Twistor Theory

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Abstract. A set of short notes on twistor theory.

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CHAPTER 1

Introduction

Twistor theory is the brainchild of Sir Roger Penrose in the 1960s. The basis of the theory is underlied in a series of papers [**1**–**5**]. The key idea is to provide a model of quantum gravity and supply a description of the physical theory on a underlying complex structure. This involves treating null rays as the fundamental objects, known as *twistors*, in which spacetime points are then derived from the resulting *Twistor Correspondence*. This requires a deep understanding of complex geometrical concepts and the physical interpretations of these abstract mathematical objects in relation to observables and physical quantities in Minkowski spacetime.

In this part, my goal is to illustrate this strong underlying connection between complex geometry and twistor theory by illustrating two important concepts - the *Twistor Correspondence* and the *Twistor Transform*. By establishing the *Twistor Transform*, one can see how the solutions to massless equations can be corresponded naturally to geometrical objects in twistor space. This allows analysis of complicated non-linear solutions in terms of sheaf cohomology classes in twistor space. This, in particular, is valuable for high helicity fields and highly non-linear equations where solutions obtained via ordinary PDE methods are often impossible to obtain.

The rundown of this part is as follows: In §2 we discuss some underlying spinor notations to be used in the rest of the essay. In §3 we define twistors and discuss some of their geometrical interpretations. We will discuss the complexified compactified Minkowski space in §4 and this leads to the discussion of the Klein/twistor correspondence in §5 and its physical interpretations in §6.

CHAPTER 2

Spinors

Here I will provide a quick tour of the 2-spinor formalism to be used in the remainder of the essay. We begin with a definition.

Definition 2.1. *(Preliminary) A* ***spinor*** *is a 2-component complex vector object that is a representation of the Lorentz group.*

There is a natural correspondence between tensors and spinors [**2**,**6**,**7**]. If we are given a Lorentz vector V^a with components (V^0, V^1, V^2, V^3) in some orthonormal frame in Minkowski spacetime, we can define the following Hermitian matrix:

$$\Phi(V^a) = V^{AA'} = \begin{pmatrix} V^{00'} & V^{01'} \\ V^{10'} & V^{11'} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} V^0 + V^3 & V^1 + iV^2 \\ V^1 - iV^2 & V^0 - V^3 \end{pmatrix} \tag{2.1}$$

where A and A' runs between $0, 1$ and $0', 1'$ respectively. The Hermitian matrix permits an $SL(2, \mathbb{C})$ transformation

$$V^{AA'} \mapsto t^A_B V^{BB'} \bar{t}^{A'}_{B'} \tag{2.2}$$

where t^A_B is an element of $SL(2, \mathbb{C})$ and $\bar{t}^{A'}_{B'} = \bar{t^A_B}$. This transformation preserves the length of the vector:

$$\det \Phi(V^a) = \frac{1}{2}\eta_{ab} V^a V^b \tag{2.3}$$

and therefore is a representation of a Lorentz transformation. In fact the following proposition holds.

Proposition 2.1. $SL(2, \mathbb{C})/\mathbb{Z}_2 \cong SO(1, 3)$

Proof. See [**2**, **7**]. □

In particular $SL(2, \mathbb{C})$ exhibits as the universal cover for the proper orthochronous Lorentz group [**7**].

When the vector V_a is null, the rank of $\Phi(V^a)$ drops to one. This allows us to identify the matrix as an outer product of 2 two-component spinors.

$$V^{AA'} = \alpha^A \bar{\alpha}^{A'} \tag{2.4}$$

This leads to the following definition.

Definition 2.2. *The* ***spin-space*** *is a complex two-dimensional vector space S with elements α^A on which $SL(2, \mathbb{C})$ acts. The complex-conjugate vector space $\bar{S} = S'$ has elements of the form $\alpha^{A'}$. They each induce a dual space S^* and S'^* with elements α_A and $\alpha_{A'}$.*

We can then develop the spin algebra as a ring of smooth complex scalar fields using the axiomatic treatment [**6**]. Higher valence spinors can be constructed by tensor multiplying the spin-spaces constructed in the definition above [**7**]. In particular, the correspondence 2.1 allows one to relate tensor and spinor components according to a standard scheme.

PROPOSITION 2.2. *Each abstract tensor index is to be equated with a pair of spinor indices, one primed and one unprimed. i.e.*

$$V^a = V^{AA'} \tag{2.5}$$

We will refer to this as the $a \mapsto AA'$ substitution rule.

PROOF. This is self-evident from the correspondence 2.1. For more details refer to [**6**]. □

A full treatment of spinor algebra can be found in §2 of [**6**] and geometrical interpretations of the tensor-spinor correspondence can be found in §1 of [**6**].

We state the following statements without proof.

PROPOSITION 2.3. *The following correspondences between tensors and spinors hold.*

(1)
$$g_{ab} = \epsilon_{AB}\epsilon_{A'B'} \tag{2.6}$$
(2)
$$g^{ab} = \epsilon^{AB}\epsilon^{A'B'} \tag{2.7}$$
(3) *The ϵ symbols are used to raise and lower spinor indices.*
(4) *The Levi-Civita connection ∇_a of M can be extended to a connection on spin-bundles if we demand*
$$\nabla_{AA'}\epsilon_{BC} = 0 = \nabla_{AA'}\epsilon_{B'C'} \tag{2.8}$$

PROOF. Refer to [**6, 7**]. □

We will see that the correct way of formulating these spin objects is the construct a spin bundle or sheaf on M (§**??**).

CHAPTER 3

Twistors and Twistor Space

In this section we will discuss the definitions of twistors. We will first present a preliminary definition of twistors and then discuss the construction of the twistor space. We will see that twistors can be realised as the spinor representation of $SU(2,2)$. We will then discuss the relationship between twistors in terms of momentum and angular momentum, and discuss its geometrical interpretations in terms of incidence of null lines and the Robinson congruence.

0.1. Definition of twistors and the twistor space. We begin with a canonical definition of twistors in the form of a twistor equation [**1**, **4**, **7**, **8**].

DEFINITION 3.1. *The **twistor equation** is the following spinor equation*

$$\nabla_{A'}{}^{(A}\Omega^{B)} = 0 \tag{3.1}$$

*Solutions to the twistor equation are **twistors**.*

Here we will explicitly find a solution to the twistor equation 3.1 [**8**]. This solution is antisymmetric in AB, so also is $\nabla^C_{C'}\nabla^A_{A'}\Omega^B$. Since in flat space the derivative operators can be commuted the latter is also antisymmetric in CB and therefore in CAB. So $\nabla_{A'}{}^{(A}\omega^{B)}$ is constant and we have

$$\nabla_{BA'}\omega^C = -i\epsilon_B{}^C\pi_{A'} \tag{3.2}$$

for some constant spinor $\pi_{A'}$. Integrating this equation will give the solution $\omega^C = x^{BA'}(-i\epsilon_B{}^C\pi_{A'}) + \tilde{\omega}^C$ for some constant spinor $\tilde{\omega}^C$ so we have the set of solutions

$$\begin{cases} \omega^A = \tilde{\omega}^A - ix^{AA'}\tilde{\pi}_{A'} \\ \pi_{A'} = \tilde{\pi}_{A'} \end{cases} \tag{3.3}$$

We see that the solutions of the twistor equation is specified by two spinor fields ω^A and $\pi_{A'}$. We can define the twistor space as the solution space [**7**]:

DEFINITION 3.2. *The **twistor space** $\mathbb{T}^\alpha$ is the solution space*[1] *of the twistor equation 3.1. It is a four-dimensional complex vector space and can be coordinated with respect to a choice of origin and a certain spin frame as*

$$Z^\alpha = (\omega^A, \pi_{A'}) \tag{3.4}$$

We see from 3.3 that the spinor ω^A alone completely defines the twistor. Under a change of origin in Minkowski space, $\pi_{A'}$ is transformed whilst ω^A remains unaffected[2]. However the twistor Z^α is supposed to be unaffected. This shows that the RHS of 3.4 is merely a *spinor representation* of the twistor Z^α [**4**], with

[1] Sometimes I will denote $\mathbb{T}$ as $\mathbb{T}^\alpha$.

[2] The opposite occurs when twistor is conformally rescaled - ω^A is transformed whilst $\pi_{A'}$ remains unaffected.

a choice of an origin and a conformal scale[3] We can further represent this spinor representation in some spinor coordinates with respect to some spinor frame[4] in terms of components:

$$Z^\alpha \leftrightarrow (\omega^0, \omega^1, \pi_{0'}, \pi_{1'}) \tag{3.5}$$

We have established the following statement:

LEMMA 3.1. *The twistor Z^α has the spinor correspondence (not Poincar-invariant)*

$$Z^\alpha \leftrightarrow (\omega^A(O), \pi_{A'}(O)) \leftrightarrow (\omega^0(O), \omega^1(O), \pi_{0'}(O), \pi_{1'}(O)) \tag{3.6}$$

where O is the choice of origin, and the second correspondence comes from a choice of an arbitrary spin frame (o^A, i^A).

We will notationally write:

$$(Z^0, Z^1, Z^2, Z^3) = (\omega^0(O), \omega^1(O), \pi_{0'}(O), \pi_{1'}(O)) \tag{3.7}$$

1. Dual twistors

The twistor space $\mathbb{T}^\alpha$ is often known as the $\begin{pmatrix}1\\0\end{pmatrix}$ twistor space. We can similarly define the dual [**7**–**9**].

DEFINITION 3.3. *We can define a real Hermitian form Φ of signature $(++--)$ in $\mathbb{T}$. In terms of the coordinates Z^a, $a = 0, 1, 2, 3$, Φ has the matrix form:*

$$\Phi \to \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix} \tag{3.8}$$

i.e.

$$\Phi(Z, Z) = Z^0\bar{Z}^2 + Z^1\bar{Z}^3 + Z^2\bar{Z}^0 + Z^3\bar{Z}^1 \tag{3.9}$$

where the bar denotes complex conjugation.

DEFINITION 3.4. *The **dual twistor space** $\mathbb{T}_\alpha$ of valence $\begin{pmatrix}0\\1\end{pmatrix}$ is the dual space with respect to Hermitian form Φ to $\mathbb{T}^\alpha$.*

LEMMA 3.2. *The **dual-twistor** $W_\alpha \in \mathbb{T}_\alpha$ has the following canonical correspondence*

$$W_\alpha \leftrightarrow (\lambda_A(O), \mu^{A'}(O)) \tag{3.10}$$

PROOF. This follows from identical treatment of $\mathbb{T}_\alpha$ as in the case in §1.1. Alternatively it follows directly from the dual property. □

DEFINITION 3.5. *We can define the **inner product** between Z^α and W_α as*

$$\langle Z, W\rangle = Z^\alpha W_\alpha \leftrightarrow \lambda_A\omega^A + \mu^{A'}\pi_{A'} \tag{3.11}$$

---

[3]This means a choice of one of the conformally related flat metrics. The spinor indices here are abstract labels [**4**].

[4]Do not confuse this with the *spinor representation* of the twistor, this is the local coordinate representation of this spinor representation.

Dual twistors can seen as solutions to the *conjugate twistor equation*:

$$\nabla_A^{(A'}\mu^{B')} = 0 \tag{3.12}$$

and the inner product can be taken directly in terms of the spinor fields ω^A and $\mu^{A'}$. From this,

LEMMA 3.3. *We can canonically identify $\mathbb{T}_\alpha$ with $\bar{\mathbb{T}}^\alpha$.*

PROOF. To see this we note that we can write the inner product $W_\alpha Z^\alpha$ as

$$[\mu^{A'}]\cdot[\omega^A] = W_\alpha Z^\alpha = \lambda_A\omega^A + \mu^{A'}\pi_{A'} = \frac{1}{2}i\left(\mu^{A'}\nabla_{BA'}\omega^B - \omega^A\nabla_{AB'}\mu^{B'}\right) \tag{3.13}$$

where the last equality follows from 3.2 and its conjugate version

$$\nabla_{AA'}\mu^{B'} = i\epsilon_{A'}{}^{B'}\lambda_A \tag{3.14}$$

Since the Hermitian form $\Phi(W,Z)$ is assumed to be non-degenerate, each solution $\mu^{A'}$ can be obtained by complex conjugating the solution ω^A. So we see that we have[5]

$$Z^\alpha = (\omega^A, \pi_{A'}) \to \bar{Z}_\alpha = (\pi_A, \omega^{A'}) \tag{3.15}$$

□

Endomorphisms of $\mathbb{T}$ that preserves the Hermitian form constitutes the group $U(2,2)$. Demanding the invariance of $\epsilon_{\alpha\beta\gamma\delta}$ gives the group $SU(2,2)$. It can be shown that this group, which has 15 parameters, is locally isomorphic to $C(1,3)$, the conformal group of Minkowski signature preserving Φ [**7**]. There is a chain of 2-1 isomorphisms[6]:

$$SU(2,2) \curvearrowright \mathbb{T} \to O(2,4) \curvearrowright Q_4 \to C(1,3) \curvearrowright Q_4 \tag{3.16}$$

This realises the second definition of twistor space:

DEFINITION 3.6. *The **twistor space** $\mathbb{T}$ is the space of the spinor representation of $SU(2,2)$.*

2. Twistors, Momentum and Angular Momentum

For any finite system in spacetime a momentum P^a and an angular momentum M^{ab} can be defined depending on the origin O. A change of origin $O \to O'$ will preserve P^a but change M^{ab}:

$$\begin{cases} P^a \mapsto P'^a = P^a \\ M^{ab} \mapsto M'^{ab} = M^{ab} - 2X^{[a}P^{b]} \end{cases} \tag{3.17}$$

where X^a is the displacement vector. The *Pauli-Lubjanski spin vector*

$$S_a = \frac{1}{2}\eta_{abcd}P^bM^{cd} \tag{3.18}$$

is also preserved under the transformation.

[5] Here we have replaced the twistor-spinor correspondence arrow $\leftrightarrow$ with the equality sign for convenience.

[6] Q_4 is the *Klein quadric* to be identified with Minkowski space in §4.

Consider the case when the system has zero rest mass. The null vector P^a corresponds to a spinor $\pi_{A'}$ defined up to a phase

$$P_a = \bar{\pi}_A \pi_{A'} \tag{3.19}$$

M^{ab} is a bivector and can be written as [**7**]:

$$M^{ab} = \phi^{AB}\epsilon^{A'B'} + \epsilon^{AB}\bar{\phi}^{A'B'} \tag{3.20}$$

The helicity s is defined as $S_d = sP_d$. This can be written as

$$S_{AA'} = s\bar{\pi}_A\pi_{A'} = -i\bar{\pi}_A\pi^{B'}\bar{\phi}_{B'A'} + i\bar{\pi}^B\phi_{BA}\pi_{A'} \tag{3.21}$$

The second equality following from the definition of the Pauli-Lubjanski vector. Multiplying both sides with $\bar{\pi}^A$ gives

$$\phi_{AB}\bar{\pi}^A\bar{\pi}_B = 0 \tag{3.22}$$

This suggests the form

$$\phi_{AB} = i\omega_{(A}\bar{\pi}_{B)} \tag{3.23}$$

for some constant ω^A. The pair (P^a, M^{ab}) can now be characterised as two spinors $(\omega^A, \pi_{A'})$. If we insist that the phase of $\pi_{A'}$ is unchanged under a change of origin, we will get 3.17 and in spinor form:

$$\begin{cases} \omega'^A = \omega^A - ix^{AA'}\pi_{A'} \\ \pi'^A = \pi^{A'} \end{cases} \tag{3.3}$$

which is the solution to the twistor equation 3.1. We therefore see that twistors have a natural interpretation as the momentum and angular momentum of massless particles.

Consider the following expression:

$$Z^\alpha \bar{Z}_\alpha = \omega^A\bar{\pi}_A + \pi_{A'}\bar{\omega}^{A'} = 2s \tag{3.24}$$

using the definitions of ϕ_{AB} and 3.21. $s \in \mathbb{R}$ is the helicity. When $s = 0$, the twistor is null and represents a real null worldline [**1**, **4**]. If s is non-zero then the twistor represents a particle with intrinsic spin and the null line is complexified.

3. Robinson Congruence and the geometrical interpretation of twistors

We have seen that a twistor corresponds to a null line in §2 by the specification of its momentum and angular momentum. We will later see that this is the content of the *twistor correspondence*. For now we will delay that discussion and consider the geometrical interpretation of a twistor.

For two null lines L and X, there are corresponding twistors $L^\alpha = (\lambda^A, \mu_{A'})$ and $X^\alpha = (\omega^A, \pi_{A'})$. If the lines intersect, say at spacetime point x^a, 3.17 then gives the intersection as

$$\omega^A\bar{\mu}_A = i\omega^A x_{AA'}\bar{\lambda}^{A'} = -\pi_{A'}\bar{\lambda}^{A'} \tag{3.25}$$

In twistor language, this is

$$X^\alpha \bar{L}_\alpha = 0 \tag{3.26}$$

Any real null line L in Minkowski space can be completely characterised by the system of all null lines which meets it [**1**]. In twistor terms, this means that knowing the set of all X^α satisfying

$$\begin{cases} X^\alpha \bar{L}_\alpha = 0 \\ X^\alpha \bar{X}_\alpha = 0 \end{cases} \tag{3.27}$$

will give the null line L up to proportionality. Therefore it is possible to see $\bar{L}_\alpha$ as a congruence of null lines in M that defines L^α or the null line L. We then have the following definition.

DEFINITION 3.7. *A general twistor $R_\alpha \in \mathbb{T}_\alpha$ can be represented as a* ***Robinson Congruence****. This is a congruence R of null lines X with twistors satisfying*

$$X^\alpha R_\alpha = 0 \tag{3.28}$$

This definition is paramount to understanding the geometrical meaning of a twistor [**1**, **7**].Writing $R_\alpha = (R_A, R^{A'})$ and $X^\alpha = (ix^{AA'}\pi_{A'}, \pi_{A'})$, the congruence equation 3.28 gives

$$\left(iR_A x^{AA'} + R^{A'}\right)\pi_{A'} = 0 \tag{3.29}$$

This gives $\pi^{A'} = k\left(iR_A x^{AA'} + R^{A'}\right)$. We can pick $x^{AA'} = (t, x, y, z)$ and an orthogonal spin frame $R_\alpha = (o_A, -a\bar{i}^{A'})$ (see [**7**]) to give

$$\pi^{A'} = ik(x - iy, ia + t - z) \tag{3.30}$$

Now choose a spacelike hyperplane $t = \tau$ and project the null vector $l^a = \bar{\pi}^A \pi^{A'}$ onto it. The integral curves of this null vector [**1**, **7**] are circles. They twist around a family of coaxial tori such that every pair of circles are linked. The twisting sense depends on the helicity s defined above - if $s > 0$, the twisting has positive screw sense (right-handed). This is where the name of *twistors* originate from and this forms the geometrical interpretation of twistors (Robinson Congruences) as local twisting circles on a spatial hypersurface.

4. Projective Twistor Space

For any constant $c \in \mathbb{C}$, we see that any solution $c\omega^\alpha$ will be a solution of the twistor solution 3.1 provided that ω^α is a solution. Any null line can be defined as a twistor up to a constant complex scaling. This means that we can identify null lines in Minkowski space as a proportionality class of twistors $[Z^\alpha]$. This defines the projective twistor space:

DEFINITION 3.8. *The* ***projective twistor space*** $\mathbb{PT}$ *is the projective version of the twistor space* $\mathbb{T}$*. It is isomorphic to* $\mathbb{P}^3$*.*

We will see that this space is intrinsically linked to a complex projective manifold in the *twistor correspondence* in §5.

CHAPTER 4

Complexified Compactified Minkowski Space

Twistors do not naturally correspond to objects in “normal” Minkowski space. In fact, twistors naturally correspond to the complexified version of Minkowski space with additional structures included at infinity. This short section explores the idea of conformal compactification and complexification with the aid additional geometrical structures. This will lead to the statement of the *twistor correspondence.*

1. Complexified Minkowski Space

We begin with a definition of *complexification.*

Definition 4.1. *Let V be a real vector space. The* ***complexification*** *of V is defined by*

$$V^{\mathbb{C}} = V \otimes_{\mathbb{R}} \mathbb{C} \tag{4.1}$$

Recall that there is a natural map of Minkowski space $\mathbb{M}$ with the space of 2×2 Hermitian matrices $\mathbb{M} \to H(2)$ by

$$x^\mu \mapsto \frac{1}{\sqrt{2}} \begin{pmatrix} x^0 + x^3 & x^1 - ix^2 \\ x^1 + ix^2 & x^0 - x^3 \end{pmatrix} \tag{4.2}$$

We can then define

Definition 4.2. *The* ***complexified Minkowski space*** *$\mathbb{M}_{\mathbb{C}}$ is defined as the complexification of $\mathbb{M}$ with the nondegenerate complex-bilinear form (extended by complex linearity)*

$$g(z, w) = z^0 w^0 - z^1 w^1 - z^2 w^2 - z^3 w^3 \tag{4.3}$$

There is a complex isomorphism:

$$\mathbb{M}_{\mathbb{C}} \to \mathbb{C}^{2\times 2} \cong \mathbb{C}^4 \tag{4.4}$$

2. Conformal Compactification

In §3 we alluded to the fact that two null lines intersect when $X^\alpha \bar{L}_\alpha = 0$. When L and X are parallel however, this condition means that L and X lie in the same null hyperplane (as null cones become null hyperplanes in the limit of spacetime infinity) [**1**]. It is therefore helpful to regard all the null lines of a null hyperplane as intersecting in a single point at infinity[1].

Definition 4.3. *We make the following identifications (Figure* **??***).*

[1]In the language of infinities, this means the identification between $\mathcal{I}^+$ and $\mathcal{I}^-$ [**8**].

(1) *Every null geodesic originates from some $A^- \in \mathcal{I}^-$ ends at $A^+ \in \mathcal{I}^+$, with the light cones at the two points spatially antipodal to each other. We identify A^+ with A^-.*
(2) *This identification makes $\mathcal{I}^+ \simeq \mathcal{I}^- = \mathcal{I}$.*
(3) *We further identify i^- with i^0 and i^0 with i^+.*

*The resulting compact conformal manifold that we construct by the means of such identifications is then known as the **conformal compactified Minkowski space**[2] $\mathbb{M}^{\#}$.*

We note that the conformal transformations of $\mathbb{M}^{\#}$ do not preserve the original Minkowski metric g of $\mathbb{M}$. By "adding" the infinities to the Minkowski space (i.e. compactifying), this allows the special conformal transformations (in addition to dilatations and Poincar) to be included in the full symmetry of the Minkowski space. This generates the entire identity connected component $C_+^{\uparrow}(1,3)$ of the fifteen-parameter group $C(1,3)$ as conformal motions of $\mathbb{M}^{\#}$.

There is another way of constructing the compactified Minkowski space $\mathbb{M}^{\#}$ [**7**, **8**]. We can consider a 6d pseudo-Euclidean space with the metric:

$$ds^2 = dT^2 + dV^2 - dW^2 - dX^2 - dY^2 - dZ^2 \tag{4.5}$$

The null cone N is given by

$$T^2 + V^2 - W^2 - X^2 - Y^2 - Z^2 = 0 \tag{4.6}$$

The intersection of N with the plane $V - W = 1$ will induce the Minkowski metric

$$ds^2 = dT^2 - dX^2 - dY^2 - dZ^2 \tag{4.7}$$

This intersection can then be canonically identified with the Minkowski space $\mathbb{M}$.

Therefore we have established that

Proposition 4.1. *The null cone generators of N i.e. points in $\mathbb{P}N$ corresponds to points in the compactified Minkowski space $\mathbb{M}^{\#}$.*

We note that lines through the origin of $\mathbb{E}^6$ are points in $\mathbb{P}^5$ and it is a fact that $\mathbb{P}N$ defines points of a quadric in $\mathbb{P}^5$. We will explicitly see how this works in the complexified case later.

We make a definition [**9**]:

Definition 4.4. *A real manifold is said to have a **conformal structure** if there exists a smoothly varying family of light cones in the tangent space at each point, defined pointwise by the vanishing of a non-degenerate quadratic form.*

We note that the pseudo-orthogonal group $O(2,4)$ preserves the metric 4.5 and N. This induces a transformation of $\mathbb{P}^5$ that sends $\mathbb{M}^{\#}$ to itself, which preserves its conformal structure and induces the group $C(1,3)$ on $\mathbb{M}^{\#}$ ($O(2,4) \to C(1,3)$ is a double cover).

[2] Another interpretation of *conformal compactification* is to choose a suitable new metric $\bar{g} = \Omega^2 g$ such that points at infinity with respect to g are at finite distance with respect to the unphysical metric $\bar{g}$, as covered in *Part III Black Holes course.*

3. Complexified Compactified Minkowski Space

DEFINITION 4.5. *The* ***complexified compactified Minkowski space*** $\mathbb{M}^{\#}_{\mathbb{C}}$ *is defined as the complexification of* $\mathbb{M}^{\#}$ *with the associated nondegenerate complex-bilinear form (metric) extended by complex linearity.*

We will henceforth denote $\mathbb{M}^{\#}_{\mathbb{C}}$ as $\mathbb{M}$ for simplicity. We now show a few properties of this space $\mathbb{M} := \mathbb{M}^{\#}_{\mathbb{C}}$:

PROPOSITION 4.2. *The complexified compactified Minkowski space is the Grassmannian* $\mathbf{G}_{(2,4)}$.

PROOF. Here I will only provide a sketch proof. The idea is that the complex Minkowski spacetime can be described as a big cell of the Grassmannian $\mathbf{G}_{(2,4)}$, the set of 2d subspaces in $\mathbb{C}^4$. To do this we can find the action of the Poincar group as the subgroup of $GL(4,\mathbb{C})$ which leaves the big cell invariant. For more details see [**10**]. □

Now we will describe the geometry of $\mathbb{M}$.

DEFINITION 4.6. *Consider the symmetric bilinear form:*

$$q(Z,W) = q_{ij}Z^iW^j \tag{4.8}$$

A ***quadric*** Q *is a projective algebraic variety in* $\mathbb{P}^n$ *given by*

$$Q = \{[Z] \in \mathbb{P}^n : q(Z,Z) = 0\} \tag{4.9}$$

PROPOSITION 4.3. *Let* V *be a 4d vector space. Consider the* ***Plcker embedding****:*

$$pl : \mathbf{G}_{(2,4)}(V) \to \mathbb{P}\left(\bigwedge^2 V\right) \cong \mathbb{P}^5 \tag{4.10}$$

where

$$pl : [Z,W] \mapsto [Z \wedge W] \tag{4.11}$$

The mapping pl *is an embedding an the image* $Q_4 = pl(\mathbf{G}_{(2,4)})$ *is a projective algebraic hypersurface of degree two in* $\mathbb{P}^5$ *given by*

$$Q_4 = \{z^{ij} : z^{12}z^{34} - z^{13}z^{24} + z^{14}z^{23} = 0\} \tag{4.12}$$

This hypersurface Q_4 *is called the* ***Klein quadric*** *in* $\mathbb{P}^5$.

PROOF. (From [**9**]). Let $\{e_i\}$ be a basis for V. For $\omega = \sum z^{ij} e_i \wedge e_j \in \Lambda^2 V$, we have to characterise those ω which are decomposable, where $\omega = Z \wedge W$ for $Z, W \in V$. It can be show that ω is decomposable if and only if $\omega \wedge \omega = 0$ (see Lemma 1.3.2 of [**9**]). Hence the map is an embedding. □

We will see that the Plcker embedding plays a crucial rule in establishing the Klein correspondence and the associated geometry in §5. As an aside, we note that by considering the affine Minkowski space $\mathbb{M}^I$ defined by taking out infinities from $\mathbb{M}$ [**9**], one can identify the Minkowski space with the intersection between the real part of $\mathbb{M}$ and $\mathbb{M}^I$. For more details see §1.6 of [**9**].

4. A First Look at Twistor Correspondence

Why have we gone through the trouble of defining $\mathbb{PT}$ and $\mathbb{M}$ in such great detail? It turns out that there is another "definition" for twistor space.

Recall the solution the twistor equation 3.1 is:

$$\Omega^A = \omega^A - ix^{AA'}\pi_{A'} \tag{3.3}$$

We can consider the points of Minkowski space at which this field vanishes [**7**]. Then we have

$$\omega^A = ix^{AA'}\pi_{A'} \tag{4.13}$$

The general solution of this equation turns out to be

$$x^{AA'} = x_0^{AA'} + \lambda^A \pi^{A'} \tag{4.14}$$

which defines a null 2-plane known as an α-plane on $\mathbb{M}$ [**7**]. We then have

PROPOSITION 4.4. *There is a correspondence (**Twistor Correspondence**) between $\mathbb{PT}$ and $\mathbb{M}$ as follows:*

$\mathbb{PT}$	$\mathbb{M}$
point	*α-plane*
line	*point*

It turns out that in the late nineteenth century Klein had already studied this correspondence in terms of complex manifolds. This correspondence, in complex geometry, is known as the *Klein Correspondence*.

CHAPTER 5

Klein Correspondence

In this section we will establish the *Klein Correspondence*. This is central to the *Twistor Correspondence* and the development of the *Twistor Transform*. We will follow a very mathematical treatment and provide the physical interpretations in the next section.

1. Flag Manifolds and Double Fibration

To establish the Klein correspondence we must generalise the notion of Grassmannians to *flag manifolds* [**11**].

Definition 5.1. *Let V be a complex vector space of dimension $n+1$. Fix a sequence of positive integers $\{0 \leq k_1 \leq k_2 \leq ... \leq k_l \leq n+1\}$. We define a* ***flag manifold*** *of type $(k_1, k_2, ..., k_l)$ with*

(5.1)
$$\mathbf{F}_{k_1 \dots k_l} = \{(W_1, ..., W_l) : W_j \text{ are subspaces } V \text{ of dimension } k_j, W_1 \subset ... \subset W_l\}$$

These manifolds are all compact complex manifolds with a covering by coordinate charts given by matrices.

We can now use the notion of flag manifolds to define the fundamental complex manifolds in twistor geometry. We start with the complex vector space $\mathbb{T}$, the twistor space. For a fixed $\mathbb{T}$, there is a canonical geometric theory derivable from it. This is the basis of twistor geometry in analogy to the role of Hilbert space in Quantum Mechanics.

Definition 5.2. *Define the* ***twistor flag manifold*** *$\mathbb{F}_{12} = \mathbf{F}_{12}(\mathbb{T})$. With the three twistor manifolds $\mathbb{F}_{12}$, $\mathbb{F}_1$ and $\mathbb{F}_2$, we have the* ***double fibration****:*

$$\begin{tikzcd} & \mathbb{F}_{12} \arrow[dl, "\mu"'] \arrow[dr, "\nu"] & \\ \mathbb{F}_1 & & \mathbb{F}_2 \end{tikzcd}$$

where $\mu(W_1, W_2) = W_1$ and $\nu(W_1, W_2) = W_2$ are natural "projection" mappings. We will canonically identify $\mathbb{F}_1 \cong \mathbb{P}^3$ to be the projective twistor space $\mathbb{PT}$, $\mathbb{F}_2 \cong \mathbf{G}_{(2,4)}$ the compactified complexified Minkowski space $\mathbb{M}$ and $\mathbb{F} = \mathbb{F}_{12}$ the ***correspondence space*** *between $\mathbb{PT}$ and $\mathbb{M}$. This gives*

$$\begin{tikzcd} & \mathbb{F} \arrow[dl, "\mu"'] \arrow[dr, "\nu"] & \\ \mathbb{PT} & & \mathbb{M} \end{tikzcd}$$

2. The Klein Correspondence

A double fibration yields a natural correspondence:

$$c : \mathbb{F}_1 \longrightarrow \mathbb{F}_2 \tag{5.2}$$

where $c(p) = \nu \circ \mu^{-1}$ for $p \in \mathbb{F}_1$. The inverse is obvious: $c^{-1}(q) = \mu \circ \nu^{-1}$ for $q \in \mathbb{F}_2$. Then

Definition 5.3. *For $A \subset \mathbb{PT}$, the **correspondence** of A under the double fibration is*

$$\tilde{A} = \nu \circ \mu^{-1}(A) \subset \mathbb{M} \tag{5.3}$$

*The **inverse correspondence** of $A \subset \mathbb{M}$ is*

$$\hat{A} = \mu \circ \nu^{-1}(A) \subset \mathbb{PT} \tag{5.4}$$

We will look at a special case of this geometric transform.

Proposition 5.1. *Let $p \in \mathbb{PT}$ and $q \in \mathbb{M}$. Then*

(1) $\tilde{p} \cong \mathbb{P}^2$
(2) $\hat{q} \cong \mathbb{P}^1$

Proof. This proof follows from [**9**]. We know that $p = S_1$ is a fixed 1d subspace of $\mathbb{T}$. Now the transform gives

$$\tilde{p} = \{W_2 \subset \mathbb{T} : \dim(W_2) = 2 \text{ and } S_1 \subset W_2\} \tag{5.5}$$

where the second condition follows from the definition of the flag manifold. Suppose $Z_0 \in S_1$ is a nonzero vector and choose a basis for $\mathbb{T}$ of the form (Z^0, Z^1, Z^2, Z^3). We let $[w_1, w_2, w_3]$ be the homogenous coordinates for $\mathbb{P}^2$ and define the map

$$w = [w_1, w_2, w_3] \mapsto S_2 = \text{span}\{Z_0, w^1 Z_1 + w^2 Z_2 + w^3 Z_3\} \tag{5.6}$$

From this we see that such $S_1 \subset S_2$ and all S_1 arise from this manner. We hence can naturally identify $W_2 \cong S_2 \cong \mathbb{P}^2$, as required.

Now for (2), we note that for $q = T_2$, where is T_2 is a fixed 2d subspace of $\mathbb{T}$, we then have

$$\hat{q} = \{W_1 \subset \mathbb{T} : W_1 \subset T_2\} \tag{5.7}$$

We know that by definition $q \cong \mathbb{C}^2$. This means that by the definition of flag manifolds, $\hat{q}$ must be isomorphic to the set of 1d[1] subspaces of $\mathbb{C}^2$. This is by definition $\mathbb{P}^1$. Hence the isomorphism is established. □

It is possible to introduce a canonical coordinate representation to the correspondence. Since $\mathbb{M}$ is identified as the Grassmannian manifold $\mathbf{G}_{(2,4)}$, we can introduce canonical Grassmann coordinates as follows [**9**]:

$$z = \left(z^{ij}\right) \in \mathbb{C}^{2\times 2} \mapsto \begin{pmatrix} iz \\ I_2 \end{pmatrix} \tag{5.8}$$

We can define a coordinate chart on $\mathbb{M}$.

[1]Here we are operating in complex dimensions.

DEFINITION 5.4. *The **affine complexified Minkowski space** $\mathbb{M}^I$:*

$$\mathbb{M}^I = \varphi\left(\mathbb{C}^{2\times 2}\right) \cong \mathbb{C}^4 \tag{5.9}$$

*Physically, this means we have removed points at infinity from $\mathbb{M}$ and the origin is now not fixed ("affine"). The pullback along the two maps yields the **affine parts** of the projective twistor space and correspondence manifold:*

$$\mathbb{PT}^I = \mu \circ \nu^{-1}(\mathbb{M}^I) \tag{5.10}$$

$$\mathbb{F}^I = \nu^{-1}(\mathbb{M}^I) \tag{5.11}$$

We then see that $\mathbb{F}^I = \mathbb{M}^I \times \mathbb{P}^1$. So for v the homogenous coordinate on $\mathbb{P}^1$, we can have the mapping

$$(z, [v]) \mapsto \left(\begin{pmatrix} iz \\ I_2 \end{pmatrix} v, \begin{pmatrix} iz \\ I_2 \end{pmatrix}\right) = (W_1, W_2) \in \mathbb{F} \tag{5.12}$$

with the projection $\mu : \mathbb{F}^I \to \mathbb{PT}^I$ given by $\mu : (z, [v]) \mapsto [izv, v]$. This then gives the coordinate system to the double fibration:

$$\begin{tikzcd} & (z,[v]) \in \mathbb{F} \arrow[dl, "\mu"'] \arrow[dr, "\nu"] & \\ {[izv, v] \in \mathbb{PT}} & & z \in \mathbb{M} \end{tikzcd}$$

It is also helpful to establish the dual correspondence using flag manifolds. We can consider

$$\begin{tikzcd} & \mathbf{F}_{23}, \mathbf{F}_{123} \arrow[dl, "\mu"'] \arrow[dr, "\nu"] & \\ \mathbf{F}_3, \mathbf{F}_{13} & & \mathbf{F}_2, \mathbf{F}_2 \end{tikzcd}$$

where we naturally identify the *dual projective twistor space* as

$$\mathbb{PT}^* = \mathbf{F}_3(\mathbb{T}) \cong \mathbf{F}_1(\mathbb{T}^*) \tag{5.13}$$

and the *ambitwistor space* $\mathbb{A} = \mathbf{F}_{13}$ with both twistor space and its dual are embedded in it. Then we have

$$\begin{tikzcd} & \mathbb{F}^* \arrow[dl, "\mu"'] \arrow[dr, "\nu"] & \\ \mathbb{PT}^* & & \mathbb{M} \end{tikzcd}$$

$$\begin{tikzcd} & \mathbb{G} \arrow[dl, "\mu"'] \arrow[dr, "\nu"] & \\ \mathbb{A} & & \mathbb{M} \end{tikzcd}$$

We state the following proposition without proof.

PROPOSITION 5.2. *Let $p \in \mathbb{M}$, $q \in \mathbb{PT}^*$, $r \in \mathbb{A}$. Then*

(1) $\hat{p}_{\mathbb{PT}^*} \cong \mathbb{P}^1$
(2) $\hat{p}_{\mathbb{A}} \cong \mathbb{P}^1 \times \mathbb{P}^1$
(3) $\tilde{q} \cong \mathbb{P}^2$
(4) $\tilde{r} \cong \mathbb{P}^1$

We are ready to consider hypersurfaces in $\mathbb{M}$. We recall that $\mathbb{M}$ is embedded in $\mathbb{P}^5$ by the Plcker embedding and is complex-analytically equivalent to $Q_4 = \{[z] : z \wedge z = 0\}$. By considering the nondegenerate bilinear mapping $\bigwedge^2 \mathbb{T} \times \bigwedge^2 \mathbb{T} \to \bigwedge^4 \mathbb{T} \cong \mathbb{C}$ we can establish an isomorphism

$$\bigwedge^2 \mathbb{T} \cong \left(\bigwedge^2 \mathbb{T}\right)^* \tag{5.14}$$

Then we can describe any hyperplane in $\mathbb{P}\left(\bigwedge^2 \mathbb{T}\right) \cong \mathbb{P}^5$ by fixed $\omega_0 \in \bigwedge^2 \mathbb{T}$:

$$H(\omega_0) = \{\omega \in \bigwedge^2 \mathbb{T} : \omega \wedge \omega_0 = 0\} \tag{5.15}$$

We want to consider intersections of the quadric.

Definition 5.5. *A* ***hyperplane section*** *s of Q_4 is the intersection of a hyperplane H with Q_4:*

$$s(\omega_0) = Q_4 \cap H(\omega_0) = \{\omega \in \bigwedge^2 \mathbb{T} : \omega = Z_1 \wedge Z_2, \omega \wedge \omega_0 = 0\} \tag{5.16}$$

Consider the set of all lines in $\mathbb{P}^5$. We want to consider the distinguished set of lines properly contained in Q_4.

Definition 5.6. *A line L is* ***null*** *if L is contained in $Q_4 \subset \mathbb{P}^5$. We call a tangent vector $X \in T_p(Q_4)$* ***null*** *if X is tangent to a null line in Q_4.*

Proposition 5.3. *Let $\bar{T}_p(Q_4)$ be the tangent plane to $p \in Q_4$. Then*

$$\bar{T}_p(Q_4) \cap Q_4 = \{\textit{null lines passing through } p\} \tag{5.17}$$

Proof. A quick way to see this is to note that for any point $r \in \bar{T}_p(Q_4) \cap Q_4$ the quadratic form of q defining Q_4 will give $q(p,r) = q(p,r) = q(r,r) = 0$. A straight line homotopy between p, r will give the same result [**9**]

$$q(tp + (1-t)r, sp + (1-s)r) = 0 \tag{5.18}$$

for any $s, t \in [0,1]$ so the line joining p and r will be properly contained in the set. □

Now we come to the important assertion:

Proposition 5.4. *For $p, q \in \mathbb{M}$,*

$$\hat{p} \cap \hat{q} \textit{ in } \mathbb{PT} \iff p, q \textit{ lie on a null line in } \mathbb{M} \tag{5.19}$$

Proof. Refer to [**9**]. Consider a point $p \in Q_4$, then $p = [Z_1 \wedge Z_2]$ for some $Z_1, Z_2 \in \mathbb{T}$. Let

$$s(p) = \{q \in \mathbb{M} : \hat{q} \cap \hat{q} \neq \emptyset\} \tag{5.20}$$

So $s(p)$ is a subset of $\mathbb{M}$ containing the point p. If $q = [Z, W] \leftrightarrow [Z \wedge W] \in Q_4$, then consider

$$Z \wedge W \wedge Z_1 \wedge Z_2 = 0 \tag{5.21}$$

This means the planes spanned by $S_2 = [Z, W]$ and $S_2' = [Z_1, Z_2]$ must intersect in a 1d subspace of $\mathbb{T}$. Then $\hat{q} = [S_2]$, $\hat{p} = [S_2']$ and

$$\hat{q} \cap \hat{p} = [S_1] \neq \emptyset \tag{5.22}$$

We have shown that (the converse is just running the argument backwards)

$$s(p) = \{q \in Q_4 : q \wedge Z_1 \wedge Z_2 = 0\} \tag{5.23}$$

Note that $\bar{T}_p(Q_4) = \{\omega : \omega \wedge Z_1 \wedge Z_2 = 0$, so in particular

$$s(p) = \bar{T}_p(Q_4) \cap Q_4 \tag{5.24}$$

The result follows from Proposition 5.3. □

DEFINITION 5.7. *Let $p \in \mathbb{PT}$ and $q \in \mathbb{PT}^*$. Then $\tilde{p}$ is an α-plane in $\mathbb{M}$. Similarly Then $\tilde{q}$ is an β-plane in $\mathbb{M}$.*

We finally have the theorem that relates null lines and planes in Q_4 to α- and β-planes.

THEOREM 5.1.
(1) *All α- and β-planes in $\mathbb{M}$ are null planes.*
(2) *Any null planes is either an α- or β-plane.*
(3) *The null lines in $\mathbb{M}$ are precisely the 5d family of lines parametrised by $\mathbb{A}$ of the form $\tilde{f}$ for $f \in \mathbb{A}$.*
(4) *Any null line in $\mathbb{M}$ is the intersection of an α- and a β-plane.*

PROOF. See Theorem 1.4.1 of [**9**]. □

Propositions 5.1, 5.2 and 5.4 together with Theorem 5.1 is the content of the *Klein correspondence*. For a full discussion refer to [**9**].

CHAPTER 6

Physical Interpretations of the Twistor Correspondence

In this short section we establish the consequences and the physical interpretations of the Klein correspondence - now interpreted in twistor geometry.

1. Summary of the Twistor Correspondence

The Klein Correspondence discussed in §5 provides a natural correspondence between the projective twistor space and the complexfied compactified Minkowski space. In an earlier paper by Penrose [**1**] this correspondence, known as the *twistor correspondence* allows identification of subspaces between the so-called C-space ($\mathbb{PT}$) and M-space ($\mathbb{M}$). We make a summary of the results in the following table:

Projective Twistor Space $\mathbb{PT}$	$\mathbb{C}$-Compactified Minkowski Space $\mathbb{M}$
point	α-plane
line	point
plane	β-plane
point on plane/ plane pencil	null geodesic $= \alpha \cap \beta$
intersection of lines	points are null separated

We see that null lines can be naturally represented as twistors and the null cone structure (conformal structure) can be deduced directly from the corresponding projective twistor space $\mathbb{PT} \cong \mathbb{P}^3$. We see that this is consistent with the massless momentum interpretation of twistors as shown in §3.

2. More Geometry

We can consider additional geometrical properties of $\mathbb{M}$ given the correspondence [**1**, **2**, **4**].

DEFINITION 6.1. *A twistor is* ***null*** *if*

$$Z^{\alpha}\bar{Z}_{\alpha} = 0 \tag{6.1}$$

i.e. it has nil helicity.

These twistors represent a null straight line, the worldline a massless particle of zero spin [**4**]. In fact, the constraint 6.1 gives a 5d submanifold N of $\mathbb{PT}$ defining these null twistors (Figure **??**). This is the space of real null lines in $\mathbb{PT}$. It is possible to show that[1]

[1]It is puzzling why the complexification process only increases the real dimensionality of the null line system from 5 to 6. This is because this "complexification" process is not the one as described above but simply a label to illustrate that lines in $\mathbb{M}^{\pm}$ have a correspondence of a point in $\mathbb{M}/\mathbb{M}_{\mathbb{R}}$ where $\mathbb{M}_{\mathbb{R}}$ is the real compactified space. The corresponding point $x^a \in \mathbb{M}$ has a complex

PROPOSITION 6.1. *The removal of N from $\mathbb{PT}$ creates two disjoint open subsets of real-dimension 6:*

$$\mathbb{PT}/N = \mathbb{PT}^+ \sqcup \mathbb{PT}^- \tag{6.2}$$

Each of the open subsets has topology $S^2 \times \mathbb{E}^4$ and they are known as the spaces of "complexified" null lines in $\mathbb{M}$.

3. Physical Consequences of the Twistor Correspondence

What is the geometrical significance to the correspondence? This is best illustrated by *Kerr's Theorem*. First, we note that the zero set of an analytic function $f(Z^\alpha)$ of twistor variable is well defined on $\mathbb{PT}$ provided that f is homogenous of some degree. This defines a complex analytic surface Q in $\mathbb{PT}$. This set Q is 4d and will intersect N in a 3d region K. This K is a null congruence in Minkowski space. Then

THEOREM 6.1. *(Kerr) Such a congruence K is shear-free. Conversely, any shear-free congruence of null lines in Minkowski is obtainable by this manner.*

PROOF. (From [**7**].) We note that given $f(Z^\alpha)$, where $Z^\alpha = (\omega^A, \pi_{A'})$ for some real point $x^{AA'}$, Z^α defines a null geodesic through this point in the direction of $\bar{\pi}_A \pi_{A'}$. Then we can define K as

$$f(ix^{AA'}\pi_{A'}, \pi_{A'}) = 0 \tag{6.3}$$

and this allows us to obtain $\pi_{A'}$ as a function of $x^{AA'}$, which defines a congruence of null lines by specifying the direction of null lines at each spacetime point. □

We remark that the shear-free condition appears as the Minkowski picture realisation of the Cauchy-Riemann equations for $\mathbb{PT}$. This illustrates the idea that the underlying complex structure provides physical interpretations, as in quantum mechanics.

We have spent a lot of time trying to establish twistors and the *Twistor Correspondence*. The reason, as it turns out, is that this correspondence allows us to compute solutions to physical equations in a straight-forward manner using algebraic geometry techniques.

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component and is future-/past-pointing, provided that x^a is causal. if the corresponding line lies in $\mathbb{PT}^\pm \cup N$ respectively [**2**].

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