Variational methods for the Navier-Stokes equations (working title)

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Contents

Chapter 1

Continuum mechanics

1.1 On $F = ma$

$$\text{Total force} = \text{change of momentum.} \tag{1.1}$$

In this form, Newton's second law of motion states that there is such a thing called "force", a non-explanatory measurement made along a trajectory of a mechanical system, which is used to measure change in momentum. Newton's three laws, namely those of inertia, force, and equilibrium, have found universal success in application to mechanical systems. Separate from any particular notion of position and space, the principles of Newtonian mechanics apply to complex systems, and as we shall see, are fundamental to the derivation of equations of fluid motion. A primary example is that of a free rigid body, say, a cube of constant density. Under the assumption of rigidity, the state of the rigid body can be encoded with six numbers. These numbers describe position and orientation, which are paired to *linear momentum* and *angular momentum.*

1.1.1 The Newtonian free particle

We suppose that we have some space, called a configuration space, describing the possible states of the mechanical system. We suppose here that this space forms a finite dimensional manifold, which is true for the typical unconstrained configuration spaces such as those of rigid body motions.

Symmetry, momenta, and inertia

Consider first the idea of a free particle in $X = \mathbb{R}^3$ describing a traditional point in space that can move by translation. One fundamental assumption is that the laws of physics should "look the same" from the point of view of each point. This is formalized by a group action, here

$$g \in G \cong (\mathbb{R}, +), \quad x \in X, \quad g \cdot x = x + g.$$

Of course, G's action has only one orbit, as in, it is transitive. Denoting by $G \cdot x$ the orbit of x under the action of G, the group action induces an equivalence relation on X:

$$x, y \in X, \quad x \sim y \Leftrightarrow G \cdot x = G \cdot y.$$

Since there is only one orbit, we see that $x \sim y$ for all $x, y \in X$, as in, we are considering each point as "the same".

Energy and time

1.1.2 The Lagrangian free particle

The kinetic energy

From $F = ma$ to the principle of least action

1.1.3 Systems of forces

Force has been defined as a non-explanatory measurement. Here, discuss physical modelling with prescribed forces.

The Euler-Lagrange equations

1.2 From $F = ma$ to $\frac{\partial \mathcal{L}}{\partial q} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}} = 0$: The Euler-Lagrange equations

1.3 From $F = ma$ to $\frac{Du}{Dt} = \frac{1}{\rho}\nabla \cdot \sigma + f$: The Cauchy momentum equations

1.4 The Cauchy momentum equations as Euler-Lagrange equations

1.5 Transport of quantity and continuity equations

1.6 The continuum hypothesis and constitutive relations

Chapter 2

The Navier-Stokes equations

Chapter 3

The finite element method

3.1 Discretizing variational principles

Bibliography

[1] Johann Bernoulli, *"Problema novum ad cujus solutionem Mathematici invitantur." (A new problem to whose solution mathematicians are invited.)*, 1696. (retrieved from wikipedia/brachistochrone_curve)

[2] A. F. Monna, *Dirichlet's principle: A mathematical comedy of errors and its influence on the development of analysis*, 1975

[3] Stig Larsson, *Partial differential equations with numerical methods*, 2003

[4] Peter Lax, *Hyperbolic Systems of Conservation Laws and the Mathematical Theory of Shock Waves*, 1973

[5] Cornelius Lanczos, *The Variational Principles of Mechanics*, 1952

[6] G. K. Batchelor, *Introduction to Fluid Dynamics*, 1967

[7] L. Gary Leal, *Advanced Transport Phenomena: Fluid Mechanics and Convectice Transport Processes*, 2007

[8] Vivette Girault, Pierre-Arnaud Raviart, *Finite Element Methods for Navier-Stokes equations*, 1986