Continuum mechanics

In a flow problem on a 2D domain Ω, the primary physical variable is the velocity field $u : \Omega \to \mathbb{R}^2$. This field evolves in accord with continuum versions of Newton's laws of motion. The Cauchy momentum equation is the $F = ma$ of continuum mechanics. It can be stated as an integral conservation law quantified over all pieces of space, $\Omega_0 \subset \Omega$:

$$\frac{d}{dt}\int_{\Omega_0(t)} \rho u(\hat{x})\, d\hat{x} = \int_{\Omega_0} F(\hat{x})\, d\hat{x} + \oint_{\partial\Omega_0} \sigma(\hat{x}) \cdot \hat{n}\, d\hat{x}.$$

This says that the rate of change of total momentum (pointwise ρu) in the piece Ω_0, as it moves with the material, is accounted for by body forces (pointwise F) and the *tractions* (pointwise $\sigma \cdot \hat{n}$ on the boundary of the piece of space), which measure internal forces due to the interaction of nearby material elements. σ, a 2×2 matrix for each point in Ω, is the *Cauchy stress tensor*, and its specification depends on the material being modelled. Conservation of mass is written as

$$\frac{d}{dt}\int_{\Omega_0(t)} \rho(\hat{x})\, d\hat{x} = 0,$$

which says that the a small piece Ω_0, as it moves with the material, has constant total mass.

The Navier-Stokes equations

The incompressible Navier-Stokes equations are a Cauchy momentum equation ($F = ma$), along with a stronger version of mass conservation called incompressibility (conservation of volume):

$$\oint_{\partial\Omega_0} u(\hat{x}) \cdot \hat{n}\, d\hat{x} = 0 \qquad \text{for all pieces } \Omega_0 \subset \Omega.$$

These equations model an incompressible Newtonian fluid with traction forces $\sigma \cdot \hat{n}$ being decomposable into two parts:

- A viscous force (which causes adjacent particles in the fluid to tend to the same speed, like a "momentum diffusion"),
- and an isotropic force which pushes small pieces of the material apart from each other, in order for the flow to obey the incompressibility constraint.

This second force is due to the pressure field $p : \Omega \to \mathbb{R}$, which is a Lagrange multiplier for the constraint. The Navier-Stokes equations, in integral "primal-dual" form, to be solved for both u and the Lagrange multiplier p, are

$$\begin{aligned} \frac{d}{dt}\int_{\Omega_0(t)} \rho u(\hat{x})\, d\hat{x} &= \int_{\Omega_0} F(\hat{x})\, d\hat{x} + \oint_{\partial\Omega_0} \mu \nabla u(\hat{x}) \cdot \hat{n}\, d\hat{x} - \oint_{\partial\Omega_0} p(\hat{x})\hat{n}\, dx \\ \text{and} \quad \oint_{\partial\Omega_0} u(\hat{x}) \cdot \hat{n}\, d\hat{x} &= 0 \qquad \text{for all pieces } \Omega_0 \subset \Omega. \end{aligned} \tag{1}$$

μ is the dynamic viscosity, which controls the momentum diffusion force.

Steady flow and the steady Stokes equations

A steady flow is one for which the velocity u does not change with time. This means that a particle moving through the fluid will always follow the same path. The left-hand side of the Navier-Stokes equations (1) describes the change of momentum in the piece Ω_0 as it moves with the material. The *Reynolds transport theorem* rephrases this as a change of momentum internal to the piece Ω_0, if it were to stay still, and the total momentum that the piece loses or gains at the boundary as it flows: (1) is rewritten as

$$\cancel{\int_{\Omega_0} \rho \frac{\partial u(\hat{x})}{\partial t}} \, d\hat{x} + \oint_{\partial\Omega_0} \rho u(u \cdot \hat{n}) \, d\hat{x} = \int_{\Omega_0} F(\hat{x}) \, d\hat{x} + \oint_{\partial\Omega_0} \mu \nabla u(\hat{x}) \cdot \hat{n} \, d\hat{x} - \oint_{\partial\Omega_0} p\hat{n} \, dx$$

for which the time-dependent component is set to zero, resulting in the steady Navier-Stokes equations. There is still a non-linear term on the left-hand side, describing the advection (the fluid is still moving, even though it is steady):

$$\text{Non-linear!} \longrightarrow \oint_{\partial\Omega_0} \rho u(u \cdot \hat{n}) \, d\hat{x} \longleftarrow \text{Non-linear!}$$

If this term is ignored, the resulting equations are linear and model a steady-state flow for which there is a well-defined velocity, but the material doesn't "actually move" — this turns out to be a useful approximation if the *Reynolds number* is small:

$$Re = \frac{\rho \, \|u\| \, L}{\mu} \ll 1 \quad \Longrightarrow \quad \text{momentum diffusion} \gg \text{momentum advection.}$$

The result is the steady Stokes equations:

$$\begin{aligned} -\oint_{\partial\Omega_0} \mu \nabla u(\hat{x}) \cdot \hat{n} \, d\hat{x} &= \int_{\Omega_0} F(\hat{x}) \, d\hat{x} - \oint_{\partial\Omega_0} p(\hat{x})\hat{n} \, dx \\ \text{and} \quad \oint_{\partial\Omega_0} u(\hat{x}) \cdot \hat{n} \, d\hat{x} &= 0 \qquad \text{for all pieces } \Omega_0 \subset \Omega. \end{aligned} \tag{2}$$

The finite element method for flow problems: A network of pieces of space

Continuum mechanics provides the foundation for understanding of fluid motion, and the Navier-Stokes equations are the primary model for the prediction of fluid behaviour in engineering. The domain and initial/boundary conditions can be arbitrarily complex. For example, fluid motion is often simulated through a digital surface model of a real-world object or system created in computer-aided design software. On the other side of the coin, fluid models are very often used in modern film, both live action and animated, and all relevant geometry can be adjusted by artists for a desired visual effect.

In applications, the Navier-Stokes equations are primarily solved on a computer, and a computer works with both finite data and finite precision. Galerkin methods are particularly amenable to computer implementation, so much so that the whole process of geometric modelling, boundary condition handling, residual minimisation, and solution reconstruction, can be automated. The finite volume method and finite element method are

Solving the Stokes equations numerically

The solution (u, p) has infinite degrees of freedom. The conservation laws are quantified over arbitrary pieces of the domain, giving an infinite number of residual equations. These residuals can be thought of as "trials", which check if a supposed solution satisfies the conservation laws. Why not choose finite degrees of freedom for approximate $(\tilde{u}, \tilde{p})$, and the same number of trial equations to go with them? Suppose $\tilde{u} \in \Phi^u$ and $\tilde{p} \in \Phi^p$, where

$$\Phi^u = \operatorname{span}\left\{\phi_1^u, \cdots, \phi_{2n_u}^u\right\}, \quad \Phi^p = \operatorname{span}\left\{\phi_1^p, \cdots, \phi_{n_p}^p\right\}$$

are, respectively, spaces of vector velocity fields and of scalar pressure fields, both compatible with the integral equations. In an attempt to find an alternative finite set of equations, the Stokes equations can be written in the *weak form*

$$\begin{aligned} &\int_\Omega \mu \nabla u(\hat{x}) \cdot \nabla \psi^u(\hat{x})\, d\hat{x} = \int_\Omega F(\hat{x}) \psi^u \, d\hat{x} + \int_\Omega p(\hat{x}) \nabla \cdot \psi^u \, dx \quad \text{for all } \psi^u \in \Psi^u \\ &\text{and} \quad -\int_\Omega u(\hat{x}) \cdot \nabla \psi^p \, d\hat{x} = 0 \quad \text{for all } \psi^p \in \Psi^p. \end{aligned} \tag{3}$$

This is a sort of "rearrangement" of the infinite system of equations which was given in terms of fluxes (analogous to a rebasis of a linear system). For the continuous PDE, the "trial spaces" Ψ^u and Ψ^p are certain generalised function spaces. For the discretized method, why not choose $\Psi^u = \Phi^u$ and $\Psi^p = \Phi^p$?

A mixed finite element method

The Stokes flow problem at the top of this poster is a Dirichlet boundary value problem, with a no-slip condition ($u = (0, 0)$) on the top and bottom walls, and unit-speed in-flow and out-flow $u = (-1, 0)$ at the left and right boundaries. This approximates the effect of motion ($\dot{x} = 1$) of the obstruction through an infinite cylinder, as long as the boundaries are far enough away.

This problem has been solved with a *finite element method*, effectively by choosing some discrete function spaces, plugging in approximations to the weak form of the Stokes equations, then rearranging everything into a large, sparse linear system. In a finite element method, "test spaces" $\Phi^{\cdot}$ and "trial spaces" $\Psi^{\cdot}$ are constructed as the spans of basis functions built on a domain tessellation (such as a triangulating mesh). These basis functions are zero except on a small number of triangles localised around some point on the mesh, a property which leads to sparsity of the resulting matrix and right-hand side.

Let the Ψ spaces be equal to the Φ spaces (called a "Bubnov–Galerkin method"). The velocity space Φ^u, for simplicity, can be a product space of scalar function spaces: $\Phi^u = \Phi^{u,s} \times \Phi^{u,s}$, with basis functions

$$\Phi^u = \operatorname{span}\left\{\phi^u_{1x}, \phi^u_{1y}, \cdots, \phi^u_{n_u x}, \phi^u_{n_u y}\right\}.$$

The boundary values can be accounted for by letting $\tilde{u}_\Gamma$ be some arbitrary approximating function which satisfies the boundary condition, and restricting Φ^u to be zero on the boundary. The resulting linear system is the one above.

$$\begin{aligned}\sum_{i=1}^{n_u}\left[u_{ix}\int_\Omega \mu\nabla\phi^u_{ix} : \nabla\psi^u_j\, d\hat{x} + u_{iy}\int_\Omega \mu\nabla\phi^u_{iy} : \nabla\psi^u_j\, d\hat{x}\right] + \sum_{i=1}^{n_p} p_i \int_\Omega \phi^p_i \nabla\cdot\psi^u_j\, d\hat{x} = \int_\Omega \nabla\tilde{u}_\Gamma : \nabla\psi^u_j\, d\hat{x},\\ \text{for } j = 1, \cdots, 2n_u.\\ \sum_{i=1}^{n_u}\left[u_{ix}\int_\Omega \psi^p_k\nabla\cdot\phi^u_{ix}\, d\hat{x} + u_{iy}\int_\Omega \psi^p_k\nabla\cdot\phi^u_{iy}\, d\hat{x}\right] = \int_\Omega \psi^p_k\nabla\cdot\tilde{u}_\Gamma\, d\hat{x},\\ \text{for } k = 1, \cdots, n_p.\end{aligned} \tag{4}$$

Since pressure is defined up to a constant, this linear system will have at most rank $2n_u + p_n - 1$, but, if it is well-formed, it can be solved in the least-squares sense for a valid solution. This is a *mixed finite element method*, as the solution is solved for a function in a "mixed finite element space" $\Phi^u \times \Phi^p$. This method is also an instance of a primal-dual method, as the mixed space is introduced in order to approximate the primal variable u and the Lagrange multiplier p simultaneously.

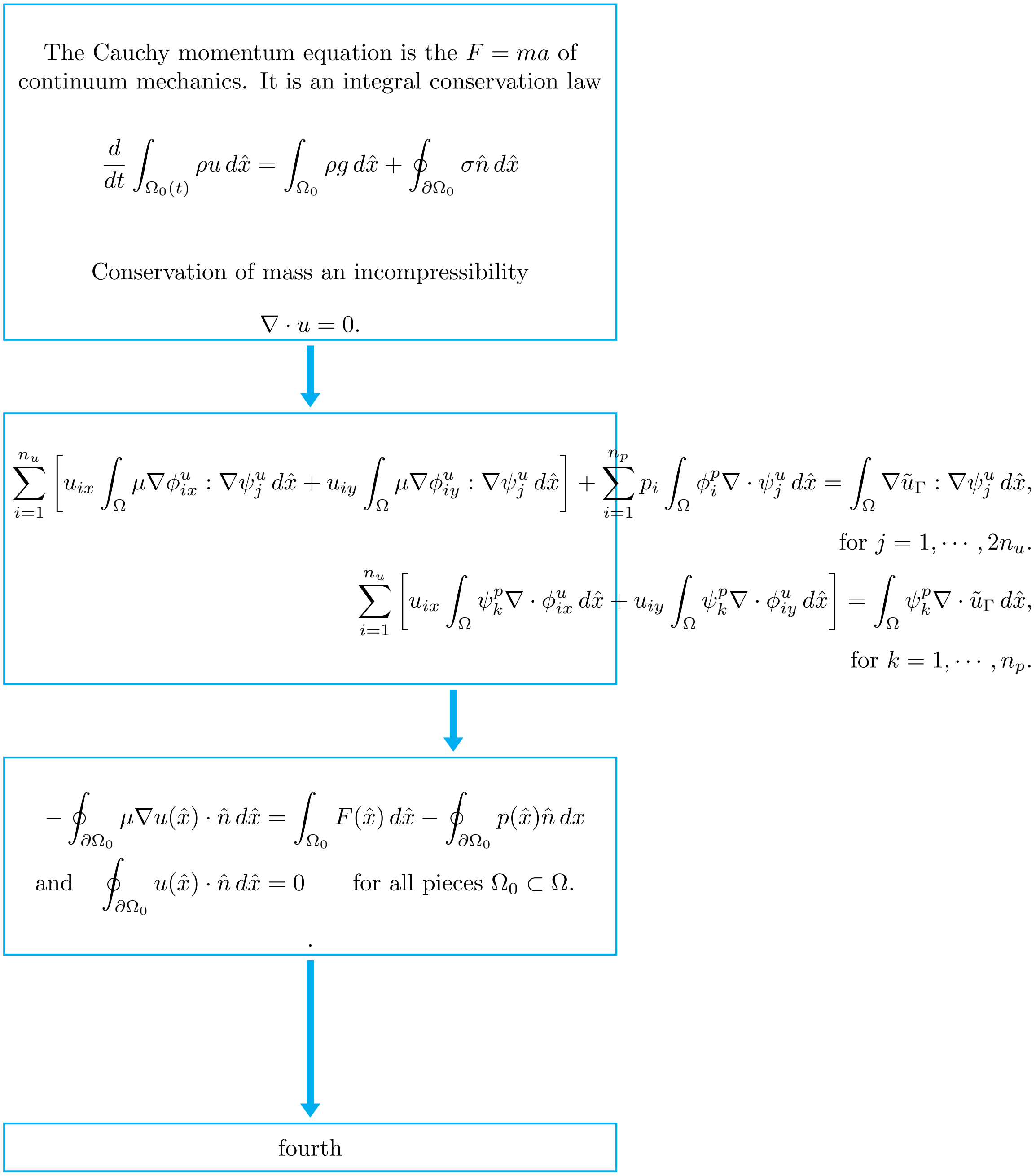
Taylor-Hood mixed finite elements and implementation

One possible stable finite element scheme uses piecewise quadratic velocity fields and piecewise linear pressure fields, called *Taylor-Hood elements.* The velocity field is sampled at midpoints of a triangulation as well as at vertices. In a routine manner, the linear system is calculated by traversing a mesh data structure to find overlapping domains (as each basis function has compact support), separating the integrals into polynomial integrals over triangles, performing a change of variables to a "reference triangle", and analytically integrating over each triangle. Once the sparse linear system is formed, it is sent to a sparse linear solver.

$\tilde{p}$

$\tilde{u}_x$

$\tilde{u}_y$



The Cauchy momentum equation is the $F = ma$ of continuum mechanics. It is an integral conservation law

$$\frac{d}{dt}\int_{\Omega_0(t)} \rho u \, d\hat{x} = \int_{\Omega_0} \rho g \, d\hat{x} + \oint_{\partial\Omega_0} \sigma \hat{n} \, d\hat{x}$$

Conservation of mass an incompressibility

$$\nabla \cdot u = 0.$$

$$\sum_{i=1}^{n_u}\left[u_{ix}\int_\Omega \mu\nabla\phi^u_{ix} : \nabla\psi^u_j \, d\hat{x} + u_{iy}\int_\Omega \mu\nabla\phi^u_{iy} : \nabla\psi^u_j \, d\hat{x}\right] + \sum_{i=1}^{n_p} p_i \int_\Omega \phi^p_i \nabla\cdot\psi^u_j \, d\hat{x} = \int_\Omega \nabla\tilde{u}_\Gamma : \nabla\psi^u_j \, d\hat{x},$$

$$\text{for } j = 1, \cdots, 2n_u.$$

$$\sum_{i=1}^{n_u}\left[u_{ix}\int_\Omega \psi^p_k \nabla\cdot\phi^u_{ix} \, d\hat{x} + u_{iy}\int_\Omega \psi^p_k \nabla\cdot\phi^u_{iy} \, d\hat{x}\right] = \int_\Omega \psi^p_k \nabla\cdot\tilde{u}_\Gamma \, d\hat{x},$$

$$\text{for } k = 1, \cdots, n_p.$$

$$-\oint_{\partial\Omega_0} \mu\nabla u(\hat{x}) \cdot \hat{n} \, d\hat{x} = \int_{\Omega_0} F(\hat{x}) \, d\hat{x} - \oint_{\partial\Omega_0} p(\hat{x})\hat{n} \, dx$$

and $\displaystyle\oint_{\partial\Omega_0} u(\hat{x}) \cdot \hat{n} \, d\hat{x} = 0$ for all pieces $\Omega_0 \subset \Omega$.

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214 vertices, 334 triangles.
1361×1361 system with 1852321 entries.
12120 non-zeroes, fill ~ 0.0065.
Reconstructed velocity.
Mesh traversal and sparse matrix assembly.
Sparse solve for coefficients of quadratic basis functions.
Triangle mesh generation.
Input: Domain Ω, boundary condition u_Γ.