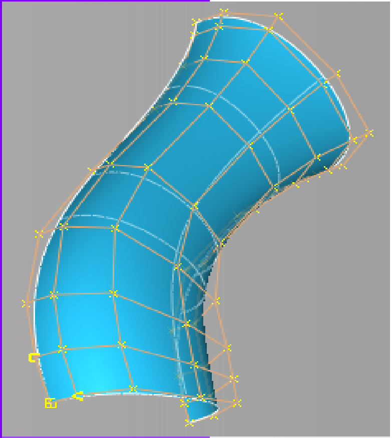
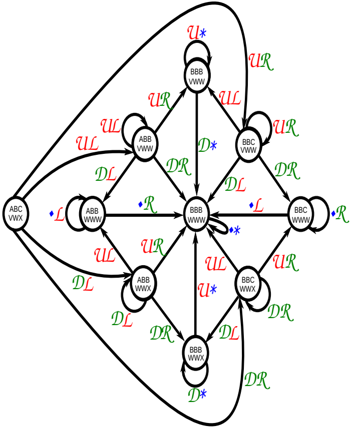
Representation of NURBS as Controlled Iterated Functions Systems





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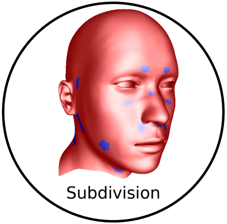
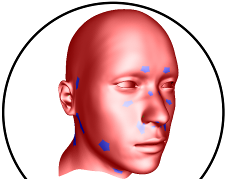
LE2I - University of Burgundy

Curves & Surfaces 2018

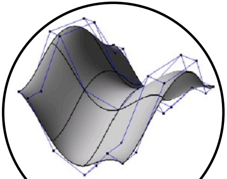
MOTIVATION



MOTIVATION

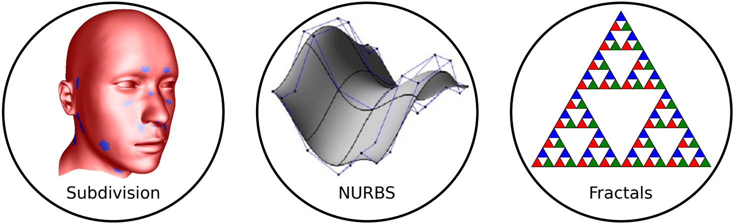


Subdivision

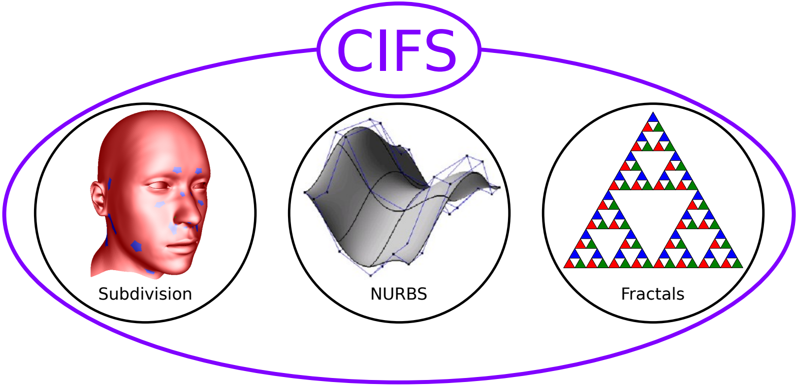


NURBS

MOTIVATION



Motivation



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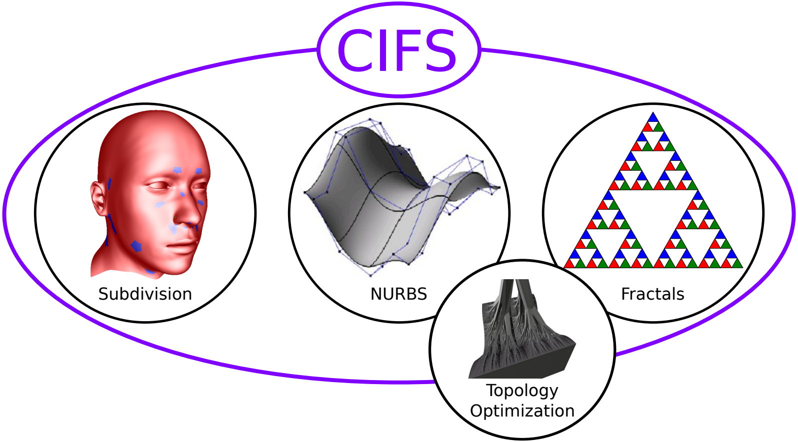
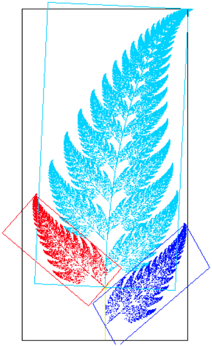


ITERATED FUNCTIONS SYSTEMS (IFS)

Definition

- An IFS is a set of contractive transformations $\{T_0 \ldots T_n\}$ iteratively applied to a compact
- After an infinity of iterations, the same self-similar structure appears whatever the starting compact is.
- This structure, called the attractor, is unique and only depends on the transformations

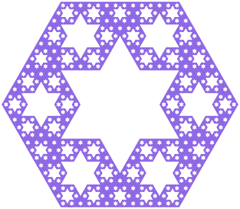


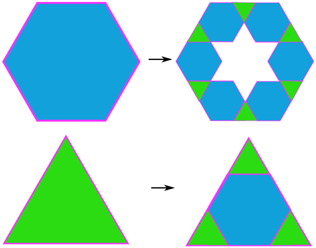
Barnsley fern
©Wikipedia

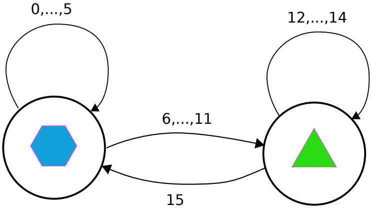
CONTROLLED IFS AUTOMATONS

CIFS automatons are created in this way :

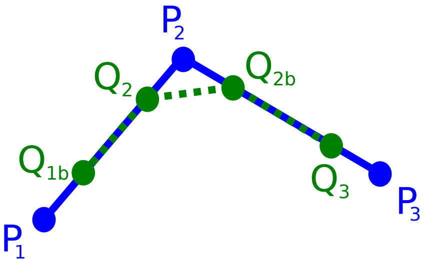
- Every state is an attractor
- Every transition is a transformation







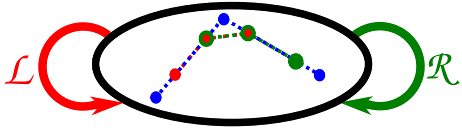
UNIFORM QUADRATIC B-SPLINES : CHAIKIN ALGORITHM



Let a uniform quadratic B-Spline defined by the control polygon $P = [P_1, P_2, P_3]$.
It is also defined by the polygon $Q = [Q_{1b}, Q_2, Q_{2b}, Q_3]$:

$$Q_i = \frac{1}{4}P_{i-1} + \frac{3}{4}P_i \qquad \text{et} \qquad Q_{ib} = \frac{3}{4}P_i + \frac{1}{4}P_{i+1}$$

UNIFORM QUADRATIC B-SPLINES : CIFS AUTOMATON



$$\mathcal{L} : P = [P_1, P_2, P_3] \mapsto Q_{\mathcal{L}} = [Q_{1b}, Q_2, Q_{2b}]$$
$$\mathcal{R} : P = [P_1, P_2, P_3] \mapsto Q_{\mathcal{R}} = [Q_2, Q_{2b}, Q_3]$$

$$M_{\mathcal{L}} = \begin{pmatrix} \frac{3}{4} & \frac{1}{4} & 0 \\ \frac{1}{4} & \frac{3}{4} & 0 \\ 0 & \frac{3}{4} & \frac{1}{4} \end{pmatrix} \qquad M_{\mathcal{R}} = \begin{pmatrix} \frac{1}{4} & \frac{3}{4} & 0 \\ 0 & \frac{3}{4} & \frac{1}{4} \\ 0 & \frac{1}{4} & \frac{3}{4} \end{pmatrix}$$

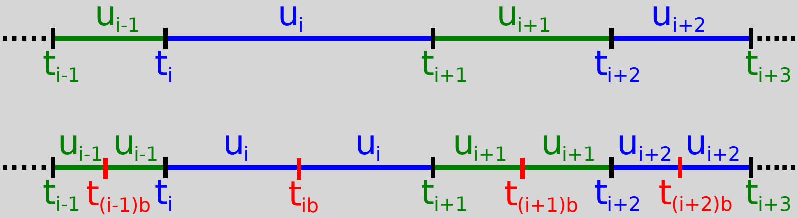
NON UNIFORM RATIONAL B-SPLINES

Definition

A NURBS of degree d composed in m pieces is defined by :

$$\begin{cases} \text{Control polygon} & : P = [P_0 \dots P_{n-1}] & n & = m + d \\ \text{Nodal vector} & : T = [t_0 \dots t_{v-1}] & v & = n + d - 1 \\ \text{Inter-nodal vector} & : U = [u_0 \dots u_{v-2}] & u_i & = t_{i+1} - t_i \end{cases}$$

Mid-node insertions ⇔ Inter-node doublings



BLOSSOMING (RAMSHAW, 1987)

Blossoming functions $\mathcal{B}(t_i \ldots t_j) = \{t_i \ldots t_j\}$

Properties

Symmetry	$\{\ldots t_i \ldots t_j \ldots\} = \{\ldots t_j \ldots t_i \ldots\}$
Diagonal	$\mathcal{C}(t) = \{t \ldots t\}$
Multi-affinity	$\{\ldots t \ldots\} = \frac{b-t}{b-a}\{\ldots a \ldots\} + \frac{t-a}{b-a}\{\ldots b \ldots\}$
Consecutivity	$P_i : \{t_i, t_{i+1} \ldots t_{i+d-1}\}$

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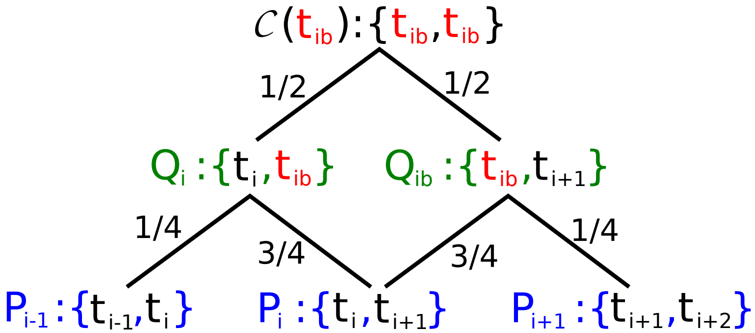
LIMIT CURVE POINT OF A UNIFORM QUADRATIC B-SPLINE

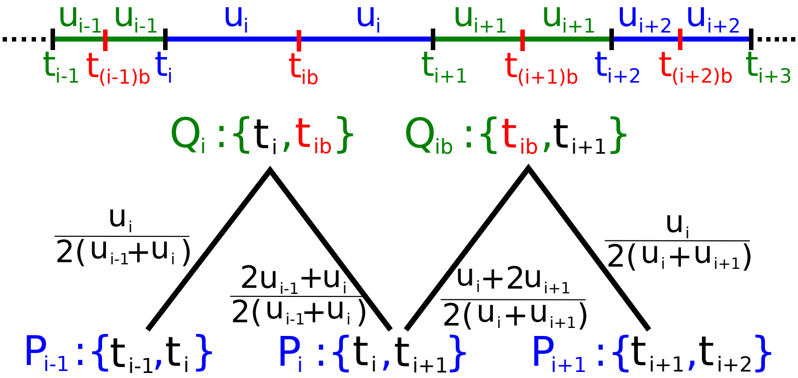
We compute $\mathcal{C}\left(t_{ib} = \frac{t_i + t_{i+1}}{2}\right)$ as a function of the control polygon P



- $\mathcal{C}(t_{ib})$ can also be defined as a function of the polygon Q
- Q is computed as a function of P thanks to Chaikin algorithm

1. Introduction
2. **Quadratic and cubic NURBS**
3. Generalization
4. Conclusion

MID-NODE INSERTIONS FOR QUADRATIC NURBS



$$\begin{cases} Q_i = \frac{u_i}{2(u_{i-1}+u_i)} P_{i-1} + \frac{2u_{i-1}+u_i}{2(u_{i-1}+u_i)} P_i \\ Q_{ib} = \frac{u_i+2u_{i+1}}{2(u_i+u_{i+1})} P_i + \frac{u_i}{2(u_i+u_{i+1})} P_{i+1} \end{cases}$$

NON-UNIFORM QUADRATIC TRANSFORMATIONS

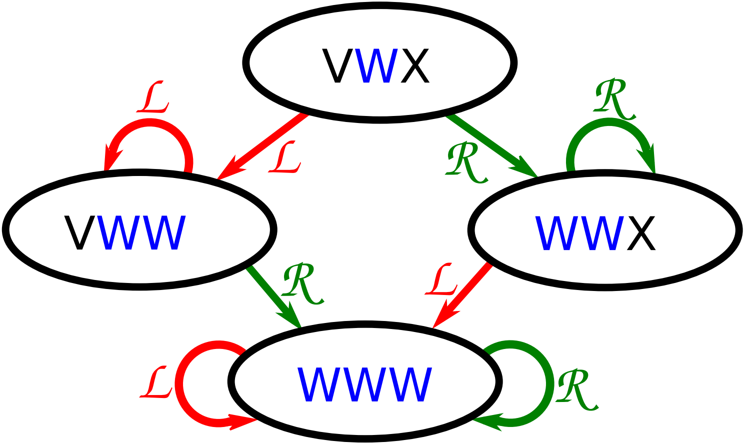
$$\mathcal{L} : \begin{cases} P = [P_1, P_2, P_3] \\ U = [u_1, u_2, u_3] \end{cases} \mapsto \begin{cases} Q_{\mathcal{L}} = [Q_{1b}, Q_2, Q_{2b}] \\ V_{\mathcal{L}} = [u_1, u_2, u_2] \end{cases}$$

$$\mathcal{R} : \begin{cases} P = [P_1, P_2, P_3] \\ U = [u_1, u_2, u_3] \end{cases} \mapsto \begin{cases} Q_{\mathcal{R}} = [Q_2, Q_{2b}, Q_3] \\ V_{\mathcal{R}} = [u_2, u_2, u_3] \end{cases}$$

$$M_{\mathcal{L}}(u_1, u_2, u_3) = \begin{pmatrix} \frac{u_1+2u_2}{2(u_1+u_2)} & \frac{u_1}{2(u_1+u_2)} & 0 \\ \frac{u_2}{2(u_1+u_2)} & \frac{2u_1+u_2}{2(u_1+u_2)} & 0 \\ 0 & \frac{u_2+2u_3}{2(u_2+u_3)} & \frac{u_2}{2(u_2+u_3)} \end{pmatrix}$$

$$M_{\mathcal{R}}(u_1, u_2, u_3) = \begin{pmatrix} \frac{u_2}{2(u_1+u_2)} & \frac{2u_1+u_2}{2(u_1+u_2)} & 0 \\ 0 & \frac{u_2+2u_3}{2(u_2+u_3)} & \frac{u_2}{2(u_2+u_3)} \\ 0 & \frac{u_3}{2(u_2+u_3)} & \frac{2u_2+u_3}{2(u_2+u_3)} \end{pmatrix}$$

QUADRATIC NURBS CIFS AUTOMATON



NON-UNIFORM CUBIC TRANSFORMATIONS

Cubic NURBS are defined at least by :

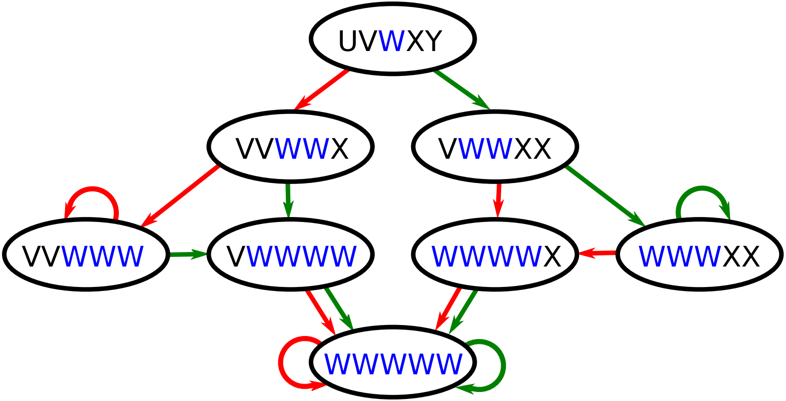
- 4 control points ;
- 6 nodes ;
- 5 inter-nodes.

This correspond to the two transformations $\mathcal{L}$ and $\mathcal{R}$:

$$\mathcal{L} : \begin{cases} P = [P_0, P_1, P_2, P_3] \\ U = [u_0, u_1, u_2, u_3, u_4] \end{cases} \mapsto \begin{cases} Q_{\mathcal{L}} = [Q_{1b}, Q_2, Q_{2b}, Q_3] \\ V_{\mathcal{L}} = [u_1, u_1, u_2, u_2, u_3] \end{cases}$$

$$\mathcal{R} : \begin{cases} P = [P_0, P_1, P_2, P_3] \\ U = [u_0, u_1, u_2, u_3, u_4] \end{cases} \mapsto \begin{cases} Q_{\mathcal{R}} = [Q_2, Q_{2b}, Q_3, Q_{3b}] \\ V_{\mathcal{R}} = [u_1, u_2, u_2, u_3, u_3] \end{cases}$$

CUBIC NURBS CIFS AUTOMATON

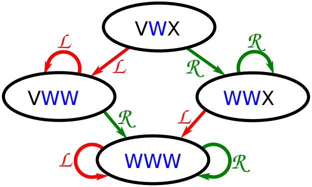


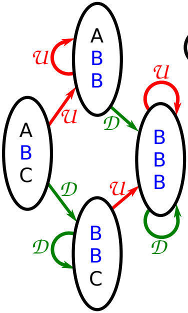
GENERALIZATION TO ANY DEGREE BY DOUBLING INTER-NODES

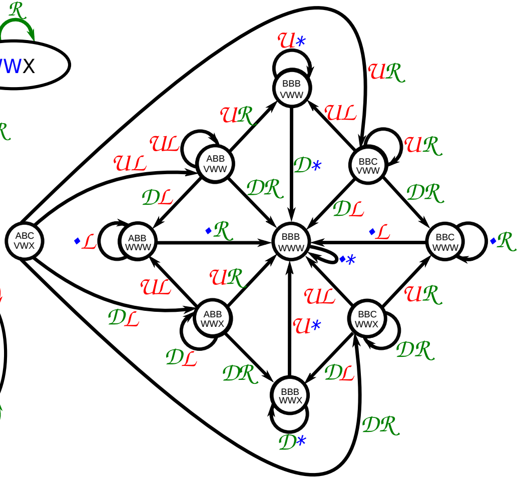
degree 1 :	W	$\mapsto$	WW
		$\mapsto$	WW
degree 2 :	VWX	$\mapsto$	VVWWXX
		$\mapsto$	VVWWXX
degree 3 :	UVWXY	$\mapsto$	UUVVWWXXYY
		$\mapsto$	UUVVWWXXYY
degree 4 :	TUVWXYZ	$\mapsto$	TTUUVVWWXXYYZZ
		$\mapsto$	TTUUVVWWXXYYZZ

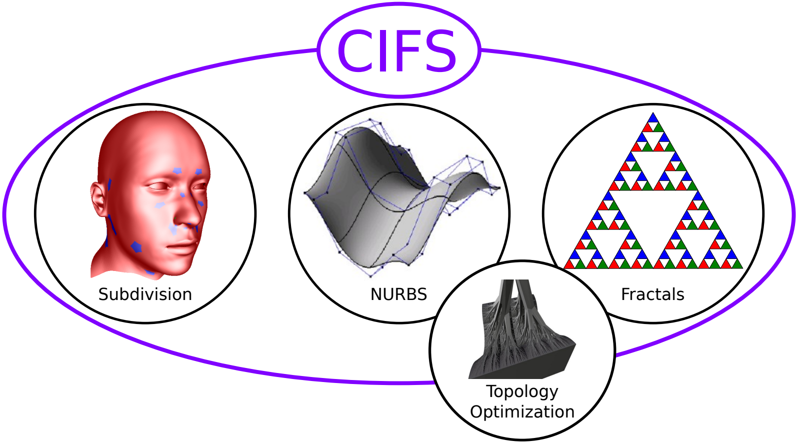
- The length of the inter-node vector is $(2d - 1)$
- The number of states of the CIFS automaton is $4(d - 1)$

GENERALIZATION TO SURFACES BY "TENSOR-PRODUCT"





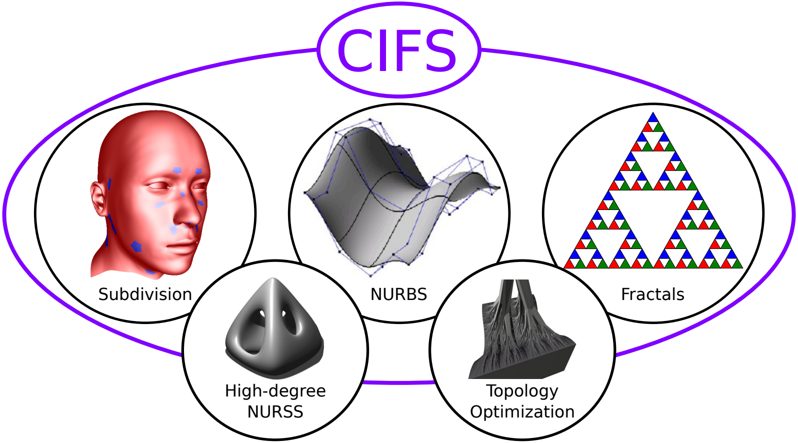




Main interest of this new formalism

- Several parametric surfaces are handle in a same way
- Common representation permits common tools

CONCLUSION



Work in progress

- 2009, Cashman : NURBS with Extraordinary Points, High-degree non-uniform Rationnal Subdivision Schemes

MAIN REFERENCES

- 1988, Barnsley : Fractals everywhere
- 2015, Sokolov : Boundary Controlled Iterated Function Systems
- 2018, Morlet : Barycentric Combinations Based Subdivision Surfaces

Thanks for your attention

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