Continuum mechanics

In a flow problem on a 2D domain Ω, the primary physical variable is the velocity field $u : \Omega \to \mathbb{R}^2$. This field evolves in accord with continuum versions of Newton's laws of motion. The Cauchy momentum equation is the $F = ma$ of continuum mechanics. It can be stated as an integral conservation law quantified over all pieces of space, $\Omega_0 \subset \Omega$:

$$\frac{d}{dt}\int_{\Omega_0(t)} \rho u(\hat{x})\, d\hat{x} = \int_{\Omega_0} F(\hat{x})\, d\hat{x} + \oint_{\partial\Omega_0} \sigma(\hat{x}) \cdot \hat{n}\, d\hat{x}.$$

This says that the rate of change of total momentum (pointwise ρu) in the piece Ω_0, as it moves with the material, is accounted for by body forces (pointwise F) and the *tractions* (pointwise $\sigma \cdot \hat{n}$ on the boundary of the piece of space), which measure internal forces due to the interaction of nearby material elements. σ, a 2×2 matrix for each point in Ω, is the *Cauchy stress tensor*, and its specification determines the properties of the material model. Conservation of mass is written as

$$\frac{d}{dt}\int_{\Omega_0(t)} \rho\, d\hat{x} = 0,$$

which simply says that the piece $\Omega_0(t)$ has constant mass.

The Stokes equations

The incompressible Navier-Stokes equations are formed by a Cauchy momentum equation ($F = ma$), along with a stronger version of mass conservation, incompressibility (conservation of volume):

$$\oint_{\partial\Omega_0} u \cdot \hat{n}\, d\hat{x} \qquad \text{for all pieces } \Omega_0 \in \Omega.$$

These equations model an incompressible Newtonian fluid with traction forces $\sigma \cdot \hat{n}$ being decomposable into two parts:

- A viscous force (which causes adjacent particles in the fluid to tend to the same speed, like a "velocity diffusion"),
- and an isotropic force which pushes small pieces of the material apart from eachother, in order for the flow to obey the incompressibility constraint.

This second force is due to the pressure field $p : \Omega \to \mathbb{R}$, a Lagrange multiplier for the incompressibility constraint.

$$\int_{\Omega_0} \rho \frac{\partial u(\hat{x})}{\partial t}\, d\hat{x} + \oint_{\Omega_0} \rho u(u \cdot \hat{n})\, d\hat{x} \int_{\Omega_0} F(\hat{x})\, d\hat{x} + \oint_{\partial\Omega_0} \sigma(\hat{x}) \cdot \hat{n}\, d\hat{x}.$$

The Cauchy momentum equation is the $F = ma$ of continuum mechanics. It is an integral conservation law

$$\frac{d}{dt} \int_{\Omega_0(t)} \rho u\, d\hat{x} = \int_{\Omega_0} \rho g\, d\hat{x} + \oint_{\partial\Omega_0} \sigma \hat{n}\, d\hat{x}$$

Conservation of mass an incompressibility

$$\nabla \cdot u = 0.$$

nice

third

fourth