Supersymmetry Class 1

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Abstract

The main topic for this sheet is spinor algebras - the whole point is to develop some skills and techniques to deal with spinors in 4d. This builds the mathematical foundation to understand QFT and SUSY concepts. This note is a summary of the topics covered in the class as well as some additional topics that I did not have time to cover.

Contents

I will begin with a summary of the class in §1. The problem sheet feedback is in §2.

1 Class 1 Summary

In this first class I would like to address two main questions.

1. What is SUSY?
2. Why should we do SUSY?

Let me highlight that the ideas I am going to present aren't new, but it is something that you should have in the back of your mind as you proceed through the course.

1.1 What is SUSY?

Supersymmetry (SUSY) is the symmetry with operator $\mathcal{Q}$ such that there is a splitting in the Hilbert space $\mathcal{H}$ as,

$$\mathcal{H} = \mathcal{H}_B \oplus \mathcal{H}_F \ , \tag{1.1}$$

where $\mathcal{H}_B$ and $\mathcal{H}_F$ indicates the Hilbert space with an even and odd number of fermionic excitations respectively. The operator $\mathcal{Q}$,

$$\mathcal{Q} : \mathcal{H}_{B,F} \to \mathcal{H}_{F,B} \tag{1.2}$$

with the following two properties:

$$\mathcal{Q}^2 = 0 \ , \tag{1.3}$$

$$\{\mathcal{Q}, \mathcal{Q}^\dagger\} = 2H \ . \tag{1.4}$$

Here H is the Hamiltonian of the theory. There are immediately two consequences of having this symmetry generated by $\mathcal{Q}$.

1. $[H, \mathcal{Q}] = 0$. This $\mathcal{Q}$ actually commutes with the Hamiltonian so it is a symmetry.
2. $\langle\Psi| H |\Psi\rangle \geq 0$ for any $|\psi\rangle \in \mathcal{H}$, which is an equality if and only if $\mathcal{Q}|\psi\rangle = 0 = \mathcal{Q}^\dagger|\psi\rangle$. This means that for a supersymmetric vacua $E_0 = 0$ [1].

There are a lot of ways to extend this structure. In particular this course focuses on discussing the theory and consequences of imposing $\mathcal{N} = 1$ supersymmetry in four-dimensions. The SUSY algebra in 4d is enhanced to the form,

$$\{\mathcal{Q}_\alpha, \mathcal{Q}^\dagger_{\dot{\alpha}}\} = 2\sigma^m_{\alpha\dot{\alpha}} P_m \ . \tag{1.5}$$

We can even have extended SUSY.

$$\{\mathcal{Q}^a_\alpha, \mathcal{Q}^{\dagger b}_{\dot{\alpha}}\} = 2\delta^{ab} P_{\alpha\dot{\alpha}} \ , \tag{1.6}$$

$$\{\mathcal{Q}^a_\alpha, \mathcal{Q}^{\beta b}\} = \epsilon_{\alpha\beta} Z^{ab} \ . \tag{1.7}$$

Formally, SUSY is formulated as odd endomorphisms of $\mathcal{H}$ with a $\mathbb{Z}_2$-grading. We are not going to pursue this formal aspect in this course.

1.2 Why SUSY?

Personally I think you should study SUSY because of 3 reasons.

1. SUSY allows us to understand QFT better.
2. SUSY leads to some amazing connections with geometry, algebra and topology.
3. SUSY builds the foundation of the most well-studied quantum gravity theory - string theory.

In this class I have tried to illustrate the first two reasons. Supersymmetry is traditionally introduced to solve the hierarchy problem and is used extensively in early phenomenological extensions of the Standard Model - you can find a discussion of the hierarchy problem in 3.

I then discussed the Wess-Zumino model. In particular, I wanted to highlight that SUSY is not just a bunch of algebraic nonsense but it is best studied in theories. The free Wess-Zumino model (§4) is the simplest four-dimensional case where supersymmetry can be manifestly analysed and this will illustrate how supersymmetry is studied in its early days of formulation.

Finally I tried to illustrate why SUSY can help us understand QFT more. In particular, you will often hear that supersymmetry will give you more control - the basic principle I wanted to quickly highlight is known as *localisation* and this is already seen in zero-dimensional supersymmetric field theories. This also illustrates how supersymmetry is linked to topological index theorems - in fact it is this observation that led to the study of four-manifolds which is now one of the most active fields of research in geometry and topology. You will find a quick summary of localisation in §5.

[1] This point, as we will see, will become important in SUSY-breaking.

There are many good resources for studying supersymmetry. In particular, I have based the discussion of the hierarchy problem on [1] and [2]. The component-field approach that I have used when discussing the Wess-Zumino model is mainly based on Martin's notes [2]. Wess and Bagger [3] is more of a book-keeping reference book for all the formulae you might want to look up, but I will provide more references as the course continues. You can find a good discussion of localisation in David Skinner's SUSY notes and in the freely-available Clay Maths monograph 'Mirror Symmetry' [4]. It is indeed a bit advanced, but I would recommend you come back to this at some point in your career.

Hopefully this is interesting and - do let me know if you have more questions.

2 Problem Sheet 1 Feedback

2.1 Question 1 - Poincaré symmetry

The main goal of this question is to find out how operators transform under the Poincaré group. There are some points to note:

(a) Some of you missed the fact that $M_{\mu\nu}$ is antisymmetric. Of course this just comes from the Lorentz algebra [2], but you should write,

$$\hat{M}_{\mu\nu} = -i(x_\mu \partial_\nu - x_\nu \partial_\mu) \ , \tag{2.1}$$

and not,

$$\hat{M}_{\mu\nu} = -2i(x_\mu \partial_\nu) \ . \tag{2.2}$$

(b) We want to check

$$U(\Lambda_1, a_1)U(\Lambda_2, a_2) = U(\Lambda_3, a_3) \ , \tag{2.3}$$

where

$$\Lambda_3 = \Lambda_1 \Lambda_2 \ , \quad a_3 = \Lambda_1 a_2 + a_1 \ . \tag{2.4}$$

The main point is that you need to make sure that the argument of the operator, i.e. $\mathcal{O}(x)$, also transforms appropriately, in particular noting that,

$$U(\Lambda_3, a_3)^{-1} \mathcal{O}(x)^{\mathcal{A}} U(\Lambda_3, a_3) = L(\Lambda_3)^{\mathcal{A}}_{\mathcal{B}} \mathcal{O}^{\mathcal{B}}(\Lambda_3^{-1} x - \Lambda_3^{-1} a_3) \ , \tag{2.5}$$

you should check that the argument of $\mathcal{O}^{\mathcal{B}}$ matches with the one obtained by carrying out two transformations U_1 and U_2 back-to-back.

(c) This part is well done. The main thing to note here is the extra term obtained,

$$[M_{\mu\nu}, \mathcal{O}^{\mathcal{A}}(x)]_{\text{new term}} = (-\mathcal{S}_{\mu\nu})^{\mathcal{A}}{}_{\mathcal{B}}\, \mathcal{O}^{\mathcal{B}}(x) \ , \tag{2.6}$$

where $\mathcal{S}_{\mu\nu}$ is a representation of the Lorentz algebra. The term exists because the field is now in a different representation space of the Poincaré group. You can think of the scalar field infinitesimla transformation as giving the relation between the field at $\Lambda^{-1}x$ versus the field at x. But since the field is now in a nontrivial representation there must be an extra part coming from 'representation space-contribution'.

2.2 Question 2 - Clifford algebra and Lorentz generators

This is just algebra. I don't really have much to say about this. If you struggle then opening any kindergarten QFT manual should save you.

[2]See Andre's course on Groups and Representations.

2.3 Question 3 - The homomorphism $SL(2,\mathbb{C}) \to SO(1,3)$

This question aims to build the homomorphism

$$\Psi : SL(2,\mathbb{C}) \to SO(1,3) \; . \tag{2.7}$$

This is sometimes known as the **spinor map**. Note that this is a two-to-one map as we will later find out that $\ker \Psi = \{\pm \mathbb{1}_2\}$. The image of the map is the identity component of $SO_{\mathbb{R}}(1,3)$, denoted typically as $SO^+_{1,3}(\mathbb{R})$. We then have,

$$PSL_2\mathbb{C} = \frac{SL_2\mathbb{C}}{\mathbb{Z}_2} \cong SO^+_{1,3}(\mathbb{R}) \; , \tag{2.8}$$

where we write $SL(2,\mathbb{C})$ as $SL_2\mathbb{C}$ and $SO(1,3)$ as $SO_{1,3}(\mathbb{R})$. Of course, since Ψ is a smooth map and that $SL(2,\mathbb{C})$ is simply-disconnected, the map will always land on the identity component of $SO(1,3)$, i.e. $SO^+(1,3)$ or $\Lambda^\uparrow_+$ [3]. This is why there seems to be an ambiguity when I define the map Ψ.

Anyways, moving on to the question. Some points to note.

- Parts (a) and (b) are generally well done. I also discussed these briefly in the class.
- Part (c) is all about spinor algebras. In particular we want to show,

$$\Lambda(A_1)^\mu{}_\nu \Lambda(A_w)^\nu{}_\rho = \Lambda(A_1 A_2)^\mu{}_\rho \; , \tag{2.9}$$

$$\Lambda(A)^\mu{}_\nu \Lambda(A)^\rho{}_\sigma \eta_{\mu\rho} = \eta_{\nu\sigma} \; . \tag{2.10}$$

  You should prove that,

$$\operatorname{Tr}\left(A_1^\dagger \bar{\sigma}^\mu A_1 \sigma_\nu\right) \operatorname{Tr}\left(A_2^\dagger \bar{\sigma}^\nu A_2 \sigma_\rho\right) \stackrel{!}{=} -2 \operatorname{Tr}\left(A_2^\dagger A_1^\dagger \bar{\sigma}^\mu A_1 A_2 \sigma_\rho\right) \tag{2.11}$$

  for the first result and similarly for the second. Don't try and cheat your way through - it's good practice.

- Part (d) is covered in the class. The main problem I have for a lot of the your answers is that most of you stated that $A = \pm\mathbb{1}_2$ implies $\Lambda^\mu{}_\nu = \eta^\mu{}_\nu$ which is saying that $\mathbb{Z}_2 \subset \ker\Psi$ but not the other way around. You need to show that $\pm\mathbb{1}_2$ are the only solutions to the kernel to complete the full argument by, for example, using the method mentioned in the class (i.e. set $A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ and imposing conditions).

- Parts (e) and (f) involve deriving infinitesimla versions of the map Ψ, i.e. the Lie algebra homomorphism,

$$\tilde{\Psi} : \mathfrak{sl}_2\mathbb{C} \to \mathfrak{so}_{1,3}\mathbb{C} \; . \tag{2.12}$$

  The key point is to first derive the infinitesimal version of $\Lambda(A)^\mu{}_\nu$,

$$\lambda^\mu{}_\nu = -\frac{1}{2}\operatorname{Tr}\left(\delta A^\dagger \bar{\sigma}^\mu \sigma_\nu\right) - \frac{1}{2}\operatorname{Tr}\left(\bar{\sigma}^\mu \delta A \sigma_\nu\right) \tag{2.13}$$

  and note that this is antisymmetric [4]. Now try and prove,

$$\operatorname{Tr}\left(\delta A \sigma^{\mu\nu}\right)^* = -\operatorname{Tr}\left(\delta A^\dagger \bar{\sigma}^{\mu\nu}\right) \; , \tag{2.14}$$

  which results in

$$\lambda_{\mu\nu} = 2 \operatorname{Re} \operatorname{Tr}\left(\delta A \sigma_{\mu\nu}\right) \; . \tag{2.15}$$

  The reverse of the map can be constructed by writing,

$$\delta A = y^{\mu\nu} \sigma_{\mu\nu} \; , \tag{2.16}$$

  and try to evaluate $\operatorname{Tr}\left(\sigma^{\mu\nu}\sigma^{\rho\sigma}\right)$ to get $\lambda_{\mu\nu} = -2y_{\mu\nu}$.

[3] This notation is typically used to indicate the proper orthochronous Lorentz group.

[4] You should check this as an exercise.

We actually haven't computed the reverse map of Ψ. This is in fact,

$$A = e^{i\phi} \frac{\sigma_\mu \Lambda^\mu{}_\nu \bar{\sigma}^\nu}{2\sqrt{\Lambda^\mu{}_\mu}} \tag{2.17}$$

with $\operatorname{tr} A = e^{i\phi} |\operatorname{tr} A|$. The phase $e^{i\phi}$ can be determined up to ± 1 by imposing $\det A = 1$ [5].

2.4 Question 4 - Spinor algebra

This question is the most important one this sheet. You shoudl be really comfortable with the spinor algebra manipulations. The important points are the following:

1. Undotted sum goes downwards from left to right, $\psi^\alpha \chi_\alpha$.
2. Dotted sum goes upwards from left to right, $\bar{\psi}_{\dot{\alpha}} \bar{\chi}^{\dot{\alpha}}$.
3. $\epsilon^{\alpha\beta} = \epsilon^{-\beta\alpha} = \epsilon_{\beta\alpha}$.
4. $\left(\sigma^\mu_{\alpha\dot{\alpha}}\right)^T = \sigma^\mu_{\dot{\alpha}\alpha}$.
5. $(\bar{\sigma}^\mu)^{\dot{\alpha}\beta} = \epsilon^{\dot{\alpha}\dot{\gamma}} \epsilon^{\beta\delta} (\sigma^\mu)_{\delta\dot{\gamma}}$.
6. $(\sigma^\mu)_{\alpha\dot{\beta}} (\bar{\sigma}_\mu)^{\dot{\gamma}\delta} = -2\delta^\delta_\alpha \delta^{\dot{\gamma}}_{\dot{\beta}}$.

You should prove all of this, and then evaluate the identities in the question again (see [5] for some identities). I've otherwise explained the other subtleties in the class.

3 The Hierarchy Problem

The hierarchy problem is one of the biggest physics problems that the Standard Model cannot resolve.

3.1 Chiral Perturbation Theory

Let's go off the discussion a bit and first illustrate some basic physics principles using the chiral perturbation theory. Recall that pions have very similar masses $m_0 = 135\text{GeV}$ and $m_\pm = 140\text{GeV}$. We can write the pions as the adjoint representation of some chiral symmetry $SU(2)$ where $\Pi = \exp \sum_i \frac{\pi_i \sigma_i}{f_\pi}$ and under $SU(2)$, $\Pi \mapsto U \Pi U$. The $SU(2)$-invariant mass term is then,

$$\mathcal{L}_{\text{mass}} = \frac{1}{2} m_\pi^2 f_\pi^2 \operatorname{tr} \Pi \mapsto \frac{1}{2} m_\pi^2 \left(\pi_0^2 + \pi_1^2 + \pi_2^2\right) + ... \tag{3.1}$$

Note that the mass parameter m_π is the spurion [6], and since it respects $SU(2)$ symmetry all quantum corrections should also respect $SU(2)$ symmetry. However the kinetic term has a charge lepton part that breaks the $SU(2)$ and shift symmetry,

$$\mathcal{L}_{\text{kin}} = \frac{1}{2} (\partial_\mu \pi_0)^2 + |(\partial_\mu + ieA_\mu)\pi_+|^2 \,, \tag{3.2}$$

so treating this as an effetive field theory there is nothing that forbids a quantum correction of the form,

$$\delta \mathcal{L}_{\text{mass}} \sim \frac{e^2}{(4\pi)^2} \Lambda^2 \pi_+ \pi_- \,. \tag{3.3}$$

[5] For the details of this construction see Hugh Osborne's Group Theory notes, §4.3.

[6] A spurion is a field that breaks a particular symmetry. The effects of breaking that symmetry will be accompanied by this spurion, that breaks a shift symmetry acting on the pions. This means that the size of the spurion should not receive any large corrections in perturbation theory - otherwise sending the spurion $\tilde{c} \to 0$ will not restore the symmetry. This means that it is technically natural for the spurion to remain small.

Here we see that if we have $\Lambda \gtrsim 750\text{GeV}$ we have a problem - the corrections will be greater thatn the observed mass splitting. The resolution is that there must be new physics popping up at the scale $E \sim 750\text{MeV}$ to tame the corrections, in fact of the form,

$$m_{\pi_\pm}^2 - m_{\pi_0}^2 \approx \frac{3e^2}{(4\pi)^2} \frac{m_\rho^2 m_{a_1}^2}{m_\rho^2 + m_{a_1}^2} \log\left(\frac{m_{a_1}^2}{m_\rho^2}\right) , \tag{3.4}$$

where ρ and a_1 are the lightest vector and axial vector resonances. Therefore - a spurion parameter that breaks symmetries lead to a hierarchy argument - it gives a scale where new physics must pop out.

Imagine if QCD is not a thing. Then we could possbily add an additioanal parameter to the action,

$$\delta\mathcal{L}_{\text{tune}} \sim \delta_m^2 \pi_+ \pi_- , \tag{3.5}$$

to fine-tune this against all the other corrections to keep the sum small - this is often called **fine-tuning** and obviously is a very unnatural way of fixing the problem.

3.2 The Higgs

The Higgs has a similar problem to the charged pions. In the Standard Model the Higgs field H is a compelx scalar with a scalar potential of form,

$$V = m_H^2 |H|^2 + \lambda |H|^4 , \tag{3.6}$$

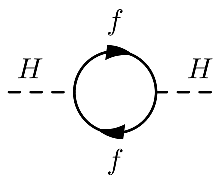
which gives the vaccum expectation value of

$$\langle H \rangle = \sqrt{-\frac{m_H^2}{2\lambda}} . \tag{3.7}$$

The problem however is that m_H^2 receives huge quantum corrections from virtual effects of all particle phenomenology that couples directly or indirectly to H, namely the term

$$\mathcal{L} \supset -\lambda_f H \bar{f} f . \tag{3.8}$$

Let us in particular calculate the one-loop contribution to m_H^2. The one-loop fermionic contribution is given by the following diagram,

$$\delta m_H^2 =$$


$$+ \dots \tag{3.9}$$

which is given by

$$-i\delta m_H^2|_{\text{top}} = (-1)N_c \int \frac{d^4k}{(2\pi)^4} \text{tr}\left[\frac{-iy_t}{\sqrt{2}} \frac{i}{\not{k} - m_t} \left(\frac{-iy_t^*}{\sqrt{2}} \frac{i}{\not{k} - m_t}\right)\right] . \tag{3.10}$$

If we use the hard-momentum cut-off, this term will evaluate to become,

$$\delta m_H^2|_{\text{top}} = -\frac{N_c|\lambda_t|^2}{8\pi^2} \left[\Lambda^2 - 3m_t^2 \log\left(\frac{\Lambda^2 + m_t^2}{m_t^2}\right) + \dots\right] \tag{3.11}$$

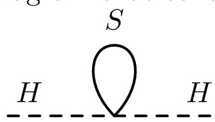
The spurion argument in the previous section gives us the following reasoning - It is not natural to assume that the corrections to the spurion m_H^2 can be large in perturbation theory. However, for the top quark $\lambda_t \sim 0.94$ and if Λ, which we have assumed to be the UV scale where new physics enters, is the Planck scale, this quantum correction to m_H^2 will be some 30 orders of magnitude larger than the original scalar boson mass-squared $m_H^2 \approx -(92.9\text{GeV})^2$. This does not make sense - it feels like the parameter that breaks the scaling-symmetry of the Higgs field now acquires a huge quantum correction which contradicts the argument we had given earlier!

There are perhaps a few ways out of this mess.

1. Pick Λ_{UV} not too large. Okay, but then we still need to make up some new physics at Λ_{UV} that both alters the propagators in the loop and cuts off the loop integral. This is hard to do with Lagrangians with no more than two derivatives.
2. Choosing a different regulator? This does not work - the quadratic sensitivity to high mass scales of the scalar boson is the reason why we have quadratic divergences. It is possible to get rid of quadratic divergences by choosing a different regulator but the hierarchy problem will still remain - there will be virtual effects (of any heavy particles) which leads to the same result.

The hierarchy problem still persists in the following two cases.

1. There exists a heavy complex scalar particle S with mass m_S coupling to Higgs with a Lagrangian term $-\lambda_S |H|^2 |S|^2$. Then the Feynman diagram that contributes will be

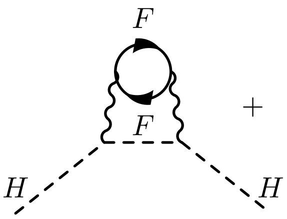
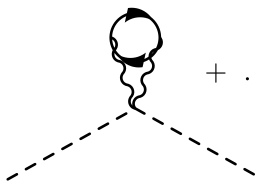
$\delta m_H^2 \;=$  $+ \; \dots$ (3.12)

which gives the contribution,

$$\delta m_H^2 = \frac{\Lambda_S}{16\pi^2}\left[\Lambda^2 - 2m_S^2 \log\left(\frac{\Lambda}{m_S}\right) + \dots\right] . \tag{3.13}$$

This correction is sensitive to the masses of the heaviest particles that H couples to. The first term can be removed by choosing dimensional regularisation instead but the m_S^2 piece will still remain no matter what.

2. The existence of a heavy fermion F that couples to the Higgs-squared mass parameter through gauge interactions. There there will be two-loop contributions similar to the form in Eq. (3.13) [2]. The diagrams are,

$\delta m_H^2 \;=$  $+$  $+ \; \dots$ (3.14)

giving us,

$$\delta m_H^2 = C_H T_F \left(\frac{g^2}{16\pi^2}\right)^2 \left[a\Lambda^2 + 24 m_F^2 \log\left(\frac{\Lambda}{m_F}\right) + \dots\right] . \tag{3.15}$$

In short, if we want to treat the Higgs boson as a fundamental particle (which we have now verified in the CERN), we must either make the assumption that none of the high-mass particles or condensates couple to H or that there must be some cancellation that kills various contributions to δM_H^2.

3.3 SUSY - a solution

This is where SUSY comes in. Suppose we have N new scalar particles ϕ_L and ϕ_R, which gives the Lagrangian,

$$\mathcal{L}_{\text{scalar}} = -\frac{\lambda}{2} H^2 \left(|\phi_L|^2 + |\phi_R|^2\right) - H\left(\mu_L |\phi_L|^2 + \mu_R |\phi_R|^2\right) - m_L^2 |\phi_L|^2 - m_R^2 |\phi_R|^2 , \tag{3.16}$$

then when $N = N_c$, $\lambda = |\lambda_t|^2$ the quadratic divergences from the fermionic loop in Eq. (3.11) will be cancelled and setting $m_t = m_L = m_R$ and $\mu_L^2 = \mu_R^2 = 2\lambda m_t^2$ then also cancels out the logarithmic divergences [1]. The key is that supersymmetry seems to give a way out of this and it is a very elegant solution.

4 The Free Wess-Zumino Model

In this section I would like to briefly discuss the free Wess-Zumino model. I think it is pretty unfair that you don't get to see any Lagrangians until the latter part of the course so this is really just a tiny teaser of what it is to come, but in fact from this really simple example we will already see some patterns emerging.

Let us start by asking the following question. Is it possible to write down the simplest four-dimensional model with supersymmetry? The simplest Lagrangian we can write down of course is something like,

$$S = \int d^4x \left(-\partial^\mu \phi^* \partial_\mu \phi + i\psi^\dagger \bar{\sigma}^\mu \partial_\mu \psi \right) , \tag{4.1}$$

where we clearly have two parts to the Lagrangian - the scalar part is a bosonic complex scalar field,

$$\mathcal{L}_B = -\partial^\mu \phi^* \partial_\mu \phi , \tag{4.2}$$

whilst the simplest fermionic part is Weyl fermion,

$$\mathcal{L}_F = i\psi^\dagger \bar{\sigma}^\mu \partial_\mu \psi . \tag{4.3}$$

Now we want to impose supersymmetry, i.e. we want the system to have a charge that acts transforms between the two Lagrangians (c.f. Eq.(1.2)). Then we must have

$$\delta\phi = \epsilon\psi \tag{4.4}$$

$$\delta\phi^* = \epsilon\psi^\dagger , \tag{4.5}$$

which gives,

$$\delta\mathcal{L}_B = -\epsilon\partial^\mu\psi\partial_\mu\phi^* - \epsilon^\dagger\partial^\mu\psi^\dagger\partial_\mu\phi . \tag{4.6}$$

We want the transformation of the fermionic Lagrangian to be up to a total derivative. To do this we set,

$$\delta\psi_\alpha = -i\left(\sigma^\mu\epsilon^\dagger\right)_\alpha \partial_\mu\phi , \tag{4.7}$$

$$\delta\psi^\dagger_{\dot{\alpha}} = -i\left(\epsilon\sigma^\mu\right)_{\dot{\alpha}} \partial_\mu\phi^* , \tag{4.8}$$

giving,

$$\delta\mathcal{L}_F = -\epsilon\bar{\sigma}^\mu\sigma^\nu\partial_\nu\psi\partial_\mu\phi^* + \psi^\dagger\bar{\sigma}^\nu\sigma^\mu\epsilon^\dagger\partial_\mu\partial_\nu\phi , \qquad = -\delta\mathcal{L}_B + \partial_\mu(\dots) . \tag{4.9}$$

That looks good. Is this a supersymmetry though? In particular, we will need to check whether the supersymmetry algebra closes. Specifically we to satisfy the algebra in Eq. (1.5), so we must have,

$$\left(\delta_{\epsilon_2}\delta_{\epsilon_1} - \delta_{\epsilon_1}\delta_{\epsilon_2}\right) X = -i\left(\epsilon_1\sigma^\mu\epsilon_2^\dagger - \epsilon_2\sigma^\mu\epsilon_1^\dagger\right)\partial_\mu X . \tag{4.10}$$

We can check this for $X = \phi, \phi^*, \psi, \psi^\dagger$, but we see that,

$$\left(\delta_{\epsilon_2}\delta_{\epsilon_1} - \delta_{\epsilon_1}\delta_{\epsilon_2}\right)\psi_\alpha = -i\left(\epsilon_1\sigma^\mu\epsilon_2^\dagger - \epsilon_2\sigma^\mu\epsilon_1^\dagger\right)\partial_\mu\psi_\alpha + i\left(\epsilon_{1\alpha}\epsilon_2^\dagger\bar{\sigma}^\mu\partial_\mu\psi - \epsilon_{2\alpha}\epsilon_1^\dagger\bar{\sigma}^\mu\partial_\mu\psi\right) . \tag{4.11}$$

The last term vanishes on-shell, i.e. when the equation of motion,

$$\bar{\sigma}^\mu\partial_\mu\psi = 0 , \tag{4.12}$$

is satisfied, the last term vanishes. However, what happens when we go off-shell? Quantum mechanically we would like to cover these cases too [7]. We use a trick called auxiliary fields where we add in the term to the Lagrangian,

$$\mathcal{L}_{\text{aux}} = F^*F . \tag{4.13}$$

[7] Since ψ_α can be virtual!

Now $F = 0$ holds on-shell. We now demand that,

$$\delta F = -i\epsilon^\dagger \bar{\sigma}^\mu \partial_\mu \psi \ , \tag{4.14}$$

$$\delta F^* = -i\partial_\mu \psi^\dagger \bar{\sigma}^\mu \epsilon \ , \tag{4.15}$$

then the SUSY algebra closes off-shell, where Eq. (4.10) is satisfied for $X = \phi, \phi^*, \psi, \psi^\dagger, F, F^*$. This works because there was originally a mismatch of degrees of freedom - on-shell the fermionic equations of motion reduces the number of degree of motion by a factor of two [8]. The auxiliary fields hence fill the missing bosonic degrees of freedom so we have a match of degrees of freedom on both side in both the on-shell and off-shell cases (see Table 4.1). This is known as the free Wess-Zumino model. It

Field	Spin	On-shell d.o.f.	Off-shell d.o.f.
ϕ, ϕ^*	0	2	2
$\psi_\alpha, \psi^\dagger_{\dot{\alpha}}$	1/2	2	4
F, F^*	0	0	2

Table 4.1: Degrees of freedom in the free Wess-Zumino model in the on-shell and off-shell cases.

actually gives the simplest supersymmetric multiplet - the chiral multiplet of four-dimensional $\mathcal{N} = 1$ supersymmetric theories. We will learn later how this can be derived from the superspace formalism.

5 Localisation

In this section we sketch out the properties of supersymmetric Lagrangians in zero dimensions. The two main ideas are:

- **Localisation.** Partition function localises around critical points of the superpotential in a supersymmetric theory.
- **Deformation Invariance.** Partition function is invariant under the change in the potential.

We will sketch out these two ideas in more detail. We will see how supersymmetric QFTs have a special property where the partition function localises to specific points in the functional space and how this potentially links to topological quantities. This section is mainly based on David Skinner's SUSY notes [9] and [4].

5.1 Localisation

The idea of localisation is simple - in a supersymmetric theory, the value of the relevant path integral reduces to a much smaller-dimensional integral. In some cases this reduces to counting contributions of certain points in the field space.

Let us illustrate this in the zero-dimensional case. Using Berezin integration rules, where

$$\int d\psi = 0 \ , \quad \int \psi d\psi = 1 \ , \tag{5.1}$$

the simplest form of a non-trivial action is of the form [10],

$$S(X, \psi_1, \psi_2) = S_0(X) - \psi^1 \psi^2 S_1(X) \ . \tag{5.2}$$

[8] To see this, consider the frame where the fermion momentum is $p^\mu = (E, 0, 0, E)$ and so the equation of motion reads,

$$\bar{\sigma}^\mu p_\mu \psi = \begin{pmatrix} 0 & 0 \\ 0 & 2p \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \ . \tag{4.16}$$

We see that then half of the fermionic degrees of freedom is projected out on-shell but off-shell it is an element of $\mathbb{C}^2$.

[9] The notes by David Skinner in Cambridge is where I have learnt a lot of mathematical physics from - they are really good and it would be a shame if you give them a miss!

[10] We will need at least two fermionic variables as the action is in the even algebra and $\psi^2 = 0$.

The partition function Z, for which we define as,

$$Z = \int \prod_i dX^i \prod_a d\psi^a e^{-S(X,\psi)} , \tag{5.3}$$

is then,

$$Z = \int dX e^{-S_0} S_1(X) , \tag{5.4}$$

using the Berezin integration rules. What is the simplest case for a supersymmetric transformation to exist? We can define a real function called the **superpotential** [11] $W : \mathcal{F} \to \mathbb{R}$ where $\mathcal{F}$ is the space of functions $x = X$ and defining, $\psi = \psi^1 + i\psi^2$ and $\bar{\psi}$ to be the conjugate variable let us write,

$$S_0(x) = \frac{1}{2}(\partial W(x))^2 , \tag{5.5}$$

$$S_1(x) = \partial^2 W(x) , \tag{5.6}$$

so

$$S(X, \psi, \bar{\psi}) = \frac{1}{2}(\partial W)^2 - \psi\bar{\psi}\partial^2 W . \tag{5.7}$$

Then the action S is invariant under the flow generated by the fermionic vector fields,

$$\mathcal{Q} = \psi \frac{\partial}{\partial x} + \partial W(x) \frac{\partial}{\partial \bar{\psi}} , \tag{5.8}$$

$$\mathcal{Q}^\dagger = \psi \frac{\partial}{\partial x} - \partial W(x) \frac{\partial}{\partial \bar{\psi}} , \tag{5.9}$$

with the nontrivial transformations being,

$$\mathcal{Q}(x) = \psi , \tag{5.10}$$

$$\mathcal{Q}(\bar{\psi}) = \partial W(x) , \tag{5.11}$$

and similarly for $\mathcal{Q}^\dagger$ [12]. These vector fields are exactly odd derivations of $C^\infty(\mathbb{R}^{1|2})$ and are the supercharges that generate supersymmetries of this zero-dimensional theory. Looking at the anticommutators, we see that,

$$\{\mathcal{Q}, \mathcal{Q}\} = 2\partial W(x)\psi \frac{\partial}{\partial \bar{\psi}} , \quad \{\mathcal{Q}^\dagger, \mathcal{Q}^\dagger\} = -2\partial W(x)\bar{\psi} \frac{\partial}{\partial \psi} , \tag{5.12}$$

$$\{\mathcal{Q}, \mathcal{Q}^\dagger\} = -\partial W(x) \left(\psi \frac{\partial}{\partial \psi} - \bar{\psi} \frac{\partial}{\partial \bar{\psi}} \right) . \tag{5.13}$$

It is a bit weird to analyse the supersymmetric algebra here but here we note two things - firstly, the RHS of Eq. (5.13) shouldn't be interpreted as the Hamiltonian - we don't have time in zero-dimensions. The second point concerns with Eq. (5.12) - we see indeed $\mathcal{Q}^2$ is not zero in general but since $\psi\partial^2 W(x) = 0$ is the equation of motion the supersymmetric algebra indeed vanishes on-shell [13].

Localisation from coordinate transformations.

Let us first try and understand localisation from a coordinate transformation perspective. In particular, we would like to evaluate the path integral,

$$\mathcal{Z} = \int e^{-S} dx d^2\psi . \tag{5.14}$$

[11] We call this the superpotential for nomenclature reasons - this will become clear when we look at other theories.

[12] But with $\mathcal{Q}^\dagger(\psi) = -\delta W(x)$.

[13] This is the similar to the case in the free Wess-Zumino model hen we missed out degrees of freedom. Turns out we have simply missed out a bosonic auxiliary field - if we include such contribution, such as using the superfield formalism in $\mathbb{R}^{0|2}$, this will allow us to reproduce the full supersymmetry algebra with $\{\mathcal{Q}, \mathcal{Q}\} = \{\mathcal{Q}^\dagger, \mathcal{Q}^\dagger\} = \{\mathcal{Q}, \mathcal{Q}^\dagger\} = 0$.

Firstly let us isolate the neighbourhoods $\mathcal{U}$ where the derivative superpotential ∂W vanishes. Taking the complement $\mathcal{U}^c$ of $\mathcal{U}$ in $\mathcal{F}$, we can change variables $(x, \psi, \bar{\psi}) \mapsto (y, \chi, \bar{\chi})$ where,

$$y = x - \frac{\psi\bar{\psi}}{\partial W} \ , \quad \chi = \psi\sqrt{\partial W} \ , \quad \bar{\chi} = \bar{\psi} \ . \tag{5.15}$$

The new measure is now,

$$dxd^2\psi = \sqrt{\partial W(y)}dyd^2\chi \ , \tag{5.16}$$

where $\mathcal{Q}(y) = 0 = \mathcal{Q}^\dagger(y)$ so y is invariant under supersymmetry [14]. We also see that the action transforms as,

$$S[y, 0, 0] = \frac{1}{2}\left(\partial W(y)\right)^2 = S[x, \psi, \bar{\psi}] \ . \tag{5.17}$$

The contribution to the path integral is surprisingly,

$$\mathcal{Z}_{\mathcal{U}^c} = \frac{1}{2\pi}\int_{\mathcal{U}^c} e^{-S[y,0,0]}\sqrt{\partial W(y)}dyd^2\chi = 0 \ , \tag{5.18}$$

due to the property of the Berezin integral. This means that the non-vanishing contributions to $\mathcal{Z}$ only comes from the neighbourhood $\mathcal{U}$ - this is exactly where the coordinate transformation breaks down as $\partial W \to 0$ means the Jacobian of the coordinate transformation is no longer invertible. This leads us to the following key observation.

Proposition 5.1 (Localisation principle.)**.** *Quantum field theories with supersymmetry generically have path integrals that localise to a vicinity of a fixed point set.*

How do we further evaluate this? Consider the case where W is a generic polynomial of degree d with $d-1$ isolated non-degenerate [15] critical points. Then around this critical point $x = x^*$, we can write,

$$W(x) = W(x^*) + \frac{\alpha_c}{2}(x - x^*)^2 + \dots \ , \tag{5.19}$$

with $\alpha_c = \partial^2 W(x^*)$. Then the action becomes,

$$S(x, \psi, \bar{\psi}) = \frac{\alpha_c^2}{2}(x - x^*)^2 - \alpha_c\bar{\psi}\psi \ , \tag{5.20}$$

and expanding the exponential in Grassmann variables in the integral will yield,

$$\begin{aligned}
\mathcal{Z} &= \sum_{x^*}\frac{1}{\sqrt{2\pi}}\int dxd^2\psi e^{-\frac{1}{2}\alpha_c^2(x-x^*)^2}\left(-1 + \alpha_c\bar{\psi}\psi\right) \\
&= \sum_{x^*}\frac{\alpha_c}{\sqrt{2\pi}}\int e^{-\frac{1}{2}\alpha_c^2(x-x^*)^2} \\
&= \sum_{x^*}\frac{\alpha_c}{|\alpha_c|} \ ,
\end{aligned} \tag{5.21}$$

which eventually leads to,

$$\mathcal{Z} = \sum_{x^*:\partial W|_{x^*}=0}\frac{\partial^2 W(x^*)}{|\partial^2 W(x^*)|} \tag{5.22}$$

This is a surprising result. We note that if d is odd then $\mathcal{Z} = 0$, and if d is even then $\mathcal{Z} = \pm 1$ as we have $d-1$ critical points. $\mathcal{Z}$ just counts the number of times the superpotential crosses $W = 0$ [16]!

There is perhaps another way to illustrate this result. To do this we will need to discuss something know as deformation invariance.

[14] In fact y is the only independent combination of $(x, \psi, \bar{\psi})$ that is supersymmetrically invariant so any invariant function will be a function of y.

[15] This means $\partial^2 W|_{x^*} \neq 0$.

[16] Or if you like, the number of kinks as a one-dimensional instanton.

5.2 Deformation Invariance

Deformation invariance can be summarised in one sentence: the path integral $\mathcal{Z}$ is sensitive only to the order of polynomial in W.

What do I mean by that? Let's suppose a quantum field theory has some symmetry G where it leaves the action and path integral measure invariant. Then the correlation functions of quantities that are variables of fields under the symmetry vanishes. To see this, let's suppose g is a field, and f is defined as the variation of ϕ under symmetry G,

$$f = \delta_G \phi \ , \tag{5.23}$$

Then the expectation value of f is,

$$\langle f \rangle = \int f e^{-S} = \int \delta_G g e^{-S} = \int \delta \left(g e^{-S} \right) = 0 \ . \tag{5.24}$$

For the present case, we can set,

$$g = \partial \rho(X) \bar{\psi} \ . \tag{5.25}$$

Now the variation of g under supersymmetry gives,

$$f = \epsilon \left(\partial \rho \partial W - \partial^2 W \psi \bar{\psi} \right) \ , \tag{5.26}$$

which leaves

$$\left\langle \partial \rho \partial W - \partial^2 W \psi \bar{\psi} \right\rangle = 0 \ . \tag{5.27}$$

Now since the action is,

$$S(X, \psi, \bar{\psi}) = \frac{1}{2} (\partial W)^2 - \psi \bar{\psi} \partial^2 W \ , \tag{5.7}$$

we can see that Eq. (5.27) gives the invariance of the correlation function,

$$\langle \delta_\rho S \rangle = 0 \ , \tag{5.28}$$

under the transformation of the superpotential $W \mapsto W + \rho$. This shows that the partition function is invariant under a change in the potential - which is true as long as ρ is small at infinity in field space when compared to h so the boundary terms in the argument will indeed vanish [17]. In particular, we can rescale $W \mapsto \lambda h$, with $\lambda \gg 1$. Then as long as,

$$\partial W e^{-\lambda^2 (\partial W)^2 / 2} \to 0 \tag{5.29}$$

when $|x| \to \infty$, boundary terms will not appear and the partition function $\mathcal{Z}$ will remain invariant. In particular, looking at the path integral $\mathcal{Z}$ now defined with this deformation parameter λ,

$$\mathcal{Z}(\lambda) = \frac{1}{\sqrt{2\pi}} \int dx d^2 \psi e^{-S_\lambda} \ , \tag{5.30}$$

where the action is now,

$$S_\lambda (X, \psi, \bar{\psi}) = \frac{\lambda^2}{2} (\partial W)^2 - \lambda \psi \bar{\psi} \partial^2 W \ , \tag{5.31}$$

we see that, given the limit in Eq. (5.29) we will have,

$$\frac{d}{d\lambda} \mathcal{Z}(\lambda) = \frac{1}{\sqrt{2\pi}} \int dx d^2 \psi \mathcal{Q}_\lambda^\dagger \left(\psi \partial W e^{-S_\lambda} \right) \ , \tag{5.32}$$

which gives zero as no boundary terms will survive. The deformation invariance of path integral allows us to deduce that for $\lambda \to 0$, $e^{-\lambda^2 (\partial W)^2 / 2}$ suppresses all contributions to the integral arbitrarily strongly apart from around the neighbourhood $\mathcal{U}$ of points $x^* : \partial W(x^*) = 0$. This is the other way to understand the localisation principle.

The deformation principle allow us to consider deformations of the superpotential $W(x)$. In particular, if $W(x)$ is a polynomial of order d, then we can deform $W(x)$ such that it has no critical points if d is odd and only one critical point if d is even - this is the same crossing phenomenon we have commented in the previous section, and we shall later see how this generalises to topological formulae in higher dimensions.

[17] ρ can be of the same order as h as long as the leading order term is smaller than that of h.

5.3 Explicit evalution

In fact, it is possible to evaluate directly the path integral $\mathcal{Z}$ in Eq. (5.14). We can write,

$$\begin{aligned}\mathcal{Z} &= \frac{1}{\sqrt{2\pi}} \int dx d^2\psi e^{-S} \\ &= \frac{1}{\sqrt{2\pi}} \int dx \partial^2 W e^{-\frac{1}{2}(\partial W)^2} \\ &= \frac{D}{\sqrt{2\pi}} \int dy y e^{-\frac{1}{2}y^2} \\ &= D\end{aligned} \tag{5.33}$$

where D is the degree of the map $x \mapsto y = \partial W(x)$. It enters the equation as the map is not one-to-one. The degree counts the number of preimages of a given point taking into account the relative orientation of each preimage with respect to its image so D is 0 and ± 1 respectively in the cases where d is odd and even, exactly as before.

One more side comment. A 'third' way of understanding localisation is to interpret the fermionic symmetry as some symmetry acting on the path integral G. In the most general case when G is freely acting, the integral over G just factors out (c.f. integration over an orbit in group theory like the Haar measure). The relevant integral here is $\int_G d\theta = 0$. However, our G, the group of fermionic symmetries parametrised by fermionic coordinate θ, has fixed locus $\mathcal{C}_0$ precisely in the open neighbourhood $\mathcal{U} \subset \mathcal{C}$ where $\mathcal{C} = \mathcal{F}$ is the space over which the integral is performed. Localisation exactly comes from these fixed points where the coordinate transformation is not well-defined as this is the fixed point of the fermionic symmetry G (generated by $\mathcal{Q}^\dagger$).

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