Supersymmetry Class 2

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Abstract

The main topic for this sheet is supersymmetric algebras and representations. This builds the foundation to understand the multiplet structure of supersymmetric actions later when we talk about the superfield formalism. To illustrate this I have therefore revisited Wigner's classification which is covered in the AQFT lectures and classes in the class as well as the supersymmetric algebras. This note is a summary of the topics covered in the class as well as some additional topics that I did not have time to cover. None of the material here is original and this is a well-developed field - the list of references is nowhere complete and representative of all the hard work many people have put in over the last couple of decades.

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As before, I will begin with a summary of the class in §1. The problem sheet feedback is in §2.

1 Class 2 Summary

The main topic for this class is supersymmetric algebras and representations. Although the sheet is indeed quite straightforward and involves a lot of algebraic manipulations, I have tried to illustrate the mathematics and physics behind each of these problems. In particular, as we went through the sheet we inevitably encountered the following topics.

1. **Conformal field theory and SCFTs.** We discussed what a conformal symmetry is, and introduced the conformal algebra. For a quick summary of CFTs and SCFTs see §3. This led us to discussing the scaling dimensions of operators.

2. **Wigner's Classification.** We then reviewed Wigner's Classification of irreducible representations of the Poincaré algebra. In particular, we discussed how the little group method works and the method of induced representations in more general terms. For a quick summary of Wigner's Classification due to Bargmann see §5. The method of induced representations is reviewed in §4.

3. **Supersymmetric multiplets.** Finally we reviewed how the supersymmetric multiplets arise. In particular, the multiplets are branched irreps of the bigger super-Poincaré group.

I have mainly followed the logic in [1, 2] as well as Bertolini's lecture notes (which a highly recommend reading) recently published as [3]. I have sneaked in a small discussion of central charges at the end of the class; an exposition could be found in §6.

2 Problem Sheet 2 Feedback

2.1 Question 1 - Super Jacobi Identities

The main point of this question is to illustrate how to use spinor identities we have developed in the lectures and the last class to evaluate expressions. The main point to highlight here is that in evaluating the super Jacobi identities you should use the following lemma.

Lemma 2.1. *The following identity holds.*

$$(\sigma_{\mu\nu})_\alpha{}^\gamma \, \epsilon_{\gamma\beta} + (\sigma_{\mu\nu})_\beta{}^\gamma \, \epsilon_{\alpha\gamma} = 0 \ . \tag{2.1}$$

Proof. This should be quite straightforward. First you should use the definition of $\sigma_{\mu\nu}$,

$$\sigma_{\mu\nu} = \frac{i}{4} \left(\sigma_\mu \bar{\sigma}_\nu - \sigma_\nu \bar{\sigma}_\mu \right) \ . \tag{2.2}$$

Now we use the following identities,

$$(\bar{\sigma}^\mu)^{\dot{\alpha}\alpha} = -\epsilon^{\alpha\beta}\,(\sigma^\mu)_{\beta\dot{\beta}}\,\epsilon^{\dot{\beta}\dot{\alpha}} \tag{2.3}$$

$$\epsilon^{\alpha\beta} = \epsilon_{\beta\alpha} \tag{2.4}$$

to show that Eq. (2.1) holds true. See Proposition 1.39 in [4] for details. □

This is a non-trivial result that you should explicitly prove in your attempt! Otherwise I will treat the attempt to be invalid.

2.2 Question 2 - Superconformal algebra

In this question we revisit the superconformal algebra and compute some commutation relations of the superconformal algebra. The main point of this question is really just to get you to be comfortable with manipulating (super-)conformal algebraic expressions. There are in fact two main points I would like to cover in more detail.

Dilatation generator and Scaling dimensions

In part (a) most of you struggled to find the commutation of the dilatation generator D with the Poincaré and superconformal supercharges Q^I_α and S^I_α, namely,

$$[D, Q^I_\alpha] = \frac{i}{2} Q^I_\alpha \,, \tag{2.5}$$

$$[D, S^I_\alpha] = -\frac{i}{2} S^I_\alpha \,. \tag{2.6}$$

The key point here is to realise that the only non-trivial identity is the super-Jacobi identity so we will use that to derive the expressions. You should be able to realise that from dimensional considerations and matching the spinor indices (Lorentz representations) on both sides that we must have,

$$[D, Q^I_\alpha] = i\lambda^{IJ} Q^J_\alpha \,, \tag{2.7}$$

where $\lambda^I \in \mathbb{C}$ a priori. Conjugating this gives,

$$[D, \bar{Q}^I_\alpha] = -i\bar{\lambda}^{IJ} Q^J_\alpha \,. \tag{2.8}$$

Now you can use the Jacobi identity - evaluating on both sides will eventually allow you to conclude that $\lambda^{IJ} = \frac{1}{2}\delta^{IJ}$ with no imaginary part[1]. This is the same with the S^I_α with just the opposite sign.

Central Charge

In part (b) you are asked to show that the central charges in the superconformal algebra must vanish. The key idea here is to use the fact that the

[1] In fact, the eigenvalues of the dilatation operator gives you the scaling dimensions which are real.

central charge is the generator of the central extension of the algebra and therefore commutes with all elements,

$$[X, Z] = 0, \quad \forall X \in \mathcal{L}\,. \tag{2.9}$$

Hence we must have $[D, Z] = 0$. I will leave you with using the Jacobi identity to further show that $Z = 0$ identically in the superconformal case.

The rest of the question is just algebraic manipulations which I don't really have anything more to say.

2.3 Question 3 - Massless supermultiplets

Massless multiplets are important in constructing the low-energy spectrum of supersymmetric theories. There are three main things to note.

Degeneracy of states

Recall that we generate the massless supermultiplet by starting with the Clifford vacuum[2] $|\Omega_\lambda\rangle$ with helicity λ. We then repeatedly hit the Clifford vacuum with raising operators to generate the entire multiplet. The key point of this question is to note that the degeneracy of the states of helicity $\lambda + \frac{1}{2}n$, where n is the number of fermionic generators acted on the Clifford vacuum, is $\binom{\mathcal{N}}{n}$ for $\mathcal{N}$-extended supersymmetry. Many of you have loosely argued that $\sum_{n \text{ even}} \binom{\mathcal{N}}{n} = \frac{1}{2}\sum_n \binom{\mathcal{N}}{n}$ without a consistent argument. The quickest way, instead, is to realise that,

$$0 = \sum_{n=0}^{\mathcal{N}} \binom{\mathcal{N}}{n} (-1)^n = (1 + (-1))^{\mathcal{N}}\,, \tag{2.10}$$

using the binomial theorem, and realising that $(-1)^n$ is indeed the eigenvalue of the fermionic number operator $(-1)^F$ acted on each level (up to a sign) so the above expression effectively captures the index $n_B - n_F$.

Multiplets as representations of $U(\mathcal{N})$

The next point to note is how the multiplet content at each level forms a representation of the maximal R-symmetry of the extended supersymmetry algebra. First recall in four-dimensions the maximal R-symmetry group of the $\mathcal{N}$-extended SUSY is $U(\mathcal{N})$ (a priori). The particles on each level of the multiplet then furnish the antisymmetric part of the Clebsch-Gordan decomposition of products of fundamental representations, as they are generated by fermionic operators which algebraic structure is isomorphic to the

[2]Note that the Clifford vacuum here is not a vacuum in the usual QFT-sense. The Clifford vacuum is only the **lowest weight state** in the super-Poincaré algbera, and is not necessarily the state with minimal energy.

exterior algebra [3]. In particular, we can use tensor notations to indicate the particles in a multiplet [4].

There is a subtlety however. The 4d R-symmetry group is not always the full $U(\mathcal{N})$ group. In the case $\mathcal{N}=2$, the R-symmetry group is in fact,

$$U(2)_R \cong U(1)_R \times SU(2)_R \ , \tag{2.11}$$

and for $\mathcal{N}=4$, the R-symmetry group is $SU(4)$. The reason for the latter is because the fermionic fields actually realise the spinor representation of Spin(6) and Spin(6) $\cong SU(4)$. The fact the $U(1)$ subgroup of $U(\mathcal{N})$ R-symmetry is sometimes not well-explained - under CPT-conjugation, the multiplet maybe self-conjugate and therefore the $U(1)$ part of the full R-symmetry group coincides with the helicity group ($\mathfrak{u}(1) \cong \mathfrak{so}(1,1)$) and therefore plays no important role. This is, for example, realised for $\mathcal{N}=4,8$ cases, so the R-symmetry group is sometimes instead listed as $SU(4)$ and $SU(8)$ respectively.

CPT-completion of 4d $\mathcal{N}=3$ vector multiplet

The second part of the question asks you to compute the four-dimensional $\mathcal{N}=3$ multiplet with the Clifford vacuum $|\Omega_{-1}\rangle$. Now if you start with the Clifford vacuum $\left|\Omega_{-1/2}\right\rangle$ instead, you will get the CPT-conjugate of the first multiplet. Therefore after CPT-completion the $\mathcal{N}=3$ multiplet doubles in size, and you can check as a part of the question that it exactly matches the $\mathcal{N}=4$ hypermultiplet (generated from $|\Omega_{-1}\rangle$). Therefore by convention this CPT-completed supermultiplet is referred to as the $\mathcal{N}=4$ multiplet and the $\mathcal{N}=3$ multiplets are forgotten. The decomposition actually occurs in general for higher extended supersymmetric multiplets - in general they can be expressed as compositions of lower multiplets. So as far as non-gravitational theories are discussed, the $\mathcal{N}=3$ multiplets are typically neglected.

2.4 Question 4 - Massive supermultiplets

I don't really have a lot to say about this question - this is just standard irrep constructions that you should be able to find in most supersymmetry textbooks. The only thing I would comment on would be how to write down the states at the second-level. Using the notation

$$a_\alpha^{(k)} = \frac{1}{\sqrt{2m}} Q_\alpha^{(k)} \ , \quad a_\alpha^{(k)\dagger} = \frac{1}{\sqrt{2m}} \bar{Q}_{\dot{\alpha}}^{(k)} \ , \tag{2.12}$$

[3] Using Young diagram notation, the particles will be represented by irreps corresponding to columns of boxes. The number of boxes corresponds to the number of raising operators acted on the Clifford vacuum (i.e. level in the multiplet).

[4] You should have learnt this in Groups and Representations, see Andre's lectures notes.

we can generate the particles at the second-level by acting two fermionic operators. In particular, since we have defined the $\alpha = 1$ operator to raise the spin by a half, spin-0 states at level two should have the form

$$\epsilon^{\alpha\beta}\left(a_\beta^{(k)}\right)^\dagger\left(a_\alpha^{(l)}\right)^\dagger|\Omega_0\rangle\ , \tag{2.13}$$

with the epsilon ensuring that the particle generated is indeed an irrep. On the other hand the spin-1 particles at the same level have the form

$$\mathcal{S}_{kl}\left[\left(a_2^{(k)}\right)^\dagger\left(a_1^{(l)}\right)^\dagger\right]|\Omega_0\rangle\ , \tag{2.14}$$

with $\mathcal{S}_{kl}$ the symmetric operator acting on the indices k and l. Most of you didn't explicitly write this and I would imagine if you are forced to write this in an exam you would miss the (anti-)symmetrising factors so I have included them here.

3 CFTs and SCFTs

We have discussed the SCFT algebra in the lectures as an extension of the SUSY algebra. Let us try and understand what the algebra means in very brief terms.

3.1 Conformal Field Theories

Here I will briefly review what a conformal field theory is. Some good references include [5, 6].

Consider the space $\mathbb{R}^{1,n-1}$ with flat metric $g_{\mu\nu}$ of signature (p, q) and the line element $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$. The cofnormal group is the subgroup of coordinate transformations that leaves the metric invariant up to a scale factor,

$$g_{\mu\nu} \mapsto \Omega(x)g_{\mu\nu}(x)\ , \tag{3.1}$$

which are angle-preserving transformations. The infinitesimal coordinate transformations are generated by $x^\mu \mapsto x^\mu + \epsilon^\mu$, which gives,

$$ds^2 \mapsto ds^2 + \left(\partial_\mu\epsilon_\nu + \partial_\nu\epsilon_\mu\right)dx^\mu dx^\nu\ , \tag{3.2}$$

where

$$\partial_\mu\epsilon_\nu + \partial_\nu\epsilon_\mu = \frac{2}{d}\left(\partial\cdot\epsilon\right)\eta_{\mu\nu}\ . \tag{3.3}$$

For the dimensions $d \geq 3$, we have to lowest order in x the generators listed in Table 3.1. The group then has dimension $\frac{1}{2}(d+2)(d+1)$. The conformal group for $\mathbb{R}^{m,n}$ is isomorphic to the group $SO(m+1, n+1)$; the inversion being an additional discrete generator not continuously connected to the

ϵ^μ	x'	Operator	Name
a^μ	$x+a$	P_μ	Translation
$\omega^\mu{}_\nu x^\nu$	Λx	$M_{\mu\nu}$	Lorentz
λx^μ	λx	D	Dilatation
$b^\mu x^2 - 2x^\mu b\cdot x$	$\frac{x+bx^2}{1+2b\cdot x+b^2x^2}$	P_μ	Special Conformal

Table 3.1: Table of the generators of the conformal group in $d \geq 3$. The first two columns give the infinitesimal and full coordinate transformations respectively, and the last two columns give the corresponding operator/generator and its name.

identity. It is the isometry group of the lightcone in $d = m+n$ dimensions. The Lie algebra $\mathfrak{so}(m+1, n+1)$ is generated by the operators/generators listed in Table 3.1, with the non-trivial non-Poincaré generators given by,

$$[M_{\mu\nu}, K_\rho] = g_{\nu\rho}K_\mu - g_{\mu\rho}K_\nu \ , \tag{3.4}$$

$$[D, P_\mu] = P_\mu \ , \tag{3.5}$$

$$[D, K_\mu] = -K_\mu \ , \tag{3.6}$$

$$[K_\mu, P_\nu] = 2\left(\eta_{\mu\nu}D - M_{\mu\nu}\right) \ . \tag{3.7}$$

In particular, the dilatation generator D generates the abelian Lie subalgebra $\mathfrak{so}(1,1)$ and therefore all other generators have a weight under it. This weight is known as the **scaling dimension**, with translations and special conformal transformations having weights $+1$ and -1 respectively. In radial quantisation, the dilatation generator D acts as the Hamiltonian and the states living in the system will be characterised by its scaling dimension and its $SO(d)$ spin. Time translations are therefore generated by dilatation [7]. We are not going into the details here, but to summarise; the unitary representations of the conformal group are generated with the highest weight state (**primary state**) defined as

$$K_\mu \left|[L]_\Delta\right\rangle = 0 \tag{3.8}$$

with K_μ acting as the lowering operator. Similarly, the **descent states** are generated by acting the primary state with P_μ and Lorentz generators. For the details see for example [8].

Two things to note here. Firstly, the case for $d = 2$ is a bit more complicated. The conformal group of the Euclidean plane for example is the group $SO(3,1)$ of Möbius transformations. This is a finite group, although looking at the Lie algebra, we will see that there is an infinite set of conformal Killing fields, which gives an infinite number of independent constraints [5].

[5] This is the origin for the claim that the 2d conformal group is infinite, see §2.4 of [6] for a detailed discussion.

For the conformal group of $\mathbb{R}^{1,1}$ this is indeed infinite as a group.

Another subtlety comes from a 'common' misconception with Weyl transformations which has the form,

$$g_{\mu\nu}(x) \mapsto \Omega(x) g_{\mu\nu} \ . \tag{3.9}$$

Here we note that a Weyl transformation is a physical change of the metric and has nothing to do with coordinate transformations, where as a conformal transformation is by definition a coordinate transformation. One has to be careful with such a distinction in quantisation in string theory.

3.2 Superconformal Symmetry

We now briefly discuss superconformal symmetries in very brief terms. The symmetry algebra of SCFTs contain both the conformal algebra $\mathfrak{so}(d,2)$ and a supersymmetry algebra involving the supercharges Q. To complete the algebra one must also include the supercharges S, where the scaling dimensions of Q and S are $\frac{1}{2}$ and $-\frac{1}{2}$ respectively. Similarly to before, we can define the irreps of the superconformal algebra in the radial quantisation scheme with K_μ acting as the lowering operators and mirroring the construction of irreps of super-Poincaré group with the **superconformal primary** defined to be the state annihilated by K, S and $\bar{S}$. For the details, please refer to [8, 9].

4 Induced Representations

Induced representations are a key part to Wigner's Classification. Here we provide a quick review of the method.

4.1 Set-up

We already know what happens when we want to restrict a representation of G to a subgroup H of G - this is how branching rules arise. The key idea of an induced representation is to do the inverse - to generate a representation for a bigger group G given a representation of a subgroup $H \subset G$.

Let us assume we are given a representation $\rho : H \to GL(V)$ where V is a vector space and H is a subgroup of G, where for $h \in H$,

$$v \overset{h}{\mapsto} \rho(h)v, \quad v \in V \ . \tag{4.1}$$

We consider the cosets G/H and represent the each coset with an element g_i such that the coset $[g_i]$ is defined as,

$$[g_i] = \{g \in G \,|\, g_i = gh, \quad h \in H\} \ . \tag{4.2}$$

Then, for any $g \in G$, we have,

$$gg_i = g_j h \tag{4.3}$$

for some $h \in H$. The number of cosets is N. We define the representation space as the product $G/H \times V$ with

$$v_i = ([g_i], v) \in G/H \otimes V \ , \tag{4.4}$$

such that

$$v_i = ([g_i], v) \overset{h}{\mapsto} ([g_j], \rho(h)v) = v_j \ , \tag{4.5}$$

under the action of $h \in H$. The representation space is therefore isomorohic to the N-fold tensor product $V^{\otimes N}$. In the space where we permute the N-copies of Vs, the representation matrices for the induced representation can then be given by $N \times N$ matrices with elements,

$$\rho_{ji}(g) = \begin{cases} \rho(h), & g_j^{-1} g g_i = h \in H \\ 0, & \text{otherwise} \end{cases} \tag{4.6}$$

You can show that this construction indeed leads to a representation (it satisfies group homomorphism $\rho(g)\rho(g') = \rho(gg')$). The dimension of the induced representation of G is $N \times \dim \rho_H$. As a sanity check, if $H = \{e\}$, you should check that the induced representation is identical to the regular representation for finite groups. So the induced representation is in fact in general reducible.

4.2 An example - dihedral group D_n

Let us illustrate the construction of induced representations using the dihedral group D_n. Recall that the dihedral group D_n is generated by elements a and b where,

$$D_n = \left\{ a, b \,\middle|\, a^n = b^2 = e, \quad ab = ba^{n-1} \right\} \ . \tag{4.7}$$

Let's choose H to be the abelian subgroup $\mathbb{Z}_n$ generated by the elements a, so we have one-dimensional representations labelled by k with

$$v \overset{a}{\mapsto} e^{\frac{2\pi k i}{n}} v \ . \tag{4.8}$$

The next step is to look at the cosets $D_n/\mathbb{Z}_n$. In fact there are two distinct ones labelled by $i = 1, 2$ where we take $g_1 = e$ and $g_2 = b$ to be the representatives. We see that under the action of a in D_n the elements $v_1 = (e, v)$ and $v_2 = (b, v)$ will transform as

$$(v_1, v_2) \overset{a}{\mapsto} \left(e^{\frac{2\pi k i}{n}} v_1, e^{-\frac{2\pi k i}{n}} v_2 \right) \ , \tag{4.9}$$

Irrep	Case	Rep of $(a^r, a^r b)$
$R_{1,1}$	$k=0$	$(1,1)$
$R_{1,2}$	$k=0$	$(1,-1)$
$R_{2,k}$	$k=1,...,\frac{n-1}{2}$	$(A_k^r, A_k^r B)$

Table 4.1: Irreps of D_n constructed from induced reps method for odd n.

but note that since acting b changes the coset but the corresponding element $h = g_j^{-1} g g_i$ does not exist the representation only acts on the first coordinate, leaving us with[6]

$$(v_1, v_2) \overset{b}{\mapsto} (v_2, v_1) \ . \tag{4.10}$$

Let us write $\vec{v} = (v_1, v_2)$ to be the representation space $V^{\otimes 2}$. Then we have,

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \overset{a}{\mapsto} A_k \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} e^{\frac{2\pi k i}{n}} & 0 \\ 0 & e^{-\frac{2\pi k i}{n}} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \ , \tag{4.11}$$

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \overset{b}{\mapsto} B \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \ . \tag{4.12}$$

We see that the matrices $A_k^n = \mathbb{1}$, $B^2 = \mathbb{1}$ and $A_k B = B A_k^{n-1}$. This gives a two-dimensional representation of D_n for each k. Now we consider the following cases.

<u>n odd</u>

In this case we see that $k = 1, ..., \frac{n-1}{2}$ gives two inequivalent two-dimensional irreps. For $k = 0$, we have $A = \mathbb{1}_2$ so we can apply an orthogonal transformation to diagonalise B - giving two 1d irreps. This is listed in Table 4.1.

<u>n even</u>

Same as in the odd case, we see that $k = 1, ..., \frac{n-1}{2}$ gives two inequivalent two-dimensional irreps. For $k = 0$, we have $A = \mathbb{1}_2$ so we can apply an orthogonal transformation to diagonalise B - giving two 1d irreps. There is an additional case with $k = \frac{n}{2}$ where $A = -\mathbb{1}_2$ so we have two more 1d irreps. All of the irreps are listed in Table 4.2. Indeed, the number of representations match the number of conjugacy classes. We can even choose

$$O = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ i & i \end{pmatrix} \tag{4.13}$$

to get

$$RA_kR^{-1} = \begin{pmatrix} \cos\left(\frac{2\pi k}{n}\right) & -\sin\left(\frac{2\pi k}{n}\right) \\ \sin\left(\frac{2\pi k}{n}\right) & \cos\left(\frac{2\pi k}{n}\right) \end{pmatrix} , \quad RBR^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \ . \tag{4.14}$$

[6]If you need more steps then see $v_1 = (e, v) \overset{b}{\mapsto} (b, v) = v_2$ and vice versa.

Irrep	Case	Rep of $(a^r, a^r b)$
$R_{1,1}$	$k=0$	$(1,1)$
$R_{1,2}$	$k=0$	$(1,-1)$
$R_{1,3}$	$k=\frac{1}{2}$	$((-1)^r,(-1)^r)$
$R_{1,4}$	$k=\frac{1}{2}$	$((-1)^r,-(-1)^r)$
$R_{2,k}$	$k=1,...,\frac{n-1}{2}$	$(A_k^r, A_k^r B)$

Table 4.2: Irreps of D_n constructed from induced reps method for even n.

5 Wigner's Classification

In this section we review the classification of irreducible unitary positive energy representation $\mathcal{H}$ of the Poincaré group P^n. The construction is first discussed by Wigner [10] and we provide a quick summary here.

5.1 Notations

We set $V = \mathbb{R}^{1,n-1}$ be a vector space with linear coordinates $x^0, ..., x^{n-1}$ and the Lorentzian metric,

$$g = (dx^0)^2 - (dx^1)^2 - ... - (dx^{n-1})^2 \ . \tag{5.1}$$

We set $x^0 = ct$, where c is the speed of light. **Minkowski spacetime** M^n is the affine space which underlies V. The **lightcone** is the cone traced out by the vectors with zero norm known as **lightlike** vectors. Vectors inside and outside the lightcone are termed **timelike** and spacelike respectively and have positive and negative norms respectively. The group of isometries of V is the orthogonal group $O(1, n-1)$, which has four components distinguished by the determinant ± 1 and whether the forward lightcone is mapped to itself (or to the backward one). The identity component is known as the proper orthochronous Lorentz group $SO^+(1, n-1)$ and has a double cober $\mathrm{Spin}(1, n-1)$. The group of isometries of M^n includes the subgroup of translations T of V, and that quotienting by T is isomorphic to $O(1, n-1)$. The **Poincaré group** P^n is the double cover of the identity component of the group of isometries, fitting into the exact sequence,

$$0 \to T \to P^n \to \mathrm{Spin}(1, n-1) \to 0 \ . \tag{5.2}$$

We can alternatively write,

$$P^n = \mathrm{Spin}(1, n-1) \ltimes T \tag{5.3}$$

5.2 Wigner's Classification

Irreducible unitary representations of the Poincaré group are uniquely characterised by two parameters, mass and helicity, so we consider the two one-by-one.

Mass

First restrict the representation to the translation subgroup T. The restricted representation of $T \hookrightarrow P^n$ decomposes into a direct sum of irreducible unitary representations. Since T is abelian, these unitary irreps are one-dimensional and are specified by characters,

$$\chi : T \to U(\mathbb{C}) = S^1 \ . \tag{5.4}$$

We label these one-dimensional irreps by a dual four-vector $p^\mu \in T^*$ known as a **4-momentum**,

$$\chi_p(x) = e^{ip\cdot x}, \quad p \in T^* \tag{5.5}$$

The **mass** is defined as the magnitude of the four-momentum vector,

$$m^2 = p \cdot p \ . \tag{5.6}$$

In other words, the unitary irreps are defined by the elements of the Pontryagin dual of T. The action of P^n on the translation group T can be deduced via the Pontryagin isomorphism,

$$\chi_{p\cdot g} = \chi_p \circ g \tag{5.7}$$

which sends $p \mapsto p \cdot g$ of the same mass.

Spin and helicity

Now restrict the irrep $\mathcal{H}$ of P^n to a rep of V, which decomposes into a direct sum of one-dimensional representations indexed by 4-momenta.

$$\mathcal{H} = \int_{p\in W} V_p \ , \tag{5.8}$$

here W indexes the set of infinitesimal characters p permuted by the action of $SO^+(1, n{-}1)$ on V^*, following the discussion $p \mapsto p\circ g$. The representation $\mathcal{H}$ is irreducible, so p form an orbit of the action (of the 'Lorentz group'). Note that m^2 is constant on each orbit, and there are two orbit types - **massless** $m = 0$ and **massive** $m > 0$ representations. In two-dimensions the massless reps further break to **left-moving** and **right-moving** reps. This transitive action on W defines a connected groupoid with objects $p \in W$ and morphisms $p \mapsto p \cdot g$. The map $p \mapsto V_p$ then defines a linear representation of this groupoid and the action of P^n on irrep $\mathcal{H}$ can be determined with this groupoid representation.

The connectedness of the groupoid means the representation is determined by the restriction to any of the automorphism groups $\mathrm{Aut}(p)$ in the groupoid, so we have the identification of W as a homogeneous space,

$$W = P^n/\mathrm{Aut}(p) \ , \tag{5.9}$$

so then V acts trivially on V_p. Define the **little group**, L_p, as

$$0 \to V \to \mathrm{Aut}(p) \to L(p) \to 1 \tag{5.10}$$

such that

$$\mathrm{Aut}(p) = L_p \ltimes V \ . \tag{5.11}$$

The little group $L_p \subset \mathrm{Spin}(1, n-1)$ is the reductive part of the compact stabiliser subgroup of p. Since the representation $\mathcal{H}$ of P^n is obtained by constructing a homogeneous complex hermitian vector bundle over the orbit, which can be extended by the direct integral construction on the groupoid representation if the action of L_p on V_p is known. This is otherwise known as the method of induced representations. For $\mathcal{H}$ to be irreducible, the rep on L_p must also be an irrep. Therefore our problem reduces to finding irreps of the little group L_p in the two types or orbits.

The mathematical basis of the construction is the functional-equivalent construction for the induced representation method illustrated for finite groups. Here in particular we focus on unitary representations and therefore the representation space $\mathcal{H}$ is the space of integrable sections, a subspace of the total space of sections of the bundle $P \times_\rho V \to G/H$ where G is the Poincaré group P^n and H is the Lorentz group. The little group method ultilises the following construction. We denote U to be the unitary induced representation of P^n on the Hilbert space $\mathcal{H}$ of integrable sections and consider the restriction of U to the abelian subgroup T. Let $t = \exp a \cdot T$ with z_0 denoting the identity coset, then,

$$U_t \cdot \psi(z_0) = D(t) \cdot \psi(z_0) = \alpha(t) \cdot \psi(z_0) \ , \tag{5.12}$$

where $\alpha(t) = \exp a \cdot \alpha_*(T) = e^{ia \cdot p_0}$. Then,

$$\left(U_t \cdot \psi\right)(z) = e^{ia \cdot \Lambda(\sigma(z)^{-1}) \cdot p_0} \cdot \psi(z) \ , \tag{5.13}$$

with Λ being the adjoint representation of some element of the Lorentz group. Here σ indicates the section and z a point in the homogeneous space $\mathrm{Spin}(1, n-1)/L(\alpha)$, where $L(\alpha)$ is the little group. Define $p = \Lambda(\sigma(z)^{-1}) \cdot p_0$, and we have identified between the points $z \in \mathrm{Spin}(1, n-1)/L(\alpha)$ and the orbit of p_0 under the adjoint action of $\mathrm{Spin}(1, n-1)$. In particular since p_0 is the fixed point of this orbit the identification is one-to-one. Therefore, each representation of T occurring in U will be uniquely indexed by the points in $\mathrm{Spin}(1, n-1)/L(\alpha)$, $z \leftrightarrow p$. The Mackey direct integral decomposition of a representation gives a more functional-analytical way of looking at this identification. We break up, similar to the construction above, the Hilbert space of integrable sections as the direct integral,

$$\mathcal{H} = \int_{\mathrm{Spin}(1,n-1)/L(\alpha)} d\nu(p) \mathcal{H}_{(p)} \ , \tag{5.14}$$

where on each $\mathcal{H}_{(p)}$, U_t is $e^{ia\cdot p}\times 1$. This is the spectral theorem for a commuting family of self-adjoint operators - since $U(T)$ consists of dim(T)-families of commuting unitary operators, this decomposes $\mathcal{H}$ over the spectrum with each individual component shown to be in 1-1 correspondence with points of $\mathrm{Spin}(1, n-1)/L(\alpha)$. We then show that this representation is irreducible by proving that any bounded linear operator commuting with the representation U of P^n is by Schur's Lemma a multiple of the identity.

The result of the above mathematical discussion implies that one can consistently obtain the unitary irreducible representations of the Poincaré group using the little group method. The irreps will be characterised by the quadratic Casimirs of P^2 and W^2, the latter of which defined as,

$$W_\mu = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma} J^{\nu\rho} P^\sigma \tag{5.15}$$

known as the **Pauli-Lubański pseudovector**. This characterises the spin/helicity of the irrep.

We now separate the discussion into two distinct cases.

Massive Case $m > 0$

Fix the mass as m. We can take the basepoint $p = (m, 0, ..., 0)$ which gives the stabiliser subgroup (little group) as $\mathrm{Spin}(n-1)$. A massive particle then corresponds to an irrep of $\mathrm{Spin}(n-1)$ as in the non-relativistic case, with the Hilbert space being

$$\mathcal{H} = L^2(\mathbb{R}^{n-1}, r)\ , \tag{5.16}$$

where r is an irrep of $\mathrm{Spin}(n-1)$. The **spin** of a representation r of $\mathrm{Spin}(n-1)$ is defined as follows. Fix a 2-plane in $\mathbb{R}^{n-1}$, and consider the double cover $\mathrm{Spin}(2) \subset \mathrm{Spin}(n-1)$ of rotations in that plane which fixes the 2-plane. The irrep r decomposes into a sum of one-dimensional irreps and $\mathrm{Spin}(2)$ as by $\lambda \mapsto \lambda^{2j}$ with $\lambda \in \mathrm{Spin}(2)$ and j a half-integer. The spin is then defined as the largest $|j|$ that occurs in the decomposition. When $n = 4$, $\mathrm{Spin}(3) \cong SU(2)$ so $|j|$ simply labels the irreps of $SU(2)$.

Massless Case $m = 0$

We consider the basepoint $(1, 1, 0, ..., 0)$. The stabiliser subgroup in this case is the double cover of orientation-preserving isometries of an $(n-2)$-dimensional Euclidean space. The helicity λ of the irrep is the label j associated to the action of $\mathrm{Spin}(2) \subset \mathrm{Spin}(n-2)$.

6 Central Charges

In this section we review how the central charges arise as a central extension of the super-Poincaré group. We first provide a lightning review of the super-

Poincaré group before discussing central charges and BPS states. The main references for this section is [1,11,12]. See [13] for mathematical background.

6.1 The super-Poincaré group

Recall the Minkowski spacetime M^n is the affine space of the underlying vector space V of translations with Lorentzian metric, and that the Poincaré group P^n is a metric-preserving cover of the component of affine symmetries of M^n connected to the identity. To define a super-spacetime, we will fix a real spin representation S with dimensions s, which has the symmetric pairing[7],

$$\tilde{\Gamma} : S \otimes S \to V \ . \tag{6.1}$$

The related pairing[8] $\Gamma : S^* \otimes S^* \to V$ is used in the supersymmetry algebra. We choose $\mathrm{Im}(\Gamma), \mathrm{Im}(\tilde{\Gamma}) \subset \overline{C}$ where $C \subset V$ is the positive cone of timelike vectors, and then note that Γ and $\tilde{\Gamma}$ are non-degenerate. Choosing a basis of V and S as $\{P_\mu\}$ and $\{Q^a\}$ respectively, we have,

$$\Gamma(Q_a, Q_b) = \Gamma^\mu_{ab} P_\mu \ . \tag{6.2}$$

The Clifford relation between Γ and $\tilde{\Gamma}$ is expressed by,

$$\Gamma^\mu_{ab} \tilde{\Gamma}^{\nu bc} + \Gamma^\nu_{ab} \tilde{\Gamma}^{\mu bc} = 2 g^{\mu\nu} \delta^c_a \ . \tag{6.3}$$

We introduce the $\mathbb{Z}_2$-graded algebra,

$$\mathcal{L} = V \oplus S^* \ , \tag{6.4}$$

with V central and the nontrivial odd bracket,

$$\{Q_a, Q_b\} = -2\Gamma^\mu_{ab} P_\mu \ . \tag{6.5}$$

The underlying supermanifold (affine space) for the corresponding super Lie group is then the **super-Minkowski spacetime**,

$$M^{n|s} = M^n \times \Pi S^* \ , \tag{6.6}$$

where Π denotes the parity-reversed vector space with the even and odd summands reversed [9].

We typically pick the coordinate basis on V and ΠS^* as x^μ and θ^a respectively, giving the global coordinates on $M^{n|s}$. The action of the Lie algebra

[7]The existence of such a symmetric equivariant pairing for real spin representations is unique to the Minkowski signature, see Notes on Spinors in [13].

[8]The Γ is uniquely determined by $\tilde{\Gamma}$ in Lorentzian signature spin representations.

[9]This is consistent with the sign rule where we introduce a sign when odd elements are interchanged and treat all structures as even.

$\mathcal{L}$ on $M^{n|s}$ gives rise to left-invariant and right-invariant vector fields with basis $\{\partial_\mu, D_a\}$ and $\{\partial_\mu, \tau_{Q_a}\}$ respectively [10]. We write,

$$D_a = \frac{\partial}{\partial \theta^a} - \Gamma^\mu_{ab}\theta^b\partial_\mu \tag{6.7}$$

$$\tau_{Q_a} = \frac{\partial}{\partial \theta^a} + \Gamma^\mu_{ab}\theta^b\partial_\mu \tag{6.8}$$

with non-trivial brackets [11]

$$[D_a, D_b] = -2\Gamma^\mu_{ab}\partial_\mu \ , \tag{6.9}$$

$$[\tau_{Q_a}, \tau_{Q_b}] = 2\Gamma^\mu_{ab}\partial_\mu \ . \tag{6.10}$$

$$\tag{6.11}$$

Note that since right-invariant vector fields give rise to left actions [12], τ_{Q_a} generates an infinitesimla left action of $P^{n|s}$.

The **super Poincaré** algebra is the graded Lie algebra,

$$\mathfrak{p}^{n|s} = (V \oplus \mathfrak{so}(V)) \oplus S^* \ , \tag{6.12}$$

its even part just the usual Poincaré algebra. The **super Poincaré group** is defined as $P^{n|s} = \mathrm{Spin}(V) \ltimes \exp(\mathcal{L})$ or equivalently,

$$1 \to \exp(\mathcal{L}) \to P^{n|s} \to \mathrm{Spin}(V) \to 1 \ . \tag{6.13}$$

There are two concepts connected to $P^{n|s}$.

1. There may be a symmetric pairing

$$S^* \otimes S^* \to \mathbb{R}^c \ , \tag{6.14}$$

i.e. $\mathrm{Sym}^2 S^*$ contains copies of the trivial representation. This leads to the extension of a new-super Lie algebra by adding for any $c' \leq c$,

$$\tilde{\mathfrak{p}}^{n|s} = \left(V \oplus \mathfrak{so}(V) \oplus \mathbb{R}^{c'}\right) \oplus S^* \ . \tag{6.15}$$

Since we can write

$$1 \to \mathbb{R}^{c'} \to \tilde{\mathfrak{p}}^{n|s} \to \mathfrak{p}^{n|s} \to 1 \ , \tag{6.16}$$

[10] ∂_μ in both as V is central.

[11] The brackets of the left-invariant D_a are given by the ones as in $\mathcal{L}$ but the right-invariant ones have a sign difference.

[12] This is because when the one-parameter subgroup generated by a vector field v acts on the left, say $g \mapsto e^{tv} \cdot g$, after differentiation g acts on the right of the vector field as

$$\left.\frac{dg'}{dt}\right|_{t=0} = ve^{tv}g = v \cdot g$$

so the vector field generated by v gives vg at $g \in G$. This means that the vector field is in fact right-invariant.

this is an extension and in particular $\mathbb{R}^{c'}$ is abelian with its image being in the centre of $\tilde{\mathfrak{p}}^{n|s}$ so such construction is known as a **central extension**. The generators of $\mathbb{R}^{c'}$ are the **central charges** [13].

2. There may exists outer automorphisms of $\mathfrak{p}^{n|s}$ which fix the Poincaré algebra (so they transform the fermionic generators). These are **infinitesimal R-symmetries** and the connected group via exponentiation is the R-symmetry group, which is compact [14].

6.2 Central Charges and BPS state

We have shown that the extended SUSY algebras can be extend by adding a central charge.

$$\{Q^a_\alpha, Q^\dagger_{\dot{\alpha}b}\} = 2\sigma^\mu_{\alpha\dot{\alpha}} P_\mu \delta^a_b \ , \tag{6.17}$$

$$\{Q^a_\alpha, Q^b_\beta\} = 2\sqrt{2}\epsilon_{\alpha\beta} Z^{ab} \ , \tag{6.18}$$

$$\{Q^\dagger_{\dot{\alpha}a}, Q^\dagger_{\dot{\beta}b}\} = 2\sqrt{2}\epsilon_{\dot{\alpha}\dot{\beta}} Z^*_{ab} \ , \tag{6.19}$$

with $\epsilon = i\sigma^2$ as before. The central charge matrix Z^{ab} is antisymmetric in a and b and can be skew-diagonalised to $\mathcal{N}/2$ eigenvalues. For the case $\mathcal{N} = 2$, we have only one Z. It is possible to define the operators,

$$A_\alpha = \frac{1}{2}\left[Q^1_\alpha + \epsilon_{\alpha\beta}\left(Q^2_\beta\right)^\dagger\right] \ , \tag{6.20}$$

$$B_\alpha = \frac{1}{2}\left[Q^1_\alpha - \epsilon_{\alpha\beta}\left(Q^2_\beta\right)^\dagger\right] \ , \tag{6.21}$$

which gives the algebra with non-trivial commutators

$$\{A_\alpha, A^\dagger_\beta\} = \delta_{\alpha\beta}(M + \sqrt{2}Z) \ , \tag{6.22}$$

$$\{B_\alpha, B^\dagger_\beta\} = \delta_{\alpha\beta}(M - \sqrt{2}Z) \ ,. \tag{6.23}$$

Let us impose the condition that the irreducible representations of the algebra must be unitary with semi-definite positive norm, which requires the unit norm state $|M, Z\rangle$ labelled by mass M and central charge Z,

$$||B^\dagger_\alpha |M, Z\rangle || \geq 0 \Longrightarrow M \geq \sqrt{2}Z \ . \tag{6.24}$$

This is a very strong constraint with two immediate consquences.

- For massless states $Z = 0$.

[13] These already arise in classical field theories due to the symplectic structure. Namely if $\mathfrak{p}^{n|s}$ is the Lie algebra of symmetries of some theory then the Lie algebra of observables is in general a central extension.

[14] The infinitesimal R-symmetries act in a quantum theory as automorphisms of the symmetry algebra and are represented projectively on the Hilbert space of the theory.

- For massive states that satisfies exactly the equality $M = \sqrt{2}Z$, the state is annihilated by half of the supercharges. The multiplet of the state is reduced to a much smaller one - this is known as the **short multiplet** where its normal $M > \sqrt{2}Z$ counterpart is known as the **long multiplet**. The construction of the short multiplet follows the normal procedure of generating supersymmetric multiplets, in particular the $\mathcal{N} = 2$ massless multiplet now reduces to the $\mathcal{N} = 1$ massive multiplets generated from the same Clifford vacuum[15].

In particular the massive states where $M = \sqrt{2}Z$ are known as BPS states. These are the states with exactly half of the supersymmetry of the system, and are interestingly related to non-perturbative effects of the system.

6.3 Multiplet content as R-symmetry representations

Since R-symmetry is an internal symmetry that effectively rotates the fermionic charges into one-another, the massless and massive supermultiplets of extended supersymmetry have states labelled by R-symmetry. In particular due to the oddness of the generators $(Q^I_{\dot{\alpha}})^\dagger$ the states in a multiplet can be represented as irreducible antisymmetric representations of the R-symmetry.

To illustrate this let us look at the $\mathcal{N} = 4$ massless supermultiplet generated from the Clifford vacuum $|\Omega_{-1}\rangle$ as an example. The states generated by the operators [16]

$$\left(a^I_1\right)^\dagger = \frac{1}{\sqrt{4E}} \left(Q^I_{\dot{1}}\right)^\dagger \tag{6.25}$$

will be the multiplet as represented schematically in Table 6.1. For example,

state (schematic)	helicity	**R**	tensor
$\|\Omega_{-1}\rangle$	-1	$\mathbf{1}$	T
$a^\dagger \|\Omega_{-1}\rangle$	$-\frac{1}{2}$	$\mathbf{4}$	T_I
$a^\dagger a^\dagger \|\Omega_{-1}\rangle$	0	$\mathbf{6}$	T_{IJ}
$a^\dagger a^\dagger a^\dagger \|\Omega_{-1}\rangle$	$\frac{1}{2}$	$\overline{\mathbf{4}}$	T^I
$a^\dagger a^\dagger a^\dagger a^\dagger \|\Omega_{-1}\rangle$	1	$\mathbf{1}$	T

Table 6.1: A list of the states, their helicities, together with their irrep in the R-symmetry group $SU(4)$ for the $\mathcal{N} = 4$ massless supermultiplet. Here I have represented the state using tensor methods in representation theory. The states are schematically illustrated and do not actually refer to the actual form of the state.

we can write the states of the form $a^\dagger |\Omega_{-1}\rangle$ in the fundamental of $SU(4)$,

[15] For example, the massless $\mathcal{N} = 2$ hypermultiplet and vector multiplet is mapped to the massive $\mathcal{N} = 1$ chiral and vector multiplet respectively.

[16] Here I have used the notation of the lectures notes where I go to the frame $p^\mu = (E, 0, 0, E)$, so the non-trivial operator relation is $\{Q^I_1, Q^{J\dagger}_{\dot{1}}\} = 4E\delta^{IJ}$.

the R-symmetry group, as a vector,

$$a^\dagger |\Omega_{-1}\rangle = \begin{pmatrix} (a_1^1)^\dagger |\Omega_{-1}\rangle \\ (a_1^2)^\dagger |\Omega_{-1}\rangle \\ (a_1^3)^\dagger |\Omega_{-1}\rangle \\ (a_1^4)^\dagger |\Omega_{-1}\rangle \end{pmatrix} \tag{6.26}$$

with the subscript labelling the extended SUSY label. The state then transforms under R-symmetry in the fundamental $a^\dagger |\Omega_{-1}\rangle \to U a^\dagger |\Omega_{-1}\rangle$ where $U \in SU(4)$. This is represented by the Young table:

$$\begin{array}{|c|}\hline \\ \hline \end{array}$$

The second level is generated by two anticommuting operators and therefore must furnish the antisymmetric representation represented by the Young diagram

$$a^\dagger |\Omega_{-1}\rangle \longleftrightarrow \begin{array}{|c|}\hline \\ \hline \\ \hline \end{array} \tag{6.27}$$

so hence the irrep **6**. This generalises to multiplets in other extended supersymmetric theories.

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