgroup $G \subset \text{GL}(n)$. Since under a change of basis the matrix elements remain homogeneous polynomials of degree r in A_{ij} , say $p(A_{ij})$, then the restricted representation will be reducible for the subgroup $G \subset \text{GL}(n)$ if there exists a certain set of homogeneous polynomials of degree r , $p_\alpha(A_{ij})$ which vanish for all $A \in G$ but not for all $B \in \text{GL}(n)$.

Now we can try and see what happens when we restrict to $\text{SU}(n)$. First note the restriction to the unimodular group $\text{SL}(n)$. Any matrix $A \in \text{GL}(n)$ can be written as $A = \alpha B$ where $\alpha = (\det A)^{1/n}$, the polynomials $p_\alpha(A_{ij})$ are simply rescaled by a factor α^r for rank- r tensors so the vanishing of polynomials $p_\alpha(B_{ij})$ will require the original polynomials $p_\alpha(A_{ij})$ to vanish. Hence we have shown the following proposition.

Proposition 8.1. *The irreducible tensors and irreps of $\text{GL}(n)$ and $\text{SL}(n)$ are the same.*

Proof. See the vanishing of polynomial argument in the paragraph above. \square

For $\text{SU}(n)$, recall that the elements of the Lie algebra of $\text{SU}(n)$, denoted by \mathfrak{su}_n , are simply the real form of the Lie algebra $\mathfrak{gl}_n \mathbb{C}$ ³³. Using the vanishing polynomial argument we can again conclude the following proposition.

Proposition 8.2. *The irreducible tensors and irreps of $\text{GL}(n)$ and $\text{SU}(n)$ are the same.*

Proof. Consider the Lie algebra of $\text{GL}(n)$, \mathfrak{gl}_n , and suppose X_{ij} be a basis of the \mathfrak{gl}_n Lie algebra, so any element of \mathfrak{gl}_n can be written as $\sum_{ij} \alpha_{ij} X_{ij}$ where $\alpha_{ij} \in \mathbb{C}$. For the Lie algebra of $\text{SU}(n)$, we will need $\alpha_{ij} \in \mathbb{R}$. Now suppose a representation of the Lie algebra basis element X_{ij} is given by the matrices x_{ij} . Then if a representation is reducible for \mathfrak{su}_n , we will need to find a basis in which the matrices $\alpha_{ij} x_{ij}$ are in reduced form for all $\alpha_{ij} \in \mathbb{R}$. Using linearity the linear forms $\alpha_{ij} x_{ij}$ must also vanish for any $\alpha_{ij} \in \mathbb{C}$ so the representation is also reducible for \mathfrak{gl}_n . The converse is the same, and we can then use the exponential map to lift the argument to the Lie groups equivalents. \square

The above proposition therefore implies that we can analyse $\text{SU}(n)$ -tensors the same way we analyse $\text{GL}(n)$ tensors - they have the same irreducible structures. We will from now on use the two terms interchangeably.

In $\text{SU}(n)$, we distinguish between lower and upper indices using the dual basis as described above for $\text{GL}(n)$ tensors. The two kinds of indices transform in under different representations in $\text{SU}(n)$, as illustrated explicitly in the following definition.

Definition 8.4. A (p, q) **tensor** in $\mathbf{n}^p \otimes \bar{\mathbf{n}}^q$ of $\text{SU}(n)$ carries p lower and q upper indices $\phi_{\mu_1 \dots \mu_p}^{\nu_1 \dots \nu_q}$. It transforms in $\text{SU}(n)$ as:

$$\phi_{\mu_1 \dots \mu_p}^{\nu_1 \dots \nu_q} \mapsto U_{\mu_1}^{\rho_1} \dots U_{\mu_p}^{\rho_p} \bar{U}^{\nu_1}_{\sigma_1} \dots \bar{U}^{\nu_q}_{\sigma_q} \phi_{\rho_1 \dots \rho_p}^{\sigma_1 \dots \sigma_q} \quad (8.9)$$

³²To see this, take the list of the tensor components as a vector and work out the transformation matrix for a few simple cases.

³³For a detailed discussion of complexifications of Lie algebras, see §9.

where $U \in \text{SU}(n)$ and \bar{U} is the complex conjugate of U . An **invariant tensor** is a tensor that is invariant under an $\text{SU}(n)$ transformation. The **dual tensor** of a (p, q) tensor is a (q, p) tensor.

In $\text{SU}(n)$, there are only three invariant tensors.

Proposition 8.3. *The invariant tensors of $\text{SU}(n)$ are the Kronecker delta δ^μ_ν and the Levi-Civita tensors $\epsilon_{\mu_1 \dots \mu_n}$ and $\epsilon^{\nu_1 \dots \nu_n}$.*

Proof. Explicit calculation. □

What do we want to do with tensors? From the argument above, we saw that tensors are extremely useful in representation theory of Lie groups and Lie algebras as they can represent irreducible representations. We have two items on our wish-list:

1. Constructing new irreps from old ones.
2. Finding branching rules.

We will find that tensors and Young diagrams play a central role in achieving these goals. As mentioned above, we will mainly focus on $\text{SU}(n)$ -tensors. There are some additional subtleties that must be dealt with when we talk about $\text{SO}(n)$ and $\text{Sp}(n)$ tensors - we will only briefly chat about this but this will not be the main focus on this section.

8.2 Building irreps using invariant tensors

Let us tackle our first goal of building irreducible representations. This is done by using the invariant tensors listed above for $\text{SU}(n)$ - namely the Levi-Civita tensors and the Kronecker Delta. The main procedure is as follows:

Proposition 8.4 (Building irreps using invariant tensors). *To build irreducible representations using invariant tensors, one follows the following procedure:*

1. *Start with a particular (p, q) tensor. For tensor products, just multiply the two tensors together.*
2. *Now build all possible contractions with the invariant tensors.*
3. *Subtract these contractions from the overall tensor and repeat step 2 until the leftover tensor is also invariant.*

To see this in action let us look at an example:

Example 8.1. Let us try and decompose ϕ_{ij} in $\text{SU}(n)$. The only invariant tensors we can use at this order is ϵ^{ab} , so we apply this to project this component out:

$$\phi_{[ij]} = \frac{1}{2} \epsilon^{ab} \phi_{ab} \epsilon_{ij} \tag{8.10}$$

Subtracting this from the tensor gives the full decomposition for rank-2 tensors:

$$T_{ij} = T_{(ij)} + T_{[ij]} \tag{8.11}$$

The above example in fact works for the tensor product $\phi_i \otimes \psi_j$, which we can just write as the rank-2 tensor $T_{ij} = \phi_i \psi_j$.

From the example above, it is therefore, in principle, possible to carry out Clebsch-Gordon decomposition by simply using invariant tensors and contracting indices whenever possible. This is in general extremely annoying and algebraically-challenging. Therefore, we will instead exploit a property of the $GL(n)$ irreps that allows us to utilise Young diagrams to perform such computations.

Remark 8.1. $SO(n)$ and $Sp(n)$ tensors are a bit more annoying. For example, in the example above for the full decomposition of $SO(n)$ one must also project out the trace part to get the full decomposition of the tensor.

8.3 Irreducible Representations of S_n and Young Diagrams

A brief interlude. I want to come back to our finite group discussion, and briefly return to symmetric groups S_n . For completeness, I am going to recall some facts from §2 and §5 related to symmetric groups S_n . First let us recall some facts about irreducible representations of S_n .

Proposition 8.5. *The number of irreducible representations of S_n is the number of conjugacy classes, which is the number $p(n)$ of partitions of n . The **partitions** of n is*

$$p(n) = \lambda_1 + \dots + \lambda_k, \lambda_1 \geq \dots \geq \lambda_k \geq 1. \quad (8.12)$$

Proof. The number of irreps is clearly equal to the number of conjugacy classes ³⁴. It remains to show that the number of conjugacy classes is equal to the number of partitions of n . This is however a corollary of Theorem 2.3. \square

To each partition of n we can associate a Young diagram, as defined in §5.

Definition 8.5. A **Young diagram** is a collection of boxes, or cells, arranged in left-justified rows, with a weakly decreasing number of boxes in each row. Listing the number of boxes in each row gives a partition $p(n)$ of integer n , where it is also the total number of boxes. Conversely, as stated above, each partition correspond to a Young diagram.

A Young diagram is sometimes called a Young frame or Ferrers diagram. To each diagram, we can number the boxes consecutively to generate Young tableau.

Definition 8.6. A **Young tableau** is a filling of a Young diagram such that the numbers are weakly increasing across each row but strictly increasing down each column. A **standard tableau** is a tableau in which entries are numbers from 1 to n , n being the total number of boxes.

Example 8.2. As an example let us look at the partitions of $n = 3$. Then we have the following partitions, Young diagrams and Standard Young Tableaux as illustrated in Table 8.1.

³⁴If you for once doubt this statement, look at Theorem 2.8 of Andre's notes.


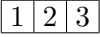





Partition	Young Diagram	Standard Young Tableaux
(3)		
(2, 1)		 
(1, 1, 1)		

Table 8.1: The Young diagrams and all possible Standard Young Tableaux for dimension-3.

Recall that Young tableaux can be used to describe projection operators for the regular representation which will then give the irreducible representations of S_n , which we have already seen in §5. Given a Standard Young tableau, we can define two subgroups of the symmetry group

$$P = P_\lambda = \{g \in S_n \mid g \text{ preserves each row}\} \quad (8.13)$$

$$Q = Q_\lambda = \{g \in S_n \mid g \text{ preserves each column}\} \quad (8.14)$$

Then in the group algebra $\mathbb{C}S_n$ we can introduce two elements corresponding to these two subgroups. In particular, we define

$$a_\lambda = \sum_{g \in P} e_g, \quad b_\lambda = \sum_{g \in Q} \text{sgn}(g) e_g. \quad (8.15)$$

The actions of $a_\lambda, b_\lambda \in \mathbb{C}S_n \rightarrow \text{End}(V^{\otimes n})$ are the following subspaces.

$$\text{Im}(a_\lambda) = \text{Sym}^{\lambda_1} V \otimes \dots \otimes \text{Sym}^{\lambda_k} V \subset V^{\otimes n}, \quad (8.16)$$

$$\text{Im}(b_\lambda) = \bigwedge^{\mu_1} V \otimes \dots \otimes \bigwedge^{\mu_l} V \subset V^{\otimes n}, \quad (8.17)$$

where μ is the conjugate partition to λ obtained by flipping the diagram along the 45° line. We can now define the following object.

Definition 8.7 (Young Projectors). The **Young symmetriser** is defined as

$$c_\lambda = a_\lambda \cdot b_\lambda \in \mathbb{C}S_n \quad (8.18)$$

Take $A = \mathbb{C}S_n$ as the group ring of S_n . If c_λ is a Young symmetriser, then the corresponding representation is $V_\lambda = A c_\lambda$. We have the following proposition.

Proposition 8.6. *An element of S_n can be written in at most one way as a product $p \cdot q$ where $p \in P$ and $q \in Q$.*

Proof. $P \cap Q = \{1\}$. So no commutations are possible. \square

This means that the form of c_λ can be written as

$$c_\lambda = \sum_{g=p \cdot q} \pm e_g, \quad (8.19)$$

with the coefficient ± 1 being the sign of q . Now we have the following lemma.

Lemma 8.1. *We have the following three statements.*

- (1) *For $p \in P$, $p \cdot a = a \cdot p = a$.*
- (2) *For $q \in Q$, $(\text{sgn}(q)q) \cdot b = b \cdot (\text{sgn}(q)q) = b$.*
- (3) *For all $p \in P$, $q \in Q$, $p \cdot c \cdot (\text{sgn}(q)q) = c$ and up to multiplication by a scalar, c is the only such element in A .*

Proof. The first two statements are obvious. So look at the third statement. Let us suppose $\sum_g n_g e_g$ satisfies the condition in (3). Then $n_{pgq} = \text{sgn}(q)n_g$ for all g, p, q and $n_{pq} = \text{sgn}(q)n_1$ where n_1 is the coefficient for the fully anti-symmetrised irrep. It is enough to show $n_g = 0$ when $g \notin PQ$. For such g it suffices to find a transposition t such that $p = t \in P$ and $q = g^{-1}tg \in Q$ so then $g = pgq$ and $n_g = -n_g$. Now suppose $T' = gT$ is the tableau obtained by replacing each entry i of T by $g(i)$. The claim is that there are two distinct integers that appear in the same row of T and in the same column of T' where the transposition t acts on. Take $p_i \in P$ and $q'_1 \in Q' = gQg^{-1}$ such that p_1T and q'_1T' have the same first row. Repeat this for the rest of the tableau to get $p \in P$ and $q' \in Q'$ such that $pT = q'T'$. Then $pT = q'gT$ so $p = q'g$. Defining $q = g^{-1}(q')^{-1}g \in Q$ we get $g = pq$. So if we don't have such distinct pair of integers t then we can write $g = pq$. \square

Before we continue let me define what it means by partitions being lexicographically-ordered.

Definition 8.8. The partitions are ordered **lexicographically** when $\lambda > \mu$ if the first non-vanishing $\lambda_i - \mu_i$ is positive.

Finally we have the following lemma.

Theorem 8.1. *The following two statements are true.*

- (1) *If $\lambda > \mu$ then for all $x \in A$, $a_\lambda \cdot x \cdot b_\mu = 0$. In particular, if $\lambda > \mu$, then $c_\lambda = c_\mu = 0$.*
- (2) *For all $x \in A$, $c_\lambda \cdot x \cdot c_\lambda$ is a scalar multiple of c_λ . In particular, $c_\lambda \cdot c_\lambda = n_\lambda c_\lambda$ for some $n_\lambda \in \mathbb{C}$.*

In particular, that statement implies some scalar multiple of c_λ is idempotent, i.e.

$$c_\lambda^2 = n_\lambda c_\lambda, \quad (8.20)$$

and the image of c_λ is an irrep V_λ of S_n , where the multiplication on $\mathbb{C}S_n$ acts on the right. Every irrep of S_n can be obtained in this way for a unique partition.

Proof. First prove (1). Take $x = g \in S_n$. Since $g \cdot b_\mu \cdot g^{-1}$ is the element constructed from gT' , where T' is the tableau used to generate b_μ , then it suffices to show that $a_\lambda \cdot b_\mu = 0$. Note $\lambda > \mu$ implies that there are two integers in the same row of T and the same column of T' . If t is the transposition of these integers, then $a_\lambda \cdot t = a_\lambda$, $t \cdot b_\mu = -b_\mu$ so we must have $a_\lambda \cdot b_\mu = a_\lambda \cdot t \cdot t \cdot b_\mu = -a_\lambda \cdot b_\mu$, as required. Part (2) immediately follows from Lemma 8.1. \square

This theorem gives a direct correspondence between conjugacy classes in S_n and irreducible representations of S_n - which is not true for general groups! We will eventually see that the image of the symmetrisers c_λ in $V^{\otimes n}$ provide essentially all the finite-dimensional irreps of $\text{GL}(V)$.

So now we have two separate unrelated things. On the one hand, we have $\text{GL}(n)$ tensors which effectively furnish irreps of $\text{GL}(n)$; on the other hand, we have irreducible representations of S_n which can be analysed using Young diagrams. Let us now make the connection between the two kinds of objects.

8.4 Schur-Weyl Duality

Let us briefly return to the group $\text{GL}(n)$. We have seen in the first section how we can decompose a rank-2 $\text{GL}(n)$ -tensor T_{ij} into two irreducible parts - namely the symmetric and anti-symmetric part.

$$T_{ij} = T_{(ij)} + T_{[ij]} = (\text{id} + \mathbf{p}) T_{ij} \quad (8.6)$$

Here $\mathbf{p} = (12)$ in cycle notation, the permutation of the two indices. The operator \mathbf{p} is obviously an element of the symmetric group S_2 so this begs the question - are the irreps of $\text{GL}(n)$ related to S_n ?

It turns out the symmetric group S_d does act on $V^{\otimes r}$ on the right and the action, as discussed above, commutes with the left action of $\text{GL}(V)$. It is therefore possible to relate the irreducible representations of $\text{GL}(n)$ with irreducible representations of the symmetric group S_n . This turns out to be an extremely powerful statement - one can now use the machinery from the representation theory of finite symmetric groups in analysing the representation theory of $\text{GL}(n)$ and its subgroups!

Let us briefly illustrate this principle by looking at Eq. (8.6) again. To obtain tensors of a particular symmetry type, we apply Young symmetrisers to the indices of the general rank- r tensor $T_{i_1 \dots i_r}$. For example, there are two types of rank-2 tensors, namely the symmetric type

$$T_{(ij)} \longleftrightarrow \boxed{i \ j}, \quad (8.21)$$

and the antisymmetric type

$$T_{[ij]} \longleftrightarrow \boxed{\begin{smallmatrix} i \\ j \end{smallmatrix}}. \quad (8.22)$$

We see that the two tensors are obtained by applying the Young symmetrisers $c_\pm = e \pm t$ where t indicates the transposition between the two indices respectively. This is true for rank- r tensors in general, with one exceptional caveat. Given a Young symmetriser of a

particular partition λ of integer n , the Young symmetriser acts on the right on the product of spaces $V^{\otimes n}$ to obtain an invariant subspace $\mathbb{S}_\lambda V^{\oplus r_\lambda}$ where r_λ is the number of standard Young tableaux obtainable from the Young diagram corresponding to the partition λ .

We now state the most important theorem in this section - the theorem allows us to relate Young diagrams with irreducible tensors we have been discussing about. Define the image of the Young symmetriser c_λ on $V^{\otimes n}$ by $\mathbb{S}_\lambda V$,

$$\mathbb{S}_\lambda V = \text{im} \left(c_\lambda \Big|_{V^{\otimes n}} \right), \quad (8.23)$$

which is a representation of $\text{GL}(V)$. Then we have the following theorem and corollary.

Theorem 8.2. *The following statements about the action of c_λ on $V^{\otimes n}$ are true.*

- (1) *Let $k = \dim V$. Then $\mathbb{S}_\lambda V$ is zero if $\lambda_{k+1} \neq 0$. If λ corresponds to the partition $\lambda = (\lambda_1 \geq \dots \geq \lambda_k \geq 0)$, then*

$$\dim \mathbb{S}_\lambda V = \prod_{1 \leq i \leq j \leq k} \frac{\lambda_i - \lambda_j + j - i}{j - i}. \quad (8.24)$$

- (2) *Let m_λ be the dimension of the irreducible representation V_λ of S_n corresponding to λ . Then*

$$V^{\otimes n} \cong \bigoplus_{\lambda} \mathbb{S}_\lambda V^{\otimes m_\lambda}. \quad (8.25)$$

- (3) *Each $\mathbb{S}_\lambda V$ is an irreducible representation of $\text{GL}(V)$.*

Proof. See §8.12 for the proof. □

In particular we have the following corollary.

Corollary 8.1 (Schur-Weyl Duality). *If $c \in \mathbb{C}S_n$ and $(\mathbb{C}S_n) \cdot c = \bigoplus_{\lambda} V_{\lambda}^{\oplus r_{\lambda}}$ as representations of S_n , then there is a corresponding decomposition of $\text{GL}(n)$ -spaces:*

$$V^{\otimes n} \cdot c = \bigoplus_{\lambda} \mathbb{S}_\lambda V^{\oplus r_{\lambda}} \quad (8.26)$$

For the details of the proofs and extended statements of the theorem and corollary I refer to §8.12 and [1]. We proceed by assuming that this is a known result, and apply this to analyse tensor representations. This is amazing. With the Schur-Weyl duality we can now proceed and analyse tensor irreps! Recall that $\text{GL}(n)$ irreducible tensors are the same as $\text{SU}(n)$ irreducible tensors. There is one extra caveat that we need to consider when we apply the Schur-Weyl duality to $\text{SU}(n)$ tensors however. Although we have shown that the invariant tensors are the same, the irreducible tensors in $\text{SU}(n)$ that descends from $\text{GL}(n)$ might not be independent.

Proposition 8.7. *The partition λ and λ' where $\lambda = (\lambda_1 \geq \dots \geq \lambda_n \geq 0)$ and $\lambda' = (\lambda_1 + s \geq \dots \geq \lambda_n + s \geq 0)$ of unimodular subgroups of $\text{GL}(n)$, such as $\text{SL}(n)$ and $\text{SU}(n)$, are equivalent.*

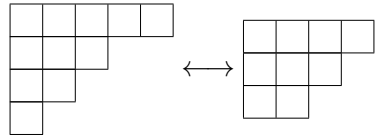
Proof. In particular, note that every column of length- n simply corresponds to the factor $(\det A)$. Since $\det A = 1$ for unimodular groups, the two partitions are equivalent. \square

In particular, we only need to consider patterns which have fewer than n rows. There is a second equivalence to relate conjugate representations.

Proposition 8.8. *The partition λ and λ' where $\lambda = (\lambda_1 \geq \dots \geq \lambda_n \geq 0)$ and $\lambda' = (\lambda_1 - \lambda_n \geq \dots \geq \lambda_1 - \lambda_2 \geq 0)$ of unimodular subgroups of $GL(n)$, such as $SL(n)$ and $SU(n)$, are equivalent.*

Proof. This can be seen from the existence of the Levi-Civita tensors, which relates the two partitions by acting on each of the columns of indices to obtain the conjugate indices. \square

The second equivalence is what we call *conjugation*. In fact, we are going to use this property extensively in our analysis of tensor irreps of $SU(n)$. In practice, the conjugate tensor irrep is obtained by looking for the Young diagram that completes the $n \times d$ rectangle, where n is the one in $SU(n)$ and d is the number of columns in the original partition. For example, the following two partitions are equivalent in $SU(5)$.


(8.27)

8.5 Tensors and Young Diagrams

Now that we have established the correspondence between tensor irreps and Young diagrams (and got the ugly maths out of the way), we can apply this technology to analyse tensors in $SU(n)$. The general rule to produce irreducible tensors is as follows:

1. Write out all partitions of r in Young diagram form for producing irreducible tensors of rank- r .
2. Write out all standard Young tableaux allowed by the rules. Each of that corresponds to an irreducible tensor.

We clearly see from the above example that Young diagrams correspond to the type of irreducible tensor you are constructing (i.e. the irrep), whilst the number of standard Young tableaux you can generate from that Young diagram would be the dimension of that irrep (see Corollary 8.1). It is often not feasible to write down all standard Young tableaux. Instead, there is a way to calculate the dimension of the tensor irrep as stated by the following proposition.

Proposition 8.9. *The dimension of an irrep in $GL(n, \mathbb{C})$ corresponding to a Young diagram can be calculated using the following algorithm.*

1. First draw two copies of the Young diagram.

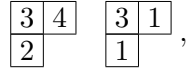
2. For the first diagram, start with n at the upper-left corner and assign numbers - for any box to the right add one, for any box below a box that already has a number in it minus one from that of the box above.
3. Compute Hook lengths using the second Young diagram.
4. Multiply all numbers in each diagram and divide the product of the numbers in the second diagram from that of the first - this is the dimension of the irrep.

Proof. Not too hard but skipped - this is a direct consequence of Theorem 8.2 and dealt with in Exercise 6.4 of [1]. \square

Example 8.3. Let us compute the dimension of



in $SU(3)$. The two diagrams are



where the first diagram corresponds to the n assignment and the second diagram is the Hook length diagram (Young tableaux with Hook lengths at each entry). Multiplying all the numbers in each diagram and dividing the numbers from 1 to 2 gives the dimension $4!/3 = 8$, as shown in the lecture notes (this is the adjoint representation of $SU(3)$).

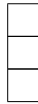
In $SU(n)$ there is a caveat - as discussed above you are allowed dual partitions (Proposition 8.8). In particular we have the following corollary on our rules on applying Young diagrams to $SU(n)$ tensors.

Proposition 8.10. *The dual irreducible representation $\bar{\mathbf{r}}$ of \mathbf{r} of $SU(n)$ can be obtained by drawing a “dual” diagram such that summing the number of boxes for each column for the two diagrams gives n .*

Proof. Direct consequence of Proposition 8.8. \square

Let us look at a few examples.

Example 8.4. In $SU(4)$ the anti-fundamental representation $\bar{\mathbf{4}}$ is

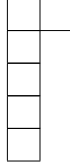


corresponding to the tensor $\phi^\mu = \epsilon^{\mu\nu\rho\sigma}\phi_{\nu\rho\sigma}$. The fundamental representation of course is



with corresponding tensor ϕ_μ . We can see that adding up the first column gives precisely 4 boxes.

Example 8.5. Since we have the Clebsch-Gordon decomposition $\mathbf{n} \otimes \bar{\mathbf{n}} = \mathbf{1} + \mathbf{adj}$, the adjoint representation of $SU(n)$ is always has the form - $(n - 1)$ boxes in the first column and 1 box in the second column, i.e. for example for adjoint of $SU(6)$ this is



Remark 8.2. From Proposition 8.7 and the above argument you can see for $SU(n)$ the number of boxes per column cannot exceed n . If the number of boxes in a column is precisely n , this means this is equivalent to the singlet representation. You can alternatively see from the Young projector - you cannot anti-symmetrise $(n + 1)$ objects non-trivially if you only have n unique objects.

Remark 8.3. When you are writing for an explicit form for the tensors, note carefully how you compose anti-symmetrisation and symmetrisation. Sometimes is best to write everything out, for example in $SU(4)$, we have the correspondence³⁵:

$$\begin{array}{|c|c|} \hline & \\ \hline & \\ \hline & \\ \hline & \\ \hline \end{array} \longleftrightarrow \phi_{\mu}^{\lambda} = \epsilon^{\lambda\nu\rho\sigma} (\phi_{\nu\rho\sigma\mu} + \phi_{\nu\rho\mu\sigma}) \quad (8.28)$$

Distributed version:

Exercise 8.1. Try Q4 of the second sheet again and find the Young tableaux and their associated tensors for the irreps **1**, **5**, **$\bar{5}$** , **15**, **10** and **24** of $SU(5)$.

This establishes the necessary ingredients in relating tensor irreps to Young diagrams.

8.6 Clebsch-Gordon decomposition from Young tableaux

Having described the tensor irreps in terms of Young diagrams and tableaux, we can now try to formulate tensor products and their Clebsch-Gordon decompositions in the same language. Recall the Clebsch-Gordon decomposition is the following:

$$\mathbb{S}_{\lambda}(V) \otimes \mathbb{S}_{\mu}(V) = \bigoplus N_{\lambda\mu\nu} \mathbb{S}_{\nu}(V) \quad (8.29)$$

the integers $N_{\lambda\mu\nu}$ describing how many times the irrep $\mathbb{S}_{\nu}(V)$ appears in the tensor product. Here we will illustrate a practical algorithm to realise this decomposition - as mentioned in the notes, this is known as the Littlewood-Richardson Rule (or the Littlewood-Richardson Theorem, see [1] for details). I am just going to state the algorithm here without proof.

Proposition 8.11. (Clebsch-Gordon decomposition from Young tableaux) To tensor A_{n-1} representations for two Young tableaux λ and μ proceeds as follows:

³⁵Note that you can use $()$ and $[]$ as symmetrising and antisymmetrising symbols, but be careful about how you write them so they are not confusing. So for this example $\phi_{(\nu[\rho]\mu\sigma]}$ is allowed.

1. Write the first Young tableaux λ ³⁶ with a s in the first row, b s in the second and so on. For example:

a	a	a
b	b	
c		

2. Now attach boxes of λ to μ one by one from the first row (labelled a s) onwards. We need to make sure that no two letters of the same type appear in the same column³⁷ and the result is always a valid Young tableau.
3. For each of Young tableaux obtained in the previous step, read all letters from right to left and top to bottom. (So you read from right to left along the first row, and then in the same direction in the second row, and so on.) This sequence, say $aaabbcabc$, must form a **lattice permutation**³⁸, i.e. we count the total number of every letter encountered at each box, and we require the number of a s to be more than or equal to the number of b s (and any subsequent letter), and so on. Otherwise we discard that Young tableau.

Proof. We need the so-called Littlewood-Richardson Theorem. You can find this in [1]. \square

This probably deserves an example:

Example 8.6. Let us look at the tensor product:

$$\square \otimes \begin{array}{|c|c|} \hline a & a \\ \hline b & \\ \hline \end{array} \quad (8.30)$$

Here I am deliberately decomposing the larger Young diagram to illustrate the rule - in practice you should always choose the smaller one. Now we do the second step:

$$\left(\begin{array}{|c|} \hline a \\ \hline \end{array} \oplus \begin{array}{|c|} \hline \\ \hline a \\ \hline \end{array} \right) \otimes \begin{array}{|c|} \hline a \\ \hline b \\ \hline \end{array} \quad (8.31)$$

Apply the second a :

$$\left(\begin{array}{|c|c|c|} \hline & a & a \\ \hline \end{array} \oplus \begin{array}{|c|c|} \hline & a \\ \hline a & \\ \hline \end{array} \oplus \begin{array}{|c|c|} \hline & a \\ \hline a & \\ \hline \end{array} \oplus \begin{array}{|c|} \hline \\ \hline a \\ \hline a \\ \hline \end{array} \right) \otimes \begin{array}{|c|} \hline b \\ \hline \end{array} \quad (8.32)$$

Now eliminate the invalid diagrams and the degenerate³⁹ ones:

$$\left(\begin{array}{|c|c|c|} \hline & a & a \\ \hline \end{array} \oplus \begin{array}{|c|c|} \hline & a \\ \hline a & \\ \hline \end{array} \right) \otimes \begin{array}{|c|} \hline b \\ \hline \end{array} \quad (8.33)$$

³⁶I would canonically choose the one with the smaller size as my first Young tableaux.

³⁷This is to ensure that the indices that are originally symmetrised do not get antisymmetrised, i.e. cancellation due to antisymmetry.

³⁸A lattice permutation is a string composed of positive integers in which every prefix (substring before that integer) contains at least as many positive integers i as integers $i + 1$. In our case just map $a \leftrightarrow 1$, $b \leftrightarrow 2$, etc.

³⁹Degenerate diagrams are the diagrams where the labels are written in the same way.

Add the third box:

$$\begin{array}{|c|c|c|c|} \hline & & & \\ \hline a & a & b & \\ \hline \end{array} \oplus \begin{array}{|c|c|c|} \hline & a & a \\ \hline b & & \\ \hline \end{array} \oplus \begin{array}{|c|c|c|} \hline & a & b \\ \hline a & & \\ \hline \end{array} \oplus \begin{array}{|c|c|} \hline & a \\ \hline a & b \\ \hline \end{array} \oplus \begin{array}{|c|} \hline a \\ \hline b \\ \hline \end{array} \quad (8.34)$$

Now according to the third rule, the first and third diagrams are invalid so we take them away. We are therefore left with:

$$\begin{array}{|c|} \hline \\ \hline \end{array} \otimes \begin{array}{|c|c|} \hline & \\ \hline & \\ \hline \end{array} = \begin{array}{|c|c|c|} \hline & & \\ \hline & & \\ \hline \end{array} \oplus \begin{array}{|c|c|} \hline & \\ \hline & \\ \hline \end{array} \oplus \begin{array}{|c|c|} \hline & \\ \hline & \\ \hline \end{array} \quad (8.35)$$

In $SU(3)$, this means that we have the decomposition (following the rule for computing dimensions):

$$\mathbf{3} \otimes \mathbf{8} = \mathbf{15} \oplus \mathbf{6} \oplus \mathbf{3} \quad (8.36)$$

recalling that three boxes in a column is just the singlet (i.e. \mathbb{C}).

8.7 Branching by index decomposition

Now we turn to branching - this is very important as we can generate subgroup representations from existing representations in the group. First recall what branching is.

Definition 8.9. Say H in a subgroup of a group G . A **branching rule** or **branching** is the restricted representation decomposition of irrep $R^{(G)}$ the form:

$$R^{(G)} \rightarrow R^{(H)} = \bigoplus R_s^{(H)} \quad (8.37)$$

where now $R_s^{(H)}$ are the irreps of H .

Branching is well-defined in the context of Lie groups. In fact, one can show that the Littlewood-Richardson Rule above allows you to give a restriction on a subrepresentation of an irrep. This requires the use of the so-called Pieri's formula [1]. Here I will present some practical way of doing branching using tensors.

The first method is index decomposition. The idea is simple - we rewrite our tensor indices as the ones in the subgroup and deduce all possible combination of these new indices under appropriate symmetrisation and anti-symmetrisation. The algorithm is as follows:

1. First write the tensor with respect to the Young diagram. We use explicitly the indices μ .
2. Replace the μ indices with indices specific to the subgroup, say a and i .
3. Compute all possible combinations and take away degenerate terms. Compute the relevant dimensions (no index - singlet field).

Example 8.7. Let us do this for one $SU(5)$ irrep and consider the canonical splitting $SU(5) \rightarrow SU(3) \times SU(2)$. We denote μ, ν as $SU(5)$ indices and replace that with a, b, \dots and i, j, \dots , the indices of $SU(3)$ and $SU(2)$. Now the branchings can be easily worked out by

specialising the SU(5) indices on a tensor to all possible combinations of SU(3) and SU(2) indices. Recall for the fundamental representation $\phi_\mu \sim \mathbf{5}$, so the branching

$$\phi_\mu \mapsto (\phi_a, \phi_i) \quad (8.38)$$

or

$$\mathbf{5} \mapsto (\mathbf{3}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{2}) \quad (8.39)$$

Now let us look at the $\mathbf{15}$ representation. This splits into ⁴⁰

$$\phi_{(\mu\nu)} \mapsto (\phi_{(ab)}, \phi_{ai}, \phi_{(ij)}) \quad (8.40)$$

or

$$\mathbf{15} \mapsto (\mathbf{6}, \mathbf{1}) \oplus (\mathbf{3}, \mathbf{2}) \oplus (\mathbf{1}, \mathbf{3}) \quad (8.41)$$

8.8 Branching using Young tableaux

There is of course another way of doing branching. Having established the 1-1 correspondence between tensor irreps and Young diagrams, one can actually try and split up possible Young diagrams from an existing Young diagram to generate branching tensors. The algorithm again is as follows:

1. Start with the Young diagram corresponding to the tensor irrep that you want to branch off.
2. Generate all possible splitting possible from the Young diagram. The Young diagram resulted must be a valid Young diagram in the respective A_n representation.
3. Finally label the splitting using dimensions.

Technically, what we are actually computing is the embedding $SU(n) \times SU(m) \subset SU(n+m)$. The number of times the original representations appears is then the total number of times that the representations R_n and R_m of SU(n) and SU(m) respectively will appear in the branching [8]. It is much easier to see this in action so let us do that for the SU(5) tensors again.

Example 8.8. Let us look at the branching of the fundamental rep of SU(5) into $SU(3) \times SU(2)$, $\mathbf{5} \mapsto (\mathbf{3}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{2})$. To do this we start with one box and try and split the diagram (we cannot) so we have

$$\square \mapsto (\square, \bullet) \oplus (\bullet, \square) \quad (8.42)$$

with the first and second entries the SU(3) and SU(2) representations respectively so we get precisely the splitting. Now look at $\mathbf{15}$. This splits into:

$$\square\square \mapsto (\square\square, \bullet) \oplus (\square, \square) \oplus (\bullet, \square\square) \quad (8.43)$$

which is precisely the required splitting

$$\mathbf{15} \mapsto (\mathbf{6}, \mathbf{1}) \oplus (\mathbf{3}, \mathbf{2}) \oplus (\mathbf{1}, \mathbf{3}) \quad (8.44)$$

⁴⁰Noting by antisymmetry that $\phi_{ai} = -\phi_{ia}$ so the two tensors (ϕ_{ai}, ϕ_{ia}) get shoved into the same irrep.

8.9 Branching with $U(1)$ factors

We want to now start considering branching that includes $U(1)$ factors. We will begin with a specific case where we are looking at a branching

$$SU(n+1) \rightarrow SU(n) \times U(1) \quad (8.45)$$

The extra $U(1)$ is added in to make it into a maximal subgroup. To proceed, we consider the subgroup $SU(n) \times U(1)$. Let us think about what happens in the tensor indices language. Let us start with the fundamental representation \mathbf{n} and consider what happens to the tensor:

$$\phi_a \rightarrow \phi_i \oplus \phi' \quad (8.46)$$

Here we note that a is the $SU(n+1)$ index and i is the $SU(n)$ index. The ϕ' indicates the tensor in $U(1)$ - there are no indices associated to $U(1)$ in this language! This means that for every box in the Young diagram (noting that each box indicates an index), it will either be assigned to the fundamental rep \mathbf{n} of $SU(n)$ or the trivial rep $\mathbf{1}$. Therefore, for a Young diagram with n boxes, j will transform like \mathbf{n} and $n-j$ will transform like singlets in $SU(n-1)$.

Now, to determine the $U(1)$ charge, we need to work out the transformations of the fundamental representations \mathbf{n} and the trivial rep in $U(1)$. Let us clear consider a more general problem of the following.

Proposition 8.12. *Consider the Lie group $SU(n+m)$ which branches into $SU(n) \times SU(m)$. Then there is a unique $U(1) \subset SU(n+m)$ subgroup which commutes with $SU(n) \times SU(m)$. The matrices that commutes with $SU(n) \times SU(m)$ are diagonal.*

Proof. Suppose V is an $SU(n+m)$ arbitrary matrix and $U \in SU(n) \times SU(m) \subset SU(n+m)$. We can write these matrices in block form:

$$V = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \quad (8.47)$$

$$U = \begin{pmatrix} U_n & 0 \\ 0 & U_m \end{pmatrix} \quad (8.48)$$

Now the commutation condition $[V, U] = 0$ implies that

$$\begin{cases} U_n A = A U_n \\ U_n B = B U_m \\ U_m C = C U_n \\ U_m D = D U_m \end{cases} \quad (8.49)$$

Now setting $U_m = 0$ in the second and third equations give $B = C = 0$. By Schur's Lemma ⁴¹, the first and fourth equation gives

$$A = \lambda_n \mathbb{1}_n \quad (8.50)$$

⁴¹In last year's notes this is incorrectly referred to the proof in Q4 of the first problem sheet. The generalised Schur's Lemma doesn't apply here - but the proof is of a similar spirit regardless.

$$D = \lambda_m \mathbb{1}_m \quad (8.51)$$

for $\lambda_n, \lambda_m \in \mathbb{C}$. Now for

$$V = \begin{pmatrix} \lambda_n \mathbb{1}_n & 0 \\ 0 & D = \lambda_m \mathbb{1}_m \end{pmatrix}$$

to be an $SU(n+m)$ matrix, then we require $|\lambda_n| = 1 = |\lambda_m|$ (from unitarity) and

$$\lambda_n^n \lambda_m^m = 1 \quad (8.52)$$

This allows us to determine the form of the matrix, for example, with the solution:

$$\begin{cases} \lambda_n = e^{-im\alpha} \\ \lambda_m = e^{in\alpha} \end{cases} \quad (8.53)$$

To show this is the unique $U(1)$ group (with freedom in choosing arbitrary charges), note that if we start with a different solution in Equation 8.53 then we will get the same answer up to a root of unity transformation. \square

The conclusion from the above proposition is that for a branching $SU(n+m) \rightarrow SU(n) \times SU(m)$, the fundamental representation $\mathbf{n} + \mathbf{m}$ breaks into \mathbf{n} and \mathbf{m} with the $U(1)$ charge m and $-n$ respectively. In our present case $SU(n+1) \rightarrow SU(n)$, we set $m = 1$ and note that $SU(1)$ does not exist - we can assign the normalised charge $\frac{1}{n}$ to the fundamental rep \mathbf{n} and $-\frac{n-1}{n}$ to the trivial rep $\mathbf{1}$. Note here that if the group is $SU(m)$ where $m = 1$, then we only allow symmetric Young diagrams (i.e. horizontal diagrams).

Example 8.9. Let us look at an example where $SU(3) \rightarrow SU(2) \times U(1)$. The fundamental rep breaks like

$$\mathbf{3} \rightarrow \mathbf{2} \oplus \mathbf{1} \quad (8.54)$$

For a Young diagram representing a tensor with n indices, we then have n boxes which then branches with m boxes that transform like doublets and $n-m$ boxes that transform like singlets. The total $U(1)$ charge is then

$$q = \frac{m}{3} - \frac{2(n-m)}{3} = -\frac{2n}{3} + m \quad (8.55)$$

Let us look at a particular example. Consider **adj** of $SU(3)$.

$$\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array} \rightarrow \left(\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array}, \bullet \right) \oplus \left(\begin{array}{|c|} \hline \square \\ \hline \end{array}, \begin{array}{|c|} \hline \square \\ \hline \end{array} \right) \oplus \left(\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array}, \begin{array}{|c|} \hline \square \\ \hline \end{array} \right) \oplus \left(\begin{array}{|c|} \hline \square \\ \hline \end{array}, \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array} \right) \quad (8.56)$$

Then using Equation 8.55 to compute charges gives

$$\mathbf{8} \rightarrow \mathbf{2}_1 \oplus \mathbf{1}_0 \oplus \mathbf{3}_0 \oplus \mathbf{2}_{-1} \quad (8.57)$$

To generalise this to the case $SU(n+m) \rightarrow SU(n) \times SU(m)$ is simple. We follow the rule in the previous subsection for the branching and then assign the $U(1)$ charges according to the rule set out in Proposition 8.12. Let us now consider some examples.

Example 8.10. Let us consider the splitting of $SU(6) \rightarrow SU(4) \times SU(2)$. Using the argument in Proposition 8.12 would give the following commuting matrix:

$$U = \text{diag}(e^{i\alpha}, e^{i\alpha}, e^{i\alpha}, e^{i\alpha}, e^{i\beta}, e^{i\beta}) \quad (8.58)$$

Imposing the special unitary condition, i.e. $\det(U) = 1$, means that $2\alpha + \beta = 0$. We can canonically pick the scaling that gives the following diagonal matrix:

$$U = \text{diag}(e^{-i\alpha}, e^{-i\alpha}, e^{-i\alpha}, e^{-i\alpha}, e^{2i\alpha}, e^{2i\alpha}) \quad (8.59)$$

Remark 8.4. If there is a splitting to more than two groups, you will in general obtain more than one possible assignments of $U(1)$ charges. For example, in the any splitting of a Lie group into two Lie subgroups case, you will always have an \mathbb{R}^* ambiguity in assigning charges. This is normally picked to have a consistent charge with the literature (so probably the simplest integer combination/ with a normalisation like in the illustrated $SU(3)$ case).

Now to generate the required $U(1)$ charges for the tensor splitting, we associate a charge⁴² to each of the indices in the subgroup. For branching $SU(n+m) \rightarrow SU(n) \times SU(m)$ we look at the fundamental:

$$\mathbf{n} + \mathbf{m} \rightarrow \mathbf{n}_m \oplus \mathbf{m}_{-n} \quad (8.60)$$

So we associate to each $SU(n)$ index a charge of m and to each $SU(m)$ index a charge of $-n$. We then add up the charges.

Example 8.11. For example, we associate a -1 charge for every $SU(4)$ index and 2 for every $SU(2)$ **lower** index in the $SU(6)$ splitting in Example 8.10. Then in the index decomposition language, we can immediately obtain:

$$\bar{\mathbf{6}} \mapsto (\bar{\mathbf{4}}, \mathbf{1})_1 \oplus (\mathbf{1}, \mathbf{2})_{-2} \quad (8.61)$$

by using the splitting:

$$\psi^\mu \mapsto (\psi^a, \psi^i) \quad (8.62)$$

Also doing that for the two index tensor case:

$$\psi_{[\mu\nu]} \mapsto (\psi_{[ab]}, \psi_{ai}, \psi_{[ij]}) \quad (8.63)$$

we get:

$$\mathbf{15} \mapsto (\mathbf{6}, \mathbf{1})_{-2} \oplus (\mathbf{4}, \mathbf{2})_1 \oplus (\mathbf{1}, \mathbf{1})_4 \quad (8.64)$$

remembering the conjugate the charge for anti-fundamental indices ⁴³.

In the Young diagram language, this is equivalent to associating a number to each of the boxes in the splittings. This will allow you to compute the charge by just summing the numbers in the Young diagram.

⁴²As in the usual case in the literature.

⁴³Remember there is some funky stuff going on when you apply the raising and lowering operators (Levi-Civita tensors). They act non-trivially to the $U(1)$ charges as they add a non-zero number of indices (n) to your tensor expression. This means that ϕ^μ is technically not the same as $\phi_{[\mu\nu\rho\sigma]}$!

Example 8.12. Let us continue with the $SU(6)$ splitting started in Example 8.11. To each box in $SU(4)$ and $SU(2)$ we associate a charge -1 and $+2$. Simply sum up in each case ($\bar{\mathbf{6}}$ and $\mathbf{15}$):

$$\begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array} \mapsto \left(\begin{array}{|c|} \hline a \\ \hline a \\ \hline a \\ \hline \end{array}, \begin{array}{|c|} \hline 2 \\ \hline 2 \\ \hline \end{array} \right) \oplus \left(\begin{array}{|c|} \hline a \\ \hline a \\ \hline a \\ \hline a \\ \hline \end{array}, \begin{array}{|c|} \hline 2 \\ \hline \end{array} \right) \quad (8.65)$$

$$\begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \end{array} \mapsto \left(\begin{array}{|c|} \hline a \\ \hline a \\ \hline \end{array}, \bullet \right) \oplus \left(\begin{array}{|c|} \hline a \\ \hline \end{array}, \begin{array}{|c|} \hline 2 \\ \hline \end{array} \right) \oplus \left(\bullet, \begin{array}{|c|} \hline 2 \\ \hline 2 \\ \hline \end{array} \right) \quad (8.66)$$

where I have used a to indicate the charge -1 . You can see the charges match up with

$$\bar{\mathbf{6}} \mapsto (\bar{\mathbf{4}}, \mathbf{1})_1 \oplus (\mathbf{1}, \mathbf{2})_{-2} \quad (8.61)$$

$$\mathbf{15} \mapsto (\mathbf{6}, \mathbf{1})_{-2} \oplus (\mathbf{4}, \mathbf{2})_1 \oplus (\mathbf{1}, \mathbf{1})_4 \quad (8.64)$$

8.10 Branching using Symmetries

In this section let us further discuss branching but using symmetry arguments. We have seen that for $SU(5)$ the antisymmetric representation branches under $SU(3) \times SU(2)$ as ⁴⁴

$$\mathbf{10} \mapsto (\bar{\mathbf{3}}, \mathbf{1}) \oplus (\mathbf{3}, \mathbf{2}) \oplus (\mathbf{1}, \mathbf{1}) \quad (8.67)$$

The question is now as follows: can we reproduce that using symmetry arguments? Let us note first that the $\mathbf{10}$ rep is the antisymmetric representation, i.e.

$$\mathbf{10} = (\mathbf{5} \otimes \mathbf{5})_A \quad (8.68)$$

where the subscript A denotes the antisymmetric part of the multiplication, i.e. after the Clebsch-Gordon decomposition is performed. The fundamental rep $\mathbf{5}$ has the branching

$$\mathbf{5} \rightarrow (\mathbf{3}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{2}) \quad (8.69)$$

under $SU(3) \times SU(2)$. We can now use symmetry arguments to obtain the branching of $\mathbf{10}$ in $SU(5)$ - we simply take the antisymmetric part as follows:

$$\begin{aligned} \mathbf{10} &= (\mathbf{5} \times \mathbf{5})_A = [((\mathbf{3}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{2})) \otimes ((\mathbf{3}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{2}))]_A \\ &= [(\mathbf{3}, \mathbf{1}) \otimes (\mathbf{3}, \mathbf{1})]_A \oplus [(\mathbf{3}, \mathbf{1}) \otimes (\mathbf{1}, \mathbf{2}) \oplus (\mathbf{1}, \mathbf{2}) \otimes (\mathbf{3}, \mathbf{1})]_A \oplus [(\mathbf{1}, \mathbf{2}) \otimes (\mathbf{1}, \mathbf{2})]_A \\ &= (\bar{\mathbf{3}}, \mathbf{1}) \oplus (\mathbf{3}, \mathbf{2}) \oplus (\mathbf{1}, \mathbf{1}) \end{aligned} \quad (8.70)$$

Note that to evaluate the second term one must regard the two $\mathbf{5}$ representations to be unique (distinguishable). This means that in order to evaluate the term we must have, say identified the first $\mathbf{5}$ as A and the last one as B and noted that ⁴⁵

$$(A, B) = \frac{1}{2} [(A, B) + (B, A)] + \frac{1}{2} [(A, B) - (B, A)] \quad (8.71)$$

⁴⁴This is in fact one of the questions on the example sheet. You should now be able to reproduce this result using the Young diagram method.

⁴⁵The bracket notation here has nothing to do with the bracket indicating branching of groups like above!

Taking the antisymmetric part of this hence gives us one copy ⁴⁶ of **5** or in our case, after taking the product, **(3, 2)**. We then obtain the result as obtained from alternative methods.

This concludes our discussion on tensor representations.

8.11 $SU(5)$ Matter Multiplet Branching

In this subsection ⁴⁷, we will explicitly work out the $SU(5)$ matter multiplet branching rules as sketched out in the notes using the methods described in this section.

Distributed version:

Exercise 8.2. Try Q4 of the second sheet again! You should now be able to proceed up to part (d) with ease. In particular, you should be able to compute the branching of $\bar{\mathbf{5}}$ and $\mathbf{10}$ and the corresponding $U(1)$ charges. Using tensor methods to construct singlets for (e) is the easiest way to proceed, for example the singlet in $\mathbf{5} \otimes \bar{\mathbf{5}}$ is $\epsilon^{abcde} \phi_a \psi_{bcde}$. In particular, use the notations $\mathbf{10} \sim \psi_{[\mu\nu]}$, $\bar{\mathbf{5}} \sim \chi^\mu$, $\mathbf{5} \sim \bar{H}_\mu$ and $\bar{\mathbf{5}} \sim H^\mu$, and the branching rule $\psi_{[\mu\nu]} = (u_{[\alpha\beta]}, Q_{\alpha i}, e_{[ij]})$, $\chi^\mu = (d^\alpha, L^i)$, $H_\mu = (t_\alpha, h_i)$, $\bar{H}^\mu = (\bar{t}_\alpha, \bar{h}_i)$. What is the physical significance of the last part of the question?

8.12 Proof of Schur-Weyl Duality

In this section we will provide a proof to Theorem 8.2. We will need a bit of machinery about semisimple algebras. Take G to be any finite group, U a right module over the group algebra $A = \mathbb{C}G$ and let

$$B = \text{Hom}_G(U, U) = \{\varphi : U \rightarrow U \mid \varphi(v \cdot g) = \varphi(v) \cdot g \ \forall u \in U, g \in G\} \quad (8.72)$$

Note that B acts on U from the left and commutes with the right action of A . B is known as the commutator algebra. If $U = \bigoplus_i U_i^{\oplus n_i}$ is an irreducible decomposition with U_i nonisomorphic irreducible right A -modules, then by Schur's Lemma we have

$$B = \bigoplus_i \text{Hom}_G(U_i^{\oplus n_i}, U_i^{\oplus n_i}) = \bigoplus_i M_{n_i}(\mathbb{C}) \quad (8.73)$$

where $M_{n_i}(\mathbb{C})$ is the ring of $n_i \times n_i$ complex matrices. If W is any left A -module, the tensor product

$$U \otimes_A W = \frac{U \otimes_{\mathbb{C}} W}{\text{subspace generated by } \{va \otimes w - v \otimes aw\}} \quad (8.74)$$

is a left B -module by acting on the first factor $b \cdot (v \otimes w) = (b \cdot v) \otimes w$. Now we have the following lemma.

Lemma 8.2. *Let U be a finite-dimensional right A -module.*

(i) *For any $c \in A$, the canonical map $U \otimes_A A c \rightarrow U c$ is an isomorphism of left B -modules.*

⁴⁶Well, one effective antisymmetric copy.

⁴⁷This subsection is for the benefit for tutors i.e. us! It will be removed from the published version of the notes as it contains solutions to the sheet.

(ii) If $W = Ac$ is an irreducible left A -module, then $U \otimes_A W = Uc$ is an irreducible left B -module.

(iii) If $W_i = Ac_i$ are the distinct irreducible left A -modules, with m_i the dimension of W_i , then

$$U \cong \bigoplus_i (U \otimes_A W_i)^{\oplus m_i} \cong \bigoplus_i (Uc_i)^{\oplus m_i} \quad (8.75)$$

is the decomposition of U into irreducible left B -modules.

Proof. By the semisimplicity of all representations of G , Ac is a direct summand of A as a left- A module. First prove (i). Consider the commutative diagram

$$\begin{array}{ccccc} U \otimes_A A & \xrightarrow{\cdot c} & U \otimes_A Ac & \xrightarrow{\hookrightarrow} & U \otimes_A A \\ \downarrow & & \downarrow & & \downarrow \\ U & \xrightarrow{\cdot c} & U \cdot c & \xrightarrow{\hookrightarrow} & U \end{array}$$

The vertical maps are the maps $v \otimes a \mapsto v \cdot a$. Now the left horizontal maps are surjective, the right ones are injective, and the outside vertical maps are isomorphisms, so the middle vertical map must be isomorphic. Now for (ii), consider first the case where U is an irreducible A -module, so $B = \mathbb{C}$. It suffices to show that $\dim U \otimes_A W \leq 1$. Identify A with a direct sum $\bigoplus_{i=1}^r M_{m_i} \mathbb{C}$ of r matrix algebras (refer to Proposition 3.29 of [1]). We can now identify W with a minimal left ideal of A . Any minimal ideal in the sum of matrix algebras is isomorphic to one which consists of r -tuples of matrices zero except for one factor, in which they are all zeros except for one column. Similarly we can identify U with a minimal right ideal of A , so same as above but with column switched out for row. Therefore any non-zero $U \otimes_A W$ can be identified with the matrices which are zero except in one row and column the same factor, otherwise it will be zero. For the general case, by Maschke's theorem we can write $U = \bigoplus_i U_i^{\oplus n_i}$ irreducible right A -modules, then $U \otimes_A W = \bigoplus_i (U_i \otimes_A W)^{\oplus n_i}$. Now since A is the left regular representation of G (by definition of group algebra), $U_i \otimes W = \mathbb{C}$ iff U_i and W correspond to the same factor of A , as above. So $U \otimes_A W = \bigoplus_i (U_i \otimes_A W)^{\oplus n_i} = \mathbb{C}^{\oplus n_k}$ for some k , and this is visibly irreducible over $B = \bigoplus_i M_{n_i}(\mathbb{C})$. Part (iii) simply follows from $A \cong \bigoplus_i W_i^{\oplus m_i}$ which gives

$$U \cong U \otimes_A A \cong \bigoplus_i (U \otimes_A W_i)^{\oplus m_i} . \quad (8.76)$$

□

We apply this lemma with $U = V^{\otimes n}$ which is a right $\mathbb{C}S_n$ -module. Set $G = S_n$. The lemma above tells us how to decompose $V^{\otimes n}$ into a B -module where B is the algebra of all endomorphisms of U that commute with all permutations of the factors. So we now have:

Lemma 8.3. *The algebra B is spanned as a linear subspace of $\text{End}(V^{\otimes n})$ by $\text{End}(V)$. A subspace of $V^{\otimes n}$ is a sub- B modules iff it is invariant by $\text{GL}(V)$; in other words a B -submodule is the same as a $\text{GL}(V)$ -submodule.*

Proof. If W is any finite-dimensional vector space, then $\text{Sym}^n W$ is the subspace of $W^{\otimes n}$ spanned by all $w^n = n!w \otimes \dots \otimes w$. Apply this to $W = \text{End}(V)$ proves the first statement as $\text{End}(V^{\otimes n}) = (V^*)^{\otimes n} \otimes V^{\otimes n} = W^{\otimes n}$, which is compatible with the actions of $G = S_n$. Since $\text{GL}(V)$ is dense in W which is either in the Euclidean or Zariski topology, the second statement also follows. \square

The proof of Theorem 8.2 then follows from the two lemmas. In particular, we define

$$\mathbb{S}_\lambda V = V^{\otimes n} \otimes_A V_\lambda, \quad (8.77)$$

where V_λ denotes the irrep corresponding to a partition λ . Then this is irreducible as a $\text{GL}(V)$ -module by Lemma . Further writing $A = \bigoplus_\lambda (V_\lambda)^{\oplus m_\lambda}$ as a left-semisimple decomposition we then have

$$V^{\oplus n} = V^{\oplus n} \otimes_A A = \bigoplus_\lambda (V^{\otimes n} \otimes_A V_\lambda)^{\oplus m_\lambda} = \bigoplus_\lambda \mathbb{S}_\lambda V, \quad (8.78)$$

which exactly gives the Schur-Weyl Duality in Corollary 8.1.

9 Complexification and Real Forms

In this section I would like to go back to the Cartan-Weyl formalism and discuss some intricacies about the Killing form. We know that the Killing form can act as a bilinear form (effectively an inner product) in weight space. But we also remarked that compact Lie algebras in fact have a negative-definite Killing form. An inner product is by definition a positive-definite bilinear form. This section aims to disentangle this seeming contradiction.

9.1 The Killing Form

First let us recall some facts about the Killing form. Recall that the Killing form is defined as follows.

Definition 9.1. Define the **adjoint** linear operator $\text{ad}_X : \mathfrak{g} \rightarrow \mathfrak{g}$ as follows:

$$\text{ad}_X : Y \mapsto [X, Y] \quad (9.1)$$

With this the **Killing form** is then defined as follows:

$$\Gamma(X, Y) = \text{tr}(\text{ad}_X \circ \text{ad}_Y) \quad (9.2)$$

where \circ denotes the composition of maps.

We note that this definition is basis-independent - one can evaluate the Killing form with respect to a specific basis which yields a real number.

The Killing form is a bilinear symmetric operation. For a semi-simple Lie algebra \mathfrak{g} , we have the following two theorems:

Theorem 9.1. *The Killing form is non-degenerate if and only if \mathfrak{g} is a semi-simple Lie algebra.*

Proof. See Theorem 4.3 of notes. □

Theorem 9.2. *For any nonzero $X \in \mathfrak{g}$, \mathfrak{g} is a compact semisimple Lie algebra if and only if $\Gamma(X, X) < 0$ (i.e. the Killing form is negative-definite).*

Proof. There are two proofs to this. Here I will only present one of them. Define

$$g_{ij} = \Gamma(X_i, X_j) = \text{Tr}(\text{ad}_{X_i} \circ \text{ad}_{X_j}) \quad (9.3)$$

where $\{X_i\}$ are a basis for Lie algebra \mathfrak{g} . Writing the matrix $\{g_{ij}\}$ as \mathcal{G} , we can then choose the coordinates on the Lie group manifold (using Sylvester's Theorem, remembering that we are operating the real basis)

$$\mathcal{G} = \text{diag}(1, 1, \dots, 1, -1, \dots, -1) \quad (9.4)$$

where 1 appears r times and -1 s times. By diagonalising, we can write

$$\mathcal{G}' = S^T \mathcal{G} S \quad (9.5)$$

where $n = r + s$ is the dimension of \mathcal{L} . We note that the metric has no zeros along the diagonal due to the non-degeneracy of the Killing form, from Theorem 9.1. Now let us write Ad_g , the adjoint representation of the Lie group G as A . Recall that the matrix Lie group $\text{O}(r, s)$ is defined by

$$\text{O}(r, s) = \{A \in \text{Mat}_{r+s}(\mathbb{R}) \mid A^T \mathcal{G} = \mathcal{G} A^{-1}\} \quad (9.6)$$

Now we show the following. Using $\{X_i\}$ as the Lie algebra basis, and x^k the coordinate with respect to the basis $\{X_k\}$. Writing a general element $X \in G$ as

$$X = x^k X_k \quad (9.7)$$

Then using

$$\text{Ad}_g(X_k) = g X_k g^{-1} = A_k^j X_j \quad (9.8)$$

then we have

$$\text{Ad}_g(X) = x^k A_k^j X_j \quad (9.9)$$

Therefore, writing out $\Gamma(X, Y) = \Gamma(\text{Ad}_g(X), \text{Ad}_g(Y))$ we get

$$\mathcal{G} = A^T \mathcal{G} A \quad (9.10)$$

i.e. exactly the definition of the matrix Lie group $\text{O}(r, s)$. Similarly, writing the adjoint representation of the Lie algebra \mathfrak{g} ad_X using the matrix B , and $\text{ad}(X) = x^k B_k^j X_j$ and the relation

$$\Gamma(\text{ad}_Z(X), Y) = -\Gamma(X, \text{ad}_Z(Y)) \quad (9.11)$$

we obtain

$$B = -\mathcal{G}^{-1} B^T \mathcal{G} \quad (9.12)$$

We now compare this with the matrix Lie algebra $\mathfrak{so}(r, s)$:

$$\mathfrak{so}(r, s) = \{A \in \text{Mat}_{r+s}(\mathbb{R}) \mid A^T \mathcal{G} = \mathcal{G} A^{-1}\} \quad (9.13)$$

So we conclude the following - the adjoint representation of G and \mathfrak{g} must be subalgebras of $\text{O}(r, s)$ and $\mathfrak{so}(r, s)$ respectively. Now consider the case where $\mathcal{G} = \pm \mathbb{1}_n$. Then

$$\begin{cases} A^T A = \mathbb{1}_n \\ B^T = -B \end{cases} \quad (9.14)$$

So the adjoint reps of G and \mathfrak{g} are subalgebras of $\text{O}(n)$ and $\mathfrak{so}(n)$. In particular, for $X \in \mathfrak{g}$ we have

$$\Gamma(X, X) = \text{Tr}(\text{ad}_X \circ \text{ad}_X) = \text{Tr}(B^2) = -\text{Tr}(B B^T) = -\sum_{ij} B_{ij} B_{ij} < 0 \quad (9.15)$$

The final sign is replaced by an equal sign only where $X = 0$ identically. Therefore $\mathcal{G} = -\mathbb{1}_n$ and we have a negative-definite Killing form. We deduce the following - a negative-definite Killing form implies that the adjoint group corresponding to the matrices of the adjoint

representation of \mathfrak{g} constitutes a closed subgroup of $\mathfrak{so}(n)$. Therefore the identity-connected component of the automorphism group of \mathfrak{g} is a subgroup of a compact group and therefore is also compact - i.e. by definition \mathfrak{g} is a compact semisimple Lie algebra [35] (Note that the adjoint map $\text{Ad} : G \rightarrow GL(\mathfrak{g})$ has kernel that is the centre⁴⁸ of G (as the elements commute through). The Lie algebra \mathfrak{g} is semi-simple, the centre is then finite so this shows that $G/Z(G)$ is compact. To show that G is compact, we will need use the Hopf-Rinow theorem to show that $(G, -\Gamma)$ is indeed a complete Riemannian manifold together with Bonnet-Myers Theorem. I refer to [36] for details.) The converse of this is obvious. (For a Lie group G the lie algebra must be compact.) \square

With this now I would like to present two main questions.

1. For a semisimple compact Lie algebra, we have shown in Theorem 9.2 that the Killing form is negative-definite. It is obvious that to obtain a positive-definite bilinear form (i.e. an inner product) on root space one can take the negative of the Killing form. But in Theorem 4.6 of the notes for complex semisimple Lie algebras we somehow seem to be able to define an inner product directly using

$$(\alpha, \beta) = \Gamma(H_\alpha, H_\beta) \quad (9.17)$$

How does this work? Is this a typo for the real compact Lie algebra case, or is the situation different for complex semisimple Lie algebras?

2. The derivation for complex semi-simple Lie algebras show that the roots are real in the subspace \mathfrak{h}_0 of the real form of the Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$. However, we know that ad_X is skew-symmetric with respect to the bilinear form (Killing form) and therefore has imaginary eigenvalues $i\alpha(X)$ (or that $\alpha(X)$ are pure imaginary). Is this a contradiction?

To answer this we will need to understand what complexification and real forms are for Lie algebras.

9.2 Complexification, Realification and Real Forms

Let us go back to the basics. First consider a vector space V .

Definition 9.2. Fix a vector space V over field k . We define the vector space $V^{\mathbb{K}} := V \otimes_k \mathbb{K}$ as follows: we let $1 \otimes \cdot c$ be the k -linear map of $V \otimes_k \mathbb{K}$ to itself where $\cdot c$ denotes multiplication by $c \in \mathbb{K}$. This denotes scalar multiplication in $V^{\mathbb{K}}$. Then $V^{\mathbb{K}}$ is the **vector space in extension field** \mathbb{K} . Alternatively, starting with a vector space W over field \mathbb{K} , we can restrict the definition of scalar multiplication to scalars in k to get a vector space over k , denoted W^k . This is called the **vector space in restriction field** k .

⁴⁸The **centre** of a group is defined as

$$Z(G) = \{z \in G \mid \forall g \in G, zg = gz\} \quad (9.16)$$

Definition 9.3. For the case $k = \mathbb{R}$ and $\mathbb{K} = \mathbb{C}$ in the above definition, and V being a real vector space, the complex vector space $V^{\mathbb{C}}$ is called the **complexification** of V . If W is a complex vector space, the real vector space $W^{\mathbb{R}}$ is the **realification** of W . Note that

$$(V^{\mathbb{C}})^{\mathbb{R}} = V \oplus iV \quad (9.18)$$

as real vector spaces⁴⁹ with i denoting a real linear transformation multiplication by i . This shows that the two operations are not inverses to each other, $(V^{\mathbb{C}})^{\mathbb{R}}$ has twice the real dimension of V , and $(W^{\mathbb{R}})^{\mathbb{C}}$ has twice the complex dimension of W .

Definition 9.4. Suppose W and V are complex and real vector spaces respectively. If

$$W^{\mathbb{R}} = V \oplus iV \quad (9.19)$$

we say V is a **real form** of the complex vector space W .

Proposition 9.1. *Any real vector space is a real form of its complexification.*

Proof. Obvious from Equation 9.18. □

Now for Lie algebras. To impose a Lie algebra structure on the \mathbb{K} vector space $\mathfrak{g} = (\mathfrak{g}_0)^{\mathbb{K}}$ the obvious thing is to define a quadra-linear map $\mathfrak{g}_0 \times \mathfrak{g}_0 \times \mathbb{K} \times \mathbb{K} \rightarrow \mathfrak{g}_0 \otimes_{\mathbb{K}} \mathbb{K}$ given by

$$(X, Y, a, b) \mapsto [X, Y] \otimes ab \in \mathfrak{g}_0 \otimes_{\mathbb{K}} \mathbb{K} \quad (9.20)$$

This extends to the k -linear map on $\mathfrak{g} \times \mathfrak{g}$ so we have the bracket product on \mathfrak{g} . The converse is similar - we can restrict to a subfield \mathbb{K} by restricting to scalar multiplication for elements in $\mathbb{K} \subset k$. Then we have the definitions:

Definition 9.5. The complex Lie algebra $(\mathfrak{g}_0)^{\mathbb{C}}$ is the **complexification** of real Lie algebra \mathfrak{g}_0 . Similarly if

$$\mathfrak{g}^{\mathbb{R}} = \mathfrak{g}_0 \oplus i\mathfrak{g}_0 \quad (9.21)$$

then \mathfrak{g}_0 is a **real form** of the complex Lie algebra \mathfrak{g} .

Proposition 9.2. *Any real Lie algebra is a real form of its complexification. The conjugation of a complex Lie algebra \mathfrak{g} with respect to a real form is a Lie algebra isomorphism of $\mathfrak{g}^{\mathbb{R}}$ with itself.*

Proof. Also clear from definitions. □

With these definitions it is immediately clear what we have to do. We have, both in the lectures and in the classes, developed a consistent method to analyse complex semisimple Lie algebras. Now given such a Lie algebra \mathfrak{g} , one can ask what is the real Lie algebra \mathfrak{g}_0 that is the real form of the original complex Lie algebra \mathfrak{g} . Once we have a consistent correspondence, we can then look at the consequences of that and the classifications of real forms.

Here I will illustrate a particular example - finding the real form of the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$.

⁴⁹We typically drop the $(\cdot)^{\mathbb{C}}$

Example 9.1. Consider the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$. We want to find a real Lie algebra \mathfrak{g}_0 such that

$$\mathfrak{g}_0 \otimes_{\mathbb{R}} \mathbb{C} = \mathfrak{sl}_2(\mathbb{C}). \quad (9.22)$$

Note that the subset of $\mathfrak{sl}_2(\mathbb{C})$ of non-semisimple matrices is a proper subalgebraic variety and therefore it cannot contain a real subspace of $\mathfrak{sl}_2(\mathbb{C})$. So we can find a real $H \in \mathfrak{g}_0$ such that $\text{ad}(H)$ acts semisimply on \mathfrak{g}_0 and is an element of the Cartan subalgebra. Now we consider the eigenspaces of $\text{ad}(H)$ acting on \mathfrak{g} . Note that there must be an eigenvalue-0 space $\mathfrak{h}_0 = \mathbb{R} \cdot H$ spanned by H . The remaining two eigenvalues must then sum to zero, which leaves two possibilities.

1. $\text{ad}(H)$ has real eigenvalues $\pm\lambda$. Rescaling by a real scalar gives $\lambda = 2$. The resulting decomposition of the vector space \mathfrak{g}_0 into eigenspaces is then

$$\mathfrak{g}_0 = \mathfrak{h}_0 \oplus \mathfrak{g}_2 \oplus \mathfrak{g}_{-2} \quad (9.23)$$

Choosing $X \in \mathfrak{g}_2$ and $Y \in \mathfrak{g}_{-2}$ and again rescaling to give $[X, Y] = H$ gives

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad (9.24)$$

This is the basis of the real form $\mathfrak{sl}_2(\mathbb{R})$.

2. $\text{ad}(H)$ has complex eigenvalues $i \pm \lambda$, where λ is a nonzero real number. Set $\lambda = 1$. The decomposition is now

$$\mathfrak{g}_0 = \mathfrak{h}_0 \oplus \mathfrak{g}_{\{1, -1\}} \quad (9.25)$$

Choose basis B and C on $\mathfrak{g}_{\{1, -1\}}$ such that

$$\begin{cases} [H, B] &= C \\ [H, C] &= -B \end{cases} \quad (9.26)$$

Note we now have two choices

$$[B, C] = \pm H \quad (9.27)$$

If the negative sign is taken then $\mathfrak{g}_0 \cong \mathfrak{sl}_2(\mathbb{R})$ and we have

$$H = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} \quad B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad C = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (9.28)$$

is the basis. If the positive sign is taken then

$$\mathfrak{g}_0 \cong \mathfrak{su}_2 = \left\{ M \mid A^\dagger = -A, \text{tr}(A) = 0 \right\} \subset \mathfrak{sl}_2(\mathbb{C}) \quad (9.29)$$

and this has basis

$$H = \begin{pmatrix} \frac{i}{2} & 0 \\ 0 & \frac{i}{2} \end{pmatrix} \quad B = \begin{pmatrix} 0 & \frac{1}{2} \\ -\frac{1}{2} & 0 \end{pmatrix} \quad C = \begin{pmatrix} 0 & \frac{i}{2} \\ \frac{i}{2} & 0 \end{pmatrix} \quad (9.30)$$

The take is as follows. Given a real form $\mathfrak{g}_0 \subset \mathfrak{g}$ of a complex semisimple Lie algebra \mathfrak{g} we can always find a real subalgebra $\mathfrak{h}_0 \subset \mathfrak{g}_0$ such that \mathfrak{h}_0 is the real form of the Cartan subalgebra \mathfrak{h} of \mathfrak{g} . \mathfrak{h}_0 is then called the **Cartan subalgebra** of \mathfrak{g}_0 . In general (rank two or higher), the root $\alpha \in \Delta$ of \mathfrak{g} on \mathfrak{h}_0 need not be all real or purely imaginary - it can be a generic complex number. The values of roots depend heavily on the choice of \mathfrak{h}_0 .

9.3 Split Forms and Compact Real Forms

The task is now simple. We want a statement of the classification of the simple real Lie algebras which complexifications are the classical/exceptional Lie algebras. We then want to go the other way, i.e. starting from a complex semisimple Lie algebra \mathfrak{g} we want to construct real forms of \mathfrak{g} . Here I will not provide a classification theorem - one can refer to [1] for a full classification (in §26). Instead, I would like to present two distinguished real forms possessed by all complex semisimple Lie algebras. This will ultimately lead us back to answering the question I set out at the start of the section.

Split and compact forms represent the two extremes of behaviour of the decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha} \mathfrak{g}_{\alpha} \quad (9.31)$$

with respect to a real subalgebra $\mathfrak{g}_0 \subset \mathfrak{g}$. We will first give the definitions of the two real forms.

Definition 9.6. Let \mathfrak{g} be a complex semisimple Lie algebra. The **split form** of \mathfrak{g} is a real form \mathfrak{g}_0 such that there exists a Cartan subalgebra $\mathfrak{h}_0 \subset \mathfrak{g}_0$ (where $\mathfrak{h}_0^{\mathbb{C}} = \mathfrak{h}$) whose action on \mathfrak{g}_0 has all real eigenvalues. This means that all the roots $\alpha \in \Delta \subset \mathfrak{h}^*$ of \mathfrak{g} assume all real values on the subspace \mathfrak{h}_0 . We then have the direct sum decomposition:

$$\mathfrak{g}_0 = \mathfrak{h}_0 \oplus \bigoplus_{\alpha} \mathfrak{j}_{\alpha} \quad (9.32)$$

where the one-dimensional eigenspaces⁵⁰ \mathfrak{j}_{α} for the action of \mathfrak{h}_0 . Each pair $\mathfrak{j}_{\pm\alpha}$ generates a subalgebra isomorphic to $\mathfrak{sl}_2(\mathbb{R})$.

The existence of the split form can be shown as follows. To construct a real (even rational) form \mathfrak{g}_0 of Lie algebra \mathfrak{g} we start with generators X_{α_i} for the simple positive roots spaces of α_i and complete this to a standard basis $(X_{\alpha_i}, Y_{\alpha_i})$ and $H_i = [X_{\alpha_i}, Y_{\alpha_i}]$ for the corresponding $\mathfrak{sl}_2(\mathbb{C})$ subspace. We then take \mathfrak{g}_0 to be the real subalgebra generated by these elements - the Cartan subalgebra $\mathfrak{h}_0 \subset \mathfrak{g}_0$ is then the real span of H_i . From this we have the following claim.

Proposition 9.3. *The split form construction uniquely characterises the real \mathfrak{g}_0 of \mathfrak{g} .*

Proof. We note that from the construction described above once \mathfrak{h} is fixed for \mathfrak{g} , the real subalgebra \mathfrak{h}_0 is uniquely determined as the span of the H_{α} for all roots $\alpha \in \Delta$. The uniqueness is guaranteed as this is the only real form \mathfrak{g}_0 of \mathfrak{g} which has a Cartan subalgebra \mathfrak{h}_0 acting on \mathfrak{g}_0 with all real eigenvalues. \square

Remark 9.1. \mathfrak{g}_0 is determined up to isomorphism and is sometimes called the natural real form of \mathfrak{g} .

There exists another more important real form in our discussion.

⁵⁰Note that $\mathfrak{j}_{\alpha} = \mathfrak{g}_{\alpha} \cap \mathfrak{g}_0$.

Definition 9.7. Let \mathfrak{g} be a complex semisimple Lie algebra. The **compact (real) form** of \mathfrak{g} is a real form \mathfrak{g}_0 such that there exists a Cartan subalgebra $\mathfrak{h}_0 \subset \mathfrak{g}_0$ where all roots $\alpha \in \Delta \subset \mathfrak{h}^*$ of \mathfrak{g} assume all purely imaginary values on the subspace \mathfrak{h}_0 . We then have the direct sum decomposition:

$$\mathfrak{g}_0 = \mathfrak{h}_0 \oplus \bigoplus_{\alpha} \mathfrak{l}_{\alpha} \quad (9.33)$$

where \mathfrak{l}_{α} are two-dimensional eigenspaces⁵¹ on which \mathfrak{h}_0 acts by rotation and generates a subalgebra isomorphic to \mathfrak{su}_2 .

Compact real forms have the following properties.

Proposition 9.4. Suppose \mathfrak{g} is a complex semisimple Lie algebra and $\mathfrak{g}_0 \subset \mathfrak{g}$ is a real form of \mathfrak{g} . Let \mathfrak{h}_0 be a Cartan subalgebra of \mathfrak{g}_0 and the complexification $\mathfrak{h} = \mathfrak{h}_0 \otimes \mathbb{C}$ the corresponding Cartan subalgebra of \mathfrak{g} . Then the following are equivalent.

1. Each root $\alpha \in \Delta \subset \mathfrak{h}^*$ of \mathfrak{g} assumes purely imaginary values on \mathfrak{h}_0 and for each root α the subalgebra of $\mathfrak{l}_{\alpha} = (\mathfrak{g}_{\alpha} \oplus \mathfrak{g}_{-\alpha}) \cap \mathfrak{g}_0 \subset \mathfrak{g}_0$ is isomorphic to $\mathfrak{su}(2)$.
2. The restriction of \mathfrak{g}_0 of the Killing form of \mathfrak{g} is negative-definite.
3. The real Lie group G_0 with Lie algebra \mathfrak{g}_0 is compact.

Proof. This is the crux of the discussion so I will present the full proof. (1) \implies (2): Consider the Killing form on $H \in \mathfrak{h}_0$. Since the roots are purely imaginary we have

$$\Gamma(H, H) = \sum_{\alpha \in \Delta} (\alpha(H))^2 < 0 \quad (9.34)$$

Since the subspaces \mathfrak{l}_{α} are orthogonal to each other with respect to Γ it remains to show that $\Gamma(Z, Z) < 0$ for an arbitrary $Z \in \mathfrak{l}_{\alpha}$. Let X and Y be generators of \mathfrak{g}_{α} and $\mathfrak{g}_{-\alpha}$ respectively with $H = [X, Y]$ a standard basis for $\mathfrak{sl}_2(\mathbb{C})$. Take generators of the algebra of \mathfrak{l}_{α} as

$$\begin{cases} H' &= iH \\ U &= X - Y \\ V &= iX + iY \end{cases} \quad (9.35)$$

Setting

$$Z = aU + bV = (a + ib)X + (-a + ib)Y \quad (9.36)$$

then we must have

$$\begin{aligned} \text{ad}(Z) \circ \text{ad}(Z) &= (a + ib)^2 \text{ad}(X) \circ \text{ad}(X) + (a - ib)^2 \text{ad}(Y) \circ \text{ad}(Y) \\ &\quad - (a^2 + b^2) (\text{ad}(X) \circ \text{ad}(Y) + \text{ad}(Y) \circ \text{ad}(X)) \end{aligned} \quad (9.37)$$

Tracing over eliminates the non-cross terms so we are left with

$$\text{tr}(\text{ad}(Z) \circ \text{ad}(Z)) = -2(a^2 + b^2) \text{tr}(\text{ad}(X) \circ \text{ad}(Y)) \quad (9.38)$$

⁵¹Note that in this case $\mathfrak{l}_{\alpha} = (\mathfrak{g}_{\alpha} \oplus \mathfrak{g}_{-\alpha}) \cap \mathfrak{g}_0$.

Now $\text{ad}(X) \circ \text{ad}(Y)$ acts by multiplication by $\frac{(n-\lambda)(n+\lambda-2)}{4} \geq 0$ on the λ -eigenspace for H (this is the factor multiplying the relevant element after raising and lowering using the operators in $\mathfrak{sl}_2\mathbb{C}$, see p. 150 of [1]). So we have proven the claim. (2) \implies (3): First note that the **adjoint form**, defined as $G/Z(G)$ where $Z(G)$ is the centre of G . If G is connected then the image of the adjoint representation $\text{Ad} : G \rightarrow \text{Aut}(\mathfrak{g})$ is the adjoint form (this is clear from the fact that $\ker(\text{Ad}) = Z(G)$). Now the adjoint form G_0 is the connected component of the identity of the group $\text{Aut}(\mathfrak{g}_0)$. In particular it is a closed subgroup of the adjoint group of \mathfrak{g} and acts faithfully on the real vector space \mathfrak{g}_0 (so preserves Γ). From Theorem 9.2 it is clear that if Γ is negative definite then G_0 is a closed subgroup of $SO_n(\mathbb{R})$ so is compact. (3) \implies (1): Now suppose G_0 is compact. Then we can construct the group-averaging positive definite inner product on \mathfrak{g}_0

$$(X, Y) = \int dg \mu(\rho(g)X, \rho(g)Y) \quad (9.39)$$

where ρ is a representation of G and μ is an appropriate measure such that this is invariant under action of G . Then for any $X \in \mathfrak{g}_0$ $\text{ad}(X)$ is represented by a skew-symmetric matrix with respect to an orthogonal basis of \mathfrak{g}_0 . Then $\Gamma(X, X) \leq 0$ and the eigenvalues of $\text{ad}(X)$ are pure imaginary and $\alpha(\mathfrak{h}_0) \subset i\mathbb{R}$ so $\bar{\alpha} = -\alpha$ for root $\alpha \in \Delta$ so (1) follows. \square

Proposition 9.5. *Every semisimple complex Lie algebra has a unique compact form.*

Proof. Define the **conjugate linear involution** action as $\sigma : \mathfrak{g} \rightarrow \mathfrak{g}$ of a complexification \mathfrak{g} of real \mathfrak{g}_0 as for $X \in \mathfrak{g}_0$ and $z \in \mathbb{C}$

$$X \otimes z \mapsto X \otimes \bar{z} \quad (9.40)$$

This is conjugate linear, Lie bracket-preserving and $\sigma^2 = 1$. Now \mathfrak{g}_0 is the fixed subalgebra of σ and conversely. Starting with the split form $\mathfrak{g}_{0,s}$ of \mathfrak{g} . Given a basis

$$\{H_i \in \mathfrak{h}, X_\alpha \in \mathfrak{g}_\alpha, Y_\alpha \in \mathfrak{g}_{-\alpha}\} \quad (9.41)$$

we can define a unique Lie algebra automorphism φ of \mathfrak{g} taking

$$\begin{cases} H_i & \mapsto -H_i \\ X_\alpha & \mapsto Y_\alpha \end{cases} \quad (9.42)$$

φ is a complex linear involution that preserves $\mathfrak{g}_{0,s}$. Note that $\sigma\varphi = \varphi\sigma$ so the fixed part $\mathfrak{g}_{0,c}$ is another real form of \mathfrak{g} with Cartan subalgebra $\mathfrak{h}_{0,c} = \mathfrak{h}^{\sigma\varphi} = i\mathfrak{h}_{0,s}$ (note that here the i appears as upon applying the automorphism φ which changes the sign of H_i the fixed algebra of \mathfrak{g} under $\sigma\varphi$ is now the purely imaginary part - i.e. complex conjugation and negative leaves pure imaginary numbers invariant in signs). Since the restriction of the Killing form to $\mathfrak{h}_{0,s}$, it is obvious that the restriction to $\mathfrak{h}_{0,c}$ is negative definite and therefore $\mathfrak{g}_{0,c}$ is the compact form of \mathfrak{g} . This construction of $\mathfrak{g}_{0,c}$ from $\mathfrak{g}_{0,s}$ is reversible so the compact form is hence unique by Proposition 9.3. \square

So what have we shown? In the discussion of complex semisimple Lie algebras we have implicitly, when discussing the root system, taken the split real form to obtain real roots. This allows us to analyse the root system in the Euclidean plane. On the other hand, if we solely consider *real* compact Lie algebras, we are forced to take effectively the compact form of the complexified Lie algebra as that is what we obtain from the Lie algebra analysis. This means that by complexifying the real compact Lie algebra one can in fact return to our discussion of the root systems, and therefore reproduce all the results as illustrated in the notes and classes. This also illustrates the extra minus sign in the Killing form. When we choose the inner product on the root space of compact real forms (and hence on real compact Lie algebras), there is an extra sign taking care of the fact that all roots are pure imaginary.

Complexification in fact is purely a mathematical tool. The physics, at the end of the day, is illustrated by a real Lie algebra. This is why the discussion of real Lie algebras and real forms of complex Lie algebras is so important.

10 Cartan-Weyl Decomposition, Root Diagrams and Weight Diagrams

This is a bit of a mixed botched section. I want to address a couple of issues and questions regarding Lie algebras in general, so have thrown basically everything I want to chat about into one giant section. Hopefully this makes sense in the grand scheme of things. I implore you to read this section by subsections (so read the ones that interest you) - the subsections are standalone explanations except when I explicitly refer to something I have talked about before.

10.1 Metric tensor and coordinates

In this subsection I would like to first address a few issues concerning the metric tensor G . Recall first from the notes the following:

Definition 10.1. The **metric tensor** G is defined as

$$G_{ij} = \frac{(\alpha_j, \alpha_j)}{2} (A^{-1})_{ij} \quad (10.1)$$

This allows us to express the Killing form of two weights w, w' :

$$(w, w')_D = a'_i G_{ij} a_j \quad (10.2)$$

in terms of their Dynkin label.

We discuss two issues at hand concerning G .

10.1.1 Root diagram coordinates

Let us consider B_2 algebra. The Cartan matrix is

$$A_{B_2} = \begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix} \quad (10.3)$$

Let us recall the following:

Proposition 10.1. *The Dynkin labels of the simple roots are the rows of the Cartan matrix.*

Proof. See notes - this is more like a statement and you can check it yourself ⁵². \square

Now let us suppose you are given the simple roots: $\alpha_1 = (1, 0)$, $\alpha_2 = (-1, 1)$. The coordinates of the simple roots here now are the coordinates in the root diagram, in a Euclidean space. We can now use the Euclidean inner product between the simple roots to compute the Cartan matrix - you should see that this returns the same Cartan matrix above. What is going on? Why can we simply take (\cdot, \cdot) as the Euclidean inner-product ⁵³?

We note that $(\cdot, \cdot)_D$ in Definition 10.1 is actually an inner product with respect to the Dynkin basis. We know from the discussion of the root systems of Lie algebras that upon

⁵²If you cannot let me know.

⁵³If you are not sure why one can simply take the Euclidean product here, have a look at [8–10]. I believe there will also be a proof in [1, 5]

restricting the Killing form of a complex semisimple Lie algebra to the $\mathfrak{h}_{\mathbb{R}}^* \in \mathfrak{h}^*$ using $\{H_\alpha\}$ as the dual basis it becomes a positive-definite bilinear form. This allows us to take it as a metric, and operate it on a Euclidean basis. We can then convert between the Euclidean inner product from the Dynkin basis inner product, as described by the metric tensor using a diagonalisation and a rescale of eigenvalues. To do this note first that the weights in the adjoint basis (excluding the zero weights) are precisely the roots. Let us take two of these roots and consider their inner product in Dynkin basis

$$(\alpha, \beta)_D = a_i G_{ij} b_j \quad (10.4)$$

Now treat the metric tensor as a positive-definite bilinear form. We can pick a basis such that this bilinear form has only 1s on the diagonal matrix δ . This means we need the orthogonal matrix and a rescaling:

$$G \mapsto O^T D O = O^T \sqrt{D} \delta \sqrt{D} O \quad (10.5)$$

Now we can consider the root basis. We note that if we effect the following transform:

$$a \mapsto O^{-1} \sqrt{D}^{-1} a \quad (10.6)$$

This effectively maps the Dynkin basis to the Euclidean basis of the root diagram... we have come full circle and got the result we wanted ⁵⁴!

10.1.2 δ as half-sum of positive roots

We saw that δ emerged a few times (e.g. the Freudenthal formula (Eq (6.20) of notes), the Weyl formula (Eq (6.21) of notes) and the quadratic Casimir formula (Theorem 6.4 of notes)). We know that $\delta = (1, 1, \dots, 1)$ for A_n - but is this true in general?

Proposition 10.2. *The **Weyl vector** δ , defined as the half-sum of all positive roots:*

$$\delta = \frac{1}{2} \sum_{\alpha \in \Delta_+} a_\alpha \quad (10.7)$$

where a_α indicates the Dynkin label of the positive root α , is equal to the sum of fundamental weights.

Proof. Firstly, you can define some element s_α as the reflection of the root on your arbitrarily chosen hyperplane. Then we must have

$$s_\alpha(\delta) = \delta - \alpha \quad (10.8)$$

and s_α permutes the positive roots apart from root α . You can also see that

$$(\delta, \alpha) = \frac{1}{2}(\alpha, \alpha) \quad (10.9)$$

⁵⁴As far as I know this discussion cannot be found anywhere - so here you go!

Now, suppose ω_i to be the set of fundamental weights such that

$$\frac{2(\omega_i, \alpha_j)}{(\alpha_j, \alpha_j)} = \delta_{ij} \quad (10.10)$$

Now if you write $\delta = a_i \omega_i$, you will get

$$(\delta, \alpha_j) = a_j (\alpha_j, \alpha_j) / 2 \quad (10.11)$$

If you then solve this for a_j , you should then get:

$$\delta = \frac{2(\delta, \alpha_j)}{(\alpha_j, \alpha_j)} \omega_j = \sum_i \omega_i \quad (10.12)$$

using Equation 10.9. □

The next obvious question would be why δ ? What is significant about this quantity? Turns out it is related to some flag varieties (see this [MO post](#) for more details). This δ is sometimes called the “Weyl vector” (see for example [here](#)), and a related question is whether δ falls into the root lattice and what the significance of that is. There is a nice summary [here](#).

There is a lot more to the theory of the Weyl vector. The whole industry of Weyl symmetry and chambers for example, is an important aspect in the theory of Lie algebras. For more information do check out [17].

10.2 Cartan-Weyl Decomposition - revisited

Another random aspect I would like to go back to is the Cartan-Weyl decomposition. Is there a way to understand the Cartan-Weyl basis, in particular, can we write the Killing form in this basis? Recall that the Cartan-Weyl decomposition is for a Lie algebra \mathfrak{g} to be written as

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Lambda} \mathfrak{l}_\alpha, \quad (10.13)$$

where Λ is the root space. Take κ to be the Killing form defined in §4.2 of the notes as the natural inner product in \mathfrak{g} . Then we know that:

1. $\kappa|_{\mathfrak{h} \times \mathfrak{h}}$ is non-degenerate.
2. For all $\alpha \in H'$, the root set of the Lie algebra \mathfrak{g} , there exists a unique $H_\alpha \in \mathfrak{h}$ such that for all $H \in \mathfrak{h}$,
$$\kappa(H_\alpha, H) = \alpha(H). \quad (10.14)$$
3. $\kappa(H, E_\alpha) = 0$ where $[H, E_\alpha] = \alpha(H)E_\alpha$.
4. $\kappa(E_\alpha, E_\beta) = 0$ for all $\alpha, \beta \in H'$ and $\underline{\alpha} \neq -\underline{\beta}$.
5. $\kappa(E_\alpha, E_{-\alpha}) \neq 0$.

I am basically just rewriting Theorem 4.6 of the notes. But note that now we have the following structure for κ in matrix form.

$$\kappa \xrightarrow{\text{in basis}} \begin{matrix} & H & E_\alpha & E_{-\alpha} & E_\beta & E_{-\beta} & \dots \\ \begin{matrix} H \\ E_\alpha \\ E_{-\alpha} \\ E_\beta \\ E_{-\beta} \\ \vdots \end{matrix} & \begin{pmatrix} \text{non-degenerate} & & & & & \\ & 0 & 1 & & & \\ & 1 & 0 & & & \\ & & & 0 & 1 & \\ & & & 1 & 0 & \\ & & & & & \ddots \end{pmatrix} \end{matrix} \quad (10.15)$$

We note the following points.

- In the \mathfrak{h} (CSA) I can pick the basis $\{H_i, i = 1, \dots, \text{rk } \mathfrak{g}\}$. But to construct the $\mathfrak{su}(2)_\alpha$ subalgebra it is often easier to construct H_α using the root structure. In general however $\{H_\alpha\}$ span \mathfrak{h} but it is not a basis as in general $\{H_\alpha\}$ are linearly-dependent.
- We can rescale the E_α to get a normalised basis, which is what I have done above. Of course it is possible to rediagonalise $\mathfrak{l}_\alpha \oplus \mathfrak{l}_{-\alpha}$ to get a diagonal basis, but this then obscures the $\mathfrak{su}(2)_\alpha$ interpretation.

This is all I want to say about Cartan-Weyl basis for now.

10.3 Weyl group and chambers

A small interlude here. I want to define something known as the Weyl group of the root system, which is a symmetry group of the root space under some natural involution action of the roots. This section is quite mathematical so you are welcomed to skip if it bothers you too much - but it will establish some of the notations that will be used later on.

The first thing to define is the Weyl group. Here we fix \mathfrak{g} a Lie algebra and \mathfrak{h} its Cartan subalgebra.

Definition 10.2. Define an involution $W_\alpha : \mathfrak{h} \rightarrow \mathfrak{h}$ by

$$W_\alpha(H) = H - 2 \frac{(\alpha, H)}{(\alpha, \alpha)} \alpha, \quad (10.16)$$

with (\cdot, \cdot) denoting the usual inner product of the root system Δ . The **Weyl group** of the root system Δ , denoted \mathfrak{W} , is then the subgroup of $GL(\mathfrak{h})$ generated by the involutions W_α where $\alpha \in \Delta$.

The geometric meaning of the involution W_α is the reflection about the hyperplane orthogonal to α . We then have the following theorem.

Theorem 10.1. *The action of \mathfrak{W} on the roots preserves the root space Δ .*

Proof. The action of reflection W_α maps a root into a root by the definition of a root system [4]. If this is unclear, have a look at Theorem 7.26 of [17]. \square

Corollary 10.1. *The Weyl group is finite.*

Proof. The roots span \mathfrak{h} so each $w \in \mathfrak{W}$ can be determined by its action on the root space Δ . In that sense \mathfrak{W} can be thought of as a subgroup of the permutation group on the roots, which is finite. \square

With this we can now define a base (note that the root system is based in Euclidean space).

Definition 10.3. A subset B of Δ is called a **base** if:

1. B is a basis of the Euclidean space E .
2. Each root β can be written as $\beta = \sum_{\alpha \in B} k_\alpha \alpha$ with integral coefficients k_α all non-negative or all non-positive.

In this sense the roots in B are known as **simple**. The **height** of a root is then defined as

$$h(\beta) = \sum_{\alpha \in B} k_\alpha . \quad (10.17)$$

This is often how the simple roots are defined [1, 4, 17]. Now we introduce a bit of terminology.

Definition 10.4. The hyperplanes P_α where $\alpha \in \Delta$ partitions E into finite many regions. The connected components of

$$E - \bigcup_{\alpha} P_\alpha \quad (10.18)$$

are called the open **Weyl chambers** of E .

Each $\gamma \in E - \bigcup_{\alpha} P_\alpha$ is called **regular**.

There is an important theorem that states the following.

Theorem 10.2. *Let $\gamma \in E$ be regular. Then the set $S(\gamma)$ of all indecomposable roots (that cannot be written as a non-negative sum of other roots) in $\Delta^+(\gamma)$ is a base of Δ and every base is obtainable in this manner.*

Proof. See §10.1 of [4]. \square

The key point from the theorem is the following statement.

Corollary 10.2. *The Weyl chambers are in natural one-to-one correspondence with bases.*

Proof. This is clear by looking at $\mathcal{W}(\gamma)$, the Weyl chamber containing γ . Then $\mathcal{W}(\gamma) = \mathcal{W}(\gamma')$ means γ, γ' lie on the same side of each hyperplane P_α . \square

Definition 10.5. The **fundamental Weyl chamber relative to base S** is the Weyl chamber $\mathcal{W}(S) = \mathcal{W}(\gamma)$ where $S = S(\gamma)$.

The Weyl group then sends one Weyl chamber onto another. In fact \mathfrak{W} permutes the bases of the root system Δ (see §10.3 of [4]).

How does this help us analyse root and weight systems? The key point to note is the following statement.

Proposition 10.3. *Every vertex of the convex hull of the weights of the representation $V = \oplus_{\alpha} V_{\alpha}$ is conjugate to α under the Weyl group \mathfrak{W} .*

Proof. The weights of V will contain a string of weights of the form

$$\alpha, \alpha + \beta, \dots, \alpha - (-(\alpha, \beta))\beta, \quad (10.19)$$

the last term being exactly the term $\mathfrak{W}_{\beta}(\alpha)$ so it is conjugate to α . \square

This seems to be not that helpful but this is now important.

Proposition 10.4. *Suppose \mathcal{W} is a Weyl chamber and $e \in E$ is a point in the Euclidean space. Then there exists exactly one point in the \mathfrak{W} -orbit of e such that it lies in the closure of the Weyl chamber \mathcal{W}*

Proof. The key points of the proof is to use the fact that the Weyl group acts transitively on the set of open Weyl chambers [4]. See also Proposition 8.29 of [17]. \square

We now define the following.

Definition 10.6. A point $\mu \in E$ is **strictly dominant relative to S** iff $\mu \in \mathcal{W}(S)$. It is **dominant relative to S** if $\mu \in \bar{\mathcal{W}}(S)$. From Proposition 10.4 for all $\mu \in E$ there exists $w \in \mathfrak{W}$ such that $w \cdot \mu$ is dominant.

This dominant point μ is in fact how we define the highest weight of the system. Typically the fundamental weights are fixed by the Cartan matrix (see below §10.5), and hence the Weyl chamber is fixed (this is known as the fundamental Weyl chamber associated to the fundamental weights). Then every point $\mu \in E$ that satisfies for all $\alpha \in \Delta$,

$$(\mu, \alpha) = 2 \frac{(\mu, \alpha)}{(\alpha, \alpha)} \in \mathbb{Z} \quad (10.20)$$

called an **integral element** is then a highest weight. I don't have time to go to all the details here, but you should look at §8 and §9 of [17] for the details (and it will be worth it).

10.4 Calculating root diagrams

With all the machinery set-up we are now ready to tackle root diagrams. Recall that root systems are often the key in analysing the Lie algebra content - constructing the root system of a Lie algebra is quite straightforward. In this subsection I would like to present an algorithm to compute the root system of a simple Lie algebra \mathfrak{g} , given the simple roots of the Lie algebra \mathfrak{g} . The goal is to compute and classify Lie algebras using the root system.

Let's therefore suppose I am given a set of simple positive roots $\{\alpha_i\}$ with $\alpha_i \in H'$, H' the root space (see notes, this is the space of functionals such that for $T \in \mathfrak{g}$, $\text{ad}(H)(T) = \alpha(H)(T)$). The Cartan matrix is given by,

$$A_{ij} = \frac{2\alpha_i \cdot \alpha_j}{\alpha_j \cdot \alpha_j}, \quad (10.21)$$

with no sum on i, j . In doing so we have secretly moved to the split form of the Lie algebra so the roots are real and can be evaluated in a Euclidean root space. The dot product here therefore is the Euclidean product. From the arguments in §10.1 and in the notes, we know that we can always pick the inner product $(\alpha, \beta) = \alpha_i \beta_j (\kappa|_{\mathfrak{h} \times \mathfrak{h}}^{-1})^{ij}$ and rotate the basis by an orthogonal transformation such that the inner product is Euclidean. It is now possible to generate the root diagram by using the following algorithm.

Proposition 10.5. *The following is an algorithm to generate all roots S from the set of simple roots Δ .*

1. Let $S = \Delta$.
2. For each pair of distinct roots, $\alpha \in \Delta$ and $\beta \in S$, determine the maximum and minimum integers n_{\pm} such that $\beta + n_{\pm}\alpha$ are roots. The useful results are:

$$(i) \quad n_+ + n_- = -\frac{2\alpha \cdot \beta}{|\beta|^2}.$$

(ii) If γ is a root, $\lambda\gamma$ the constant multiple of γ is a root if and only if $\lambda = \pm 1$.

(iii) If $\alpha, \beta \in \Delta$, then $\alpha - \beta$ is not a root.

(iv) Any root string has at most length of 4.

(v) For $\alpha, \beta \in \Delta$, the α -string through β

$$S_{\alpha, \beta} = \{\beta + \rho\alpha \in S \mid \rho \in \mathbb{Z}\} \quad (10.22)$$

has length

$$1 - \frac{2\alpha \cdot \beta}{|\alpha|^2} \in \mathbb{N}. \quad (10.23)$$

(vi) If $\alpha \cdot \beta \leq 0$ then $\alpha \pm \beta$ is a root.

3. Repeat (2) until exhaustion. The positive root set is S .

4. The full root set is then $S \cup (-S)$.

Proof. The proof is basically just utilising Proposition 6.1 of the notes together with a lot of root system facts. I will refer to [37] for the details.

The proof of (i) is a bit more complicated (Proposition 3 of [37]). The idea is to consider the vector space

$$V = \bigoplus_{n \in \mathbb{Z}} \mathbb{C}_{\underline{\beta} + n\underline{\alpha}}, \quad (10.24)$$

and notice that this space is reducible (since this space is one-dimensional and invariant under the $\mathfrak{su}(2)_\alpha$ action). This immediately means that there exists integers p, q with $p \leq 0 \leq q$ such that $\underline{\beta} + n\underline{\alpha}$ is a root iff $p \leq n \leq q$. Now constrain using the action of $[H_\alpha, \cdot]$.

(iv) and (vi) comes from Theorem 6.1 of the notes. (ii), (iii) and (v) follows from the intermediate steps in constructing root strings, see [37] for details (Definition 3, Theorem 1 and Lemma 12). The multiple property is proved in §8.3 of [4]. \square

This is a bit complicated to explain without going into full details so let us illustrate this with an example.

Example 10.1. Let us look at $B_2 \cong \mathfrak{so}(5)_\mathbb{C}$. The simple roots in this case is $\underline{\alpha}_1 = (1, 0)$ and $\underline{\alpha}_2 = (-1, 1)$. Starting with $\Delta = \{\underline{\alpha}_1, \underline{\alpha}_2\}$, we have $\underline{\alpha}_1 \cdot \underline{\alpha}_1 = 1$, $\underline{\alpha}_1 \cdot \underline{\alpha}_2 = 2$ and $\underline{\alpha}_2 \cdot \underline{\alpha}_2 = -1$ (remember we are using the Euclidean inner product here). So setting $S = \Delta$, we can then look at the $\underline{\alpha}_1$ -string through $\underline{\alpha}_2$, $\underline{\alpha}_2 + n\underline{\alpha}_1$.

- $\underline{\alpha}_2 - \underline{\alpha}_1$ is not a root by (iii) of Proposition 10.5. So $n_- = 0$.
- From (i) and (v) of Proposition 10.5 we have $n_+ = 2$.
- So we add $\underline{\alpha}_2 + \underline{\alpha}_1$ and $\underline{\alpha}_2 + 2\underline{\alpha}_1$ to set S .

Now we look at $S = \{\underline{\alpha}_1, \underline{\alpha}_2, \underline{\alpha}_1 + \underline{\alpha}_2, 2\underline{\alpha}_1 + \underline{\alpha}_2\}$, then we have:

- $\underline{\alpha}_1$ -strings through any $s \in S$ will not lead to new roots since $2\underline{\alpha}_1$ is not a new root by (ii) of Proposition 10.5 and the others is the same as the root string $S_{\underline{\alpha}_1, \underline{\alpha}_2}$.
- $S_{\underline{\alpha}_2, \underline{\alpha}_1}$ gives $n_- = 0$ and $n_+ = 1$, so generates $\underline{\alpha}_1 + \underline{\alpha}_2$, which we already have.
- $S_{\underline{\alpha}_2, \underline{\alpha}_1 + \underline{\alpha}_2}$ is redundant again.
- $S_{\underline{\alpha}_2, \underline{\alpha}_1 + 2\underline{\alpha}_2}$ gives $n_+ = 0 = n_-$ since both $2\underline{\alpha}_1$ and $2(\underline{\alpha}_1 + \underline{\alpha}_2)$ are not roots.

So we deduce by exhaustion that there are no more roots. The full root set Φ is therefore,

$$\Phi = \{\pm\underline{\alpha}_1, \pm\underline{\alpha}_2, \pm(\underline{\alpha}_1 + \underline{\alpha}_2), \pm(2\underline{\alpha}_1 + \underline{\alpha}_2)\} \quad (10.25)$$

giving the root diagram in Figure 10.1.

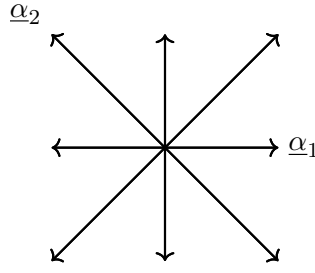


Figure 10.1: Root Diagram of B_2 .

Distributed version:

Exercise 10.1. Repeat the analysis of the root system of B_2 to the other three rank-2 Lie algebras, A_2 , D_2 and G_2 . In each case,

1. Write down the Cartan matrix and metric tensor. Invert to find the simple roots $\underline{\alpha}_1$ and $\underline{\alpha}_2$ in the Euclidean basis.
2. Find the root system.
3. Draw the root system and verify these are exactly as illustrated in p.118 of the notes.

10.5 Weight diagrams

Now we move on to weight diagrams. In particular I want to introduce an algorithm that replaces the one on p.127 of the notes (as I find it quite confusing to check what “all the weights” is) to find all the weights given the highest weight Λ .

Proposition 10.6. *Let us suppose d_Λ is the irreducible representation of \mathfrak{g} with highest weight Λ . Let $|\Lambda\rangle$ be the Dynkin index of the highest weight. To generate all the weights, we use the following algorithm.*

1. Set the set of weights $S = \{\underline{\Lambda}\}$.
2. For each element $\underline{\mu} \in S$, write,

$$\underline{\mu} = \sum \mu_i \underline{\omega}_i, \quad (10.26)$$

where $\underline{\omega}_i$ are the fundamental weights of \mathfrak{g} . Then we construct all weights of the form

$$\underline{\mu} - \sum_{i=1}^r m_i \underline{\alpha}_i \quad (10.27)$$

where $\underline{\alpha}_i$ are the simple roots from S for all $0 \leq m_i \leq \lambda_i$ where λ_i is the Dynkin index of $\underline{\mu}$ and $m_i \in \mathbb{N}$. We explicitly use the Dynkin-index basis of the simple roots $\underline{\alpha}_i$.

3. Exhaust the previous step to generate the full weight system.

Proof. This is known as the highest weight system and it is covered in most Lie algebra books, see for example, [9, 17]. \square

Again let us illustrate this with an example.

Example 10.2. Consider G_2 with the Cartan matrix,

$$A_{G_2} = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix} \quad (10.28)$$

and consider its fundamental irrep where $|\Lambda\rangle = |1, 0\rangle$. Then go through the algorithm.

- Start with $|1, 0\rangle$. We can apply $E_{-\alpha_1}$ once to get

$$|1, 0\rangle \xrightarrow{-\alpha_1} |-1, 1\rangle .$$

- Dynkin index of the second entry is +1 so apply $E_{-\alpha_2}$ once to get

$$|-1, 1\rangle \xrightarrow{-\alpha_2} |2, -1\rangle .$$

- Dynkin index of the first entry is +2 so apply $E_{-\alpha_1}$ twice to get

$$|2, -1\rangle \xrightarrow{-\alpha_1} |0, 0\rangle \xrightarrow{-\alpha_1} |-2, 1\rangle .$$

- $|0, 0\rangle$ terminates so we can only apply $E_{-\alpha_2}$ to $|-2, 1\rangle$ once to get

$$|-2, 1\rangle \xrightarrow{-\alpha_2} |1, -1\rangle .$$

- Finally apply once of $E_{-\alpha_1}$ to get

$$|1, -1\rangle \xrightarrow{-\alpha_1} |-1, 0\rangle .$$

At this point we need to check degeneracies,. But there are none with the same $\underline{H} = (H_1, H_2)$ eigenvalues so there is no degeneracy. This is the **7** rep of G_2 .

To draw the weight diagram we can rotate it back into standard Euclidean basis. The fundamental weights are defined as

$$\underline{\mu}_i = B_{ij}\alpha_j , \quad (10.29)$$

where B_{ij} is the inverse of the Cartan matrix. So using

$$\alpha_1 = (1, 0) \quad (10.30)$$

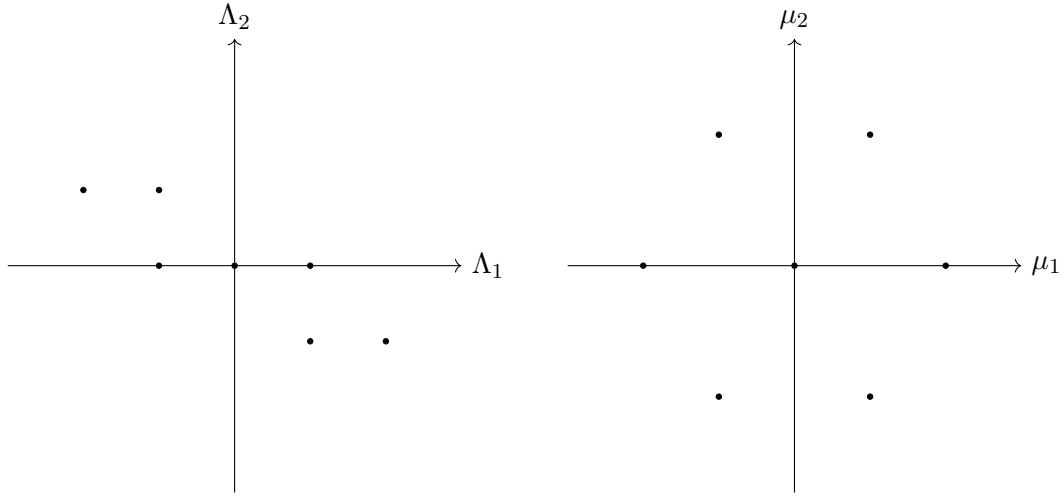
$$\alpha_2 = \frac{1}{2}(-3, \sqrt{3}) \quad (10.31)$$

We get

$$\underline{\mu}_1 = \frac{1}{2}(1, \sqrt{3}) \quad (10.32)$$

$$\underline{\mu}_2 = (0, \sqrt{3}) . \quad (10.33)$$

The weight diagram of **7** of G_2 is then sketched out in Figure 10.2.



(a) Weight diagram of the fundamental representation of G_2 in the Dynkin basis. (b) Weight diagram of the fundamental representation of G_2 in the Euclidean basis.

Figure 10.2: The weight diagram of the fundamental representation of G_2 in different basis. Note that the two weight diagrams are not scaled with respect to each other.

Distributed version:

Exercise 10.2. Repeat for the irrep of G_2 defined by the highest weight $|\underline{\Lambda}\rangle = |0, 1\rangle$. Plot the weight diagram. Compare this with the G_2 root diagram. What is this representation of G_2 ?

Distributed version:

Exercise 10.3. Recall that in Theorem 6.3 of the notes we have the Weyl formula:

$$\dim(\lambda) = \prod_{\alpha \in \Delta_+} \frac{(\lambda + \delta, \alpha)}{(\delta, \alpha)}, \quad (10.34)$$

with $\delta = \frac{1}{2} \sum_{\alpha \in \Delta_+} \alpha$ or with Dynkin labels all being one. Calculate the dimension of the spinor representation **16** of \mathfrak{so}_{10} in Q4 of Sheet 3 using this formula now that we have all the relevant elements defined.

10.6 Projection matrices and branching

The final little piece of information concerns the analysis of branching rules,

$$R^{(G)} \rightarrow R^{(H)} = \bigoplus_s R_s^{(H)}, \quad (10.35)$$

where $R_s^{(H)}$ are irreps of H (see Eq. (1.38) of the notes). It is often easier to analyse the embedding of H in G in terms of a **projection matrix** that takes the roots and weights of \mathfrak{g} into \mathfrak{h} , the Lie algebras of G and H respectively. The existence of these matrices can

be trivially proven as the roots of \mathfrak{h} must be identified with a subset of that of \mathfrak{g} since by definition of branching $H \subset G$ as a subgroup.

It is perhaps interesting to look at the effect of the Weyl group defined in §10.3 on the projection matrices. Clearly, the Weyl-reflected root diagrams give the same embedding but the coordination of the weight space of G will now be different. This normally means the projection matrix will just have different entries, as we will need to fix some initial coordinates to analyse the projection matrices of G onto H .

The question now becomes how we choose such a coordination. This is basically done by requiring the highest weight of an irreducible representation of \mathfrak{g} to be projected to that of \mathfrak{h} , say the fundamental rep. Let us look at some examples to illustrate this point.

Example 10.3. Let us analyse the branching projection matrix of $SU(3) \rightarrow SU(2) \times U(1)$. To do this we first normalise the $U(1)$ charge to $3Y$, Y being the hypercharge so P is an integer matrix. We want to project

$$P \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} b_1 \\ u_1 \end{pmatrix}, \quad (10.36)$$

where a_i , b_i and u_i are the weights of $SU(3)$, $SU(2)$ and $U(1)$ respectively. In particular we have also nominally set b_1 to be $2I_3$ of $SU(3)$. Apply this to

$$\mathbf{3} \rightarrow \mathbf{2}_{-1} \oplus \mathbf{1}_{-2} \quad (10.37)$$

we then are able to fix

$$P = \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}. \quad (10.38)$$

It is now obvious to check that this projection matrix also works for the adjoint rep $\mathbf{8}$ of $SU(3)$, the branching rule exactly gives

$$\mathbf{8} \rightarrow \mathbf{3}_0 \oplus \mathbf{2}_1 \oplus \mathbf{2}_{-1} \oplus \mathbf{1}_0. \quad (10.39)$$

Distributed version:

Exercise 10.4. Work out the projection matrix of P_5 for the branching rule of $SU(5) \rightarrow SU(3) \times SU(2)$. You may need the branching rules

$$\mathbf{5} \rightarrow (\mathbf{1}, \mathbf{3}) \oplus (\mathbf{2}, \mathbf{1}) \quad (10.40)$$

$$\bar{\mathbf{5}} \rightarrow (\mathbf{1}, \bar{\mathbf{3}}) \oplus (\mathbf{2}, \mathbf{1}) \quad (10.41)$$

$$\mathbf{10} \rightarrow (\mathbf{1}, \mathbf{1}) \oplus (\mathbf{2}, \mathbf{3}) \oplus (\mathbf{1}, \bar{\mathbf{3}}) \quad (10.42)$$

$$\bar{\mathbf{10}} \rightarrow (\mathbf{1}, \mathbf{1}) \oplus (\mathbf{2}, \bar{\mathbf{3}}) \oplus (\mathbf{1}, \mathbf{3}). \quad (10.43)$$

Also work out the $U(1)$ dual vector. Verify all of this with the data you are given in Q3 of Sheet 3.

11 Subalgebras, Subgroups and Dynkin Diagrams

In this short section I would like to highlight why the algorithm provided in §6.3 in the notes on finding subalgebras of Lie algebras \mathfrak{g} using Dynkin and extended Dynkin diagrams. Subalgebras and subgroups are often important as in physics symmetry breaking often requires one to find the subgroup of an existing group under some physical processes.

11.1 Subalgebras and Extended Dynkin Diagrams

Let us first, for a complete discussion, recall the definitions of a subalgebra.

Definition 11.1. A **Lie subalgebra** \mathfrak{h} of Lie algebra \mathfrak{g} is a subspace $\mathfrak{h} \subset \mathfrak{g}$ that is closed under the bracket operation, i.e. if $X, Y \in \mathfrak{h}$, then $[X, Y] \in \mathfrak{h}$.

Similary, we have

Definition 11.2. A **Lie subgroup** H of Lie group G is a subset $H \subset G$ that is also a group.

We recall from the notes the following definitions.

Definition 11.3. A **maximal subalgebra** is a subalgebra \mathfrak{h} of \mathfrak{g} such that there are no other subalgebras contained between \mathfrak{g} and \mathfrak{h} . i.e. A subalgebra $\mathfrak{j} \subset \mathfrak{g}$ such that

$$\mathfrak{h} \subset \mathfrak{j} \subset \mathfrak{g} \quad (11.1)$$

does not exist. Here we use the \subset notation to denote **proper** containment., i.e. the containment excludes the trivial containment $\mathfrak{g} \subseteq \mathfrak{g}$.

We also define the notion of a regular subalgebra.

Definition 11.4. A **regular subalgebra** \mathfrak{g}' of a Lie algebra \mathfrak{g} is a subalgebra with the property that given the root space decomposition of \mathfrak{g} as

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_{\alpha} \quad (11.2)$$

we can write the root space decomposition of \mathfrak{g}' as

$$\mathfrak{g}' = \mathfrak{h}' \oplus \bigoplus_{\alpha \in \Delta'} \mathfrak{g}_{\alpha} \quad (11.3)$$

such that $\mathfrak{h}' \subset \mathfrak{h}$ as a subalgebra and $\Delta' \subset \Delta$ is a subset of the roots.

Let us illustrate this with an example.

Example 11.1. Consider $A_3 \oplus A_1 \subset A_5$, or $\mathfrak{g}' = \mathfrak{sl}_4 \oplus \mathfrak{sl}_2 \subset \mathfrak{sl}_6 = \mathfrak{g}$. We can take the Cartan subalgebra \mathfrak{h}' to be the diagonal matrices so clearly $\mathfrak{h}' \subset \mathfrak{h}$. We note that $\dim(\mathfrak{h}') + 1 = \dim(\mathfrak{h})$ as there is a one-dimensional subspace of \mathfrak{h} generated by the elements

$$H = \text{diag}(1, 1, 1, 1, -2, -2) \quad (11.4)$$

which indicates the extra $\mathfrak{u}(1)$ factor⁵⁵. The root vectors of \mathfrak{g}' are just the root vectors of \mathfrak{g} with non-zero elements on the two diagonal blocks. We have therefore shown that \mathfrak{g}' is a regular subalgebra of \mathfrak{g} .

Example 11.2. Let us look at a counter example - a non-regular subalgebra. Consider $\mathfrak{g}' = A_2 \oplus A_1 \subset A_5 = \mathfrak{g}$. Using the Pauli matrices $\{\sigma_i\}_{i=1,2,3}$ and the Gell-Mann matrices $\{\lambda_j\}_{j=1,\dots,8}$ as basis of \mathfrak{sl}_2 and \mathfrak{sl}_3 respectively and defining

$$\begin{cases} \sigma_0 = \mathbb{1}_2 \\ \lambda_0 = \mathbb{1}_3 \end{cases} \quad (11.5)$$

then every 6×6 matrix can be written as

$$\{\sigma_i \otimes \lambda_j\}_{i=0,\dots,3; j=0,\dots,8} \quad (11.6)$$

Note that $[\sigma_0 \otimes \lambda_i, \sigma_j \otimes \lambda_0] = 0$. So the Cartan subalgebra of \mathfrak{g}' is generated by

$$\{\sigma_3 \otimes \lambda_0, \sigma_0 \otimes \lambda_3, \sigma_0 \otimes \lambda_8\} \quad (11.7)$$

So $\mathfrak{h}' \subset \mathfrak{h}$. However, looking at the root vectors:

$$\{\sigma_{\pm} \otimes \lambda_0, \sigma_0 \otimes t_{\pm}, \sigma_0 \otimes u_{\pm}, \sigma_0 \otimes v_{\pm}\} \quad (11.8)$$

where $t_{\pm} = \frac{1}{2}(\lambda_1 \pm \lambda_2)$, $u_{\pm} = \frac{1}{2}(\lambda_4 \pm \lambda_5)$ and $v_{\pm} = \frac{1}{2}(\lambda_6 \pm \lambda_7)$, we see that $\Delta' \subsetneq \Delta$. So this is not a regular subalgebra of \mathfrak{g} .

In the lecture notes, we have set out an algorithm that allows us to compute regular subalgebras using Dynkin diagrams [39]. In [39], two notions of subalgebras are introduced.

Definition 11.5. *R-subalgebras* are subalgebras that are contained in some regular subalgebra of Lie algebra \mathfrak{g} . Otherwise the subalgebra is known as an *S-subalgebra*.

Before we continue again let us recall the definitions of simple and semisimple in the context of Lie algebras.

Definition 11.6. An **abelian** Lie algebra is a Lie algebra that satisfies $[\mathfrak{g}, \mathfrak{g}] = 0$.

Definition 11.7. A **simple** Lie algebra is a Lie algebra that contains no proper ideals and is not abelian.

Definition 11.8. A direct sum of simple Lie algebras is known as a **semisimple** Lie algebra.

Definition 11.9. A **reductive** Lie algebra is the direct sum of a simple and an abelian Lie algebra.

⁵⁵The reason why this is not in \mathfrak{h}' is intuitively quite clear. \mathfrak{g}' contains a tensor sum of two algebras A_3 and A_1 should be independently generated. If this element is contained in \mathfrak{h}' then the two algebras are no longer linearly independent of each other.

Semi-simple Lie algebra	Dynkin Diagram
A_n	
B_n	
C_n	
D_n	
E_6	
E_7	
E_8	
F_4	
G_2	

Table 11.1: The list of Dynkin diagrams for semi-simple Lie algebras.

We would like to classify all reductive subalgebras. This requires us to look for all maximal regular subalgebras as well as maximal S -subalgebras. We note that a maximal subalgebra may not be semisimple - here we restrict to the cases where all maximal subalgebras are semisimple. We state without proof the following result.

Lemma 11.1. *Every reductive subalgebra of a Lie algebra \mathfrak{g} that contains an abelian ideal is an R -subalgebra.*

Proof. See Dynkin's original paper [39]. □

As it turns out the R -subalgebra case is much more easily dealt with and from now on we will focus on this (i.e. identifying regular subalgebras). Let us begin by considering $\mathfrak{g}' \subset \mathfrak{g}$ which is a regular subalgebra and $\Delta' \subset \Delta$ be the set of roots of \mathfrak{g}' (where $\Pi' \subset \Delta'$ is the set of simple roots). Recall that if $\alpha', \beta' \in \Pi'$, then $\alpha' - \beta' \notin \Delta'$. (In fact $\alpha' - \beta' \notin \Delta$ otherwise this will lead to a contradiction where $\alpha' - \beta' \in \Delta'$.) This means the following:

Proposition 11.1. *To find regular subalgebras of \mathfrak{g} , we set to find sets of $\Pi' \in \Delta$ such that if $\alpha', \beta' \in \Pi'$ then*

$$\alpha' - \beta' \notin \Delta \quad (11.9)$$

Then the subalgebra will be generated by the set of elements $\{e_{\pm\alpha'}, h_{\alpha'}\}$ where $\alpha' \in \Pi'$.

Proof. This is clear from the analysis above. □

To do this Dynkin therefore introduced the notion of Extended Dynkin Diagrams (see Figures 11.1 and ??) with the help of adding in a new root.

Definition 11.10. An **extended Dynkin diagram** is formed with the root set together with the most negative root. For the root set Δ of Lie algebra \mathfrak{g} , the new root set is

$$\tilde{\Delta} = \Delta \cup \{-\theta\} \quad (11.10)$$

Semi-simple Lie algebra	Extended Dynkin Diagram
\tilde{A}_n	
\tilde{B}_n	
\tilde{C}_n	
\tilde{D}_n	
\tilde{E}_6	
\tilde{E}_7	
\tilde{E}_8	
\tilde{F}_4	
\tilde{G}_2	

Table 11.2: The list of extended (affine) Dynkin diagrams for semi-simple Lie algebras.

where θ is the highest (positive) root of \mathfrak{g} . This root system is called the **extended Π -system** where as the original root system is known as the **Π -system**.

Using this we can now restate Theorem 6.5 of the notes.

Theorem 11.1. *Up to isomorphism all maximal semisimple subalgebras of \mathfrak{g} are obtained by considering all root subsets*

$$\tilde{S} \subset \Delta \cup \{-\theta\} \quad (11.11)$$

where θ is the highest root of \mathfrak{g} . In particular we have the following algorithm regarding the use of extended Dynkin diagrams.

1. Regular semisimple maximal subalgebras \mathfrak{g}' can be obtained by considering the Dynkin diagram that has one node removed from the extended Dynkin diagram of \mathfrak{g} , with the following exceptions that are not maximal

- (a) $F_4, \Delta \cup \{-\theta\} \setminus \{\alpha_3\}$: $A_3 \oplus A_1 \hookrightarrow B_4 \hookrightarrow F_4$.
- (b) $E_7, \Delta \cup \{-\theta\} \setminus \{\alpha_3\}$: $A_3 \oplus A_3 \oplus A_1 \hookrightarrow D_6 \oplus A_1 \hookrightarrow E_7$.
- (c) $E_8, \Delta \cup \{-\theta\} \setminus \{\alpha_3\}$: $A_3 \oplus D_5 \hookrightarrow D_8 \hookrightarrow E_8$.
- (d) $E_8, \Delta \cup \{-\theta\} \setminus \{\alpha_5\}$: $A_5 \oplus A_2 \oplus A_1 \hookrightarrow E_6 \oplus A_2 \hookrightarrow E_8$.
- (e) $E_8, \Delta \cup \{-\theta\} \setminus \{\alpha_6\}$: $A_7 \oplus A_1 \hookrightarrow E_7 \oplus A_1 \hookrightarrow E_8$.

2. Reductive semisimple maximal subalgebras $\mathfrak{g}'' \oplus \mathfrak{u}(1)_{\mathbb{C}}$ is obtained by removing one node from the Dynkin diagram of \mathfrak{g} .

Proof. First consider the regular semisimple maximal subalgebras. The extended Π -system of Lie algebra \mathfrak{g} has the following property. Choosing the lowest root $\alpha_0 = -\theta$,

where θ is the highest root, means that $\alpha_0 - \alpha_j$ for any root α_j is not a root. This means that

$$\frac{2(\alpha_0, \alpha_j)}{(\alpha_0, \alpha_0)} \in \mathbb{Z} \quad (11.12)$$

$$\frac{2(\alpha_0, \alpha_j)}{(\alpha_j, \alpha_j)} \in \mathbb{Z} \quad (11.13)$$

so the extended Π -system obeys the same relations as a Π -system with an extra linear relation between the roots - the system is linearly-dependent. Now removing one root from the extended Π -system leaves a linearly-independent system, with the roots $\{\alpha_i\}$ obeying the relation

$$\frac{2(\alpha_i, \alpha_j)}{(\alpha_j, \alpha_j)} = -(n_+ - n_-) \quad (11.14)$$

This means that the resulting algebra is still a Lie algebra, and the roots are simple roots of a regular maximal subalgebra of \mathfrak{g} . The system may not be indecomposable so the resulting algebra may not be semisimple, but we have restricted to the semisimple case (i.e. discard all non-semisimple algebras obtained from this method). The exceptional cases are discovered later - one can follow the discussions in [9, 38, 40]. My choice of root conventions follow [9]. For the reductive case, notice that these are parabolic subalgebras and are algebraically generated by the Cartan-Weyl basis of \mathfrak{g} without e_α . Removing $e_{-\alpha}$ as well from the Cartan-Weyl basis⁵⁶, we see that there is an extra h_α left over which generates an additional $\mathfrak{u}(1)_\mathbb{C}$ subalgebra. Removing the set $\{e_{\pm\alpha}, h_\alpha\}$ gives the semisimple Lie algebra which is a subalgebra of this reductive subalgebra of the form $\mathfrak{g}'' \oplus \mathfrak{u}(1)_\mathbb{C}$. \square

To obtain S -subalgebras one must follow a different approach - this is discussed in §8.4 of [9].

We also note that promoting the subalgebra to the subgroup level requires us to again consider global arguments as discussed in earlier sections and in [1]. In particular, discrete quotients may affect our results. As an example, the maximal subalgebra of E_8 is $E_6 \oplus A_2$, but in the group level the maximal subgroup of E_8 is $E_6 \times SU(3) \setminus (\mathbb{Z}/3\mathbb{Z})$.

11.2 Symmetry Breaking in Physics

Symmetry breaking is ubiquitous in physics. Often times due to some physical effect a high-energy gauge group G will be broken down to a smaller symmetry H . The principle is as follows:

Proposition 11.2. *Let G be the high-energy gauge group in which the subgroup $G' \subset G$ is broken. Then the low-energy symmetry will be the commutant of G' , i.e. the group H where $G \supset G' \times H$ times the abelian part that commutes with G' .*

Proof. This is related to Goldstone's Theorem. See Advanced Quantum Field Theory. \square

⁵⁶Or Chevalley basis.

If the centre of the group G is discrete, or that the maximal subgroup has a discrete quotient, then its effects will be picked out by global topological effects that are perturbatively inaccessible. In particular in the perturbative regime we can operate solely on the algebra level.

We are told that the world we know is governed by symmetries. Symmetries naturally provide invariants known as conserved quantities of the space by Noether's theorem, which in the end allow us easier access to the system as these conserved quantities, such as energy and momentum, provide physical laws that systems must obey provided that the symmetry exists. However, it is not true that all symmetries (that we have invented) are obeyed for all the possible physical models. One can, for example, imagine writing down a theory that contains terms that do not respect the symmetry. On the other hand, one can also imagine that the symmetry is 'broken' by some mechanism, such as a phase transition, such that the physical state that appears at first seems to do not respect the original symmetry of the theory. What we have just described above are the two 'senses' of the symmetry breaking in physics, summarised below:

- **Explicit symmetry breaking.** This is when the equations of motion explicitly violates the symmetry being considered. Typically, if the theory has a Lagrangian description, this is illustrated by an explicit set of terms in the Lagrangian that vary under the symmetry being considered. A common example is explicit supersymmetry breaking, where terms that violate supersymmetry (SUSY) are written down as 'small violations' of a supersymmetric theory (for example, say the MSSM) and its phenomenological effects are analysed.
- **Spontaneous broken symmetry.** This is typically manifested when there is a degeneracy in the vacua ⁵⁷

In your AQFT course you will encounter the latter type of broken symmetry and study that in more detail. In particular, the Higgs mechanism is in fact an explicit example of spontaneous broken symmetry where part of the Standard Model gauge symmetry is broken to the $U(1)$ gauge group,

$$SU(2)_L \times U(1)_Y \rightarrow U(1)_{\text{EM}} . \quad (11.15)$$

This mechanism is in fact the reason why the quarks and leptons acquire a mass via something known as the Yukawa term - as the Higgs boson acquires a VEV, the Yukawa term provides a Dirac-type coupling to the mass matrices and hence the massless quarks and charge leptons acquire the necessary mass terms. Another typical example is in string theory - to study Heterotic $E_8 \times E_8$ string compactifications, one picks the first E_8 as the gauge group and choose some relevant vector bundle with an associated gauge group $G \subset E_8$ such that the low-energy gauge group is the commutant of G in E_8 . A typical

⁵⁷For an extremely insightful physics discussion, see [41]. The symmetry is not realised as symmetry transformations of the physical states of the theory, and in particular they transform the vacuum state. However, the full theory still contains the full symmetry being considered.

choice is a rank-5 line bundle which gives $G \simeq SU(5)$ so the commutant is some $SU(5)$ with a bundle of abelian factors (which may have phenomenological consequences) [42, 43]. I will leave this discussion here for you to ponder and think about after your AQFT and Strings II courses.

12 Casimirs

In the lectures and problem sheets we have encountered something known as the quadratic Casimir. The quadratic Casimir turns out to be very useful at identifying irreducible representations of \mathfrak{su}_3 . Is that the only invariant combination we can find in a Lie algebra - are there any other objects of a similar property that we can construct? We will aim to answer this question in this subsection.

To understand the theory of Casimirs, let us take a step back and look at the structure of Lie algebras and look at a more general structure. We will deduce new Casimir operators in that manner.

12.1 Universal Enveloping Algebra

First let us recall what the definition of an algebra is (for completeness).

Definition 12.1. An **algebra** is a ring R together with the action of a field \mathbb{F} on R such that multiplication and addition of the ring is compatible with the field action.

An associative algebra is an algebra that satisfies the associative property.

As alluded before the Lie algebra is an algebra (clearly, hence the name). It is important to note that it is possible to construct a Lie algebra from any associative algebra \mathfrak{U} by simply considering the same vector space and defining the Lie bracket as the commutator with respect to the associative product in \mathfrak{U} , i.e. if $a, b \in \mathfrak{U}$, we can construct the Lie bracket,

$$[a, b] = a \cdot b - b \cdot a, \quad (12.1)$$

where \cdot is the associative product in \mathfrak{U} .

The important statement is that we can go in the opposite direction - given a Lie algebra \mathfrak{g} , it is possible to construct an associative algebra \mathfrak{U} which contains \mathfrak{g} as a subspace and for which the commutator that naturally exists in \mathfrak{U} will reproduce the Lie bracket for \mathfrak{g} when restricted to \mathfrak{g} . This motivates the following theorem.

Theorem 12.1. For any Lie algebra \mathfrak{g} , there exists an associative algebra with identity denoted $U(\mathfrak{g})$ together with the map $\iota : \mathfrak{g} \rightarrow U(\mathfrak{g})$ such that the following properties hold:

- $\iota([X, Y]) = \iota(X)\iota(Y) - \iota(Y)\iota(X)$ for all $X, Y \in \mathfrak{g}$.
- $\iota(X)$ generates the algebra $U(\mathfrak{g})$, so it is itself the smallest subalgebra with identity containing all $\iota(X)$.
- (Universal property) Suppose \mathcal{A} is an associative algebra with identity and $j : \mathfrak{g} \rightarrow \mathcal{A}$ is a linear map such that $j([X, Y]) = j(X)j(Y) - j(Y)j(X)$ for all $X, Y \in \mathfrak{g}$. then there exists a unique algebra homomorphism $\phi : U(\mathfrak{g}) \rightarrow \mathcal{A}$ such that $\phi(1) = 1$ and $\phi(\iota(X)) = j(X)$ for all $X \in \mathfrak{g}$. In other words, the map j splits through the homomorphism ϕ ; a universal enveloping algebra $U(\mathfrak{g})$ is a universal initial object such that $\iota_U : \mathfrak{l} \rightarrow U(\mathfrak{l})$ in the category of enveloping algebras of \mathfrak{l} .

The pair $(U(\mathfrak{g}), \iota)$ is known as a **universal enveloping algebra** of \mathfrak{g} .

Proof. Noting that the tensor product on vector spaces is associative, define the **tensor algebra** $T(\mathfrak{g})$ over \mathfrak{g} as

$$T(\mathfrak{g}) = \bigoplus_{k=0}^{\infty} \mathfrak{g}^{\otimes k} . \quad (12.2)$$

Each element of $T(\mathfrak{g})$ has finite linear combinations of elements in $\mathfrak{g}^{\otimes k}$ for each k in the sum. $T(\mathfrak{g})$ is then an associative algebra with identity if we define the product of two elements simply by using the tensor product, i.e. for $u_I = u_{i_1} \otimes \dots \otimes u_{i_k}$ and $v_J = v_{j_1} \otimes \dots \otimes v_{j_l}$ we have

$$u_I \cdot v_J = u_I \otimes v_J \quad (12.3)$$

and extend the product by linearity. Then the identity element $1 \in \mathbb{C} = \mathfrak{g}^{\otimes 0}$ is the multiplicative identity for $T(\mathfrak{g})$, and the associativity of the tensor product assures that $T(\mathfrak{g})$ is an associative algebra.

Now we need to show that $T(\mathfrak{g})$ has the universal property. We construct the map $\phi : T(\mathfrak{g}) \rightarrow \mathcal{A}$ for any associative algebra with identity \mathcal{A} by restricting ϕ to $\mathfrak{g}^{\otimes k}$ to be the unique linear map $\mathfrak{g}^{\otimes k} \rightarrow \mathcal{A}$ such that

$$\phi(X_i \otimes \dots \otimes X_k) = j(X_1) \dots j(X_k) , \quad (12.4)$$

for any linear map $j : \mathfrak{g} \rightarrow \mathcal{A}$. It is straightforward to check that ϕ is an algebra homomorphism and if ϕ is to be an algebra homomorphism that agrees with j on \mathfrak{g} then this is the unique form.

Now we construct $U(\mathfrak{g})$ as a quotient of $T(\mathfrak{g})$. We take the smallest **two-sided ideal** of $T(\mathfrak{g})$, \mathfrak{j} , where for all $\alpha \in T(\mathfrak{g})$ and $\beta \in \mathfrak{j}$, $\alpha\beta, \beta\alpha \in \mathfrak{j}$, to contain all the elements of form

$$X \otimes Y - Y \otimes X - [X, Y] , \quad (12.5)$$

where $X, Y \in \mathfrak{g}$. So \mathfrak{j} is the space of elements of the form

$$\sum_{j=1}^N \alpha_j (X_j \otimes Y_j - Y_j \otimes X_j - [X_j, Y_j]) \beta_j , \quad (12.6)$$

where $X_j, Y_j \in \mathfrak{g}$ and $\alpha_j, \beta_j \in T(\mathfrak{g})$. We form the quotient vector space $T(\mathfrak{g})/\mathfrak{j}$ and we can check that the kernel of ϕ will contain all the elements of the form in Eq.(12.5) and is a two-sided ideal so $\ker(\phi) \supset \mathfrak{j}$. Hence the map $\phi : T(\mathfrak{g}) \rightarrow \mathcal{A}$ factors through $U(\mathfrak{g})$ which gives the desired homomorphism. The uniqueness of $U(\mathfrak{g})$ is guaranteed by the fact that it is spanned by products of elements of \mathfrak{g} . \square

12.2 Casimirs and Invariant Tensors

With the universal enveloping algebra defined, we can now construct something known as the Casimir which belongs to the centre of $U(\mathfrak{g})$.

Definition 12.2. A **Casimir element** is a distinguished element of the centre of the universal enveloping algebra $\mathcal{Z}(U(\mathfrak{g}))$ of Lie algebra \mathfrak{g} .

A prototype example of a Casimir element is the quadratic Casimir, as defined in the lectures as follows.

Definition 12.3. The **quadratic Casimir operator** of a semi-simple Lie algebra \mathfrak{g} is defined as

$$\mathcal{C}_2 = \kappa^{ij} T_i T_j, \quad (12.7)$$

where κ^{ij} is the Killing form and T_i are the generators of the Lie algebra \mathfrak{g} . It is straightforward to show that the value of the quadratic Casimir operator is independent of the choice of the basis (see Proposition 10.5 of [17]). To check that \mathcal{C}_2 indeed is a distinguished element of $\mathcal{Z}(U(\mathfrak{g}))$ is also straightforward - it is straightforward to check $[X_i, \mathcal{C}_2] = 0$ for all $X_i \in \mathfrak{g}$, so since $U(\mathfrak{g})$ is generated by the elements of \mathfrak{g} we then see that \mathcal{C}_2 commutes with every element of $U(\mathfrak{g})$ and hence is in the centre of $U(\mathfrak{g})$.

It is now straightforward to generalize this notion to different orders known as the ‘higher-order Casimir operators’.

Definition 12.4. The **higher-order Casimir operators** form a distinguished basis of the centre $\mathcal{Z}(U(\mathfrak{g}))$, given by homogeneous polynomials,

$$\mathcal{C}_n = d_{a_1 \dots a_n} T^{a_1} \dots T^{a_n} \quad (12.8)$$

where T^a are the generators of \mathfrak{g} with $d_{a_1 \dots a_n}$ being the suitable invariant tensors of the adjoint representation. The **order** n of the Casimir operator \mathcal{C}_n is the order n of the polynomial.

Recall that representation matrices in the adjoint representation are just the structure constants. Since the Casimir operators are distinguished elements of the centre of the universal enveloping algebra $U(\mathfrak{g})$, they are homogeneous polynomials in the generators of \mathfrak{g} and therefore constitute a maximal set of algebraically independent elements of the centre. It also follows that they are invariant tensors of the adjoint representation (as they can be represented by structure constants effectively).

Proposition 12.1. *The Casimir operators are in one-to-one correspondence with the set of tensors that generate the space of adjoint representation invariant tensors.*

Proof. To show this let us look at the polynomial,

$$P = c\mathbb{1} + \sum_a t_a T^a + \sum_{a,b} t_{ab} T^a T^b + \dots, \quad (12.9)$$

where T^a are the generators of \mathfrak{g} . Suppose this polynomial belongs to the centre of $U(\mathfrak{g})$. Then it is sufficient that all the coefficients $t_{a_1 \dots a_n}$ are invariant tensors of the adjoint representations, as the G -invariance of the element requires the coefficients to all be in the adjoint representation such that the transformation leaves it in the universal enveloping algebra $U(\mathfrak{g})$ ⁵⁸. The converse is also true. \square

⁵⁸To further see this point, look at §17 of [9] or [44].

We have deduced that all Casimir operators of \mathfrak{g} are of the form

$$\mathcal{C}_n = d_{a_1 \dots a_n} T^{a_1} \dots T^{a_n} \quad (12.8)$$

To see this explicitly, note that for the quadratic Casimir, which is,

$$\mathcal{C}_2 = \kappa_{ab} T^a T^b, \quad (12.7)$$

we have that the Killing form $\kappa_{ab} \propto \sum_{c,d} f^{ac}_d f^{bd}_c$. We however note that although other elements of the centre $\mathcal{Z}(U(\mathfrak{g}))$ can be obtained by taking the tensors $\text{tr}(\text{ad}_{T^i} \circ \dots \circ \text{ad}_{T^n})$, it is not necessarily true that they are all algebraically independent so they do not typically provide Casimir operators. There is however an independent high-order Casimir operator for each order (which for the case A_n you can see [9]).

There is clearly a lot more to the theory of invariant tensors and Casimirs. I implore you to read §14 and §17 of [9] for more on that.

13 The Standard Model

In Q2-3 of Sheet 3 we have been looking at the weight systems and branchings of $SU(5)$. We have shown after (some) work:

$$\mathbf{10} \oplus \bar{\mathbf{5}} \rightarrow (\mathbf{3}, \mathbf{2})_{1/3} \oplus (\bar{\mathbf{3}}, \mathbf{1})_{-4/3} \oplus (\mathbf{1}, \mathbf{1})_2 \oplus (\mathbf{1}, \mathbf{2})_{-1} \oplus (\bar{\mathbf{3}}, \mathbf{1})_{2/3} \quad (13.1)$$

where $U_Y(1)$ charge is attached as a subscript. This branching is precisely the representation for one standard model family. To see this, let us first recall what the Standard Model is.

The Standard Model is a chiral quantum field theory that is gauged under local transformations of the Standard Model gauge group:

$$G_{SM} = SU(3)_C \times SU(2)_L \times U(1)_Y \quad (13.2)$$

The $SU(3)_C$ being the colour gauge symmetry, $SU(2)_L$ the electroweak symmetry on left-handed particles and $U(1)$ typically known as the hypercharge. We denote the fundamental gauge fields as G_μ^A , W_μ^a and B_μ , which after spontaneous symmetry breaking we will write G_μ^A , W_μ^\pm , Z_μ and A_μ . The Lagrangian for the kinetic and self-interaction terms of the gauge bosons is then:

$$\mathcal{L}^{\text{gauge}} = -\frac{1}{4} (G_{\mu\nu}^A)^2 - \frac{1}{4} (W_{\mu\nu}^a)^2 - \frac{1}{4} (B_{\mu\nu})^2 - \Theta_G G_{\mu\nu}^A \tilde{G}_{\mu\nu}^A - \Theta_W W_{\mu\nu}^a \tilde{W}_{\mu\nu}^a - \Theta_B B_{\mu\nu} \tilde{B}_{\mu\nu} \quad (13.3)$$

Note that here the topological θ -terms are allowed *a priori*. The fermionic matter content can be summarised below:

$$\begin{aligned} (\mathbf{3}, \mathbf{2}, \frac{1}{6}) : \quad Q_L^i &= \left\{ \begin{pmatrix} u_L \\ d_L \end{pmatrix} \begin{pmatrix} c_L \\ s_L \end{pmatrix} \begin{pmatrix} t_L \\ b_L \end{pmatrix} \right\} \\ (\bar{\mathbf{3}}, \mathbf{1}, \frac{2}{3}) : \quad u_R^i &= \{u_R, c_R, t_R\} \\ (\bar{\mathbf{3}}, \mathbf{1}, -\frac{1}{3}) : \quad d_R^i &= \{d_R, s_R, b_R\} \\ (\mathbf{1}, \mathbf{2}, -\frac{1}{2}) : \quad L_L^i &= \left\{ \begin{pmatrix} \nu_{e,L} \\ e_L \end{pmatrix} \begin{pmatrix} \nu_{\mu,L} \\ \mu_L \end{pmatrix} \begin{pmatrix} \nu_{\tau,L} \\ \tau_L \end{pmatrix} \right\} \\ (\mathbf{1}, \mathbf{1}, -1) : \quad e_R^i &= \{e_R, \mu_R, \tau_R\} \\ (\mathbf{1}, \mathbf{1}, 0) : \quad \nu_R^i &= \{\nu_{e,R}, \nu_{\mu,R}, \nu_{\tau,R}\} \end{aligned} \quad (13.4)$$

where the right-handed neutrinos can be neglected from the model. There are two pieces in the Lagrangian for the fermions. The first is the kinetic term:

$$\mathcal{L}_F^{\text{kinetic}} = i\bar{L}_L^i \not{D} L_L^i + i\bar{Q}_L^i \not{D} Q_L^i + i\bar{e}_R^i \not{D} e_R^i + i\bar{\nu}_R^i \not{D} \nu_R^i + i\bar{U}_R^i \not{D} u_R^i + i\bar{d}_R^i \not{D} d_R^i \quad (13.5)$$

The covariant derivative is written as:

$$D_\mu = \partial_\mu - ig_s G_\mu^A T^A - ig W_\mu^a T^a - ig' Y B_\mu \quad (13.6)$$

The Yukawa couplings are

$$\mathcal{L}_F^{\text{Yukawa}} = -y_{ij}^d \bar{Q}_L^i H d_R^j - y_{ij}^u \bar{Q}_L^i \tilde{H} u_R^j + y_{ij}^e \bar{L}^i H e_R^j \left(+ y_{ij}^\nu \bar{L}_L^i H \nu_R^j \right) + (\text{h.c.}) \quad (13.7)$$

The Higgs field is a complex scalar doublet in the system:

$$H = \begin{pmatrix} H_+ \\ H_0 \end{pmatrix} \quad (13.8)$$

with indices $(\mathbf{1}, \mathbf{2}, \frac{1}{2})$, with Lagrangian

$$\mathcal{L}^{\text{Higgs}} = D_\mu H (D^\mu H)^\dagger + m^2 |H|^2 - \lambda |H|^4 \quad (13.9)$$

So the Standard Model Lagrangian is given by, in general,

$$\mathcal{L}_{SM} = \mathcal{L}^{\text{gauge}} + \mathcal{L}_F^{\text{kinetic}} + \mathcal{L}_F^{\text{Yukawa}} + \mathcal{L}^{\text{Higgs}} \quad (13.10)$$

14 Grand Unified Theories

Having clarified what the Standard Model is, let us have a look at what Grand Unified Theories are.

The idea of Grand Unified Theories is to propose an underlying gauge group which contains G_{SM} as a subgroup. The G_{GUT} must hence be at least rank 4, and must contain:

$$G_{GUT} \supset G_{SM} = SU(3)_c \times SU(2)_L \times U(1)_Y \quad (14.1)$$

The motivation comes from the unification of the gauge couplings into a single one. Consider the one-loop renormalisation equation for gauge couplings.

$$\frac{1}{\alpha_a(Q^2)} = \frac{1}{\alpha_a(M^2)} + \frac{b_a}{4\pi} \log \frac{M^2}{Q^2} \quad (14.2)$$

with

$$b = -\frac{11}{3}N + \frac{2}{3}C(R)n_f + \frac{1}{3}C(R)n_s \quad (14.3)$$

For the Standard Model this is

$$\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} 0 \\ -\frac{22}{3} \\ -11 \end{pmatrix} + \frac{4}{3}N_{gen} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + N_{Higgs} \begin{pmatrix} \frac{1}{10} \\ \frac{1}{6} \\ 0 \end{pmatrix} \quad (14.4)$$

where N_{gen} and N_{Higgs} is the number of generations and Higgs respectively. We can look at the evolution of the of the couplings. For non-SUSY (supersymmetric) one-loop renormalisations, the couplings do not match precisely together at high energies. However, for MSSM models, the couplings exactly match - the gauge couplings join together to become one single gauge coupling at some energy scale Λ_{GUT} . This is what we are really considering when we talk about Grand Unified Theories - these are supersymmetric theories that represent the “unified” theory when the gauge couplings of QCD and EW join together at high energies.

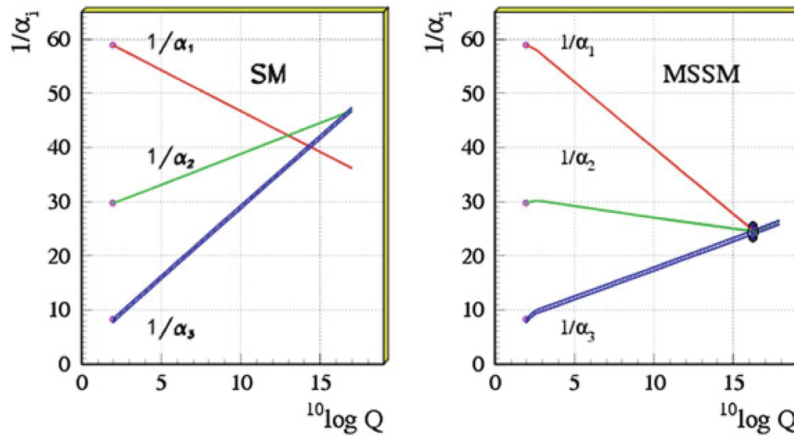


Figure 14.1: The evolution of the inverse couplings α_i with respect to the logarithmic energy scale $\log Q$. Figure taken from [45].

We in addition have to assume that there are no further relevant degrees of freedom in the intermediate scales. This is the so-called *desert-hypothesis*. We note that the full GUT multiplets do not modify the running of the relative gauge couplings at one-loop accuracy, so their introduction does not spoil the unification of couplings.

Here we consider two prominent GUT groups - $SU(5)$ and $SO(10)$.

14.1 $SU(5)$ GUT Group

$SU(5)$ is the unique compact simple Lie group of rank 4 admitting complex representations. It contains G_{SM} as a maximal subgroup. Let us look at some properties of this GUT group. Properties of $SU(5)$ GUT:

1. The number of gauge bosons is $5^2 - 1 = 24$. There are 12 Standard Model and 12 extra gauge bosons labelled (X_r^\pm, Y_r^\pm) transforming as $SU(2)_L$ doublets and $SU(3)_c$ triplets.
2. The abelian subgroup commuting with the generators of $SU(3) \times SU(2)$ is

$$T_{24} = \sqrt{\frac{3}{5}} \begin{pmatrix} -\frac{1}{3} & & & & \\ & -\frac{1}{3} & & & \\ & & -\frac{1}{3} & & \\ & & & \frac{1}{2} & \\ & & & & \frac{1}{2} \end{pmatrix} = \sqrt{\frac{3}{5}} \frac{Y}{2} \quad (14.5)$$

3. Each quark-lepton generation fits nicely in the reducible representation $\bar{\mathbf{5}} \oplus \mathbf{10}$. The generators are:

$$T_A = \begin{pmatrix} \frac{1}{2}\lambda_A & 0 \\ 0 & 0 \end{pmatrix} \quad (14.6)$$

for $A = 1, \dots, 8$ and

$$T_A = \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{2}\sigma_{A-20} \end{pmatrix} \quad (14.7)$$

for $A = 21, 22, 23$. The generators labelled $A_9, \dots, 20$ correspond to the $SU(5)/G_{SM}$ generators. The explicit embedding is given by

$$\bar{\mathbf{5}} = \begin{pmatrix} d_1^c \\ d_2^c \\ d_3^c \\ e^- \\ \nu_e \end{pmatrix} \quad (14.8)$$

and

$$\mathbf{10} = \begin{pmatrix} 0 & u_3^c & u_2^c & u_1 & d_1 \\ & 0 & u_1^c & u_2 & d_2 \\ & & 0 & u_3 & d_3 \\ & & & 0 & e^+ \\ & & & & 0 \end{pmatrix} \quad (14.9)$$

where $d^c = D_R^1$, $(\nu_e, e^-) = L^1$, $u^c = U_R^1$, $(u, d) = Q_L^1$, $e^+ = E_R^1$. This assignment is free of anomalies.

4. The fermionic part of the Lagrangian is

$$\mathcal{L}_f = \bar{\mathbf{5}}_\alpha^\dagger (i\bar{\sigma}_\mu D^\mu)_\alpha{}^\beta \bar{\mathbf{5}}_\beta + \frac{1}{2} \mathbf{10}^{\alpha\beta\dagger} (i\bar{\sigma}_\mu D^\mu)_{\gamma\delta}^{\alpha\beta} \mathbf{10}^{\gamma\delta} \quad (14.10)$$

where $D^\mu = \partial^\mu + ig_5 T_A A_A^\mu$.

5. To break $SU(5)$ to the Standard Model spontaneously we add the scalar field ‘‘GUT-Higgs’’ transforming in the adjoint **24**:

$$\mathcal{L}_\Sigma = \text{tr}(D_\mu \Sigma)^2 - \text{tr}(\Sigma^2 - M_G^2) - (\lambda H^\dagger \Sigma H + \bar{\lambda} \bar{H} \Sigma \bar{H}^\dagger) - m_H^2 H^\dagger H - m_{\bar{H}}^2 \bar{H}^\dagger \bar{H} \quad (14.11)$$

Under $SU(5)$ this transforms as:

$$\Sigma \mapsto U \Sigma U^\dagger \quad (14.12)$$

and acquires a large VEV

$$\langle \Sigma \rangle = \begin{pmatrix} 2 & & & & \\ & 2 & & & \\ & & 2 & & \\ & & & -3 & \\ & & & & -3 \end{pmatrix} v \quad (14.13)$$

so the X, Y gauge masses $M_{X,Y}^2 \simeq \alpha_{GUT} v^2$.

The $SU(5)$ GUT theory has many implications⁵⁹:

1. Charge Quantisation: $\text{tr } Q_{EM} = 0$ since $Q_{EM} \in SU(5)$ so

$$Q_{d^c} = -\frac{1}{3} Q_{e^-} \quad (14.14)$$

2. Gauge Coupling Relation: Because of the unification of gauge couplings we obtain relations between couplings. For example, we get:

$$g_1^2 = \frac{3}{5} g_2^2 \quad (14.15)$$

3. Fermionic Mass Relations: $SU(5)$ relates Yukawa couplings of d and e^- , giving

$$\frac{m_s}{m_d} = \frac{m_\mu}{m_e} \quad (14.16)$$

This is obviously incompatible (you can check this) with the observed values. Georgi and Jarlskog [46] used $SU(5)$ Clebschs to fix this in an ingenious way - you can find out more in [45].

⁵⁹I am too lazy to use the Feynman diagram package, might update this later please forgive me...

4. Nucleon Decay: X and Y gauge bosons allows baryon number violating decay modes, for example: $p \rightarrow \pi^0 e^+$.

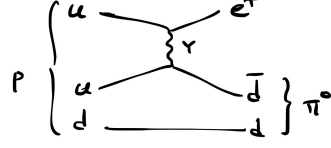


Figure 14.2: Feynman diagram of the decay $p \rightarrow \pi^0 e^+$.

This is suppressed by the large M_Y mass. The simplest $SU(5)$ GUT is therefore excluded as

$$\tau_{p \rightarrow \pi^0 e^+} \simeq 4 \times 10^{29 \pm 7} \text{years} < \tau_{p \rightarrow \pi^0 e^+, \text{bound}} = 6.6 \times 10^3 \text{years} \quad (14.17)$$

A detailed calculation can be found in [45].

14.2 $SO(10)$ GUT Group

The $SO(10)$ is one of the two rank-5 compact simple Lie groups admitting complex representations. One family of Standard Model quarks and leptons fits nicely in the irrep **16**:

$$\psi_{16} = (\nu_e \ u_1 \ u_2 \ u_3; \ e^- \ d_1 \ d_2 \ d_3; \ d_3^c \ d_2^c \ d_1^c \ e^+; \ u_3^c \ u_2^c \ u_1^c \ \nu_R) \quad (14.18)$$

including the right-handed neutrino. $SO(10)$ in fact contains $SU(5) \times U(1)$ as a maximal subgroup, with the splitting (which you computed last sheet)

$$\mathbf{45} \rightarrow \mathbf{24} \oplus \mathbf{10} \oplus \bar{\mathbf{10}} \oplus \mathbf{1} \quad (14.19)$$

Breaking of the $SO(10)$ can proceed in many steps, one of them being, for example:

$$SO(10) \rightarrow SU(5) \times U(1) \rightarrow G_{SM} \quad (14.20)$$

This requires adjoint scalars Φ_{45} and also extra scalars ϕ_{16} in the **16** irrep. The minimal embedding of the electroweak Higgs doublet is in a multiplet of scalars H_{10} in **10**. Since

$$\mathbf{10} \rightarrow \mathbf{5} \oplus \bar{\mathbf{5}} \quad (14.21)$$

which gives (H_u, H_d) , the Yukawa terms are

$$\mathcal{L} = Y^{ij} \bar{\psi}_{16}^i \psi_{16}^j H_{10} + h.c. \quad (14.22)$$

so we have unification of Yukawa couplings at the GUT scale. Of course, this is not really physical as this implies that there are no CKM mixings. One can consider improved models in Georgi [47].

15 The sequel: AQFT

We are not done - in fact, this is only the beginning. In fact, there is a lot more to symmetries, Lie algebras and physics. This course is just the teaser to let you peek at that wonderful mystery land of theoretical and mathematical physics. There are many, many more things to chat about, such as affine Lie algebras, gauge bundles, axions, string theory...

You are lucky though. The natural sequel to this course (and the culmination of the QFT course together with all the knowledge you have learnt so far) in the Advanced Quantum Field Theory course in Hilary Term. In particular, you will encounter three main ideas:

1. **Functional methods in QFT.** There is of course, a similar formulation of QFT as the path integral formulation of quantum mechanics. The path integral gives one access to information that cannot be accessed from the perturbative Hamiltonian formulation of QFT, such as the theory of anomalies which are fundamental in understanding low-energy effective field theories of high energy theories.
2. **Renormalisation.** This is a huge topic (and a big mess) that I have completely skipped in these set of notes ⁶⁰. Renormalisation in QFT typically covers two main ideas - regularisation, the idea of removing infinities by introducing cut-offs in a theory, and renormalisation, the idea of removing dependencies on the cut-offs you have introduced in regularisation to compute physical quantities. The actual theory takes about an entire lecture course to explain (and possibly a whole lifetime to really comprehend) but I will leave it here ⁶¹.
3. **Gauge theories.** Gauge theory is an extremely difficult topic. One can simply view gauges as redundancies and this is in fact a very useful way of thinking about it in the context of quantum field theory, but the ‘correct’ mathematical definition should be a connection on a gauge bundle. This probably takes about two lecture courses to explain ⁶², so I will just leave it for a later date or a later note.

There is way to much to do and learn but you will survive. The list of things you would want to learn just grows. I, for now, should stop typing this and return to actual research to learn more...

⁶⁰But, who knows, maybe I will have something on the side to supplement this supplementary note.

⁶¹If you are interested, the best sets of notes I have found (on top of canonical QFT textbooks) are [48, 49].

⁶²One on the physics notion and how you encounter and deal with them in QFTs, one on the mathematical formulation (connection theory) and then one to wrap everything together.

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⁶³Kit, Thomas, Riccardo and Ethan were tutors in previous years.

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