SIAM Review
Vol. 14, No. 1, January 1972

THE PADÉ TABLE AND ITS RELATION TO CERTAIN ALGORITHMS OF NUMERICAL ANALYSIS*

W. B. GRAGG†

To Heinz Rutishauser, in memoriam

Abstract. The algebraic theory of the Padé table of a formal power series is presented with a natural notation which indicates possible extensions to Laurent series. The theory is related to bigradient determinants, the epsilon and eta algorithms, and to a variant of the quotient-difference algorithm. Normality criteria for the Padé table, which provide existence theorems for the algorithms, are developed.

CONTENTS

1. Introduction. The Padé table of a formal power series

$$C(z) = \sum_{m=0}^{\infty} c_m z^m$$

is a doubly infinite array of rational functions

$$r_{mn}(z) = \frac{p_{mn}(z)}{q_{mn}(z)} = \frac{a_0 + a_1 z + \cdots + a_m z^m}{b_0 + b_1 z + \cdots + b_n z^n}$$

determined in such a manner that the Maclaurin expansion of r_{mn} agrees with $C(z)$ as far as possible. The power series $C(z)$ is said to be normal if, for each pair (m, n), this agreement is exactly through the power z^{m+n}. The original concept of the Padé table is due to Frobenius who developed the basic algorithmic aspects of the theory. The attribution to Padé, whose thesis came later, seems to be due to his treatment of certain abnormal cases which may arise. See Theorem 3.2.

There are interesting connections between the Padé table and other, not obviously related, areas of analysis. The bibliography provides a sample of some of these areas. Perhaps the most significant connection is that with the analytic

* Received by the editors March 30, 1970, and in revised form January 25, 1971. This invited paper was prepared in part and published under Contract DA-49-092-ARO-110 with the U.S. Army Research Office.

† Department of Mathematics, University of California, San Diego, La Jolla, California 92037.
The research and preparation of this paper were supported in part by the National Science Foundation under Grant GP-9304.

theory of continued fractions, a subject of earlier origin, which lists among its contributors Chebyshev, A. A. Markov and Stieltjes. The so-called associated and corresponding continued fractions for the power series $C(z)$ are, respectively, the superdiagonal $\{r_{n,n+1}\}_0^\infty$ and the staircase $\{r_{nn}, r_{n,n+1}\}_0^\infty$ consisting of the diagonal and superdiagonal in the Padé table. Moreover, since the origin of the theory of orthogonal polynomials lies within the theory of continued fractions, it is not surprising that the Padé table has connections with the former. If the coefficients

$$c_m = \int_{-\infty}^{+\infty} t^m \, d\mu(t)$$

are moments of a nondecreasing function μ with infinitely many points of increase, then the reciprocal polynomials

$$q_{n,n+1}^*(z) \equiv z^{n+1} q_{n,n+1}(z^{-1}), \qquad n \geqq 0,$$

of the denominators $q_{n,n+1}$ are the polynomials orthogonal with respect to μ. To see this compare the determinant expressions of Theorem 3.3 with those in Szegö, p. 27.

There are also connections with Euclid's algorithm for finding the greatest common divisor of two polynomials and with a generalization of the criterion of Routh and Hurwitz for counting the number of zeros of a polynomial in the left half-plane. These are closely related to the material of § 4.

The modern stimulus for interest in these areas has come from two principal sources. In his thesis D. Shanks introduced a class of nonlinear sequence transformations, the $e_m(s_n)$ transformations, and related them to the Padé table. Shanks' transformations were given as certain determinantal quotients, and hence were not readily computable. By appealing to a classical, but little known, determinant identity, P. Wynn showed how to compute the $e_m(s_n)$ transformations recursively. This resulted in Wynn's epsilon algorithm, of which Bauer's eta algorithm is a computationally more stable variant. The first step of each of these algorithms is mathematically equivalent to the well-known Aitken Δ^2-process. Several years later Wynn, on eliminating certain auxiliary values from the epsilon algorithm, found a fundamental identity (Theorem 5.5) among the Padé fractions themselves. This identity, which was overlooked by Frobenius, completes the set of those given by him. Thus it is natural to refer to it as "the missing identity of Frobenius."

The second modern development is the quotient-difference algorithm of H. Rutishauser. The original problem, suggested by E. Stiefel, was that of determining the characteristic roots of a finite square matrix A from the values of the Schwarz constants

$$s_m = y_0^T A^m x_0, \qquad m \geqq 0,$$

which are readily computed as a byproduct of the power method iteration

$$x_{m+1} = Ax_m, \qquad m \geqq 0.$$

Rutishauser observed that this is in essence a special case of Hadamard's problem, namely that of determining the poles of the rational function

$$f(\lambda) = y_0^T(\lambda I - A)^{-1} x_0 = \sum_{m=0}^{\infty} \frac{s_m}{\lambda^{m+1}}$$

from the coefficients in its Taylor expansion about $\lambda = \infty$, and applied the theory of continued fractions to the problem. Now there is a relation between formal J(acobi)-fractions (= the continued fractions associated with the above power series) and tridiagonal matrices J. The successive approximants of the J-fraction are the (1, 1)-elements of the matrices $(\lambda I_n - J_n)^{-1}$, with J_n the nth principal submatrix of J. This led to Rutishauser's matrix interpretation of the quotient-difference algorithm as an LR-algorithm for tridiagonal matrices and, on generalization, to the corresponding algorithm for full matrices.

The latter is as follows. Let $A \equiv A^{(1)}$ be the given matrix. If $A^{(k)}$ is known, factor

$$A^{(k)} = L^{(k)}R^{(k)}$$

into the product of a (unit) left triangular matrix $L^{(k)}$ and a right triangular matrix $R^{(k)}$. Then let

$$A^{(k+1)} = R^{(k)}L^{(k)}$$

and repeat indefinitely. Under certain hypotheses, which include the assumption of the existence of the factorization at each step and the requirement that the moduli of the roots of A be distinct, the matrices $A^{(k)}$ converge to a right triangular matrix $A^{(\infty)}$ whose diagonal consists of the roots of A. A number of refinements are possible.

Now the LR-factorization of a matrix A is the significant part of Gaussian elimination. The factorization exists if the leading principal submatrices of A have nonnull determinants. Practically there is the need for maximal pivot selection. The QR-algorithm ($\equiv UR$-algorithm) of Francis and Kublanovskaja avoids these problems by replacing the LR-factorization above by a UR-factorization: $A = UR$ with U unitary and R right triangular. The next iterate is then $A' = RU$. Again, important refinements make this a practical computing procedure. The UR-algorithm has been studied by Parlett and Wilkinson (see the bibliography) and is the method of choice for the nonsymmetric matrix eigenvalue problem. Hence by generalization and stabilization, the theory of continued fractions, of which that of the Padé table is an extension, has led to significant advances in the numerical solution of the algebraic eigenvalue problem.

The contents of the present paper are as follows. In § 2 certain determinant identities are developed which will have application in the sequel. The most significant are the Cauchy-Binet formula and Sylvester's determinant identity. Section 3 presents the fundamental existence and uniqueness theory for Padé fractions, an extension of the theorem of Padé on block structure of the table, and gives determinant formulas for the entries. The Padé table of a quotient of two formal power series is studied in § 4. This illuminating concept provides connections with resultants and a more general theorem of Trudi on the greatest common divisor of two polynomials. It also leads to a rather natural, but useful, duality relation between the Padé table of $C(z)$ and that of its reciprocal series $D(z)$. In § 5 the classical identities of Frobenius are established, together with Wynn's addition already referred to. Section 6 presents the algebraic basis for the epsilon and eta algorithms.

Normality criteria and sign patterns associated with the Padé tables of certain classes of power series occupy § 7. The specific connection between Pólya frequency series and the quotient-difference algorithm is apparently new. Algorithms for this class of series seem to be more numerically stable than for the class of Stieltjes series (see the example of § 9). These results should be of interest to numerical analysts. The material of § 8 on the asymptotic behavior of elements associated with the Padé table of a meromorphic function includes a refinement (Theorem 8.3(b)) of a classical theorem of Montessus de Ballore. It finds application in § 9 where a variant of the quotient-difference algorithm is developed. The duality theory, which was pointed out for the original quotient-difference algorithm by Henrici, has purposely been emphasized. In particular the corollary of Theorem 9.3 contains a dual *LR*-, *RL*-characterization of the new algorithm, and provides a convenient computational check which will be useful in studying stability properties for the two classes of series mentioned above.

2. Some determinant identities. In this section are collected some classical facts about determinants. Proofs of well-known results are omitted. Those given are instructional.

The set of $m \times n$ matrices over the complex number field $\mathscr{C}$ is denoted by $\mathscr{M}_{mn}$, and $\mathscr{M}_n$ is the set of n square matrices. In general the jth column of $A \in \mathscr{M}_{mn}$ is a_j, and the (i, j) element is $a_{ij}: A = (a_1, a_2, \cdots, a_n) = (a_{ij})$. An exception is the *unit matrix* $I = I_n = (e_1, e_2, \cdots, e_n) = (\delta_{ij}) \in \mathscr{M}_n$. The *transpose* of $A \in \mathscr{M}_{mn}$ is $A^T = (a'_{ij}) \in \mathscr{M}_{nm}$ with $a'_{ij} = a_{ji}$, and if $m = n$ the *determinant* of A satisfies $\det A = \det A^T$. For $A \in \mathscr{M}_n$, by definition,

$$p_A(z) \equiv \det(A + zI_n),$$

and

$$q_A(z) \equiv \det(zI_n - A) = (-1)^n p_A(-z)$$

is the *characteristic polynomial* of A.

The set of *multi-indices* α with k elements taken from $\{1, 2, \cdots, m\}$ is

$$\Delta_{mk} \equiv \{\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_k) : 1 \leqq \alpha_1 < \alpha_2 < \cdots < \alpha_k \leqq m\};$$

Δ_{mk} contains $\binom{m}{k}$ elements and is empty if $m < k$. The *length* of $\alpha \in \Delta_{mk}$ is $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_k$, and $\alpha' = (\alpha'_1, \alpha'_2, \cdots, \alpha'_{m-k}) \in \Delta_{m,m-k}$ is the *complement* of α with respect to $\{1, 2, \cdots, m\}$. If $A = (a_{ij}) \in \mathscr{M}_{mn}$, $\alpha \in \Delta_{mk}$, and $\beta \in \Delta_{nl}$, then

$$A(\alpha, \beta) \equiv A\begin{pmatrix} \alpha_1, \alpha_2, \cdots, \alpha_k \\ \beta_1, \beta_2, \cdots, \beta_l \end{pmatrix} \equiv (a_{\alpha_i \beta_j}) \in \mathscr{M}_{kl}$$

is the *submatrix* of A formed from rows α_i and columns β_j. The *index* of $A(\alpha, \beta)$ is

$$\begin{aligned} p(\alpha, \beta) &\equiv \sum_{i=1}^{k-1} [\alpha_{i+1} - (\alpha_i + 1)] + \sum_{j=1}^{l-1} [\beta_{j+1} - (\beta_j + 1)] \\ &= (\alpha_k - \alpha_1 - k + 1) + (\beta_l - \beta_1 - l + 1); \end{aligned}$$

$A(\alpha, \beta)$ is a *connected* submatrix of A if $p(\alpha, \beta) = 0$. By convention $\det A(\alpha, \beta) = 1$ if $\alpha \in \Delta_{m0}$, $\beta \in \Delta_{n0}$.

The *adjugate* of $A \in \mathscr{M}_n$ is $A^A = (b_{ij}) \in \mathscr{M}_n$ with

$$b_{ij} = (-1)^{i+j} \det A\begin{pmatrix} 1, \cdots, j-1, j+1, \cdots, n \\ 1, \cdots, i-1, i+1, \cdots, n \end{pmatrix}.$$

It satisfies $AA^A = A^A A = (\det A)I_n$. Hence if A is nonsingular $A^{-1} = (\det A)^{-1}A^A$.

THEOREM 2.1 (On the characteristic polynomial). *If $A \in \mathscr{M}_n$, then*

$$p_A(z) = \sum_{k=0}^{n} \left(\sum_{\alpha_k \in \Delta_{nk}} \det A(\alpha_k', \alpha_k') \right) z^k.$$

THEOREM 2.2 (Jacobi's theorem on subdeterminants of A^A). *If $A \in \mathscr{M}_n$ and $\alpha, \beta \in \Delta_{nk}$, then*

$$\det A^A(\alpha, \beta) = (-1)^{|\alpha| + |\beta|} \det A(\beta', \alpha').$$

THEOREM 2.3 (Cauchy-Binet formula). *Let $A \in \mathscr{M}_{mn}$, $B \in \mathscr{M}_{mk}$, and $C \in \mathscr{M}_{kn}$ with $A = BC$. If $\alpha \in \Delta_{ml}$ and $\beta \in \Delta_{nl}$, then*

$$\det A(\alpha, \beta) = \sum_{\gamma \in \Delta_{kl}} \det B(\alpha, \gamma) \det C(\gamma, \beta).$$

The special case $l = 1$ corresponds to ordinary matrix multiplication, and the case $k = l = m = n$ states that $\det A = \det B \cdot \det C$. If $k < l$, the sum is empty and $\det A(\alpha, \beta) = 0$.

THEOREM 2.4 (Laplace expansion). *Let $A \in \mathscr{M}_m$ and $\alpha \in \Delta_{mk}$. Then*

$$\det A = \sum_{\beta \in \Delta_{mk}} (-1)^{|\alpha| + |\beta|} \det A(\alpha, \beta) \det A(\alpha', \beta').$$

THEOREM 2.5 (Extensional identity). *Let $a, b, c, d \in \mathscr{M}_{n1}$ and $M \in \mathscr{M}_{n,n-2}$. Then*

$$\det(a, b, M) \det(c, d, M) = \det(a, c, M) \det(b, d, M) - \det(a, d, M) \det(b, c, M).$$

Proof. By elementary operations,

$$\begin{aligned} \det\begin{pmatrix} M & a & b & c & d & 0 \\ 0 & a & b & c & d & M \end{pmatrix} &= \det\begin{pmatrix} M & a & b & c & d & 0 \\ -M & 0 & 0 & 0 & 0 & M \end{pmatrix} \\ &= \det\begin{pmatrix} M & a & b & c & d & 0 \\ 0 & 0 & 0 & 0 & 0 & M \end{pmatrix} \\ &= \det(M, a, b) \det(0, 0, M) = 0. \end{aligned}$$

On the other hand, by the Laplace expansion with $m = 2n$, $k = n$, and $\alpha = (1, 2, \cdots, n)$,

$$\begin{aligned}\det\begin{pmatrix} M & a & b & c & d & 0 \\ 0 & a & b & c & d & M\end{pmatrix} &= \det(M, a, b)\det(c, d, M) \\ &\quad - \det(M, a, c)\det(b, d, M) \\ &\quad + \det(M, a, d)\det(b, c, M) \\ &\quad + \det(M, b, c)\det(a, d, M) \\ &\quad - \det(M, b, d)\det(a, c, M) \\ &\quad + \det(M, c, d)\det(a, b, M) \\ &= 2[\det(a, b, M)\det(c, d, M) \\ &\qquad - \det(a, c, M)\det(b, d, M) \\ &\qquad + \det(a, d, M)\det(b, c, M)],\end{aligned}$$

the last equality again following by elementary operations.

THEOREM 2.6 (Sylvester's determinant identity). *Let* $A' \in \mathcal{M}_n$, $n \geqq 2$, *be partitioned*

$$\begin{aligned}A' = \begin{pmatrix} a_{11} & a_1^T & a_{12} \\ a_1' & A & a_2' \\ a_{21} & a_2^T & a_{22}\end{pmatrix} = \begin{pmatrix} A_{11} & * \\ * & a_{22}\end{pmatrix} = \begin{pmatrix} * & A_{12} \\ a_{21} & *\end{pmatrix} = \begin{pmatrix} * & a_{12} \\ A_{21} & *\end{pmatrix} \\ = \begin{pmatrix} a_{11} & * \\ * & A_{22}\end{pmatrix}\end{aligned}$$

with $a_{ij} \in \mathscr{C}$, $i, j = 1, 2$. *Then*

$$\det A \cdot \det A' = \det A_{11} \cdot \det A_{22} - \det A_{21} \cdot \det A_{12}.$$

Proof 1. Put $a = e_1$, $b = e_n$, and $(c, M, d) = A'$ in the extensional identity

$$\begin{aligned}\det(a, M, b)\det(c, M, d) &= \det(a, M, d)\det(c, M, b) \\ &\quad - \det(a, c, M)\det(M, d, b).\end{aligned}$$

Proof 2. If the matrices B_{ii} are square and B_{11} is nonsingular, then

$$B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22}\end{pmatrix} = \begin{pmatrix} B_{11} & 0 \\ B_{21} & I\end{pmatrix}\begin{pmatrix} I & B_{11}^{-1}B_{12} \\ 0 & B_{22} - B_{21}B_{11}^{-1}B_{12}\end{pmatrix}.$$

Consequently

$$\det B = \det B_{11} \cdot \det(B_{22} - B_{21}B_{11}^{-1}B_{12}).$$

Hence if A is nonsingular,

$$\det A_{ij} = \pm\det\begin{pmatrix} A & a_j' \\ a_i^T & a_{ij}\end{pmatrix} = \pm\det A \cdot (a_{ij} - a_i^T A^{-1} a_j'),$$

with the negative sign only possible if $i \neq j$. Then similarly,

$$\begin{aligned}\det A \cdot \det A' &= \det A \cdot \det \begin{pmatrix} A & a_1' & a_2' \\ a_1^T & a_{11} & a_{12} \\ a_2^T & a_{21} & a_{22} \end{pmatrix} \\ &= (\det A)^2 \det \left[\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} - \begin{pmatrix} a_1^T \\ a_2^T \end{pmatrix} A^{-1}(a_1', a_2') \right] \\ &= (\det A)^2 \det (a_{ij} - a_i^T A^{-1} a_j')_{i,j=1}^2 \\ &= \det A_{11} \cdot \det A_{22} - \det A_{21} \cdot \det A_{12}.\end{aligned}$$

This last rational identity holds even if A is singular.

THEOREM 2.7 (Vandermonde determinant). *Let*

$$V(z_1, z_2, \cdots, z_n) \equiv (z_j^{i-1}) \in \mathscr{M}_n.$$

Then

$$\begin{aligned}v(z_1, z_2, \cdots, z_n) &\equiv \det V(z_1, z_2, \cdots, z_n) \\ &= \prod_{i<j} (z_j - z_i)\end{aligned}$$

and

$$v(z_1^{-1}, z_2^{-1}, \cdots, z_n^{-1}) = \sigma_n \frac{v(z_1, z_2, \cdots, z_n)}{(z_1 z_2 \cdots z_n)^{n-1}}$$

with

$$\sigma_n \equiv (-1)^{n(n-1)/2}.$$

THEOREM 2.8 (On totally positive matrices). *Let all square* connected *submatrices of* $A \in \mathscr{M}_{mn}$ *have positive determinants. Then* all *subdeterminants of* A *are positive.*

Proof. It will be shown by induction on k, and for fixed k, on the index $p(\alpha, \beta) = (\alpha_k - \alpha_1 - k + 1) + (\beta_k - \beta_1 - k + 1)$ of $A(\alpha, \beta)$, that

$$\det A(\alpha, \beta) > 0, \qquad \alpha \in \Delta_{mk}, \quad \beta \in \Delta_{nk}.$$

By hypothesis this is true if $p(\alpha, \beta) = 0$, and in particular if $k = 1$. Thus let $1 < k \leqq \min(m, n)$, $0 < q \leqq m + n - 2k$, and assume that

$$\det A(\gamma, \delta) > 0, \qquad \gamma \in \Delta_{ml}, \quad \delta \in \Delta_{nl},$$

when $l = k - 1$, and when both $l = k$ and $p(\gamma, \delta) \leqq q - 1$.

Let $\alpha \in \Delta_{mk}$ and $\beta \in \Delta_{nk}$ with $p(\alpha, \beta) = q$. Suppose that β_{k+1}, $\beta_1 < \beta_{k+1} < \beta_k$, is an integer omitted in the sequence $\beta = (\beta_1, \beta_2, \cdots, \beta_k)$. In the extensional identity

$$\begin{aligned}\det (M, c, e_k) \det (a, M, b) = \det (a, M, c) \det (M, b, e_k) \\ - \det (M, b, c) \det (a, M, e_k)\end{aligned}$$

put

$$(a, M, b, c) = A\begin{pmatrix}\alpha_1, \alpha_2, \cdots, \alpha_k \\ \beta_1, \beta_2, \cdots, \beta_{k+1}\end{pmatrix}$$

to obtain

$$\begin{aligned}\det A\begin{pmatrix}\alpha_1, \cdots, \alpha_{k-2}, \alpha_{k-1} \\ \beta_2, \cdots, \beta_{k-1}, \beta_{k+1}\end{pmatrix} \det A\begin{pmatrix}\alpha_1, \cdots, \alpha_k \\ \beta_1, \cdots, \beta_k\end{pmatrix} \\ = \det A\begin{pmatrix}\alpha_1, \cdots, \alpha_{k-1}, \alpha_k \\ \beta_1, \cdots, \beta_{k-1}, \beta_{k+1}\end{pmatrix} \det A\begin{pmatrix}\alpha_1, \cdots, \alpha_{k-1} \\ \beta_2, \cdots, \beta_k\end{pmatrix} \\ - \det A\begin{pmatrix}\alpha_1, \cdots, \alpha_{k-1}, \alpha_{k-1} \\ \beta_2, \cdots, \beta_k, \beta_{k+1}\end{pmatrix} \det A\begin{pmatrix}\alpha_1, \cdots, \alpha_{k-1} \\ \beta_1, \cdots, \beta_{k-1}\end{pmatrix}.\end{aligned}$$

The number of interchanges required to restore $(\beta_2, \cdots, \beta_{k-1}, \beta_{k+1})$ to its natural order is the same as that for $(\beta_1, \cdots, \beta_{k-1}, \beta_{k+1})$, and one less than that for $(\beta_2, \cdots, \beta_k, \beta_{k+1})$. Moreover the indices of the matrices so restored are at most $q - 1$. Hence on division by

$$\det A\begin{pmatrix}\alpha_1, \cdots, \alpha_{k-2}, \alpha_{k-1} \\ \beta_2, \cdots, \beta_{k-1}, \beta_{k+1}\end{pmatrix}$$

the induction hypothesis gives $\det A(\alpha, \beta) > 0$. If the integers $(\beta_1, \beta_2, \cdots, \beta_k)$ are consecutive, then since $q > 0$, $(\alpha_1, \alpha_2, \cdots, \alpha_k)$ will omit an integer α_{k+1} with $\alpha_1 < \alpha_{k+1} < \alpha_k$, and the above argument can be applied to A^T.

THEOREM 2.9 (The determinant of a sum of matrices). *Let* $A_k = (a_1^{(k)}, a_2^{(k)}, \cdots, a_n^{(k)}) \in \mathscr{M}_n$, $k = 1, 2, \cdots, n$. *Then*

$$\det\left(\sum_{k=1}^{m} A_k\right) = \sum_{k \in \Phi_{mn}} \det(a_1^{(\varkappa_1)}, a_2^{(\varkappa_2)}, \cdots, a_n^{(\varkappa_n)})$$

with Φ_{mn} *the set of functions* $\varkappa$ *from* $\{1, 2, \cdots, n\}$ *to* $\{1, 2, \cdots, m\}$.

3. The Padé and c-tables of a formal power series. The class $\mathscr{P}$ of *formal power series* over $\mathscr{C}$ consists of all expressions of the form

$$C(z) \equiv c_0 + c_1 z + c_2 z^2 + \cdots \equiv \sum_{m=0}^{\infty} c_m z^m$$

with coefficients $c_m \in \mathscr{C}$. Convergence of the infinite series for complex $z \neq 0$ is not implied. Somewhat more precisely, $\mathscr{P}$ could be thought of as the set of infinite sequences $(c_0, c_1, c_2, \cdots)$, $c_m \in \mathscr{C}$. The *sum* of

$$\sum_{m=0}^{\infty} b_m z^m \quad \text{and} \quad \sum_{m=0}^{\infty} c_m z^m \quad \text{is} \quad \sum_{m=0}^{\infty} (b_m + c_m) z^m,$$

and the *scalar multiple* of

$$\sum_{m=0}^{\infty} c_m z^m \quad \text{by} \quad c \in C \quad \text{is} \quad \sum_{m=0}^{\infty} (c c_m) z^m.$$

With these definitions $\mathscr{P}$ is an infinite-dimensional vector space over $\mathscr{C}$, with additive identity $0 = 0 + 0z + 0z^2 + \cdots$. The algebraic dimension of $\mathscr{P}$ is $\mathbf{c}$, the power of the continuum.

The (Cauchy) product of

$$\sum_{m=0}^{\infty} b_m z^m \quad \text{and} \quad \sum_{m=0}^{\infty} c_m z^m \quad \text{is} \quad \sum_{m=0}^{\infty} a_m z^m$$

with

$$a_m \equiv \sum_{k=0}^{m} b_{m-k} c_k, \qquad m \geqq 0.$$

With this product $\mathscr{P}$ is a commutative semigroup with multiplicative identity $1 = 1 + 0z + 0z^2 + \cdots$. It is easy to show that the cancellation law holds, and that multiplication is distributive over addition. Hence $\mathscr{P}$, together with the operations of addition and multiplication, is an *integral domain*. The *units* of $\mathscr{P}$ are the power series

$$C(z) = \sum_{m=0}^{\infty} c_m z^m \quad \text{with} \quad c_0 \neq 0.$$

Each unit $C(z)$ has a *reciprocal*

$$C^{-1}(z) = D(z) = \sum_{m=0}^{\infty} d_m z^m$$

such that $C(z)D(z) = 1$. The coefficients of $D(z)$ are determined by

$$d_0 = c_0^{-1},$$
$$d_m = -c_0^{-1} \sum_{k=0}^{m-1} c_{m-k} d_k, \qquad m \geqq 1.$$

Corresponding to the formal power series $C(z) \in \mathscr{P}$ is the *semicirculant matrix*

$$C \equiv \begin{pmatrix} c_0 & & & & \\ c_1 & c_0 & & & \\ c_2 & c_1 & c_0 & & \\ c_3 & c_2 & c_1 & c_0 & \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \equiv (c_{i-j}) \in \mathscr{M}_\infty .$$

Observe that by definition

$$c_{-1} = c_{-2} = c_{-3} = \cdots \equiv 0.$$

The sum and product of two such matrices are defined, as is the scalar multiple of C, by $c \in \mathscr{C}$. Moreover this correspondence is an *isomorphism* with respect to these three operations, for the (i, j) element of $A = BC$ is

$$\sum_{k=0}^{\infty} b_{i-k}c_{k-j} = \sum_{k=j}^{i} b_{i-k}c_{k-j} = \sum_{l=0}^{i-j} b_{i-j-l}c_l = a_{i-j},$$

and the other two assertions are trivial. Finally, the *composition* of $C(z)$ with $c \in \mathscr{C}$ is

$$C(cz) = \sum_{m=0}^{\infty} (c_m c^m) z^m.$$

In particular the coefficients of $C(-z)$ are $(-1)^m c_m$.

Let $C(z) \in \mathscr{P}$ and let m and n be nonnegative integers. The complex rational form

$$\frac{u(z)}{v(z)} = \frac{u_0 + u_1 z + \cdots + u_m z^m}{v_0 + v_1 z + \cdots + v_n z^n}$$

is a *Padé form* of type (m, n) for $C(z)$ if $v \neq 0$ and

$$C(z)v(z) - u(z) = O(z^{m+n+1}).$$

The (algebraic) O-symbol indicates that the right side is a power series beginning exactly with a power $z^{m+n+k+1}$, $0 \leqq k \leqq +\infty$; $k = +\infty$ means that $Cv - u = 0$. This is equivalent to the linear system of $m + n + 1$ equations in the $m + n + 2$ unknowns $u_0, \cdots, u_m, v_0, \cdots, v_n$:

$$\begin{array}{ll} S_{mn}^u & \\ & \displaystyle\sum_{j=0}^{n} c_{i-j}v_j = \begin{cases} u_i, & i = 0, 1, \cdots, m, \\ 0, & i = m+1, \cdots, m+n. \end{cases} \\ S_{mn}^v & \end{array}$$

THEOREM 3.1 (Frobenius). *There always exist Padé forms of type* (m, n) *for* $C(z)$. *Each such form is a representation of the same rational function* r_{mn}. *The reduced representation*

$$r_{mn}(z) = \frac{p_{mn}(z)}{q_{mn}(z)}$$

is possible with p_{mn} *and* q_{mn} *relatively prime and* $p_{mn}(0) = c_0$, $q_{mn}(0) = 1$.

Proof. There always exist nontrivial solutions $(u_0, \cdots, u_m, v_0, \cdots, v_n)^T$ of S_{mn}. With such a solution $v \neq 0$, for otherwise S_{mn}^u would imply $u = 0$. Likewise, if $v(z) = O(z^\lambda)$, $v_\lambda \neq 0$, then $u(z) = O(z^\lambda)$, $u_\lambda = c_0 v_\lambda$. Hence the reduced representation of u/v may be normalized as required.

If u/v and u'/v' are two Padé forms of type (m, n) for $C(z)$, then

$$Cv - u = O(z^{m+n+1}), \qquad Cv' - u' = O(z^{m+n+1}),$$

and

$$uv' - u'v = (Cv' - u')v - (Cv - u)v' = O(z^{m+n+1}).$$

The left side is a polynomial of degree at most $m + n$, the right side a power series beginning with a power $z^{m+n+k+1}$, $k \geqq 0$. Hence both sides vanish, and the rational functions determined by u/v and u'/v' are identical. This completes the proof.

The uniquely determined rational function $r_{mn} = p_{mn}/q_{mn}$ is the Padé fraction of type (m, n) for $C(z)$. The doubly infinite array shown in Table 1:

TABLE 1

	0	0	0	0	
∞	r_{00}	r_{01}	r_{02}	r_{03}	$\cdots$
∞	r_{10}	r_{11}	r_{12}	r_{13}	$\cdots$
∞	r_{20}	r_{21}	r_{22}	r_{23}	$\cdots$
∞	r_{30}	r_{31}	r_{32}	r_{33}	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	

is the (extended) Padé table for $C(z)$. The first column contains the partial sums

$$C_m(z) \equiv \sum_{k=0}^{m} c_k z^k$$

of $C(z)$. The auxiliary values will be explained subsequently (§ 5).

Also associated with $C(z)$ are the nontrivial connected submatrices

$$C_{mn} \equiv (c_{m+i-j})_{i,j=1}^{n}, \qquad n \geqq 1,$$

of the semicirculant matrix C. Their determinants are

$$c_{mn} \equiv \det C_{mn}, \qquad c_{m0} \equiv 1,$$

and the array shown in Table 2:

TABLE 2

	$\frac{1}{2}$	0	0	0	$\cdots$
$\frac{1}{2}$	c_{00}	c_{01}	c_{02}	c_{03}	$\cdots$
0	c_{10}	c_{11}	c_{12}	c_{13}	$\cdots$
0	c_{20}	c_{21}	c_{22}	c_{23}	$\cdots$
0	c_{30}	c_{31}	c_{32}	c_{33}	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	

is the (extended) c-table for $C(z)$. Evidently

$$c_{m0} = 1, \quad c_{m1} = c_m, \quad c_{0n} = c_0^n.$$

Example. The Padé table for

$$C(z) = 1 + z + z^2 + 2z^3 + 3z^4 + 4z^5 + \cdots$$

is given in Table 3:

TABLE 3

$$
\begin{array}{lllllllll}
r_{00} & r_{01} = & r_{02} & r_{03} = & r_{04} = & r_{05} & r_{06} & r_{07} & \cdots \\
 & \parallel & \parallel & \parallel & \parallel & \parallel & & & \\
r_{10} & r_{11} = & r_{12} & r_{13} = & r_{14} = & r_{15} & r_{16} = & r_{17} & \cdots \\
 & & & \parallel & \parallel & \parallel & \parallel & \parallel & \\
r_{20} & r_{21} & r_{22} & r_{23} = & r_{24} = & r_{25} & r_{26} = & r_{27} & \cdots \\
r_{30} & r_{31} & r_{32} = & r_{33} = & r_{34} = & r_{35} = & r_{36} = & r_{37} = & \cdots \\
 & & \parallel & \parallel & \parallel & \parallel & \parallel & \parallel & \\
r_{40} & r_{41} & r_{42} = & r_{43} = & r_{44} = & r_{45} = & r_{46} = & r_{47} = & \cdots \\
 & & \parallel & \parallel & \parallel & \parallel & \parallel & \parallel & \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots &
\end{array}
$$

with the nontrivial Padé fractions r_{mn} given by

$$r_{01}(z) = \frac{1}{1-z}, \qquad r_{21}(z) = \frac{1 - z - z^2}{1 - 2z},$$

$$r_{31}(z) = \frac{1 - z/2 - z^2/2 + z^3/3}{1 - 3z/2}, \qquad r_{41}(z) = \frac{1 - z/3 - z^2/3 + 2z^3/3 + z^4/3}{1 - 4z/3},$$

$$r_{22}(z) = \frac{1 - z^2}{1 - z - z^2}, \qquad r_{32}(z) = \frac{1 - z + z^3}{1 - 2z + z^2},$$

$$r_{03}(z) = \frac{1}{1 - z - z^3}, \qquad r_{06}(z) = \frac{1}{1 - z - z^3 + z^6},$$

$$r_{16}(z) = \frac{1 - z}{1 - 2z + z^2 - z^3 + z^4 + z^6}, \quad r_{07}(z) = \frac{1}{1 - z - z^3 + z^6 + z^7}.$$

As can be readily verified, $C(z)$ is the Maclaurin expansion of the rational function $r_{32}(z)$. The c-table for $C(z)$ is given in Table 4:

TABLE 4

$m \backslash n$	0	1	2	3	4	5	6	7	8	$\cdots$
0	1	1	1	1	1	1	1	1	1	$\cdots$
1	1	1	0	1	0	0	1	-1	1	$\cdots$
2	1	1	-1	1	0	0	1	0	1	$\cdots$
3	1	2	1	1	1	1	1	1	1	$\cdots$
4	1	3	1	0	0	0	0	0	0	$\cdots$
5	1	4	1	0	0	0	0	0	0	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	

The next result is an extension of a theorem of Padé. It shows that the block structure exhibited by the previous example holds in general when $C(z)$ is a unit,

$c_0 \neq 0$. Equal Padé fractions occur in square blocks in the Padé table. The c-table has a corresponding block structure, with groups of determinants $c_{\mu\nu}$ occurring in maximal square blocks of the form

$$\begin{array}{c} n \\ | \\ m - \end{array} \tag{3.1} \underbrace{\left.\begin{matrix} * & * & * & \cdots & * \\ * & 0 & 0 & \cdots & 0 \\ * & 0 & 0 & \cdots & 0 \\ * & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ * & 0 & 0 & \cdots & 0 \end{matrix}\right\}}_{k+1} k+1, \qquad * \neq 0.$$

The theorem also characterizes the Padé forms of type (μ, ν) for $C(z)$ and gives a formula for the ranks of the linear systems $S_{\mu\nu}$ and $S^{v}_{\mu\nu}$.

THEOREM 3.2. *Let p/q be a Padé fraction for the power series $C(z) \in \mathscr{P}$, $c_0 \neq 0$. Let the degrees of p and q be m and n, respectively, and let the power series $C(z)p(z) - q(z)$ begin exactly with the power $z^{m+n+k+1}$. Then the following statements are true*:

(a) $k \geqq 0$.

(b) $r_{\mu\nu} = p/q$ *if and only if*

$$m \leqq \mu \leqq m + k \quad \text{and} \quad n \leqq \nu \leqq n + k. \tag{3.2}$$

For (μ, ν) satisfying (3.2):

(c) *u/v is a Padé form of type (μ, ν) for $C(z)$ if and only if*

$$u(z) = z^{\lambda_{\mu\nu}} d(z)p(z), \qquad v(z) = z^{\lambda_{\mu\nu}} d(z)q(z),$$

with

$$\lambda_{\mu\nu} = \max\{0, (\mu - m) + (\nu - n) - k\}$$

and $d \neq 0$ of degree at most

$$\varkappa_{\mu\nu} = \begin{cases} \left[\dfrac{k}{2}\right] - \max\left\{\left|\mu - m - \left[\dfrac{k}{2}\right]\right|, \left|\nu - n - \left[\dfrac{k}{2}\right]\right|\right\}, & k < +\infty, \\ \min\{\mu - m, \nu - n\}, & k = +\infty. \end{cases}$$

(d) $$\mu + \nu - \operatorname{rank} S_{\mu\nu} = \nu - \operatorname{rank} S^{v}_{\mu\nu} = \varkappa_{\mu\nu}.$$

(e) $$\begin{aligned} &c_{\mu n} \neq 0, && m \leqq \mu \leqq m + k, \\ &c_{m\nu} \neq 0, && n \leqq \nu \leqq n + k, \\ &c_{\mu\nu} = 0, && m < \mu \leqq m + k \quad \text{and} \quad n < \nu \leqq n + k. \end{aligned}$$

Example. For $k = 4$ the corresponding blocks in the $\varkappa$- and λ-tables are as given in Tables 5 and 6:

TABLE 5

$\varkappa$-block

$\mu - m$ \ $\nu - n$	0	1	2	3	4
0	0	0	0	0	0
1	0	1	1	1	0
2	0	1	2	1	0
3	0	1	1	1	0
4	0	0	0	0	0

TABLE 6

λ-block

$\mu - m$ \ $\nu - n$	0	1	2	3	4
0	0	0	0	0	0
1	0	0	0	0	1
2	0	0	0	1	2
3	0	0	1	2	3
4	0	1	2	3	4

Proof. Let $r_{\mu\nu} = p/q$ and let u/v be a Padé form of type (μ, ν). Then

$$\begin{aligned} &\deg u \leqq \mu, \qquad \deg v \leqq \nu, \\ &Cv - u = O(z^{\mu+\nu+1}). \end{aligned} \tag{3.3}$$

Since $c_0 \neq 0$ and $(u_0, \cdots, u_m, v_0, \cdots, v_n)^T \neq 0$, it follows from the form of $S_{\mu\nu}$ that $v \neq 0$ *and* $u \neq 0$. Remove the greatest common divisor from u and v to obtain

$$\begin{aligned} u(z) &= z^{\lambda}(d_0 + d_1 z + \cdots + d_{\varkappa} z^{\varkappa}) p(z), \\ v(z) &= z^{\lambda}(d_0 + d_1 z + \cdots + d_{\varkappa} z^{\varkappa}) q(z), \end{aligned} \tag{3.4}$$

with $d_0 d_{\varkappa} \neq 0$. Then

$$z^{\lambda}(Cq - p) = O(z^{\mu+\nu+1}).$$

There follow the inequalities

$$\begin{aligned} &\varkappa \geqq 0, \qquad \lambda \geqq 0, \\ &\varkappa + \lambda + m \leqq \mu, \qquad \varkappa + \lambda + n \leqq \nu, \\ &\mu + \nu \leqq m + n + k + \lambda, \end{aligned}$$

or equivalently,

$$\begin{aligned} &\varkappa \geqq 0, \\ &\lambda \geqq \max\{0, (\mu - m) + (\nu - n) - k\}, \\ &\varkappa + \lambda \leqq \min\{\mu - m, \nu - n\}; \end{aligned} \tag{3.5}$$

see Fig. 1.

Conversely, let the integers μ, ν be such that there exist integers $\varkappa$, λ satisfying (3.5). For arbitary $d(z) = d_0 + d_1 z + \cdots + d_{\varkappa} z^{\varkappa} \neq 0$ define u and v by (3.4). Then (3.3) holds, u/v is a Padé form of type (μ, ν), and $r_{\mu\nu} = p/q$.

The problems of whether $r_{\mu\nu} = p/q$ and the general structure of Padé forms of type (μ, ν) are thus reduced to statements involving the inequalities (3.5).

Since p/q is a Padé fraction for $C(z)$ there exist integers $\varkappa$, λ, μ, ν satisfying (3.5). Then

$$k \geqq (\mu - m) + (\nu - n) - \lambda \geqq 2\varkappa + \lambda \geqq 0,$$

proving statement (a).

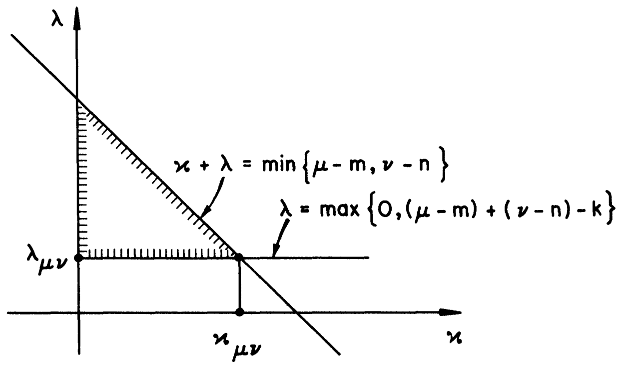

FIG. 1

Further, $r_{\mu\nu} = p/q$ if and only if there exists a solution $(\varkappa, \lambda)$ of (3.5). This is possible if and only if

$$\max\{0, (\mu - m) + (\nu - n) - k\} \leqq \min\{\mu - m, \nu - n\},$$

that is, if and only if (3.2) holds. This proves (b).

The most general Padé form of type (μ, ν) is obtained when λ is minimized and $\varkappa$ maximized subject to the inequalities (3.5). An easy computation shows that the optimal values are $\lambda = \lambda_{\mu\nu}$ and $\varkappa = \varkappa_{\mu\nu}$. This proves (c). Statement (d) follows since the general solutions of $S_{\mu\nu}$ and $S^{v}_{\mu\nu}$ contain $\varkappa_{\mu\nu} + 1$ parameters.

Now $c_{\mu\nu} = 0$ if and only if $S^{v}_{\mu\nu}$ has a nontrivial solution with $v_0 = 0$. But it follows from (c) that

$$\varkappa_{\mu\nu} + \lambda_{\mu\nu} = \min\{\mu - m, \nu - n\}$$

is the maximal integer λ for which $v(z) = O(z^{\lambda})$ in some Padé form of type (μ, ν). Hence $c_{\mu\nu} = 0$ if and only if $\min\{\mu - m, \nu - n\} > 0$. This proves (e), completing the proof of Theorem 3.2.

A number of results follow easily from Theorem 3.2. For example,

$$r_{mn} = r_{m+1,n+1} \Leftrightarrow c_{m+1,n+1} = 0.$$

The implication $\Rightarrow$ is a direct consequence of (e). But the reverse implication $\Leftarrow$ is also an (indirect) consequence of (e). For if $r_{mn} \neq r_{m+1,n+1}$, then r_{mn} and $r_{m+1,n+1}$ belong to different blocks in the Padé table. Moreover, $r_{m+1,n+1}$ must lie on the left or upper boundary of its block. Hence $c_{m+1,n+1} \neq 0$ by (e). More generally, if the c-table contains the (maximal) block (3.1), then (b) holds. The following criteria, among others, are easily seen to be sufficient for the former.

COROLLARY 1. *Either of the criteria* (α), (β), *or* (γ) *is sufficient for* (b) *to hold.*

$$(\alpha) \qquad \begin{aligned} &c_{m+1,n}c_{m,n+1} \neq 0, \\ &c_{m+l,n+l} = 0, \qquad l = 1, 2, \cdots, k, \\ &c_{m+k+1,n+k+1} \neq 0; \end{aligned}$$

$$c_{m+1,n}c_{m,n+1} \neq 0,$$

$$(\beta) \qquad c_{m+l,n+1} = 0, \qquad l = 1, 2, \cdots, k,$$

$$c_{m+k+1,n+1} \neq 0;$$

$$c_{m+1,n}c_{m,n+1} \neq 0,$$

$$(\gamma) \qquad c_{m+1,n+l} = 0, \qquad l = 1, 2, \cdots, k,$$

$$c_{m+1,n+k+1} \neq 0.$$

In each case if $k = 0\,(+\infty)$ *the second (third) condition is dropped.*

The square r- and c-blocks described in Theorem 3.2 are called *blocks of order* k. The Padé fraction r_{mn} is *normal* if the r-block containing it is of order $k = 0$; that is, r_{mn} occurs exactly once in the Padé table. The power series $C(z)$ is *normal* if all its Padé fractions are normal; that is, no two are equal.

COROLLARY 2. *The following statements are equivalent*:

(i) *the Padé fraction* $r_{mn} = p_{mn}/q_{mn}$ *is normal*;

(ii) *the degrees of* p_{mn} *and* q_{mn} *are* m *and* n, *respectively, and the power series expansion of* $C(z)q_{mn}(z) - p_{mn}(z)$ *begins exactly with the power* z^{m+n+1};

(iii) *the determinants*

$$c_{mn} \qquad c_{m,n+1}$$

$$c_{m+1,n} \qquad c_{m+1,n+1}$$

do not vanish. The power series $C(z)$ *is normal if and only if*

$$c_{mn} \neq 0, \qquad m \geqq 0, \qquad n \geqq 0;$$

in particular each coefficient $c_{m1} = c_m$ *must not vanish.*

Deserving of emphasis is an important special case of Theorem 3.2.

COROLLARY 3. *The formal power series* $C(z)$ *is the Maclaurin expansion of an irreducible rational function* $r(z) = p(z)/q(z)$, *with* p *and* q *of degrees* M *and* N, *respectively, if and only if*

$$c_{mN} \neq 0, \qquad m \geqq M,$$

$$c_{Mn} \neq 0, \qquad n \geqq N,$$

$$c_{mn} = 0, \qquad m > M \quad \text{and} \quad n > N.$$

It is appropriate to call such a power series (M, N)-*normal* if, in addition,

$$c_{mn} \neq 0, \qquad 0 \leqq m \leqq M \quad \text{or} \quad 0 \leqq n \leqq N.$$

The Padé table of an (M, N)-normal power series has exactly one block of order $k > 0$, and this block is infinite with its upper left element $r = r_{MN}$ as the common entry.

There also occur power series $C(z)$ which are *seminormal* in the sense that

$$c_{mn} \neq 0, \qquad m + n \text{ odd}.$$

The blocks of a seminormal power series are of order at most one, and by Theorem 3.2 (d) all systems S_{mn} are of maximal rank.

The final theorem of this section provides explicit determinant representations for all (distinct) Padé fractions in the table.

THEOREM 3.3. *Let the rank of the linear system S_{mn} be maximal. Then r_{mn} has the* (*not necessarily reduced*) *representation*

$$r_{mn}(z) = u_{mn}(z)/v_{mn}(z)$$

with

$$u_{mn}(z) \equiv \det \begin{pmatrix} C_m(z) & zC_{m-1}(z) & \cdots & z^n C_{m-n}(z) \\ c_{m+1} & c_m & \cdots & c_{m-n+1} \\ \vdots & \vdots & & \vdots \\ c_{m+n} & c_{m+n-1} & \cdots & c_m \end{pmatrix}, \tag{3.6}$$

and

$$v_{mn}(z) \equiv \det \begin{pmatrix} 1 & z & \cdots & z^n \\ c_{m+1} & c_m & \cdots & c_{m-n+1} \\ \vdots & \vdots & & \vdots \\ c_{m+n} & c_{m+n-1} & \cdots & c_m \end{pmatrix} \tag{3.7}$$

$$\equiv \det (C_{mn} - zC_{m+1,n}). \tag{3.8}$$

Moreover,

$$C(z)v_{mn}(z) - u_{mn}(z)$$

$$= (-1)^n \sum_{k=1}^{\infty} \det \begin{pmatrix} c_{m+1} & c_m & \cdots & c_{m-n+1} \\ \vdots & \vdots & & \vdots \\ c_{m+n} & c_{m+n-1} & \cdots & c_m \\ c_{m+n+k} & c_{m+n+k-1} & \cdots & c_{m+k} \end{pmatrix} z^{m+n+k}. \tag{3.9}$$

Proof. It will be shown that (3.6) and (3.7) imply (3.9). From (3.7),

$$C(z)v_{mn}(z) = \det \begin{pmatrix} C(z) & zC(z) & \cdots & z^n C(z) \\ c_{m+1} & c_m & \cdots & c_{m-n+1} \\ \vdots & \vdots & & \vdots \\ c_{m+n} & c_{m+n-1} & \cdots & c_m \end{pmatrix}.$$

For $i = 1, 2, \cdots, n$, multiply the $(i + 1)$th row of (3.6) by z^{m+1} and add to the first row. There results

$$u_{mn}(z) = \det \begin{pmatrix} C_{m+n}(z) & zC_{m+n-1}(z) & \cdots & z^n C_m(z) \\ c_{m+1} & c_m & \cdots & c_{m-n+1} \\ \vdots & \vdots & & \vdots \\ c_{m+n} & c_{m+n-1} & \cdots & c_m \end{pmatrix}.$$

Subtraction, followed by a change of indices and permutation of the first row to the last, gives

$$C(z)v_{mn}(z) - u_{mn}(z)$$

$$= \det \begin{pmatrix} \sum_{k=m+n+1}^{\infty} c_k z^k & \sum_{k=m+n}^{\infty} c_k z^{k+1} & \cdots & \sum_{k=m+1}^{\infty} c_k z^{k+n} \\ c_{m+1} & c_m & \cdots & c_{m-n+1} \\ \vdots & \vdots & & \vdots \\ c_{m+n} & c_{m+n-1} & \cdots & c_m \end{pmatrix}$$

$$= (-1)^n \det \begin{pmatrix} c_{m+1} & c_m & \cdots & c_{m-n+1} \\ \vdots & \vdots & & \vdots \\ c_{m+n} & c_{m+n-1} & & c_m \\ \sum_{k=1}^{\infty} c_{m+n+k} z^{m+n+k} & \sum_{k=1}^{\infty} c_{m+n+k-1} z^{m+n+k} & \cdots & \sum_{k=1}^{\infty} c_{m+k} z^{m+n+k} \end{pmatrix}$$

which is equivalent to (3.9). By the rank assumption $v_{mn} \neq 0$.

To prove (3.8) multiply the jth column of (3.7) by z and subtract from the $(j+1)$th, $j = 1, 2, \cdots, n$. Then expand the resulting determinant along the first row.

COROLLARY 1. *If $c_{mn} \neq 0$, then the reduced representation*

$$r_{mn}(z) = p_{mn}(z)/q_{mn}(z),$$

with

$$p_{mn}(z) = \frac{u_{mn}(z)}{c_{mn}} = c_0 + \cdots + \frac{c_{m,n+1}}{c_{mn}} z^m,$$

$$q_{mn}(z) = \frac{v_{mn}(z)}{c_{mn}} = 1 + \cdots + (-1)^n \frac{c_{m+1,n}}{c_{mn}} z^n,$$

and

$$C(z) - r_{mn}(z) = (-1)^n \frac{c_{m+1,n+1}}{c_{mn}} z^{m+n+1} + O(z^{m+n+2}).$$

Proof. From (3.6), (3.7), and (3.9),

$$u_{mn}(z) = c_0 c_{mn} + \cdots + c_{m,n+1} z^m,$$

$$v_{mn}(z) = c_{mn} + \cdots + (-1)^n c_{m+1,n} z^n,$$

and

$$C(z)v_{mn}(z) - u_{mn}(z) = (-1)^n c_{m+1,n+1} z^{m+n+1} + O(z^{m+n+2}).$$

Since $c_{mn} \neq 0$, r_{mn} lies on the left or upper boundary of its block in the Padé table. By Theorem 3.2(c), $\varkappa_{mn} = \lambda_{mn} = 0$, and $u_{mn}(z) = c_{mn} p_{mn}(z)$, $v_{mn}(z) = c_{mn} q_{mn}(z)$.

COROLLARY 2. *The alternative representations*

$$u_{mn}(z) = \sigma_{n+1} z^{-mn} \det (\Delta^{i+j} C_{m-n}(z))_{i,j=0}^{n} \tag{3.10}$$

and

$$v_{mn}(z) = \sigma_{n+1} z^{-mn} \det (\Delta^{i+j} C_{m-n}(z))_{i,j=1}^{n} \tag{3.11}$$

are valid, with $\sigma_n \equiv (-1)^{n(n-1)/2}$, $C_m(z) \equiv 0$ *if* $m < 0$, *and* forward differences

$$\Delta^k s_n \equiv \sum_{l=0}^{k} (-1)^{k-l} \binom{k}{l} s_{n+l}, \qquad k \geqq 0.$$

Also if the Hankel determinants

$$c_n^{(m)} \equiv \det (c_{m+i+j})_{i,j=0}^{n-1}, \qquad n \geqq 1,$$

then

$$c_n^{(m)} = \det (\Delta^{i+j} c_m)_{i,j=0}^{n-1} \tag{3.12}$$

and

$$c_{mn} = \sigma_n c_n^{(m-n+1)}, \qquad c_n^{(m)} = \sigma_n c_{m+n-1,n}. \tag{3.13}$$

Proof. For $j = 0, 1, \cdots, n$ divide the jth column of (3.7) by z^j. Then multiply the ith row by z^{m+i}, $i = 1, 2, \cdots, n$. This gives, on reversing the columns,

$$v_{mn} = \sigma_{n+1} z^{-mn} \det \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \Delta C_{m-n} & \Delta C_{m-n+1} & \cdots & \Delta C_m \\ \Delta C_{m-n+1} & \Delta C_{m-n+2} & \cdots & \Delta C_{m+1} \\ \vdots & \vdots & \cdot & \vdots \\ \Delta C_{m-1} & \Delta C_m & \cdots & \Delta C_{m+n-1} \end{pmatrix}.$$

Now since

$$\Delta^{k+1} s_n = \Delta^k s_{n+1} - \Delta^k s_n, \qquad k \geqq 0,$$

one has, by a sequence of row operations,

$$v_{mn} = \sigma_{n+1} z^{-mn} \det \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \Delta C_{m-n} & \Delta C_{m-n+1} & \cdots & \Delta C_m \\ \Delta^2 C_{m-n} & \Delta^2 C_{m-n+1} & \cdots & \Delta^2 C_m \\ \vdots & \vdots & & \vdots \\ \Delta^n C_{m-n} & \Delta^n C_{m-n+1} & \cdots & \Delta^n C_m \end{pmatrix}.$$

The result (3.11) follows from this by the corresponding sequence of column operations, and expansion along the first row. The proofs of (3.10), (3.12) and (3.13) are similar.

4. The Padé table of the quotient of two formal power series. Now let

$$C(z) = A(z)/B(z), \qquad b_0 = 1.$$

The requirement

$$C(z)v(z) - u(z) = O(z^{m+n+1})$$

is equivalent to

$$A(z)v(z) - B(z)u(z) = O(z^{m+n+1})$$

and hence to the linear system

$$\sum_{j=0}^{n} a_{i-j}v_j - \sum_{j=0}^{m} b_{i-j}u_j = 0, \qquad i = 0, 1, \cdots, m + n.$$

A number of interesting consequences result from this formulation. The first is an easy, but substantial, generalization of Theorem 3.3.

THEOREM 4.1. *The determinants* c_{mn} *are* bigradients *in the coefficients of* $A(z)$ *and* $B(z)$:

$$c_{mn} = (b|a)_{mn} \equiv \det\begin{pmatrix} b_0 & 0 & \cdots & 0 & a_0 & 0 & \cdots & 0 \\ b_1 & b_0 & \cdots & 0 & a_1 & a_0 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ b_{m+n-1} & b_{m+n-2} & \cdots & b_n & a_{m+n-1} & a_{m+n-2} & \cdots & a_m \end{pmatrix}. \tag{4.1}$$

Moreover,

$$u_{mn}(z) = (-1)^{n+1}\det\begin{pmatrix} b_0 & 0 & \cdots & 0 & a_0 & 0 & \cdots & 0 \\ b_1 & b_0 & \cdots & 0 & a_1 & a_0 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ b_{m+n} & b_{m+n-1} & \cdots & b_n & a_{m+n} & a_{m+n-1} & \cdots & a_m \\ 1 & z & \cdots & z^m & 0 & 0 & \cdots & 0 \end{pmatrix}, \tag{4.2}$$

$$v_{mn}(z) = (-1)^{n}\det\begin{pmatrix} b_0 & 0 & \cdots & 0 & a_0 & 0 & \cdots & 0 \\ b_1 & b_0 & \cdots & 0 & a_1 & a_0 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ b_{m+n} & b_{m+n-1} & \cdots & b_n & a_{m+n} & a_{m+n-1} & \cdots & a_m \\ 0 & 0 & \cdots & 0 & 1 & z & \cdots & z^n \end{pmatrix} \tag{4.3}$$

and

$$A(z)v_{mn} - B(z)u_{mn}(z)$$

$$(4.4) \quad = (-1)^n \det \begin{pmatrix} b_0 & 0 & \cdots & 0 & a_0 & 0 & \cdots & 0 \\ b_1 & b_0 & \cdots & 0 & a_1 & a_0 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \\ b_{m+n} & b_{m+n-1} & \cdots & b_n & a_{m+n} & a_{m+n-1} & \cdots & a_m \\ B(z) & zB(z) & \cdots & z^m B(z) & A(z) & zA(z) & \cdots & z^n A(z) \end{pmatrix}.$$

$$= (-1)^n \sum_{k=1}^{\infty} \det \begin{pmatrix} b_0 & 0 & \cdots & 0 & a_0 & 0 & \cdots & 0 \\ b_1 & b_0 & \cdots & 0 & a_1 & a_0 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ b_{m+n} & b_{m+n-1} & \cdots & b_n & a_{m+n} & a_{m+n-1} & \cdots & a_m \\ b_{m+n+k} & b_{m+n+k-1} & \cdots & b_{n+k} & a_{m+n+k} & a_{m+n+k-1} & \cdots & a_{m+k} \end{pmatrix}.$$

Proof. Let $u(z)$ and $v(z)$ be the right sides of (4.2) and (4.3), respectively. As in the proof of Theorem 3.3, $A(z)v(z) - B(z)u(z)$ is equal to the determinant expressions of (4.4). Since $b_0 = 1$, by expansion along the first and last rows, $u(0) = (-1)^{n+1+m+n+1+m}a_0(b|a)_{mn} = c_0(b|a)_{mn}$. Similarly $v(0) = (b|a)_{mn}$. Hence (4.1) will imply (4.2), (4.3), and (4.4). Now observe that if $B(z) = 1$, then $A(z) = C(z)$, and so in general

$$(b|a)_{mn} = \det \begin{pmatrix} I_m & * \\ 0 & C_{mn} \end{pmatrix} = c_{mn}.$$

This completes the proof.

The *reciprocal polynomial* of a polynomial

$$u(z) = u_0 + u_1 z + \cdots + u_l z^l,$$

of degree at most l, is

$$u^*(z) = u_0 z^l + u_1 z^{l-1} + \cdots + u_l = z^l u(z^{-1}),$$

the name indicating that (if $u_0 u_l \neq 0$) the zeros of u^* are the reciprocals of those of u. Clearly $(u^*)^* = u$; that is, the operation of forming the reciprocal polynomial is involutory.

The next two theorems are concerned with (monic) polynomials

$$p^*(z) = a_0 z^M + a_1 z^{M-1} + \cdots + a_M = \prod_{m=1}^{M} (z - \zeta_m)$$

and

$$q^*(z) = b_0 z^N + b_1 z^{N-1} + \cdots + b_N = \prod_{n=1}^{N} (z - \pi_n).$$

Their reciprocal polynomials

$$p(z) = \sum_{m=0}^{\infty} a_m z^m, \qquad q(z) = \sum_{n=0}^{\infty} b_n z^n$$

are special formal power series: $a_m \equiv 0, m > M$, and $b_n \equiv 0, n > N$.

THEOREM 4.2. *The* resultant *of* p^* *and* q^* *is*

$$R(p^*, q^*) \equiv (b|a)_{MN} = \prod_{m,n=1}^{M,N} (\zeta_m - \pi_n).$$

Consequently p^* *and* q^* *have a common zero if and only if* $R(p^*, q^*) = 0$.

Proof. Let $M = 2$ and $N = 3$. On taking determinants of both sides of

$$\begin{pmatrix} 1 & \zeta_1 & \zeta_1^2 & \zeta_1^3 & \zeta_1^4 \\ 1 & \zeta_2 & \zeta_2^2 & \zeta_2^3 & \zeta_2^4 \\ 1 & \pi_1 & \pi_1^2 & \pi_1^3 & \pi_1^4 \\ 1 & \pi_2 & \pi_2^2 & \pi_2^3 & \pi_2^4 \\ 1 & \pi_3 & \pi_3^2 & \pi_3^3 & \pi_3^4 \end{pmatrix} \begin{pmatrix} b_0 & 0 & a_0 & 0 & 0 \\ b_1 & b_0 & a_1 & a_0 & 0 \\ b_2 & b_1 & a_2 & a_1 & a_0 \\ b_3 & b_2 & 0 & a_2 & a_1 \\ 0 & b_3 & 0 & 0 & a_2 \end{pmatrix}$$

$$= \begin{pmatrix} q^*(\zeta_1) & \zeta_1 q^*(\zeta_1) & 0 & 0 & 0 \\ q^*(\zeta_2) & \zeta_2 q^*(\zeta_2) & 0 & 0 & 0 \\ 0 & 0 & p^*(\pi_1) & \pi_1 p^*(\pi_1) & \pi_1^2 p^*(\pi_1) \\ 0 & 0 & p^*(\pi_2) & \pi_2 p^*(\pi_2) & \pi_2^2 p^*(\pi_2) \\ 0 & 0 & p^*(\pi_3) & \pi_3 p^*(\pi_3) & \pi_3^2 p^*(\pi_3) \end{pmatrix}$$

one finds

$$v(\zeta_1, \zeta_2, \pi_1, \pi_2, \pi_3)(b|a)_{23} = v(\zeta_1, \zeta_2) v(\pi_1, \pi_2, \pi_3) \prod_{m=1}^{2} q^*(\zeta_m) \prod_{n=1}^{3} p^*(\pi_n).$$

Now observe that

$$\begin{aligned} v(\zeta_1, \zeta_2, \pi_1, \pi_2, \pi_3) = (\zeta_2 - \zeta_1) \; & (\pi_1 - \zeta_1)(\pi_2 - \zeta_1)(\pi_3 - \zeta_1) \\ & (\pi_1 - \zeta_2)(\pi_2 - \zeta_2)(\pi_3 - \zeta_2) \\ & \qquad\quad (\pi_2 - \pi_1)(\pi_3 - \pi_1) \\ & \qquad\qquad\qquad (\pi_3 - \pi_2) \\ = v(\zeta_1, \zeta_2) v(\pi_1, \pi_2, \pi_3) & \prod_{m,n=1}^{2,3} (\pi_n - \zeta_m) \end{aligned}$$

and that

$$\prod_{m=1}^{2} q^*(\zeta_m) = \prod_{m,n=1}^{2,3} (\zeta_m - \pi_n), \qquad \prod_{n=1}^{3} p^*(\pi_n) = \prod_{m,n=1}^{2,3} (\pi_n - \zeta_m).$$

The general case is entirely similar. This completes the proof.

Theorems 4.1 and 3.2 may be applied to extend this result and provide an *explicit formula* for the greatest common divisor of p^* and q^*

THEOREM 4.3 (Trudi). *Let d^* be the (monic) greatest common divisor of p^* and q^*. Then the degree of d^* is $k = M - n = N - n$, with m and n determined by*

$$0 = (b|a)_{MN} = (b|a)_{M-1,N-1} = \cdots = (b|a)_{m+1,n+1} \neq (b|a)_{mn}. \tag{4.5}$$

Moreover if $C(z)$ is the Maclaurin expansion of $p(z)/q(z)$, then

$$\begin{aligned}(b|a)_{mn}d^*(z) &= (-1)^{n-1}[p^*(z)v^*_{m-1,n-1}(z) - q^*(z)u^*_{m-1,n-1}(z)]\\ &= \det\begin{pmatrix} b_0 & 0 & \cdots & 0 & a_0 & 0 & \cdots & 0\\ b_1 & b_0 & \cdots & 0 & a_1 & a_0 & \cdots & 0\\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots\\ b_{m+n-2} & b_{m+n-3} & \cdots & b_{n-1} & a_{m+n-2} & a_{m+n-3} & \cdots & a_{m-1}\\ z^{m-1}q^*(z) & z^{m-2}q^*(z) & \cdots & q^*(z) & z^{n-1}p^*(z) & z^{n-2}p^*(z) & \cdots & p^*(z)\end{pmatrix}\\ &= \sum_{l=0}^{k} \det\begin{pmatrix} b_0 & 0 & \cdots & 0 & a_0 & 0 & \cdots & 0\\ b_1 & b_0 & \cdots & 0 & a_1 & a_0 & \cdots & 0\\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots\\ b_{m+n-2} & b_{m+n-3} & \cdots & b_{n-1} & a_{m+n-2} & a_{m+n-3} & \cdots & a_{m-1}\\ b_{m+n+l-1} & b_{m+n+l-2} & \cdots & b_{n+l} & a_{m+n+l-1} & a_{m+n+l-2} & \cdots & a_{m+l}\end{pmatrix} z^{k-l}.\end{aligned} \tag{4.6}$$

Proof. Consider the Padé table for $C(z)$. By Theorem 3.2(c) the reduced representation for $r_{MN} = p/q$ is $r_{mn} = p_{mn}/q_{mn}$, where

$$r_{MN} = r_{M-1,N-1} = \cdots r_{mn} \neq r_{m-1,n-1}.$$

This occurs if and only if

$$0 = c_{MN} = c_{M-1,N-1} = \cdots c_{m+1,n+1} \neq c_{mn},$$

that is, if and only if (4.5) holds. The reciprocal polynomial of d^* is the greatest common divisor of p and q, and satisfies $p = dp_{mn}$, $q = dq_{mn}$, that is,

$$(b|a)_{mn}p = du_{mn}, \qquad (b|a)_{mn}q = dv_{mn}.$$

By Corollary 1 of Theorem 3.3,

$$\begin{aligned}(b|a)_{mn}[pv_{m-1,n-1} - qu_{m-1,n-1}] &= (b|a)_{mn}d[u_{mn}v_{m-1,n-1} - v_{mn}u_{m-1,n-1}] \\ &= (b|a)_{mn}d[(u_{mn} - Cv_{mn})v_{m-1,n-1} \\ &\quad + v_{mn}(Cv_{m-1,n-1} - u_{m-1,n-1})] \\ &= (-1)^{n-1}[(b|a)_{mn}]^2 d[z^{m+n-1} + O(z^{m+n})] \\ &= (-1)^{n-1}[(b|a)_{mn}]^2 dz^{m+n-1},\end{aligned}$$

since the polynomial in the second bracket is of degree at most $m + n - 1$. Hence

$$(b|a)_{mn}d(z)z^{m+n-1} = (-1)^{n-1}[p(z)v_{m-1,n-1}(z) - q(z)u_{m-1,n-1}(z)],$$

and (4.6) follows from this and Theorem 4.1 (equation (4.4)) on taking reciprocal polynomials.

The final theorem of this section provides a useful *duality relation* between the Padé and c-tables of a unit $C(z)$ and those of its reciprocal series.

THEOREM 4.4. *If $D(z)$ is the reciprocal of $C(z)$ and $c_0 = 1$, then*

$$c_{mn} = (-1)^{mn}d_{nm}. \tag{4.7}$$

Moreover, if $\hat{u}_{mn}$, $\hat{v}_{mn}$, and $\hat{r}_{mn}$ are the Padé elements of type (m, n) for $D(z)$, then

$$u_{mn} = (-1)^{mn}\hat{v}_{nm}, \qquad v_{mn} = (-1)^{mn}\hat{u}_{nm}, \tag{4.8}$$

and the pointwise product

$$r_{mn} \cdot \hat{r}_{nm} = 1. \tag{4.9}$$

Proof 1. Let $C(z) = A(z)/B(z)$ with $a_0 = b_0 = 1$ so that $D(z) = B(z)/A(z)$. Permutation of the a- and b-blocks in the matrix of (4.1) requires mn column interchanges. Hence

$$c_{mn} = (b|a)_{mn} = (-1)^{mn}(a|b)_{nm} = (-1)^{mn}d_{nm}.$$

Similarly with (4.2),

$$u_{mn} = (-1)^{n+1+(m+1)(n+1)+m}\hat{v}_{nm} = (-1)^{mn}\hat{v}_{nm},$$

and this is also equivalent to the second equality of (4.8). The assertion (4.9) about the rational functions r_{mn} and $\hat{r}_{nm}$ requires a little care using Theorem 3.2. By (4.7) the block structures of the Padé table for $C(z)$ and the transpose of that for $D(z)$ are identical. From (4.8) and Theorem 3.3, (4.9) holds if r_{mn} and $\hat{r}_{nm}$ lie on the left or upper boundaries of their blocks. Hence it holds in general.

Proof 2. This proof does not use determinants. First suppose that $C(z)$ is normal. Multiply $D\hat{v}_{nm} - \hat{u}_{nm} = O(z^{m+n+1})$ by $-C$ to obtain $C\hat{u}_{nm} - \hat{v}_{nm} = O(z^{m+n-1})$. Also $Cv_{mn} - u_{mn} = O(z^{m+n+1})$. Since $\hat{v}_{nm}$ and $\hat{u}_{nm}$ are polynomials of degrees at most m and n, respectively, and since the rank of S_{mn} is maximal, there exist scalars k_{mn} such that

$$u_{mn} = k_{mn}\hat{v}_{nm}, \qquad v_{mn} = k_{mn}\hat{u}_{nm}.$$

Comparison of the constant and leading terms in the first of these equations, using Corollary 1 of Theorem 3.3, gives

$$c_{mn} = k_{mn}d_{nm}, \qquad c_{m,n+1} = (-1)^m k_{mn} d_{n+1,m}.$$

Hence $D(z)$ is also normal, and since $c_{m0} = d_{0m} = 1$,

$$k_{mn} = \frac{c_{mn}}{d_{nm}} = (-1)^m k_{m,n-1} = (-1)^{mn} k_{m0} = (-1)^{mn}.$$

The rational identities (4.7) and (4.8) now hold even if $C(z)$ is not normal, and (4.9) is proved as above.

Proof 3. An alternative proof of (4.7) may be obtained by applying Theorem 2.2 to the matrix $A = C^A_{0,m+n} = C^{-1}_{0,m+n} = D_{0,m+n}$. Thus

$$\begin{aligned} c_{mn} &= \det C_{0,m+n}\begin{pmatrix} m+1, m+2, \cdots, m+n \\ 1 \quad , \quad 2 \quad , \cdots, \quad n \end{pmatrix} \\ &= (-1)^{mn} \det D_{0,m+n}\begin{pmatrix} n+1, n+2, \cdots, n+m \\ 1 \quad , \quad 2 \quad , \cdots, \quad m \end{pmatrix} \\ &= (-1)^{mn} d_{nm}, \end{aligned}$$

which completes the proof.

Theorem 4.4 provides the entries in the first row of the Padé table for $C(z)$ as

$$r_{0n}(z) = \frac{1}{D_n(z)} = \frac{1}{d_0 + d_1 z + \cdots + d_n z^n}.$$

Also, the second row of the c-table contains the values

$$c_{1n} = (-1)^n d_n.$$

This gives an explicit formula for the coefficients of the reciprocal series.

COROLLARY (Wronski).

$$D(z) = \sum_{n=0}^{\infty} (-1)^n \det \begin{pmatrix} c_1 & 1 & 0 & \cdots & 0 & 0 \\ c_2 & c_1 & 1 & \cdots & 0 & 0 \\ c_3 & c_2 & c_1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ c_{n-1} & c_{n-2} & c_{n-3} & \cdots & c_1 & 1 \\ c_n & c_{n-1} & c_{n-2} & \cdots & c_2 & c_1 \end{pmatrix} z^n.$$

5. The identities of Frobenius. The basic algorithmic aspects of the Padé table of the formal power series

$$C(z) = \sum_{m=0}^{\infty} c_m z^m, \qquad c_0 = 1,$$

are developed in this section. They consist of a hierarchy of identities among the determinants c_{mn}, numerators and denominators u_{mn} and v_{mn}, and finally the rational functions r_{mn} themselves. If $C(z)$ is normal, such relations permit, in principle, the recursive construction of these Padé elements. Several of the results follow directly from Sylvester's determinant identity (Theorem 2.6). Proofs of others are simply verifications based on variations of the uniqueness argument of Theorem 3.1. This was the original approach of Frobenius.

THEOREM 5.1. *The quadratic identity*

$$\begin{array}{ccc} & * & \\ * & \circledast & * \\ & * & \end{array} \qquad c_{m-1,n}c_{m+1,n} + c_{m,n-1}c_{m,n+1} = c_{mn}^2$$

is valid in the extended c-table.

Proof. Apply Sylvester's identity to $A' = C_{m,n+1}$. The extended c-values are uniquely determined by the requirement that this identity and the duality relation (4.7) hold in the extended table. This completes the proof.

If $C(z)$ is normal, this and the boundary values

$$c_{m0} = c_{0n} = 1, \quad c_{m1} = c_m, \quad c_{1n} = (-1)^n d_n$$

permit the construction of the c-table by successive addition of upward sloping diagonals. Within a diagonal the computation is from bottom to top if the c_m are known, and vice versa if the d_n are known.

The natural extensions of the u- and v-tables, corresponding to those of the c- and Padé tables, are obtained by defining

$$\begin{aligned} &u_{m,-1}(z) \equiv z^m, \qquad && v_{m,-1}(z) \equiv 0, \qquad && m \geqq 0, \\ &u_{-1,n}(z) \equiv 0, && v_{-1,n}(z) \equiv (-z)^n, && n \geqq 0. \end{aligned}$$

Observe that the duality relations (4.8) continue to hold in the extended tables.

The next two theorems are concerned with identities among elements in w-tables, where

$$w_{mn}(z) \equiv P(z)u_{mn}(z) + Q(z)v_{mn}(z)$$

and $P(z)$ and $Q(z)$ are arbitrary formal power series. Three obvious choices for the pair $(P(z), Q(z))$ are $(1, 0)$, $(0, 1)$, and $(-1, C(z))$.

THEOREM 5.2. *There exists, among any three elements occupying distinct positions in the w-table, a linear homogeneous relation with polynomial coefficients. In particular the triangle formulas*:

$$\begin{array}{cc} \circledast & * \\ * & \end{array} \qquad c_{m,n+1}w_{m+1,n} - c_{m+1,n}w_{m,n+1} = c_{m+1,n+1}zw_{mn},$$

$$\begin{array}{cc} * & \circledast \\ & * \end{array} \qquad c_{mn}w_{m+1,n} - c_{m+1,n}w_{mn} = c_{m+1,n+1}zw_{m,n-1},$$

$$\begin{array}{cc} * & \\ \circledast & * \end{array} \qquad c_{m,n+1}w_{mn} - c_{mn}w_{m,n+1} = c_{m+1,n+1}zw_{m-1,n},$$

$$\begin{array}{cc} & * \\ * & \circledast \end{array} \qquad c_{m+1,n}w_{m-1,n} + c_{m,n+1}w_{m,n-1} = c_{mn}w_{mn},$$

and the formulas relating three elements in a row, column, or diagonal:

$$*\circledast* \qquad \begin{aligned} &c_{mn}c_{m+1,n}w_{m,n+1} - c_{m,n+1}c_{m+1,n+1}zw_{m,n-1} \\ &\qquad = (c_{m+1,n}c_{m,n+1} - c_{mn}c_{m+1,n+1}z)w_{mn}, \end{aligned}$$

$$\begin{matrix}*\\ \circledast\\ *\end{matrix} \qquad \begin{aligned} &c_{mn}c_{m,n+1}w_{m+1,n} + c_{m+1,n}c_{m+1,n+1}zw_{m-1,n} \\ &\qquad = (c_{m,n+1}c_{m+1,n} + c_{mn}c_{m+1,n+1}z)w_{mn}, \end{aligned}$$

$$\begin{matrix}* & & \\ & \circledast & \\ & & *\end{matrix} \qquad \begin{aligned} &c_{mn}^2w_{m+1,n+1} - c_{m+1,n+1}^2z^2w_{m-1,n-1} \\ &\qquad = [c_{mn}c_{m+1,n+1} + (c_{m,n-1}c_{m+1,n+2} - c_{m-1,n}c_{m+2,n+1})z]w_{mn}, \end{aligned}$$

$$\begin{matrix} & & *\\ & \circledast & \\ * & & \end{matrix} \qquad \begin{aligned} &c_{m+1,n}^2w_{m-1,n+1} - c_{m,n+1}^2w_{m+1,n-1} \\ &\qquad = [(c_{m-1,n+1}c_{m+2,n} - c_{m+1,n-1}c_{m,n+2}) - c_{m+1,n}c_{m,n+1}z]w_{mn}, \end{aligned}$$

are valid.

Proof. First let $w = v$. Apply Sylvester's identity to

$$v_{m,n+1}(z) = \det\begin{pmatrix} 1 & z & \cdots & z^{n+1} \\ c_{m+1} & c_m & \cdots & c_{m-n} \\ \vdots & \vdots & & \vdots \\ c_{m+n+1} & c_{m+n} & \cdots & c_m \end{pmatrix}$$

to prove $\overset{*}{\circledast}*$. Likewise $\begin{matrix}\circledast & *\\ * & \end{matrix}$ follows from

$$(-1)^{n+1}v_{m,n+1}(z) = \det\begin{pmatrix} c_{m+1} & c_m & \cdots & c_{m-n} \\ c_{m+2} & c_{m+1} & \cdots & c_{m-n+1} \\ \vdots & \vdots & & \vdots \\ c_{m+n+1} & c_{m+n} & \cdots & c_m \\ 1 & z & \cdots & z^{n+1} \end{pmatrix}.$$

Now apply the duality relations (4.7) and (4.8) to the corresponding identities for the reciprocal series $D(z)$ to show that they also hold for $w = u$. Since they are linear and homogeneous (and P and Q are independent of m and n) they must also hold for general $w = Pu + Qv$.

The remaining identities, and in fact identities relating elements of any three distinct positions in the w-table, may be built up from $\begin{matrix}* & \\ * & *\end{matrix}$ and $\begin{matrix}* & *\\ * & \end{matrix}$ in an inductive manner by forming appropriate linear combinations of existing relations and using Theorem 5.1 to simplify the resulting expressions. Thus to prove $*\overset{*}{\circledast}$ consider $\overset{*}{*}\circledast$ and $\overset{*}{*}\overset{*}{\odot}$, that is,

$$c_{mn}w_{m,n-1} - c_{m,n-1}w_{mn} = c_{m+1,n}zw_{m-1,n-1}$$

and

$$c_{m-1,n}w_{m,n-1} - c_{m,n-1}w_{m-1,n} = c_{mn}zw_{m-1,n-1}.$$

Multiply the former by c_{mn}, the latter by $c_{m+1,n}$, and subtract to get

$$(c_{mn}^2 - c_{m-1,n}c_{m+1,n})w_{m,n-1} + c_{m,n-1}c_{m+1,n}w_{m-1,n} = c_{m,n-1}c_{mn}w_{mn}.$$

Application of Theorem 1 to the term in parentheses, and cancellation of $c_{m,n-1}$, proves $*\overset{*}{\circledast}$. Further details are similar, and hence are omitted.

THEOREM 5.3. *The quadratic identity*

$$\begin{matrix} & * & \\ * & \circledast & * \\ & * & \end{matrix} \qquad w_{m-1,n}w_{m+1,n} + w_{m,n-1}w_{m,n+1} = w_{mn}^2$$

is also valid in the extended w-table.

Proof. Let

$$\begin{matrix} & (m-1,n) & & & N & \\ (m,n-1) & (m,n) & (m,n+1) \equiv & W & C & E. \\ & (m+1,n) & & & S & \end{matrix}$$

Eliminate c_{mn}, $c_{m,n+1}$, $c_{m+1,n}$ and $c_{m+1,n+1}$ from the four triangle identities to get

$$\det\begin{pmatrix} 0 & -w_S & w_E & zw_C \\ w_S & 0 & -w_C & -zw_N \\ -w_E & w_C & 0 & -zw_N \\ -zw_C & zw_W & zw_N & 0 \end{pmatrix} = z^2(w_N \cdot w_S + w_W \cdot w_E - w_C^2)^2 = 0.$$

THEOREM 5.4. *Let (m,n) and (m',n') correspond to distinct positions in the Padé table, and*

$$\mu \equiv \min(m+n, m'+n'), \qquad \nu \equiv \max(m'+n, m+n').$$

Then the cross product

$$u_{m'n'}v_{mn} - u_{mn}v_{m'n'} \equiv e_{mn}^{(m',n')}$$

is a polynomial of degree at most ν, and is divisible by the power $z^{\mu+1}$. In particular the following relations, corresponding to neighboring positions in the Padé table, are valid:

$$\underset{*}{\circledast} \qquad u_{m+1,n}v_{mn} - u_{mn}v_{m+1,n} = (-1)^n c_{m+1,n}c_{m+1,n+1}z^{m+n+1},$$

$$\circledast\, * \qquad u_{m,n+1}v_{mn} - u_{mn}v_{m,n+1} = (-1)^n c_{m,n+1}c_{m+1,n+1}z^{m+n+1},$$

$$\begin{matrix}\circledast & \\ & *\end{matrix} \qquad u_{m+1,n+1}v_{mn} - u_{mn}v_{m+1,n+1} = (-1)^n c_{m+1,n+1}^2 z^{m+n+1},$$

$$\begin{matrix}\odot & * \\ * & \end{matrix} \qquad u_{m,n+1}v_{m+1,n} - u_{m+1,n}v_{m,n+1} = (-1)^n c_{m+1,n+1}^2 z^{m+n+2}.$$

Proof. The degree of $e_{mn}^{(m',n')}$ is obviously at most ν, and since

$$\begin{aligned} e_{mn}^{(m',n')} &= (Cv_{mn} - u_{mn})v_{m'n'} - (Cv_{m'n'} - u_{m'n'})v_{mn} \\ &= O(z^{m+n+1}) + O(z^{m'+n'+1}) = O(z^{\mu+1}), \end{aligned}$$

it is divisible by $z^{\mu+1}$. The listed identities correspond to the four cases in which $\mu + 1 = \nu$; that is, $e_{mn}^{(m',n')}$ is a scalar multiple of $z^{\mu+1}$. From Corollary 1 of Theorem 3.3,

$$Cv_{mn} - u_{mn} = (-1)^n c_{m+1,n+1} z^{m+n+1} + O(z^{m+n+2})$$

and

$$v_{mn} = c_{mn} + O(z).$$

The first three identities follow since, in these cases, $\mu = m + n < m' + n'$. The fourth requires Theorem 5.1:

$$\begin{aligned} e_{m+1,n}^{(m,n+1)} &= (-1)^n(c_{m+2,n+1}c_{m,n+1} + c_{m+1,n+2}c_{m+1,n})z^{m+n+2} \\ &= (-1)^n c_{m+1,n+1}^2 z^{m+n+2}. \end{aligned}$$

This completes the proof.

The final "missing identity of Frobenius" was discovered by Wynn in 1966.

THEOREM 5.5. *If the Padé fraction r_{mn} is normal, then the identity*

$$\begin{array}{ccc} & * & \\ * & \circledast & * \\ & * & \end{array} \qquad \frac{1}{r_{m+1,n} - r_{mn}} + \frac{1}{r_{m-1,n} - r_{mn}} = \frac{1}{r_{m,n+1} - r_{mn}} + \frac{1}{r_{m,n-1} - r_{mn}}$$

is valid.

Proof. If r_{mn} is normal, then, by Theorem 3.2, none of the Padé fractions occurring in the alleged identity are equal. Moreover each linear system $S_{\mu\nu}$ is of maximal rank so that $u_{\mu\nu} = r_{\mu\nu}v_{\mu\nu} \neq 0$. Again let

$$\begin{array}{cccccccc} & (m-1,n) & & & & N & \\ (m,n-1) & (m,n) & (m,n+1) & \equiv & W & C & E \\ & (m+1,n) & (m+1,n+1) & & & S & SE \end{array}$$

From Theorem 4,

$$\begin{array}{ll} \begin{array}{c} * \\ \circledast \end{array} & v_N v_C(r_N - r_C) = (-1)^n c_C c_E z^{m+n}, \\ \begin{array}{c} \circledast \\ * \end{array} & v_S v_C(r_S - r_C) = (-1)^n c_S c_{SE} z^{m+n+1}, \\ \circledast\ * & v_E v_C(r_E - r_C) = (-1)^n c_E c_{SE} z^{m+n+1}, \\ *\ \circledast & v_W v_C(r_C - r_W) = -(-1)^n c_C c_S z^{m+n}. \end{array}$$

Hence

$$v_N v_S(r_N - r_C)(r_S - r_C) + v_W v_E(r_W - r_C)(r_E - r_C) = 0.$$

From Theorem 3,

$$\begin{aligned} v_N v_S r_N r_S + v_W v_E r_W r_E &= u_N u_S + u_W u_E \\ &= u_C^2 = r_C^2 v_C^2 = r_C^2 (v_N v_S + v_W v_E), \end{aligned}$$

that is,

$$v_N v_S (r_N r_S - r_C^2) + v_W v_E (r_W r_E - r_C^2) = 0.$$

Consequently,

$$\begin{aligned} 0 &= \det \begin{pmatrix} r_N r_S - r_C^2 & r_W r_E - r_C^2 \\ (r_N - r_C)(r_S - r_C) & (r_W - r_C)(r_E - r_C) \end{pmatrix} \\ &= r_C \det \begin{pmatrix} (r_N - r_C) + (r_S - r_C) & (r_W - r_C) + (r_E - r_C) \\ (r_N - r_C)(r_S - r_C) & (r_W - r_C)(r_E - r_C) \end{pmatrix} \end{aligned}$$

by subtraction of rows. This is equivalent to the stated result.

Wynn's identity permits the recursive construction of a normal Padé table from the partial sums

$$r_{m0}(z) = C_m(z) = \sum_{k=0}^{\infty} c_k z^k$$

according to the scheme in Table 7.

TABLE 7

$$\begin{array}{c|cccccc} & 0 & 0 & 0 & 0 & 0 & 0 \quad \cdots \\ \hline \infty & C_0 & r_{01} & r_{02} & r_{03} & r_{04} & \cdots \\ \infty & C_1 & r_{11} & r_{12} & r_{13} & & \\ \infty & C_2 & r_{21} & r_{22} & & & \\ \infty & C_3 & r_{31} & & & & \\ \infty & C_4 & & & & & \\ \infty & \vdots & & & & & \\ \vdots & & & & & & \end{array}$$

The dual algorithm begins with the partial sums

$$r_{0n}^{-1}(z) = D_n(z) = \sum_{k=0}^{n} d_k z^k$$

of the reciprocal series $D(z)$ and constructs the upward sloping diagonals from top to bottom.

6. The epsilon and eta algorithms. A special case of the power series

$$C(z) = c_0 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots, \qquad c_0 = 1,$$

is the infinite series

$$C(1) \equiv c_0 + c_1 + c_2 + c_3 \cdots .$$

The partial sums

$$s_m = C_m(1) \equiv c_0 + c_1 + c_2 + \cdots + c_m$$

form an infinite sequence $\{s_m\}_0^\infty$. Conversely, with any infinite sequence $\{s_m\}_0^\infty$, one may associate the formal power series $C(z)$ by defining

$$c_0 = s_0, \qquad c_m = \Delta s_{m-1}, \qquad\qquad m \geqq 1.$$

Then $s_m = C_m(1)$.

The *epsilon algorithm* is a (nonlinear) *sequence transformation*. One constructs the triangular array in Table 8:

TABLE 8

$$\begin{array}{c|cccccc}
 & \varepsilon_1^{(0)} & & & & & \\
\varepsilon_0^{(1)} & \downarrow & \varepsilon_2^{(0)} & & & & \\
 & \varepsilon_1^{(1)} & & \varepsilon_3^{(0)} & & & \\
\varepsilon_0^{(2)} & \downarrow & \varepsilon_2^{(1)} & & \varepsilon_4^{(0)} & & \\
 & \varepsilon_1^{(2)} & & \varepsilon_3^{(1)} & & \varepsilon_5^{(0)} & \\
\varepsilon_0^{(3)} & \downarrow & \varepsilon_2^{(2)} & & \varepsilon_4^{(1)} & \vdots & \ddots \\
 & \varepsilon_1^{(3)} & & \varepsilon_3^{(2)} & \vdots & & \\
\varepsilon_0^{(4)} & \downarrow & \varepsilon_2^{(3)} & \vdots & & & \\
 & \varepsilon_1^{(4)} & \vdots & & & & \\
\vdots & \vdots & & & & &
\end{array}$$

(when possible) from the boundary values

$$\varepsilon_0^{(m)} = 0, \qquad \varepsilon_1^{(m)} = s_m,$$

and the *rhombus rules*

$$(\varepsilon_{n+1}^{(m)} - \varepsilon_{n-1}^{(m+1)})(\varepsilon_n^{(m+1)} - \varepsilon_n^{(m)}) = 1.$$

At the mth stage the upward sloping diagonal $\varepsilon_0^{(m)}, \varepsilon_1^{(m-1)}, \cdots, \varepsilon_m^{(0)}$ is added to the ε-array. In practice this requires storage of only the latest upward sloping diagonal.

The following theorem provides explicit formulas for the values $\varepsilon_n^{(m)}$ and relates them to the Padé table of the formal power series $C(z)$ corresponding to the sequence $\{s_m\}_0^\infty$.

THEOREM 6.1 (Shanks, Wynn). *If the indicated quantities exist, then*

$$\varepsilon_{2n+1}^{(m)} = \frac{\det(\Delta^{i+j}s_m)_{i,j=0}^n}{\det(\Delta^{i+j}s_m)_{i,j=1}^n} = r_{m+n,n}(1)$$

and

$$\varepsilon_{2n+2}^{(m)} = \frac{\det(\Delta^{i+j+1}s_m)_{i,j=1}^n}{\det(\Delta^{i+j+1}s_m)_{i,j=0}^n}.$$

Proof. Let the array of rational functions in Table 9 be constructed from the boundary conditions $r_{m0}(z) = C_m(z)$ by the rhombus rules

$$\begin{array}{ccc} & N & \\ W & & E \\ & S & \end{array} \qquad (r_S - r_N)(r'_E - r'_W) = (r'_S - r'_N)(r_E - r_W) = 1.$$

TABLE 9

$$
\begin{array}{c|cccccccccc}
 & 0 & & 0 & & 0 & & 0 & & & \\
\hline
0 & & r'_{00} & & r'_{01} & & r'_{02} & & r'_{03} & \cdots & \\
 & r_{00} & & r_{01} & & r_{02} & & r_{03} & \cdots & & \\
0 & & r'_{10} & & r'_{11} & & r'_{12} & & r'_{13} & \cdots & \\
 & r_{10} & & r_{11} & & r_{12} & & r_{13} & \cdots & & \\
0 & & r'_{20} & & r'_{21} & & r'_{22} & & r'_{23} & \cdots & \\
 & r_{20} & & r_{21} & & r_{22} & & r_{23} & \cdots & & \\
0 & & r'_{30} & & r'_{31} & & r'_{32} & & r'_{33} & \cdots & \\
 & r_{30} & & r_{31} & & r_{32} & & r_{33} & \cdots & & \\
\vdots & & \vdots & & \vdots & & \vdots & & & & \\
 & \vdots & & \vdots & & \vdots & & \vdots & & &
\end{array}
$$

Then r_{mn} is the Padé fraction of type (m, n) for $C(z)$. For, consider the "constellation"

$$
\begin{array}{ccccc}
 & & r_N & & \\
 & r'_{NW} & & r'_{NE} & \\
r_W & & r_C & & r_E. \\
 & r'_{SW} & & r'_{SE} & \\
 & & r_S & &
\end{array}
$$

From the rhombus rules,

$$
\begin{aligned}
\frac{1}{r_N - r_C} + \frac{1}{r_S - r_C} &= (r'_{NW} - r'_{NE}) + (r'_{SE} - r'_{SW}) \\
&= (r'_{NW} - r'_{SW}) + (r'_{SE} - r'_{NE}) \\
&= \frac{1}{r_W - r_C} + \frac{1}{r_E - r_C}.
\end{aligned}
$$

Hence the functions r'_{mn} may be eliminated to give the "star identity" of Theorem 5.5 for the Padé table. By Corollary 2 of Theorem 3.3,

$$
r_{mn}(z) = \frac{\det(\Delta^{i+j} C_{m-n}(z))_{i,j=0}^{n}}{\det(\Delta^{i+j} C_{m-n}(z))_{i,j=1}^{n}}, \tag{6.1}
$$

and the duality theorem, Theorem 4.4, gives

$$
r_{nm}(z) = \frac{\det(\Delta^{i+j} D_{m-n}(z))_{i,j=1}^{n}}{\det(\Delta^{i+j} D_{m-n}(z))_{i,j=0}^{n}}, \tag{6.2}
$$

with $D_n(z)$ the nth partial sum of the reciprocal series $D(z)$. Now eliminate the functions r_{mn} from the above array in the same way. Observe that the functions r'_{mn} are obtained from the r'_{m0} exactly as the r_{nm} are obtained from the r_{0n}. Since

$$r'_{m0}(z) = \frac{1}{\Delta C_{m-1}(z)}, \qquad r_{0m}(z) = \frac{1}{D_m(z)},$$

it follows from (6.2) that

$$r'_{mn}(z) = \frac{\det\,(\Delta^{i+j}\Delta C_{m-n-1}(z))_{i,j=1}^{n}}{\det\,(\Delta^{i+j}\Delta C_{m-n-1}(z))_{i,j=0}^{n}}. \tag{6.3}$$

The theorem is a consequence of (6.1) and (6.3), on setting $s_m = C_m(1)$, $\varepsilon_{2n+1}^{(m)} = r_{m+n,n}(1)$, and $\varepsilon_{2n+2}^{(m)} = r'_{m+n+1,n}(z)$.

The *eta algorithm* is a (nonlinear) *series transformation.* One constructs the triangular array in Table 10 (when possible) from the boundary values

$$\eta_0^{(m)} = \infty, \qquad \eta_1^{(m)} = c_m, \tag{6.4}$$

TABLE 10

$$\begin{array}{c|cccccc}
 & \eta_1^{(0)} & & & & & \\
\eta_0^{(1)} & & \eta_2^{(0)} & & & & \\
 & \eta_1^{(1)} & & \eta_3^{(0)} & & & \\
\eta_0^{(2)} & & \eta_2^{(1)} & & \eta_4^{(0)} & & \\
 & \eta_1^{(2)} & & \eta_3^{(1)} & & \eta_5^{(0)} & \\
\eta_0^{(3)} & & \eta_2^{(2)} & & \eta_4^{(1)} & & \ddots \\
 & \eta_1^{(3)} & & \eta_3^{(2)} & \vdots & & \\
\eta_0^{(4)} & & \eta_2^{(3)} & \vdots & & & \\
\vdots & \eta_1^{(4)} & \vdots & & & & \\
 & \vdots & & & & &
\end{array}$$

and the rhombus rules

$$\eta_{2n+1}^{(m)} + \eta_{2n}^{(m)} = \eta_{2n}^{(m+1)} + \eta_{2n-1}^{(m+1)}, \tag{6.5}$$

$$\frac{1}{\eta_{2n+2}^{(m)}} + \frac{1}{\eta_{2n+1}^{(m)}} = \frac{1}{\eta_{2n+1}^{(m+1)}} + \frac{1}{\eta_{2n}^{(m+1)}}. \tag{6.6}$$

The next theorem is the analogue of Theorem 6.1 for the eta algorithm.

THEOREM 6.2 (Bauer). *If the indicated quantities exist, then*

$$\begin{aligned}
\eta_{2n+1}^{(m)} &= \frac{\det\,(\Delta^{i+j}c_m)_{i,j=0}^{n}\,\det\,(\Delta^{i+j}c_{m+1})_{i,j=0}^{n-1}}{\det\,(\Delta^{i+j+1}c_m)_{i,j=0}^{n-1}\,\det\,(\Delta^{i+j+1}c_{m+1})_{i,j=0}^{n-1}},\\
\eta_{2n+2}^{(m)} &= \frac{\det\,(\Delta^{i+j}c_m)_{i,j=0}^{n}\,\det\,(\Delta^{i+j}c_{m+1})_{i,j=0}^{n}}{\det\,(\Delta^{i+j+1}c_m)_{i,j=0}^{n}\,\det\,(\Delta^{i+j+1}c_{m+1})_{i,j=0}^{n-1}},
\end{aligned} \tag{6.7}$$

and

$$(6.8)\qquad \begin{aligned} r_{m+n,n}(1) &= \sum_{k=0}^{m-1} \eta_1^{(k)} + \sum_{k=0}^{2n+1} \eta_k^{(m)}, \\ r_{m+n,n+1}(1) &= \sum_{k=0}^{m-1} \eta_1^{(k)} + \sum_{k=0}^{2n+2} \eta_k^{(m)}. \end{aligned}$$

Proof. Let

$$\eta_{2n+1}^{(m)} \equiv r_{m+n,n}(1) - r_{m+n-1,n}(1),$$

$$\eta_{2n+2}^{(m)} \equiv r_{m+n,n+1}(1) - r_{m+n,n}(1).$$

The boundary values (6.4), rhombus rules (6.5), (6.6), and the assertion (6.8) follow directly from this definition; (6.6) is Wynn's star identity with $z = 1$. Also the identities $**$ and $\overset{*}{*}$ of Theorem 5.4 provide

$$\eta_{2n+1}^{(m)} = (-1)^n \frac{c_{m+n,n}c_{m+n,n+1}}{v_{m+n-1,n}(1)v_{m+n,n}(1)},$$

$$\eta_{2n+2}^{(m)} = (-1)^{n+1} \frac{c_{m+n,n+1}c_{m+n+1,n+1}}{v_{m+n,n}(1)v_{m+n,n+1}(1)}.$$

From Corollary 2 of Theorem 3.3,

$$c_{m+n,n} = \sigma_n c_n^{(m+1)} = \sigma_n \det(\Delta^{i+j}c_{m+1})_{i,j=0}^{n-1}$$

and

$$\begin{aligned} v_{m+n,n}(1) &= \sigma_{n+1} \det(\Delta^{i+j}s_m)_{i,j=1}^{n} \\ &= \sigma_{n+1} \det(\Delta^{i+j+2}s_m)_{i,j=0}^{n-1} \\ &= \sigma_{n+1} \det(\Delta^{i+j+1}c_m)_{i,j=0}^{n-1} \end{aligned}$$

since $\Delta s_m = c_{m+1}$. The assertion (6.7) follows since

$$\sigma_n = (-1)^{n(n-1)/2} = (-1)^{1+2+\cdots+(n-1)}.$$

This completes the proof.

Observe that (6.8) states that the lower "almost triangular" half of the Padé table shown in Table 11 is obtained by summing the eta scheme, first along the

TABLE 11

$$\begin{array}{cccccc} r_{00} & r_{01} & & & & \\ r_{10} & r_{11} & r_{12} & & & \\ r_{20} & r_{21} & r_{22} & r_{23} & & \\ r_{30} & r_{31} & r_{32} & r_{33} & r_{34} & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}$$

first column, and then along the downward sloping diagonals. The summed eta array has the form shown in Table 12.

TABLE 12

$$
\begin{array}{ccccc}
r_{00} & & & & \\
 & r_{01} & & & \\
r_{10} & & r_{11} & & \\
 & r_{11}// & & r_{12} & \\
r_{20} & & r_{21} & & r_{22} \\
 & r_{21}// & & r_{22}// \vdots & \ddots \\
r_{30} & & r_{31} & \vdots & \\
 & r_{31}// & \vdots & & \\
r_{40} & \vdots & & & \\
\vdots & & & &
\end{array}
$$

Hence some elements of the Padé table are repeated.

7. Normality criteria: Pólya frequency, Stieltjes, and nonnegative definite power series. In this section three classes of formal power series are investigated with regard to normality properties of the Padé tables of their members.

The power series $C(z)$ is a *Pólya frequency* (PF) *series* if *all* subdeterminants of the corresponding semicirculant matrix C are nonnegative.

LEMMA 7.1. (a) *If $A(z)$ and $B(z)$ are PF series, then so is their product $A(z)B(z)$.*

(b) *If $C(z)$ is a PF series and $C(z)D(z) = 1$, then $D'(z) \equiv D(-z)$ is also a PF series.*

Proof. (a) Let $\alpha, \beta \in \Delta_{\infty k}$. By the Cauchy-Binet formula (Theorem 2.3),

$$\det (AB)(\alpha, \beta) = \sum_{\gamma \in \Delta_{\infty k}} \det A(\alpha, \gamma) \det D(\gamma, \beta) \geqq 0.$$

(b) If $\alpha, \beta \in \Delta_{\infty k}$, then $\alpha, \beta \in \Delta_{nk}$ for some integer $n \geqq 1$. Since

$$C_{0n}D_{0n} = I_n, \qquad \det C_{0n} = c_0^n > 0,$$

Jacobi's theorem, Theorem 2.2, gives

$$\begin{aligned}
\det D'(\alpha, \beta) &= \det ((-1)^{\alpha_i - \beta_j} d_{\alpha_i - \beta_j})_{i,j=1}^k \\
&= (-1)^{|\alpha| - |\beta|} \det D_{0n}(\alpha, \beta) \\
&= (-1)^{|\alpha| - |\beta|} c_0^{-n} \det C_{0n}^A(\alpha, \beta) \\
&= c_0^{-n} \det C_{0n}(\beta', \alpha') \geqq 0.
\end{aligned}$$

THEOREM 7.1 (Schoenberg, Karlin). *Let $C(z)$ be the Maclaurin expansion of the meromorphic function*

$$f(z) = e^{\gamma z} \frac{\prod_{m=1}^{\infty} (1 + \alpha_m z)}{\prod_{n=1}^{\infty} (1 - \beta_n z)} \tag{7.1}$$

with $\alpha_m \geqq 0$, $\beta_n \geqq 0$, $\gamma \geqq 0$ and $\sum (\alpha_k + \beta_k) < \infty$. Then $C(z)$ is a Pólya frequency series. Moreover if M (N) is the number of positive α_m (β_n), then:

(a) *if $\gamma = 0$ and $M + N < \infty$, then $C(z)$ is (M, N)-normal:*

$$c_{mn} > 0, \qquad 0 \leqq m \leqq M \quad \textit{or} \quad 0 \leqq n \leqq N,$$
$$c_{mn} = 0, \qquad m > M \quad \textit{and} \quad n > N;$$

(b) *if $\gamma > 0$ or $M + N = \infty$, then $C(z)$ is normal:*

$$c_{mn} > 0, \qquad m \geqq 0, \qquad n \geqq 0.$$

Proof. Let $A_\alpha(z) = 1 + \alpha z$, $\alpha \geqq 0$. The nonnull subdeterminants of the matrix A_α are $\alpha^k \geqq 0$, $k = 0, 1, 2, \cdots$. Hence $A_\alpha(z)$ is a PF series. From Lemma 7.1(b), $B_\beta(z) = \sum \beta^n z^n = 1/A_\beta(-z)$ is also a PF series for $\beta \geqq 0$. By repeated application of Lemma 7.1(a), the Maclaurin expansion $C^{(s)}(z) = \sum c_m^{(s)} z^m$ of

$$f^{(s)}(z) = \left(1 + \frac{\gamma z}{s}\right)^s \frac{\prod_{m=1}^{s} (1 + \alpha_m z)}{\prod_{n=1}^{s} (1 - \beta_n z)}$$

is a PF series for every integer $s \geqq 0$. Since $f^{(s)}(z) \to f(z)$ uniformly for $|z| \leqq \varepsilon < 1/\max(\beta_n)$, $s \to \infty$, the coefficients $c_m^{(s)} \to c_m$ as $s \to \infty$. Consequently $\lim_{s \to \infty} \det C^{(s)}(\alpha, \beta) = \det C(\alpha, \beta) \geqq 0$ for $\alpha, \beta \in \Delta_{\infty k}$, proving the first assertion.

Let $E_\gamma(z) = \sum_0^\infty e_m(\gamma) z^m = \sum_0^\infty (\gamma^m/m!) z^m$. By induction using Theorem 5.1,

$$e_{mn}(\gamma) = \frac{(m-1)!!(n-1)!!}{(m+n-1)!!} \gamma^{mn}$$

with

$$k!! \equiv \prod_{j=0}^{k} j!, \qquad (-1)!! \equiv 1.$$

Also

$$\det A_\alpha \begin{pmatrix} k+1, k+2, \cdots, & k+n \\ k \quad, k+1, \cdots, k+n-1 \end{pmatrix} = \alpha^n, \qquad k \geqq 1, \quad n \geqq 1.$$

Let $m \leqq M$. By neglecting some nonnegative terms in the repeated application of the Cauchy-Binet formula,

$$c_{mn} = \det C\begin{pmatrix} m+1, m+2, \cdots, m+n \\ 1 \quad , \quad 2 \quad , \cdots, \quad n \end{pmatrix}$$

$$\geqq \det E_\gamma(m+1, m+2, \cdots, m+n; \delta_0) \det A_{\alpha_1}(\delta_0, \delta_1) \cdots \det A_{\alpha_m}(\delta_{m-1}, \delta_m)$$

with $\delta_i \in \Delta_{m+n,n}$, $\delta_m = (1, 2, \cdots, n)$. If $\gamma = 0$, take

$$\delta_i = (m+1-i, m+2-i, \cdots, m+n-i), \qquad i = 0, 1, \cdots, m,$$

to obtain

$$c_{mn} \geqq (\alpha_1\alpha_2 \cdots \alpha_m)^n > 0, \qquad n \geqq 0.$$

From this and the duality theorem, Theorem 4.4, if $n \leqq N$, then

$$c_{mn} = (-1)^{mn} d_{nm} = d'_{nm} \geqq (\beta_1\beta_2 \cdots \beta_n)^m > 0, \qquad m \geqq 0.$$

This proves the first part of (a) and Corollary 3 of Theorem 3.2 completes the proof of (a). If $\gamma > 0$, put $\delta_i \equiv (1, 2, \cdots, n)$, $0 \leqq i \leqq m$, to find

$$c_{mn} \geqq e_{mn}(\gamma) > 0, \qquad m \geqq 0, \quad n \geqq 0.$$

This completes the proof.

The converse of the first assertion of Theorem 1 was conjectured by Schoenberg and proved by Edrei. Consequently the representation (7.1) characterizes a Pólya frequency series.

The formal power series $C(z)$ is a *nonnegative definite* series if the coefficients are *moments* of a nondecreasing function μ:

$$c_m = \int_{-\infty}^{+\infty} t^m \, d\mu(t), \qquad m \geqq 0.$$

It is a *Stieltjes series* if, in addition, $\mu(t)$ is constant for $-\infty < t < 0$:

$$c_m = \int_0^{\infty} t^m \, d\mu(t), \qquad m \geqq 0. \tag{7.2}$$

For positive definite series it is appropriate to consider, instead of the determinants c_{mn}, the Hankel determinants

$$c_n^{(m)} \equiv \det C_n^{(m)} \equiv \det (c_{m+i+j})_{i,j=1}^n.$$

The matrices $C_n^{(m)}$, $m \geqq 0$, $n \geqq 1$, are the connected submatrices of the *Hankel matrix*

$$C^R \equiv \begin{pmatrix} c_0 & c_1 & c_2 & c_3 & \cdots \\ c_1 & c_2 & c_3 & c_4 & \cdots \\ c_2 & c_3 & c_4 & c_5 & \cdots \\ c_3 & c_4 & c_5 & c_6 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix} = (c_{i+j})_{i,j=0}^{\infty} \in \mathcal{M}_\infty. \tag{7.3}$$

Of course

$$c_{mn} = \sigma_n c_n^{(m-n+1)}, \qquad c_n^{(m)} = \sigma_n c_{m+n-1,n},$$

with the sign factor

$$\sigma_n \equiv (-1)^{n(n-1)/2}$$

arising from a reversal of the columns. On dropping the signs σ_n the c-table becomes Table 13

TABLE 13

$c_0^{(0)}$	$c_1^{(-1)}$	$c_2^{(-2)}$	$c_3^{(-3)}$	$c_4^{(-4)}$	$\cdots$
$c_0^{(1)}$	$c_1^{(0)}$	$c_2^{(-1)}$	$c_3^{(-2)}$	$c_4^{(-3)}$	$\cdots$
$c_0^{(2)}$	$c_1^{(1)}$	$c_2^{(0)}$	$c_3^{(-1)}$	$c_4^{(-2)}$	$\cdots$
$c_0^{(3)}$	$c_1^{(2)}$	$c_2^{(1)}$	$c_3^{(0)}$	$c_4^{(-1)}$	$\cdots$
$c_0^{(4)}$	$c_1^{(3)}$	$c_2^{(2)}$	$c_3^{(1)}$	$c_4^{(0)}$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	

with the boundary values

$$c_0^{(m)} \equiv 1, \quad c_1^{(m)} = c_m, \quad m \geqq 0,$$

and

$$c_n^{(-n)} \equiv 0, \qquad n \geqq 1.$$

Also the Frobenius identity of Theorem 5.1 becomes

$$c_n^{(m-1)} c_n^{(m+1)} - c_{n-1}^{(m+1)} c_{n+1}^{(m-1)} = (c_n^{(m)})^2. \tag{7.4}$$

THEOREM 7.2. *Let $C(z)$ be a Stieltjes series. Then*

$$c_n^{(m)} \geqq 0, \qquad m \geqq 0, \quad n \geqq 0.$$

Moreover if the function μ has exactly $N \leqq \infty$ points of increase, then $C(z)$ is $(N-1, N)$-normal:

$$c_n^{(m)} > 0, \qquad m \geqq 0, \quad 0 \leqq n \leqq N,$$

$$\sigma_m c_n^{(m)} > 0, \qquad m \leqq 0 < m + n \leqq N,$$

$$c_n^{(m)} = 0, \qquad m + n > N, \quad n > N.$$

In particular $C(z)$ is normal if and only if μ has infinitely many points of increase.

Proof. Let $m \geqq 0, n \geqq 1$. The quadratic forms

$$\sum_{i,j=0}^{n-1} c_{m+i+j}\xi_i\bar{\xi}_j = \int_0^\infty t^m \left| \sum_{i=0}^{n-1} \xi_i t^i \right|^2 d\mu(t)$$

are nonnegative; hence $c_n^{(m)} \geqq 0$. Moreover $c_n^{(m)} = 0$ if and only if there exists a polynomial $\chi(t) = \xi_0 + \xi_1 t + \cdots + \xi_{n-1}t^{n-1} \not\equiv 0$ with $\chi(t) = 0$ at every point of increase of $\mu(t)$. Since $\chi(t)$ has at most $n-1$ zeros, $c_n^{(m)} = 0$ if and only if $N \leqq n$

-1. That is, for $m \geqq 0, n \geqq 0, c_n^{(m)} > 0$ if and only if $n \leqq N$. The proof of the second assertion is completed by applying Theorem 3.2(e) and (7.4) to determine the (strict) signs of the remaining entries in the modified c-table (Table 13). For $N = 3$, see Table 14.

TABLE 14

+	0	0	0	0	0	0	0	0	⋯
+	+	−	−	+	+	−	−	+	⋯
+	+	+	−	−	+	+	−	−	⋯
+	+	+	+	−	−	+	+	−	⋯
+	+	+	+	0	0	0	0	0	⋯
+	+	+	+	0	0	0	0	0	⋯
+	+	+	+	0	0	0	0	0	⋯
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	

COROLLARY 1. *Let $C(z)$ be the Maclaurin expansion of $-f'(z)/f(z)$ with*

$$f(z) = \prod_{n=1}^{N} \left(1 - \frac{z}{\pi_n}\right)^{m_n},$$

distinct $\pi_n > 0$, integers $m_n \geqq 1$, and $\sum_1^N m_n \pi_n^{-1} < \infty$. Then $C(z)$ is an $(N-1, N)$-normal Stieltjes series.

Proof. The logarithmic derivative

$$\frac{f'(z)}{f(z)} = \sum_{n=1}^{N} \frac{m_n}{z - \pi_n} = -\sum_{n=1}^{N} \frac{m_n}{\pi_n} \frac{1}{1 - z/\pi_n}.$$

If $|z| < \min(\pi_n)$, then by the geometric series,

$$\frac{f'(z)}{f(z)} = -\sum_{n=1}^{N} \frac{m_n}{\pi_n} \sum_{m=0}^{\infty} \left(\frac{z}{\pi_n}\right)^m = -\sum_{m=0}^{\infty} \left(\sum_{n=1}^{N} m_n \pi_n^{-m-1}\right) z^n.$$

Consequently,

$$c_m = \sum_{n=1}^{N} m_n \pi_n^{-m-1} = \int_0^\infty t^m \, d\mu(t)$$

with

$$\mu(t) = \sum_{n=1}^{N} m_n \pi_n^{-1} h(t - \pi_n^{-1})$$

and h the Heaviside function

$$h(t) = \begin{cases} 0, & t < 0, \\ \frac{1}{2}, & t = 0, \\ 1, & t > 0. \end{cases}$$

The corollary follows from this and Theorem 7.2.

COROLLARY 2. *Let $C(z)$ be a normal Stieltjes series. Then all subdeterminants of the matrix C^R in* (7.3) *are positive.*

Proof. This is a direct consequence of Theorems 7.2 and 2.8.

Similar results may be established for nonnegative definite series. However in this case it is possible that some coefficients c_{2m+1} vanish.

THEOREM 7.3. *Let $C(z)$ be a nonnegative definite power series. Then*

$$c_n^{(2m)} \geqq 0, \qquad m \geqq 0, \quad n \geqq 0.$$

Moreover if the function μ has exactly $N \leqq \infty$ points of increase, then $C(z)$ is $(N-1, N)$-seminormal:

$$\begin{aligned} c_n^{(2m)} &> 0, \quad m \geqq 0, \quad 0 \leqq n \leqq N, \\ (-1)^m c_n^{(2m)} &> 0, \quad m \leqq 0 < 2m + n \leqq N, \\ c_n^{(m)} &= 0, \quad m + n > N, \quad n > N. \end{aligned}$$

In particular $C(z)$ is seminormal if and only if μ has infinitely many points of increase.

Proof. The quadratic forms

$$\sum_{i,j=0}^{n-1} c_{2m+i+j}\xi_i\bar{\xi}_j = \int_{-\infty}^{+\infty} t^{2m} \left| \sum_{i=0}^{n-1} \xi_i t^i \right|^2 d\mu(t)$$

are nonnegative for $m \geqq 0$, $n \geqq 1$. The remainder of the proof follows that of Theorem 7.2. Apart from the infinite square block of null $c_n^{(m)}$, only those with even superscript may be determined from (7.4). For $N = 3$, see Table 15.

TABLE 15

$$\begin{array}{|ccccccccccc}
+ & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
\hline
+ & + & - & - & + & + & - & - & + & \cdots \\
+ & \cdot & + & \cdot & - & \cdot & + & \cdot & - & \cdots \\
+ & + & \cdot & + & \cdot & - & \cdot & - & \cdot & \cdots \\
+ & \cdot & + & \cdot & 0 & 0 & 0 & 0 & 0 & \cdots \\
+ & + & \cdot & + & 0 & 0 & 0 & 0 & 0 & \cdots \\
+ & \cdot & + & \cdot & 0 & 0 & 0 & 0 & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots &
\end{array}$$

This completes the proof.

Converses of the first assertions of Theorems 7.2 and 7.3 are related to the classical moment problems of Stieltjes and Hamburger, respectively. For the former, the conditions

$$c_n^{(m)} \geqq 0, \qquad m = 0, 1, \quad n \geqq 0,$$

are sufficient for the existence of a nondecreasing function μ satisfying (7.2). The corresponding criterion for the Hamburger case is

$$c_n^{(0)} \geqq 0, \qquad n \geqq 0.$$

For seminormal power series null determinants c_{mn}, if they occur, are necessarily isolated. The following extension of Theorem 5.1 permits the recursive construction of the c-table in such cases.

THEOREM 7.4. *The identity*

$$\begin{aligned} c_{EE}c_{NW}c_{SW} + c_{WW}c_{NE}c_{SE} &= c_{SS}c_{NW}c_{NE} + c_{NN}c_{SW}c_{SE} \\ &\quad + (c_{NN}c_{SS} - c_{WW}c_{EE} - c_Nc_S + c_Wc_E)c_C, \end{aligned} \tag{7.5}$$

relating the entries of the constellation

$$\begin{array}{ccccc} & & c_{NN} & & \\ & c_{NW} & c_N & c_{NE} & \\ c_{WW} & c_W & c_C & c_E & c_{EE} \\ & c_{SW} & c_S & c_{SE} & \\ & & c_{SS} & & \end{array}$$

in the extended c-table, is valid. In particular if $c_C = 0$, *then*

$$c_{EE}c_{NW}c_{SW} + c_{WW}c_{NE}c_{SE} = c_{SS}c_{NW}c_{NE} + c_{NN}c_{SW}c_{SE}.$$

Proof. One has

$$\begin{aligned} c_N^2 &= c_{NN}c_C + c_{NW}c_{NE}, & c_E^2 &= c_{NE}c_{SE} + c_Cc_{EE}, \\ c_S^2 &= c_Cc_{SS} + c_{SW}c_{SE}, & c_C^2 &= c_Nc_S + c_Wc_E. \\ c_W^2 &= c_{NW}c_{SW} + c_{WW}c_C, \end{aligned}$$

Consequently,

$$\begin{aligned} (c_Nc_S - c_Wc_E)c_C^2 &= (c_Nc_S - c_Wc_E)(c_Nc_S + c_Wc_E) \\ &= c_N^2c_S^2 - c_W^2c_E^2 \\ &= (c_{NN}c_C + c_{NW}c_{NE})(c_Cc_{SS} + c_{SW}c_{SE}) \\ &\quad - (c_{NW}c_{SW} + c_{WW}c_C)(c_{NE}c_{SE} + c_Cc_{EE}) \\ &= (c_{NN}c_{SS} - c_{WW}c_{EE})c_C^2 \\ &\quad - (c_{EE}c_{NW}c_{SW} + c_{WW}c_{NE}c_{SE} - c_{SS}c_{NW}c_{NE} - c_{NN}c_{SW}c_{SE})c_C, \end{aligned}$$

and cancellation of c_C proves (7.5).

8. The Padé table of a meromorphic function: simple zeros and poles. In this and the next section it is assumed that the power series $C(z)$ is the Maclaurin expansion of a function f which is meromorphic for $|z| < R$, $0 < R \leqq \infty$. Hence for sufficiently small z,

$$f(z) = c_0 + c_1z + c_2z^2 + \cdots, \qquad c_0 = 1.$$

Let f have simple zeros ζ_m ordered

$$0 < |\zeta_1| \leqq |\zeta_2| \leqq |\zeta_3| \leqq \cdots < R$$

and simple poles π_n ordered

$$0 < |\pi_1| \leqq |\pi_2| \leqq |\pi_3| \leqq \cdots < R.^1$$

Let M be the number of zeros and N the number of poles: $0 \leqq M \leqq \infty$, $0 \leqq N \leqq \infty$. If $M < \infty$ ($N < \infty$), it is convenient to choose ζ_{M+1} (π_{N+1}) such that $|\zeta_M| < \zeta_{M+1} < R$ ($|\pi_N| < \pi_{N+1} < R$).

The theorems of this section provide asymptotic formulas for the elements associated with the Padé table of $C(z)$, both as $m \to \infty$ and n remains fixed, and as $n \to \infty$ while m is fixed. Only the former case needs to be considered since corresponding results hold for the latter by easy applications of the duality theorem, Theorem 4.4.

Denote by

$$\alpha_n = \operatorname{Res} f(\pi_n), \qquad 1 \leqq n < N + 1,$$

the residue of f at the pole π_n. The functions

$$f^{(s)}(z) \equiv f(z) - \sum_{k=1}^{s} \frac{\alpha_k}{z - \pi_k} \equiv \sum_{m=0}^{\infty} c_m^{(s)} z^m, \qquad 0 \leqq s < N + 1,$$

are holomorphic for $|z| < |\pi_{s+1}|$. As in the proof of Corollary 1 of Theorem 7.2,

$$\sum_{k=1}^{s} \frac{\alpha_k}{z - \pi_k} = - \sum_{m=0}^{\infty} \left(\sum_{k=1}^{s} \alpha_k \pi_k^{-m-1} \right) z^m, \qquad |z| < |\pi_1|.$$

Consequently, on comparing coefficients of like powers of z,

$$c_m = - \sum_{k=1}^{s} \alpha_k \pi_k^{-m-1} + c_m^{(s)}, \qquad m \geqq 0. \tag{8.1}$$

From Theorem 3.3, equation (3.8), for $m \geqq n - 1$,

$$v_{mn}(z) = (-1)^n \det \left(\sum_{k=1}^{s} \alpha_k \left(1 - \frac{z}{\pi_k} \right) \pi_k^{-m-1-(i-j)} + \Delta_z c_{m+i-j}^{(s)} \right)_{i,j=1}^{n} \tag{8.2}$$

with

$$\Delta_z c_m^{(s)} \equiv z c_{m+1}^{(s)} - c_m^{(s)}.$$

---

[1] There can be only a finite number of zeros (poles) in each closed disk $|z| \leqq \rho < R$, for otherwise the zeros (poles) would have a point of accumulation in the disk and f would vanish identically (not be meromorphic) there. By choosing a countable sequence $\rho_k \to R$ there can be at most a countable number of zeros (poles) in $|z| < R$.

For $0 \leqq s < N + 1$, let

$$U_k \equiv (u_1^{(k)}, u_2^{(k)}, \cdots, u_n^{(k)}) \equiv \left(\alpha_k\left(1 - \frac{z}{\pi_k}\right)\pi_k^{-m-1-(i-j)}\right) \in \mathscr{M}_n, \quad k = 1, 2, \cdots, s,$$

$$\tilde{U}_{s+1} \equiv (u_1^{(s+1)}, u_2^{(s+1)}, \cdots, u_n^{(s+1)}) \equiv (\Delta_z c_{m+i-j}^{(s)}) \in \mathscr{M}_n,$$

$$V_s \equiv \left(\alpha_j\left(1 - \frac{z}{\pi_j}\right)\pi_j^{-i}\right) \in \mathscr{M}_{ns},$$

and

$$W_s \equiv (\pi_i^{-m+j-1}) \in \mathscr{M}_{sn}.$$

Then from Theorem 2.9,

$$v_{mn}(z) = (-1)^n \det\left(\sum_{k=1}^{s} U_k + \tilde{U}_{s+1}\right)$$

$$= (-1)^n \sum_{\varkappa \in \Phi_{s+1,n}} \det(u_1^{(\varkappa_1)}, u_2^{(\varkappa_2)}, \cdots, u_n^{(\varkappa_n)}) \tag{8.3}$$

$$= (-1)^n \left[\det\left(\sum_{k=1}^{s} U_k\right) + \sum_{\varkappa \in \Phi_{s+1,n} - \Phi_{sn}} \det(u_1^{(\varkappa_1)}, u_2^{(\varkappa_2)}, \cdots, u_n^{(\varkappa_n)})\right]$$

$$= (-1)^n \left[\det(V_s W_s) + \sum_{\varkappa \in \Phi_{s+1,n} - \Phi_{sn}} \det(u_1^{(\varkappa_1)}, u_2^{(\varkappa_2)}, \cdots, u_n^{(\varkappa_n)})\right]. \tag{8.4}$$

THEOREM 8.1. *Let f be a rational function. Then for $0 < n \leqq N$ and*

$$m \geqq \max(0, M - N + 1) + n - 1,$$

$$v_{mn}(z) = \sigma_{n+1} \sum_{\gamma \in \Delta_{Nn}} \alpha_{\gamma_1}\alpha_{\gamma_2} \cdots \alpha_{\gamma_n} v^2(\pi_{\gamma_1}, \pi_{\gamma_2}, \cdots, \pi_{\gamma_n})$$

$$\cdot \left(1 - \frac{z}{\pi_{\gamma_1}}\right)\left(1 - \frac{z}{\pi_{\gamma_2}}\right) \cdots \left(1 - \frac{z}{\pi_{\gamma_n}}\right)(\pi_{\gamma_1}\pi_{\gamma_2} \cdots \pi_{\gamma_n})^{-m-n}.$$

Proof. Let $s = N$ and observe that $f^{(N)}$ is a polynomial of degree at most $M - N$. Hence $c_m^{(N)} = 0$, $m \geqq M - N + 1$, and for $m \geqq \max(0, M - N + 1) + n - 1$,

$$v_{mn}(z) = (-1)^n \det(V_N W_N).$$

By the Cauchy-Binet formula (Theorem 2.3),

$$\det(V_N W_N) = \sum_{\gamma \in \Delta_{Nn}} \det V_N\begin{pmatrix} 1, 2, \cdots, n \\ \gamma_1, \gamma_2, \cdots, \gamma_n \end{pmatrix} \det W_N\begin{pmatrix} \gamma_1, \gamma_2, \cdots, \gamma_n \\ 1, 2, \cdots, n \end{pmatrix}.$$

From Theorem 2.7,

$$
\begin{aligned}
\det V_N\begin{pmatrix} 1, 2, \cdots, n \\ \gamma_1, \gamma_2, \cdots, \gamma_n \end{pmatrix} &= \det\left(\alpha_{\gamma_j}\left(1 - \frac{z}{\pi_{\gamma_j}}\right)\pi_{\gamma_j}^{-i}\right)_{i,j=1}^{n} \\
&= \alpha_{\gamma_1}\alpha_{\gamma_2}\cdots\alpha_{\gamma_n}\left(1 - \frac{z}{\pi_{\gamma_1}}\right)\left(1 - \frac{z}{\pi_{\gamma_2}}\right)\cdots\left(1 - \frac{z}{\pi_{\gamma_n}}\right) \\
&\quad\cdot(\pi_{\gamma_1}\pi_{\gamma_2}\cdots\pi_{\gamma_n})^{-1}v(\pi_{\gamma_1}^{-1}, \pi_{\gamma_2}^{-1}, \cdots, \pi_{\gamma_n}^{-1}) \\
&= \sigma_n\alpha_{\gamma_1}\alpha_{\gamma_2}\cdots\alpha_{\gamma_n}v(\pi_{\gamma_1}, \pi_{\gamma_2}, \cdots, \pi_{\gamma_n}) \\
&\quad\cdot\left(1 - \frac{z}{\pi_{\gamma_1}}\right)\left(1 - \frac{z}{\pi_{\gamma_2}}\right)\cdots\left(1 - \frac{z}{\pi_{\gamma_n}}\right)(\pi_{\gamma_1}\pi_{\gamma_2}\cdots\pi_{\gamma_n})^{-n}.
\end{aligned}
$$

Similarly,

$$
\det W_N\begin{pmatrix} \gamma_1, \gamma_2, \cdots, \gamma_n \\ 1, 2, \cdots, n \end{pmatrix} = v(\pi_{\gamma_1}, \pi_{\gamma_2}, \cdots, \pi_{\gamma_n})(\pi_{\gamma_1}\pi_{\gamma_2}\cdots\pi_{\gamma_n})^{-m}.
$$

A combination of these results provides the theorem.

THEOREM 8.2. (a) *If* $n < N + 2$, *then*

$$
v_{mn}(z) = O((\pi_1\pi_2\cdots\pi_n)^{-m}), \qquad m \to \infty.
$$

(b) *If* $n < N + 1$ *and* $z \neq \pi_k$, $1 \leqq k \leqq n$, *then*

$$
\begin{aligned}
v_{mn}(z) &= \sigma_{n+1}\alpha_1\alpha_2\cdots\alpha_n v^2(\pi_1, \pi_2, \cdots, \pi_n) \\
&\quad\cdot\left(1 - \frac{z}{\pi_1}\right)\left(1 - \frac{z}{\pi_2}\right)\cdots\left(1 - \frac{z}{\pi_n}\right)(\pi_1\pi_2\cdots\pi_n)^{-m-n} \\
&\quad\cdot\left[1 + O\left(\left(\frac{\pi_n}{\pi_{n+1}}\right)^m\right)\right], \qquad m \to \infty.
\end{aligned}
\tag{8.5}
$$

(c) *If* $1 \leqq k \leqq n < N + 1$, *then*

$$
v_{mn}(\pi_k) = O(\pi_1\cdots\pi_{k-1}\pi_{k+1}\cdots\pi_{n+1})^{-m}, \qquad m \to \infty.
$$

Proof. One first requires an estimate for the coefficients $c_m^{(s)}$ as $m \to \infty$. If $|\pi_s| = \cdots = |\pi_r| < |\pi_{r+1}|$ with $r > s$, or if $|\pi_s| < |\pi_{s+1}| = \cdots = |\pi_r|$ with r maximal, then the Maclaurin expansion $\sum_0^\infty c_m^{(r)}z^m$ is convergent for $z = \pi_r$. Hence $c_m^{(r)}\pi_r^m \to 0$ as $m \to \infty$, that is, $c_m^{(r)} = o(\pi_r^{-m}) = o(\pi_{s+1}^{-m})$ as $m \to \infty$. Consequently from (8.1),

$$
c_m^{(s)} = -\sum_{k=s+1}^{r} \alpha_k\pi_k^{-m-1} + c_m^{(r)} = O(\pi_{s+1}^{-m}), \qquad m \to \infty, \tag{8.6}
$$

that is, $c_m^{(s)}\pi_{s+1}^m$ is bounded as $m \to \infty$.

Now consider (8.3) with $s = n - 1$. If $\varkappa \in \Phi_{nn}$ is such that $\varkappa_i = \varkappa_j \neq n$ for some pair i, j, then the columns $u_i^{(\varkappa_i)}$ and $u_j^{(\varkappa_j)}$ are proportional and $\det(u_1^{(\varkappa_1)}, u_2^{(\varkappa_2)}, \cdots, u_n^{(\varkappa_n)}) = 0$. Using (8.6) and factoring $(\pi_{\varkappa_1}\pi_{\varkappa_2}\cdots\pi_{\varkappa_n})^{-m}$ from the columns of the remaining determinants, one sees that the dominant factors are $(\pi_1\pi_2\cdots\pi_n)^{-m}$. This proves (a). Statement (b) follows from (8.4) with $s = n$. As in the proof of Theorem 8.1, $\det(V_nW_n) = \det V_n \cdot \det W_n$ is equal to the right side of (8.5), apart from the remainder term. The latter is estimated as above with the additional observation that if $\varkappa \in \Phi_{n+1,n} - \Phi_{nn}$, then at least one $\varkappa_i = n + 1$. There results

$$\sum_{\varkappa \in \Phi_{n+1,n} - \Phi_{nn}} \det (u_1^{(\varkappa_1)}, u_2^{(\varkappa_2)}, \cdots, u_n^{(\varkappa_n)}) = O((\pi_1 \cdots \pi_{n-1}\pi_{n+1})^{-m}), \qquad m \to \infty,$$

completing the proof. To prove (c) note that if $z = \pi_k$, then $U_k = 0$ and (8.3) with $s = n$ provides similarly

$$\begin{aligned} v_{mn}(\pi_k) &= (-1)^n \sum_{\varkappa \in \Phi_{n+1,n}, \varkappa_j \neq k} \det (u_1^{(\varkappa_1)}, u_2^{(\varkappa_2)}, \cdots, u_n^{(\varkappa_n)}) \\ &= O((\pi_1 \cdots \pi_{k-1}\pi_{k+1} \cdots \pi_{n+1})^{-m}), \qquad m \to \infty. \end{aligned}$$

COROLLARY 1. (a) *If* $n < N + 2$, *then*

$$c_{mn} = O((\pi_1\pi_2 \cdots \pi_n)^{-m}), \qquad m \to \infty.$$

(b) *If* $n < N + 1$, *then*

$$\begin{aligned} c_{mn} = \sigma_{n+1}\alpha_1\alpha_2 \cdots \alpha_n v^2(\pi_1, \pi_2, \cdots, \pi_n)(\pi_1\pi_2 \cdots \pi_n)^{-m-n} \\ \cdot [1 + O((\pi_n/\pi_{n+1})^m)], \qquad m \to \infty. \end{aligned} \tag{8.7}$$

Proof. Put $z = 0$ in Theorem 8.2.

COROLLARY 2. *If* $n < N + 1$ *and* $|\pi_n| < |\pi_{n+1}|$, *then*

$$c_{mn} \sim \sigma_{n+1}\alpha_1\alpha_2 \cdots \alpha_n v^2(\pi_1, \pi_2, \cdots, \pi_n)(\pi_1\pi_2 \cdots \pi_n)^{-m-n}, \qquad m \to \infty,$$

and

$$\lim_{m \to \infty} q_{mn}(z) = q_n(z) \equiv \prod_{k=1}^{n} \left(1 - \frac{z}{\pi_k}\right).$$

Proof. The remainder terms in (8.5) and (8.7) tend to zero as $m \to \infty$. In particular $c_{mn} \neq 0$ for m sufficiently large. By Corollary 1 of Theorem 3.3, $q_{mn}(z) = v_{mn}(z)/c_{mn} \to q_n(z)$ as $m \to \infty$.

The first statement of the following theorem is due to Montessus de Ballore. The second seems to be new.

THEOREM 8.3. (a) *If* $n < N + 1$ *and* $|\pi_n| < |\pi_{n+1}|$, *then*

$$f(z) = \lim_{m \to \infty} r_{mn}(z)$$

uniformly on compact subsets of

$$D_n \equiv \{z : |z| < |\pi_{n+1}|, z \neq \pi_k, 1 \leqq k \leqq n\}.$$

(b) *If* $n < N + 2$, $|\pi_n| < |\pi_{n+1}| < |\pi_{n+2}|$, *and* $z \in D_n$, *then*

$$f(z) - r_{mn}(z) \sim \frac{\alpha_{n+1}}{z - \pi_{n+1}} \frac{q_n^2(\pi_{n+1})}{q_n^2(z)} \left(\frac{z}{\pi_{n+1}}\right)^{m+n+1}, \qquad m \to \infty.$$

Proof. From the Frobenius identity $\overset{*}{*}$,

$$r_{mn}(z) - r_{m-1,n}(z) = \frac{c_{m,n+1}}{c_{m-1,n}} \frac{(-1)^n z^{m+n}}{q_{mn}(z)q_{m-1,n}(z)}$$

for $m \geqq m_0$. From Corollary 8.1 (a),

$$c_{m,n+1} = O((\pi_1\pi_2 \cdots \pi_{n+1})^{-m}), \qquad m \to \infty.$$

From Corollary 8.2,

$$c_{m-1,n} \sim \sigma_{n+1}\alpha_1\alpha_2 \cdots \alpha_n v^2(\pi_1, \pi_2, \cdots, \pi_n)(\pi_1\pi_2 \cdots \pi_n)^{-m-n+1},$$

$$q_{mn}(z)q_{m-1,n}(z) \to q_n^2(z), \qquad m \to \infty.$$

Hence if Δ_n is any compact subset of D_n, then

$$r_{mn}(z) - r_{m-1,n}(z) = O((z/\pi_{n+1})^m)$$

uniformly for $z \in \Delta_n$ and $m \geqq m_1$. By the theorems of Weierstrass, the limit function

$$f_n(z) = \lim_{m\to\infty} r_{mn}(z) = \lim_{m\to\infty} \sum_{k=0}^{m} [r_{km}(z) - r_{k-1,m}(z)]$$

exists uniformly for $z \in \Delta_n$ and is holomorphic there. Moreover

$$\frac{d^k}{dz^k} f_n(z) = \lim_{m\to\infty} \frac{d^k}{dz^k} r_{mn}(z), \qquad z \in D_n, \quad k \geqq 0.$$

Since by construction

$$\frac{d^k}{dz^k} r_{mn}(0) = \frac{d^k}{dz^k} f(0), \qquad 0 \leqq k \leqq m + n, \quad m \geqq m_0,$$

the uniqueness of the Maclaurin expansion shows that $f_n = f$. This completes the proof of (a).

To prove (b) observe that the additional hypotheses imply

$$c_{m,n+1} \sim \sigma_{n+2}\alpha_1 \cdots \alpha_n\alpha_{n+1}v^2(\pi_1, \cdots, \pi_n, \pi_{n+1})(\pi_1 \cdots \pi_n\pi_{n+1})^{-m-n-1},$$

$$m \to \infty.$$

Consequently if $z \in D_n$,

$$r_{mn}(z) - r_{m-1,n}(z) \sim -\frac{\alpha_{n+1}}{\pi_{n+1}} \frac{q_n^2(\pi_{n+1})}{q_n^2(z)} \left(\frac{z}{\pi_{n+1}}\right)^{m+n}, \qquad m \to \infty,$$

and

$$f(z) - r_{mn}(z) = \sum_{k=m+1}^{\infty} [r_{kn}(z) - r_{k-1,n}(z)]$$

$$\sim -\frac{\alpha_{n+1}}{\pi_{n+1}} \frac{q_n^2(\pi_{n+1})}{q_n^2(z)} \left(\frac{z}{\pi_{n+1}}\right)^{m+n+1} \sum_{k=0}^{\infty} \left(\frac{z}{\pi_{n+1}}\right)^k$$

$$= \frac{\alpha_{n+1}}{z - \pi_{n+1}} \frac{q_n^2(\pi_{n+1})}{q_n^2(z)} \left(\frac{z}{\pi_{n+1}}\right)^{m+n+1}, \qquad m \to \infty.$$

9. A quotient-difference algorithm. By Corollary 2 of Theorem 8.2, if $n < N + 1$ and $|\pi_n| < |\pi_{n+1}|$, then

$$c_{m-1,n}/c_{mn} \to \pi_1\pi_2 \cdots \pi_n, \qquad m \to \infty.$$

If, in addition, $|\pi_{n-1}| < |\pi_n|$, then

$$\frac{c_{m,n-1}}{c_{m+1,n-1}} \to \pi_1\pi_2 \cdots \pi_{n-1}, \qquad m \to \infty;$$

consequently,

$$\pi_{mn} \equiv \frac{c_{m-1,n}c_{m+1,n-1}}{c_{m,n-1}c_{mn}} \to \pi_n, \qquad m \to \infty.$$

Let $D(z)$ be the reciprocal series of $C(z)$. From the duality theorem, Theorem 4.4, if $m < M + 1$ and $|\zeta_{m-1}| < |\zeta_m| < |\zeta_{m+1}|$, then

$$\begin{aligned}\zeta_{mn} &\equiv \frac{d_{n-1,m}d_{n+1,m-1}}{d_{n,m-1}d_{nm}} \\ &= -\frac{c_{m,n-1}c_{m-1,n+1}}{c_{m-1,n}c_{mn}} \to \zeta_m, \qquad n \to \infty.\end{aligned}$$

TABLE 16

$$\begin{array}{c|ccccccccc}
 & 0 & & 0 & & 0 & & 0 & & \\
\hline
0 & & \zeta_{11} & & \zeta_{12} & & \zeta_{13} & & \zeta_{14} & \cdots \\
 & \pi_{11} & & \pi_{12} & & \pi_{13} & & \pi_{14} & & \cdots \\
0 & & \zeta_{21} & & \zeta_{22} & & \zeta_{23} & & \zeta_{24} & \cdots \\
 & \pi_{21} & & \pi_{22} & & \pi_{23} & & \pi_{24} & & \cdots \\
0 & & \zeta_{31} & & \zeta_{32} & & \zeta_{33} & & \zeta_{34} & \cdots \\
 & \pi_{31} & & \pi_{32} & & \pi_{33} & & \pi_{34} & & \cdots \\
0 & & \zeta_{41} & & \zeta_{42} & & \zeta_{43} & & \zeta_{44} & \cdots \\
 & \pi_{41} & & \pi_{42} & & \pi_{43} & & \pi_{44} & & \cdots \\
\vdots & & \vdots & & \vdots & & \vdots & & \vdots & \\
 & \vdots & & \vdots & & \vdots & & \vdots & &
\end{array}$$

The array in Table 16 is the *$\pi\zeta$-table* for the power series $C(z)$. Observe that the $\pi\zeta$-table for $D(z)$ is the transpose of that for $C(z)$.

THEOREM 9.1. *In the $\pi\zeta$-table the boundary conditions*

$$\pi_{m1} = c_{m-1}/c_m, \qquad \zeta_{1n} = d_{n-1}/d_n$$

and rhombus rules

$$\pi_{mn}\zeta_{mn} \equiv \rho_{mn} \equiv \pi_{m,n+1}\zeta_{m+1,n},$$

$$\pi_{m,n+1} + \zeta_{m+1,n} \equiv \sigma_{mn} \equiv \pi_{m+1,n+1} + \zeta_{m+1,n+1}$$

are valid. Moreover,

$$\rho_{mn} = -\frac{c_{m-1,n+1}c_{m+1,n-1}}{c_{nm}^2}.$$

Proof. The boundary conditions follow directly from the definitions; also

$$\begin{aligned}
\pi_{mn}\zeta_{mn} &= -\frac{c_{m-1,n}c_{m+1,n-1}}{c_{m,n-1}c_{mn}}\,\frac{c_{m,n-1}c_{m-1,n+1}}{c_{m-1,n}c_{mn}} \\
&= -\frac{c_{m-1,n+1}c_{m+1,n-1}}{c_{mn}^2} \\
&= -\frac{c_{m-1,n+1}c_{m+1,n}}{c_{mn}c_{m,n+1}}\,\frac{c_{m+1,n-1}c_{m,n+1}}{c_{mn}c_{m+1,n}} \\
&= \pi_{m,n+1}\zeta_{m+1,n}.
\end{aligned}$$

The additive rhombus rule requires two applications of Theorem 5.1:

$$\begin{aligned}
\pi_{m,n+1} + \zeta_{m+1,n} &= \frac{c_{m-1,n+1}c_{m+1,n}}{c_{mn}c_{m,n+1}} - \frac{c_{m+1,n-1}c_{m,n+1}}{c_{mn}c_{m+1,n}} \\
&= \frac{c_{m-1,n+1}c_{m+1,n+1}c_{m+1,n}^2 - c_{m+1,n-1}c_{m+1,n+1}c_{m,n+1}^2}{c_{mn}c_{m,n+1}c_{m+1,n}c_{m+1,n+1}} \\
&= \frac{(c_{m,n+1}^2 - c_{mn}c_{m,n+2})c_{m+1,n}^2 - (c_{m+1,n}^2 - c_{mn}c_{m+2,n})c_{m,n+1}^2}{c_{mn}c_{m,n+1}c_{m+1,n}c_{m+1,n+1}} \\
&= \frac{c_{mn}c_{m+2,n}c_{m,n+1}^2 - c_{mn}c_{m,n+2}c_{m+1,n}^2}{c_{mn}c_{m,n+1}c_{m+1,n}c_{m+1,n+1}} \\
&= \frac{c_{m,n+1}c_{m+2,n}}{c_{m+1,n}c_{m+1,n+1}} - \frac{c_{m+1,n}c_{m,n+2}}{c_{m,n+1}c_{m+1,n+1}} = \pi_{m+1,n+1} + \zeta_{m+1,n+1},
\end{aligned}$$

completing the proof.

The consequences of Theorem 9.1 are illustrated in Table 16. Beginning with the coefficients c_m of a normal power series $C(z)$ the rhombus rules permit the recursive construction of the $\pi\zeta$-table by successive addition of upward sloping diagonals. The dual algorithm applies when the coefficients of $D(z)$ are known. In practice only two one-dimensional arrays of storage are required, one for each of the latest upward sloping π- and ζ-diagonals.

Example. The reciprocal of the function

$$f(z) = J_0(2\sqrt{z}) = \sum_{m=0}^{\infty} \frac{(-1)^m}{(m!)^2} z^m$$

satisfies the hypotheses of Theorem 7.1(b),

$$\alpha_m = \gamma = 0, \qquad \beta_n = \frac{4}{j_{0n}^2} \sim \frac{4\pi}{n^2}, \qquad n \to \infty.$$

Hence $C_1(z) = J_0^{-1}(2\sqrt{z})$ is a normal Pólya frequency series. The first seven columns of the $\pi\zeta$-table for $C_1(z)$ are given in Table 17.

TABLE 17

	−1.0000		−4.0000		−9.0000	
1.0000	∧	3.0000	∧	5.0000	∧	7.0000
∧	−0.3333	∧	−2.4000	∧	−6.4286	∧
1.3333	∧	5.0667	∧	9.0286	∧	13.016
∧	−0.0877	∧	−1.3468	∧	−4.4592	∧
1.4211	∧	6.3258	∧	12.141	∧	18.079
∧	−0.0197	∧	−0.7017	∧	−2.9946	∧
1.4408	∧	7.0078	∧	14.434	∧	22.242
∧	−0.0041	∧	−0.3407	∧	−1.9433	∧
1.4448	∧	7.3445	∧	16.036	∧	25.579
∧	−0.0008	∧	−0.1560	∧	−1.2184	∧
1.4456	∧	7.4997	∧	17.099	∧	28.178
∧	−0.0002	∧	−0.0684	∧	−0.7393	∧
1.4458	∧	7.5680	∧	17.770	∧	30.145
∧	−0.0000	∧	−0.0291	∧	−0.4358	∧
1.4458	∧	7.5971	∧	18.176	∧	⋮
∧	−0.0000	∧	−0.0122	∧	⋮	
1.4458	∧	7.6093	∧	⋮		
∧	−0.0000	∧	⋮			
1.4458	∧	⋮				
∧	⋮					
⋮						

Also from Corollary 1 of Theorem 7.2, the Maclaurin expansion, $C_2(z)$, of $-f'(z)/f(z)$ is a normal Stieltjes series. The corresponding entries in the $\pi\zeta$-table of $C_2(z)$ are given in Table 18.

The algorithm of Theorem 9.1 has a useful matrix interpretation. With the elements

$$0 = \zeta_{k+1,0}, \pi_{k1}, \zeta_{k1}, \pi_{k-1,2}, \cdots, \pi_{1k}, \zeta_{1k}, \pi_{1,k+1} = 0$$

of the kth upward sloping diagonal in the $\pi\zeta$-table, and the definitions

$$\sigma_{mn} \equiv \pi_{m,n+1} + \zeta_{m+1,n}, \qquad \rho_{mn} \equiv \pi_{mn}\zeta_{mn},$$

TABLE 18

	-2.0000		6.0000	$>$	4.0000	$>$
2.0000		-8.0000		2.0000	$>$	0.2500
$\vee$	0.5000		-24.000		32.000	$>$
1.5000	$\vee$	16.500		-54.000		16.000
$\vee$	0.0455	$\vee$	7.3333		-108.00	
1.4545	$\vee$	9.2121	$\vee$	61.333		-192.00
$\vee$	0.0072	$\vee$	1.1014	$\vee$	34.500	
1.4474	$\vee$	8.1178	$\vee$	27.935	$\vee$	162.50
$\vee$	0.0013	$\vee$	0.3201	$\vee$	5.9308	$\vee$
1.4461	$\vee$	7.7990	$\vee$	22.324	$\vee$	64.431
$\vee$	0.0002	$\vee$	0.1118	$\vee$	2.0549	$\vee$
1.4459	$\vee$	7.6875	$\vee$	20.381	$\vee$	47.566
$\vee$	0.0000	$\vee$	0.0422	$\vee$	0.8805	$\vee$
1.4458	$\vee$	7.6453	$\vee$	19.543	$\vee$	41.384
$\vee$	0.0000	$\vee$	0.0165	$\vee$	0.4158	$\vee$
1.4458	$\vee$	7.6288	$\vee$	19.143	$\vee$	$\vdots$
$\vee$	0.0000	$\vee$	0.0066	$\vee$	$\vdots$	
1.4458	$\vee$	7.6223	$\vee$	$\vdots$		
$\vee$	0.0000	$\vee$	$\vdots$			
1.4458	$\vee$	$\vdots$				
$\vee$	$\vdots$					
$\vdots$						

form the tridiagonal matrices

$$J_\pi^{(k)} \equiv \begin{pmatrix} \sigma_{k0} & \rho_{k1} & & & & \\ 1 & \sigma_{k-1,1} & \rho_{k-1,2} & & & \\ & 1 & \sigma_{k-2,2} & \rho_{k-2,3} & & \\ & & \ddots & \ddots & \ddots & \\ & & & 1 & \sigma_{1,k-1} & \rho_{1k} \\ & & & & 1 & \sigma_{0k} \end{pmatrix},$$

and

$$J_\zeta^{(k)} \equiv (J_\pi^{(k)})^T.$$

Let the bidiagonal matrices

$$R_\pi^{(k)} \equiv \begin{pmatrix} 1 & \zeta_{k1} & & & \\ & 1 & \zeta_{k-1,2} & & \\ & & \ddots & \ddots & \\ & & & 1 & \zeta_{1k} \\ & & & & 1 \end{pmatrix}, \qquad L_\pi^{(k)} \equiv \begin{pmatrix} \pi_{k1} & & & & \\ 1 & \pi_{k-1,2} & & & \\ & \ddots & \ddots & & \\ & & 1 & \pi_{1k} & \\ & & & 1 & \pi_{0,k+1} \end{pmatrix}$$

and

$$L_\zeta^{(k)} \equiv \begin{pmatrix} 1 & & & & \\ \pi_{k1} & 1 & & & \\ & \ddots & \ddots & & \\ & & \pi_{2,k-1} & 1 & \\ & & & \pi_{1k} & 1 \end{pmatrix}, \qquad R_\zeta^{(k)} \equiv \begin{pmatrix} \zeta_{k+1,0} & 1 & & & \\ & \zeta_{k1} & 1 & & \\ & & \ddots & \ddots & \\ & & & \zeta_{2,k-1} & 1 \\ & & & & \zeta_{1k} \end{pmatrix}.$$

Then the following result is a direct consequence of the boundary conditions and rhombus rules.

COROLLARY. *The quotient-difference algorithm of Theorem* 9.1 *is equivalent to the algorithms*:

(π): *Let* $J_\pi^{(0)} = 0$. *For* $k \geqq 1$ *factor*

$$K_\pi^{(k)} \equiv \begin{pmatrix} J_\pi^{(k-1)} & 0 \\ e_k^T & 0 \end{pmatrix} = R_\pi^{(k)} L_\pi^{(k)}$$

subject to $\zeta_{1k} = d_{k-1}/d_k$; *then let*

$$J_\pi^{(k)} = L_\pi^{(k)} R_\pi^{(k)}.$$

(ζ): *Let* $J_\zeta^{(0)} = 0$. *For* $k \geqq 1$ *factor*

$$K_\zeta^{(k)} \equiv \begin{pmatrix} J_\zeta^{(k-1)} & e_k \\ 0 & 0 \end{pmatrix} = L_\zeta^{(k)} R_\zeta^{(k)}$$

subject to $\pi_{k1} = c_{k-1}/c_k$; *then let*

$$J_\zeta^{(k)} = R_\zeta^{(k)} L_\zeta^{(k)}.$$

The next theorem provides convergence results for certain rows and columns of the $\pi\zeta$-table when $C(z)$, and hence also $D(z)$, is the Maclaurin expansion of a meromorphic function with simple zeros and poles.

THEOREM 9.2. *If* $n < N + 2$ *and* $|\pi_n| < |\pi_{n+1}|$, *then*

$$\zeta_{mn} = O((\pi_n/\pi_{n+1})^m) \to 0, \qquad m \to \infty,$$

and

$$\rho_{mn} = O((\pi_n/\pi_{n+1})^m) \to 0, \qquad m \to \infty.$$

If $n < N + 1$ *and* $|\pi_{n-1}| < |\pi_n| < |\pi_{n+1}|$, *then*

$$\pi_{mn} = \pi_n + O(\theta_n^m) \to \pi_n, \qquad m \to \infty,$$

and

$$\sigma_{m,n-1} = \pi_n + O(\theta_n^m) \to \pi_n, \qquad m \to \infty,$$

with

$$\theta_n \equiv \max\left(\left|\frac{\pi_{n-1}}{\pi_n}\right|, \left|\frac{\pi_n}{\pi_{n+1}}\right|\right).$$

The duals of these statements also hold. If $m < M + 2$ *and* $|\zeta_m| < |\zeta_{m+1}|$, *then*

$$\pi_{mn} = O((\zeta_m/\zeta_{m+1})^n) \to 0, \qquad n \to \infty,$$

and

$$\rho_{mn} = O((\zeta_m/\zeta_{m+1})^n) \to 0, \qquad n \to \infty.$$

If $m < M + 1$ *and* $|\zeta_{m-1}| < |\zeta_m| < |\zeta_{m+1}|$, *then*

$$\zeta_{mn} = \zeta_m + O(\tau_m^n) \to \zeta_m, \qquad n \to \infty,$$

and

$$\sigma_{m-1,n} = \zeta_m + O(\tau_m^n) \to \zeta_m, \qquad n \to \infty,$$

with

$$\tau_m \equiv \max\left(\left|\frac{\zeta_{m-1}}{\zeta_m}\right|, \left|\frac{\zeta_m}{\zeta_{m+1}}\right|\right).$$

Proof. From Corollary 1 of Theorem 8.2,

$$\begin{aligned}\zeta_{mn} &= O\left[\left(\frac{\pi_1\pi_2\cdots\pi_{n-1}\cdot\pi_1\pi_2\cdots\pi_{n+1}}{\pi_1\pi_2\cdots\pi_n\cdot\pi_1\pi_2\cdots\pi_n}\right)^{-m}\right]\\ &= O\left(\left(\frac{\pi_n}{\pi_{n+1}}\right)^m\right) \to 0, \qquad m \to \infty;\end{aligned}$$

likewise for ρ_{mn} as $m \to \infty$. Similarly,

$$\begin{aligned}\pi_{mn} &= \pi_n\frac{[1 + O((\pi_n/\pi_{n+1})^m)][1 + O((\pi_{n-1}/\pi_n)^m)]}{[1 + O((\pi_{n-1}/\pi_n)^m)][1 + O((\pi_n/\pi_{n+1})^m)]}\\ &= \pi_n + O(\theta_n^m), \qquad m \to \infty,\end{aligned}$$

and the fourth assertion follows from these, thus completing the proof.

Now let the polynomials

$$p_{mn}^{(0)}(z) \equiv u_{mn}(z)/c_{m,n+1}, \qquad q_{mn}^{(0)}(z) \equiv (-1)^n(v_{mn}(z)/c_{m+1,n}).$$

From Corollary 1 of Theorem 3.3, $p_{mn}^{(0)}\,(q_{mn}^{(0)})$ is the numerator (denominator) of the Padé fraction r_{mn}, normalized so that its leading coefficient is unity. It is also natural to define, as in § 3,

$$p_{k,-1}^{(0)}(z) \equiv z^k \equiv q_{-1,k}^{(0)}(z), \qquad k \geqq 0.$$

THEOREM 9.3. *The triangle identities*

$$\begin{array}{ll} \circledast & * \\ * & \end{array} \qquad \begin{aligned} p_{m+1,n}^{(0)}(z) &= zp_{mn}^{(0)}(z) - \zeta_{m+1,n+1}p_{m,n+1}^{(0)}(z), \\ q_{m,n+1}^{(0)}(z) &= zq_{mn}^{(0)}(z) - \pi_{m+1,n+1}q_{m+1,n}^{(0)}(z), \end{aligned}$$

$$\begin{array}{ll} * & \\ \circledast & * \end{array} \qquad \begin{aligned} p_{mn}^{(0)}(z) &= p_{m,n-1}^{(0)}(z) + \pi_{m,n+1}p_{m-1,n}^{(0)}(z), \\ q_{mn}^{(0)}(z) &= q_{m-1,n}^{(0)}(z) + \zeta_{m+1,n}q_{m,n-1}^{(0)}(z), \end{aligned}$$

and the identities relating three consecutive polynomials along an upward sloping diagonal

$$\begin{array}{lll} & & * \\ & \circledast & \\ * & & \end{array} \qquad \begin{aligned} p_{m+1,n-1}^{(0)}(z) &= (z - \sigma_{mn})p_{mn}^{(0)}(z) - \rho_{m,n+1}p_{m-1,n+1}^{(0)}(z), \\ q_{m-1,n+1}^{(0)}(z) &= (z - \sigma_{mn})q_{mn}^{(0)}(z) - \rho_{m+1,n}q_{m+1,n-1}^{(0)}(z), \end{aligned}$$

are valid.

Proof. These follow from the definitions and the corresponding Frobenius identities of Theorem 5.2.

Either pair of triangle identities permits the recursive construction of the polynomials

$$1 = q_{k0}^{(0)}, q_{k-1,1}^{(0)}, \cdots, q_{0k}^{(0)}, q_{-1,k+1}^{(0)} = z^{k+1}$$

and

$$z^{k+1} = p_{k+1,-1}^{(0)}, p_{k0}^{(0)}, \cdots, p_{1,k-1}^{(0)}, p_{0k}^{(0)} = 1$$

from those of the previous diagonals in their respective tables, and the entries of the kth diagonal in the $\pi\zeta$-table. The following corollary is a consequence of the third pair of identities.

COROLLARY. *The polynomials* $q_{k-n,n}^{(0)}$, $n = 0, 1, \cdots, k + 1$, *and* $p_{m,k-m}^{(0)}$, $m = 0, 1, \cdots, k + 1$, *are the characteristic polynomials of the submatrices*

$$J_{\pi}^{(k)}\begin{pmatrix} 1, 2, \cdots, n \\ 1, 2, \cdots, n \end{pmatrix} \quad \textit{and} \quad J_{\zeta}^{(k)}\begin{pmatrix} k + 2 - m, k + 3 - m, \cdots, k + 1 \\ k + 2 - m, k + 3 - m, \cdots, k + 1 \end{pmatrix}$$

of $J_{\pi}^{(k)}$ *and* $J_{\zeta}^{(k)}$, *respectively. In particular,*

$$\det(zI_{k+1} - J_{\pi}^{(k)}) = z^{k+1} = \det(zI_{k+1} - J_{\zeta}^{(k)})$$

and the trace

$$\tau^{(k)} \equiv \sum_{l=0}^{k} \sigma_{k-l,l} = \sum_{l=0}^{k} (\zeta_{k-l+1,l} + \pi_{k-l,l+1}) \equiv 0, \qquad k = 0, 1, 2, \cdots.$$

Proof. The first assertion is true for $n = 0, 1$. If it is assumed to hold up to n, then by Laplace expansions along the last column and row,

$$\begin{aligned}
&\det\left[zI_{n+1} - J_\pi^{(k)}\begin{pmatrix}1,2,\cdots,n+1\\1,2,\cdots,n+1\end{pmatrix}\right]\\
&\quad= (z-\sigma_{k-n,n})\det\left[zI_n - J_\pi^{(k)}\begin{pmatrix}1,2,\cdots,n\\1,2,\cdots,n\end{pmatrix}\right]\\
&\qquad-\rho_{k-n+1,n}\det\left[zI_{n-1} - J_\pi^{(k)}\begin{pmatrix}1,2,\cdots,n-1\\1,2,\cdots,n-1\end{pmatrix}\right]\\
&\quad= (z-\sigma_{k-n,n})q_{k-n,n}^{(0)}(z) - \rho_{k-n+1,n}q_{k-n+1,n-1}^{(0)}(z)\\
&\quad= q_{k-n-1,n+1}^{(0)}(z).
\end{aligned}$$

Similarly for the polynomials $p_{m,k-m}^{(k)}$ by Laplace expansions along the first column and row. From Theorem 2.1 the trace of $J_\pi^{(k)}$ is equal to the sum of its characteristic roots. Since each such root is null, $\tau^{(k)} = 0$, completing the proof.

The last fact provides a convenient check on the numerical stability of the computation. The next theorem, on the convergence of the polynomials $p_{mn}^{(0)}$ and $q_{mn}^{(0)}$, is a consequence of the definitions and Theorem 8.2, as in the proof of Theorem 9.2.

THEOREM 9.4. *Let $n < N + 1$ and $|\pi_n| < |\pi_{n+1}|$. If $z \neq \pi_k$, $1 \leqq k \leqq n$, then*

$$\begin{aligned}
q_{mn}^{(0)}(z) &= q_n^{(0)}(z) + O\left(\left(\frac{\pi_n}{\pi_{n+1}}\right)^m\right)\\
&\to q_n^{(0)}(z) \equiv \prod_{k=1}^{n}(z-\pi_k), \qquad m\to\infty;
\end{aligned}$$

for $1 \leqq k \leqq n$,

$$q_{mn}^{(0)}(\pi_k) = O\left(\left(\frac{\pi_k}{\pi_{n+1}}\right)^m\right) \to 0, \qquad m\to\infty.$$

The dual statements also hold. Let $m < M + 1$ and $|\zeta_m| < |\zeta_{m+1}|$. If $z \neq \zeta_k$, $1 \leqq k \leqq m$, then

$$\begin{aligned}
p_{mn}^{(0)}(z) &= p_m^{(0)}(z) + O\left(\left(\frac{\zeta_m}{\zeta_{m+1}}\right)^n\right)\\
&\to p_m^{(0)}(z) \equiv \prod_{k=1}^{m}(z-\zeta_k), \qquad n\to\infty;
\end{aligned}$$

for $1 \leqq k \leqq m$,

$$p_{mn}^{(0)}(\zeta_k) = O\left(\left(\frac{\zeta_k}{\zeta_{k+1}}\right)^n\right) \to 0, \qquad n\to\infty.$$

Generalizations of the polynomials $p_{mn}^{(0)}$ and $q_{mn}^{(0)}$ are useful for the determination of equimodular zeros and poles. Observe that if

$$0 < |\pi_1| = \cdots = |\pi_{n_1}| < |\pi_{n_1+1}| = \cdots = |\pi_{n_2}| < \cdots,$$

then by Theorem 9.2,

$$\rho_{mn_k} \to 0, \qquad m \to \infty, \quad k = 1,2,3,\cdots,$$

and as $m \to \infty$, the matrix

$$J_\pi^{(m)}\begin{pmatrix}1,2,\cdots,N\\1,2,\cdots,N\end{pmatrix}$$

becomes block lower bidiagonal with diagonal blocks of order $\nu_k \equiv n_{k+1} - n_k$:

$$J_\pi^{(m)}\begin{pmatrix}n_k+1, n_k+2,\cdots,n_{k+1}\\n_k+1,n_k+2,\cdots,n_{k+1}\end{pmatrix}, \qquad k = 0,1,2,\cdots,$$

and subdiagonal blocks

$$e_1 e_{\nu_k}^T \in \mathscr{M}_{n_{k+1},n_k}, \qquad k = 1,2,3,\cdots.$$

For example, with $n_1 = 1$, $n_2 = 3$ and $n_3 = N = 5$:

$$J_\pi^{(m)}\begin{pmatrix}1,2,3,4,5\\1,2,3,4,5\end{pmatrix} \sim \left(\begin{array}{c|cc|cc} * & 0 & & & \\ \hline 1 & * & * & & \\ & 1 & * & 0 & \\ \hline & & 1 & * & * \\ & & & 1 & * \end{array}\right), \qquad m \to \infty.$$

Unless $\nu_k = 1$, the elements of the diagonal blocks do not converge as $m \to \infty$. However it will now be shown that the characteristic polynomials of these blocks do converge to the monic polynomials with zeros $\pi_{n_k+1}, \pi_{n_k+2}, \cdots, \pi_{n_{k+1}}$. For this purpose define the polynomials

$$p_{mn}^{(k)}(z) \equiv \det\left[zI_{m-k} - J_\zeta^{(m+n)}\begin{pmatrix}k+n+2,k+n+3,\cdots,m+n+1\\k+n+2,k+n+3,\cdots,m+n+1\end{pmatrix}\right]$$

and

$$q_{mn}^{(k)}(z) \equiv \det\left[zI_{n-k} - J_\pi^{(m+n)}\begin{pmatrix}k+1,k+2,\cdots,n\\k+1,k+2,\cdots,n\end{pmatrix}\right].$$

THEOREM 9.5. *The polynomials $p_{mn}^{(k)}$ and $q_{mn}^{(k)}$ satisfy the recursion relations*

$$p_{k-1,n}^{(k)}(z) \equiv 0, \qquad p_{kn}^{(k)}(z) = 1,$$

$$p_{m+1,n-1}^{(k)}(z) = (z - \sigma_{mn})p_{mn}^{(k)}(z) - \rho_{m,n+1} r_{m-1,n+1}^{(k)}(z),$$

and

$$q_{m,k-1}^{(k)}(z) \equiv 0, \qquad q_{mk}^{(k)}(z) = 1,$$

$$q_{m-1,n+1}^{(k)}(z) = (z - \sigma_{mn})q_{mn}^{(k)}(z) - \rho_{m+1,n}q_{m+1,n-1}^{(k)}(z).$$

If $m < M + 1$ *and*

$$|\zeta_k| < |\zeta_{k+1}| \leqq \cdots \leqq |\zeta_m| < |\zeta_{m+1}|,$$

then

$$\lim_{n\to\infty} p_{mn}^{(k)}(z) = p_m^{(k)}(z) \equiv \prod_{l=k+1}^{m} (z - \zeta_l).$$

Dually, if $n < N + 1$ *and*

$$|\pi_k| < |\pi_{k+1}| \leqq \cdots \leqq |\pi_n| < |\pi_{n+1}|,$$

then

$$\lim_{m\to\infty} q_{mn}^{(k)}(z) = q_n^{(k)}(z) \equiv \prod_{l=k+1}^{n} (z - \pi_l).$$

Proof. The recursion relations follow directly from the definitions, as in the proof of the corollary of Theorem 9.3. From that corollary,

$$q_{mn}^{(0)}(z) = \det\left[zI_n - J_\pi^{(m+n)}\begin{pmatrix}1, 2, \cdots, n\\1, 2, \cdots, n\end{pmatrix}\right].$$

If the matrices B_{ii} are square, and B_{11} is nonsingular, then as in the proof of Theorem 2.6,

$$\begin{aligned}(\det B_{11})^{-1} \det\begin{pmatrix}B_{11} & B_{12}\\B_{21} & B_{22}\end{pmatrix} &= \det(B_{22} - B_{21}B_{11}^{-1}B_{12})\\ &= \det[B_{22} - (\det B_{11})^{-1}B_{21}B_{11}^{A}B_{12}].\end{aligned}$$

Now apply this with

$$B_{11} = zI_k - J_\pi^{(m+n)}\begin{pmatrix}1, 2, \cdots, k\\1, 2, \cdots, k\end{pmatrix}, \qquad B_{12} = \rho_{m+n-k+1,k}e_ke_k^T,$$

$$B_{21} = e_1e_k^T, \qquad B_{22} = zI_{n-k} - J_\pi^{(m+n)}\begin{pmatrix}k+1, k+2, \cdots, n\\k+1, k+2, \cdots, n\end{pmatrix},$$

and observe that the (k, k) element of B_{11}^A is

$$\det\left[zI_{k-1} - J_\pi^{(m+n)}\begin{pmatrix}1, 2, \cdots, k-1\\1, 2, \cdots, k-1\end{pmatrix}\right] = q_{m+n+k-1,k-1}^{(0)}(z).$$

There results

$$\frac{q_{mn}^{(0)}(z)}{q_{m+n-k,k}^{(0)}(z)} = \det\left[zI_{n-k} - J_\pi^{(m+n)}\begin{pmatrix}k+1, k+2, \cdots, n\\k+1, k+2, \cdots, n\end{pmatrix} - \varepsilon_{mn}^{(k)}(z)e_1e_1^T\right]$$

with

$$\varepsilon_{mn}^{(k)}(z) \equiv \frac{\rho_{m+n+k-1,k}q_{m+n-k+1,k-1}^{(0)}(z)}{q_{m+n-k,k}^{(0)}(z)}.$$

By Theorems 9.1 and 8.2,

$$
\begin{aligned}
\rho_{mk} q^{(0)}_{m,k-1}(z) &= (-1)^k \frac{c_{m-1,k+1}c_{m+1,k-1}}{c^2_{mk}} \frac{v_{m,k-1}(z)}{c_{m+1,k-1}} \\
&= (-1)^k \frac{c_{m-1,k+1}v_{m,k-1}(z)}{c^2_{mk}} \\
&= O((\pi_k/\pi_{k+1})^m) \to 0, \qquad m \to \infty.
\end{aligned}
$$

From Theorem 8.4, if $z \neq \pi_1, \pi_2, \cdots, \pi_k$, then

$$\frac{q^{(0)}_{mn}(z)}{q^{(0)}_{m+n-k,k}(z)} \to q^{(k)}_n(z), \qquad e^{(k)}_{mn}(z) \to 0, \qquad m \to \infty.$$

Hence, by the continuity of the determinant as a function of its elements,

$$q^{(k)}_{mn}(z) \to q^{(k)}_n(z), \qquad m \to \infty,$$

provided $z \neq \pi_1, \pi_2, \cdots, \pi_k$. Since the $q^{(k)}_{mn}$ are polynomials of degree at most $n - k$, this holds for unrestricted values of z. This completes the proof.

A useful simplification in the quotient-difference algorithm of Theorem 9.1 occurs if

$$C(z) = p(z) = c_0 + c_1 z + \cdots + c_M z^M$$

is a polynomial of degree M. In this case, by Corollary 3 of Theorem 3.2,

$$c_{Mn} \neq 0, \qquad n \geqq 0,$$

and

$$c_{mn} = 0, \qquad m > M \quad \text{and} \quad n > 0.$$

The $\pi\zeta$-table then has the form given in Table 19.

TABLE 19

$$
\begin{array}{c|ccccccccc}
 & 0 & & 0 & & 0 & & 0 & & \cdots \\
\hline
0 & & \zeta_{11} & & \zeta_{12} & & \zeta_{13} & & \zeta_{14} & \cdots \\
 & \pi_{11} & & \pi_{21} & & \pi_{22} & & \pi_{23} & & \cdots \\
0 & & \zeta_{21} & & \zeta_{22} & & \zeta_{23} & & \zeta_{24} & \cdots \\
 & \pi_{21} & & \pi_{22} & & \pi_{23} & & \pi_{24} & & \cdots \\
\vdots & & \vdots & & \vdots & & \vdots & & \vdots & \\
 & \vdots & & \vdots & & \vdots & & \vdots & & \\
0 & & \zeta_{M-1,1} & & \zeta_{M-1,2} & & \zeta_{M-1,3} & & \zeta_{M-1,4} & \cdots \\
 & \pi_{M-1,1} & & \pi_{M-1,2} & & \pi_{M-1,3} & & \pi_{M-1,4} & & \cdots \\
0 & & \zeta_{M1} & & \zeta_{M2} & & \zeta_{M3} & & \zeta_{M4} & \cdots \\
\hline
 & \pi_{M1} & & 0 & & 0 & & 0 & & \\
\end{array}
$$

It may be constructed by successive addition of columns, working from left to right. This is a *progressive* quotient-difference algorithm. The dual algorithm applies when $D(z)$ is a polynomial.

The quotient difference algorithm developed here is a variant of, and for general formal power series, distinct from, the original one of Rutishauser. Its relation to the Padé table and the corresponding duality theory is perhaps more natural.

REFERENCES

M. AISSEN, I. J. SCHOENBERG AND A. WHITNEY, *On the generating functions of totally positive sequences. I*, J. Analyse Math., 2 (1952), pp. 93–103.

A. C. AITKEN, *On Bernoulli's numerical solution of algebraic equations*, Proc. Roy. Soc. Edinburgh, 46 (1962), pp. 289–305.

———, *Further numerical studies in algebraic equations and matrices*, Ibid., 51 (1931), pp. 80–90.

N. I. AKHIEZER, *The Classical Moment Problem and Some Related Questions in Analysis*, Hafner, New York, 1965.

CHR. ANDERSON, *The qd-algorithm as a method for finding the roots of a polynomial equation when all roots are positive*, Tech. Rep. CS9, Computer Sci. Div., Stanford Univ., 1964.

G. A. BAKER, JR., *The theory and application of the Padé approximant method*, Advances in Theoretical Physics, vol. 1, K. A. Brueckner, ed., Academic Press, New York, 1965, pp. 1–58.

———, *Best error bounds for Padé approximants to convergent series of Stieltjes*, J. Mathematical Phys., 10 (1969), pp. 814–820.

G. A. BAKER, JR. AND JOHN L. GAMMEL, eds., *The Padé Approximant in Theoretical Physics*, Academic Press, New York, 1970.

HANS BANDEMER, *Über die Konvergenz des Quotienten-Differenzen-Algorithmus von Rutishauser für Eigenwertprobleme gewisser linearer Operatoren*, Math. Nachr., 27 (1964), pp. 353–375.

ERWIN H. BAREISS, *Resultant procedure and the mechanization of the Graeffe process*, J. Assoc. Comput. Mach., 7 (1960), pp. 346–386.

F. L. BAUER, *Beiträge zur Entwicklung numerischer Verfahren für programmgesteuerte Rechenanlagen. I: Quadratisch konvergente Durchführung der Bernoulli-Jacobischen Methode zur Nullstellenbestimmung von Polynomen. II: Direkte Faktorisierung eines Polynoms*, Sitz.-Ber. Bayr. Akad. Wiss. Munich, 1954, pp. 275–303; 1956, pp. 163–203.

———, *The quotient-difference and epsilon algorithms*, On Numerical Approximation, R. E. Langer, ed., University of Wisconsin Press, Madison, 1959, pp. 361–370.

———, *The g-algorithm*, J. Soc. Indust. Appl. Math., 8 (1960), pp. 1–17.

———, *Nonlinear sequence transformations*, Approximation of Functions, H. L. Garabedian, ed., Elsevier, Amsterdam, 1965, pp. 134–151.

———, *QD-method with Newton shift*, Tech. Rep. 56, Computer Sci. Dept., Stanford Univ., 1967, 6 pp.

F. L. BAUER AND E. FRANK, *Note on formal properties of certain continued fractions*, Proc. Amer. Math. Soc., 9 (1958), pp. 340–347.

ODOARDO BRUGIA, *A noniterative method for the partial fraction expansion of a rational function with high order poles*, this Review, 7 (1965), pp. 381–387.

G. E. COLLINS, *Subresultants and reduced polynomial remainder sequences*, J. Assoc. Comput. Mach., 14 (1967), pp. 128–142.

GERMUND DAHLQUIST, SVEN-ÅKE GUSTAFSON AND KÁROLY SIKLÓSI, *Convergence acceleration from the point of view of linear programming*, BIT, 5 (1965), pp. 1–16.

P. DIENES, *The Taylor Series*, Clarendon Press, Oxford, 1931.

A. EDREI, *On the generating functions of totally positive sequences. II*, J. Analyse Math., 2 (1952), pp. 104–109.

———, *On the generating function of a doubly infinite, totally positive sequence*, Trans. Amer. Math. Soc., 74 (1953), pp. 367–383.

J. G. F. FRANCIS, *The QR transformation: a unitary analogue to the LR transformation*, Comput. J., 4 (1961/62), pp. 265–271, 332–345.

EVELYN FRANK, *Corresponding type continued fractions*, Amer. J. Math., 68 (1946), pp. 89–108.

G. FROBENIUS, *Über Relationen zwischen den Näherungsbruchen von Potenzreihen*, J. für Math., 90 (1881), pp. 1–17.

I. GARGANTINI AND P. HENRICI, *A continued fraction algorithm for the computation of higher transcendental functions in the complex plane*, Math. Comp., 21 (1967), pp. 18–29.

WALTER GAUTSCHI, *Efficient computation of the complex error function*, SIAM J. Numer. Anal., 7 (1970), pp. 187–198.

———, *Computational aspects of three-term recurrence relations*, this Review, 9 (1967), pp. 24–82.

MICHAEL GOLOMB, *Zeros and poles of functions defined by Taylor series*, Bull. Amer. Math. Soc., 49 (1943), pp. 581–592.

GENE H. GOLUB, *Bounds for eigenvalues of tridiagonal symmetric matrices computed by the LR method*, Math. Comp., 16 (1962), pp. 438–445.

GENE H. GOLUB AND JOHN H. WELSCH, *Calculation of Gquss quadrature rules*, Math. Comp., 23 (1969), pp. 221–230.

WILLIAM B. GRAGG, *Truncation error bounds for g-fractions*, Numer. Math., 11 (1968), pp. 370–379.

———, *Truncation error bounds for π-fractions*, Bull. Amer. Math. Soc., 76 (1970), pp. 1091–1094.

WILLIAM B. GRAGG AND A. S. HOUSEHOLDER, *On a theorem of Koenig*, Numer. Math., 8 (1966), pp. 465–468.

S.-Å. GUSTAFSON, *On the computational solution of a class of generalized moment problems*, SIAM J. Numer. Anal., 7 (1970), pp. 343–357.

S.-Å. GUSTAFSON, K. O. KORTANEK AND W. ROM, *Non-Chebyshevian moment problems*, Ibid., 7 (1970), pp. 335–342.

J. HADAMARD, *Essai sur l'étude des fonctions données par leur développement de Taylor*, J. Math. Pures Appl., 8 (1892), pp. 101–186.

F. HAUSDORFF, *Momentenprobleme für ein endliches Intervall*, Math. Z., 16 (1923), pp. 220–248.

E. HELLINGER, *Zur Stieltjes Kettenbruchtheorie*, Math. Ann., 86 (1922), pp. 18–29.

P. HENRICI, *The Quotient-Difference-Algorithm*, Nat. Bur. Standards Appl. Math. Ser., 49 (1958), pp. 23–46.

———, *Some applications of the quotient-difference algorithm*, Proc. Symposium Appl. Math., vol. 15, Amer. Math. Soc., Providence, R.I., 1963, pp. 159–183.

———, *Bounds for eigenvalues of certain tridiagonal matrices*, J. Soc. Indust. Appl. Math., 11 (1963), pp. 281–290. Errata: 12 (1964), p. 497.

———, *The quotient-difference algorithm*, Mathematical Methods for Digital Computers, vol. II, A. Ralston and H. S. Wilf, eds., John Wiley, New York, 1967, pp. 35–62.

P. HENRICI AND PIA PFLUGER, *Truncation error estimates for Stieltjes fractions*, Numer. Math., 9 (1966), pp. 120–138.

P. HENRICI AND BRUCE O. WATKINS, *Finding zeros of a polynomial by the QD-algorithm.*, Comm. ACM, 8 (1965), pp. 570–574.

JONAS T. HOLDEMAN, JR., *A method for the approximation of functions defined by formal series expansions in orthogonal polynomials*, Math. Comp., 23 (1969), pp. 275–288.

A. S. HOUSEHOLDER, *Bigradients and the problem of Routh and Hurwitz*, this Review, 10 (1968), pp. 56–66. Errata: p. 438.

———, *Bezoutiants, elimination and localization*, this Review, 12 (1970), 73–78.

———, *The Numerical Treatment of a Single Nonlinear Equation*, McGraw-Hill, New York, 1970.

———, *The Padé table, the Frobenius identities, and the qd algorithm*, Linear Algebra Appl., 4 (1971), pp. 161–174.

A. S. HOUSEHOLDER AND G. W. STEWART, III, *Bigradients, Hankel determinants, and the Padé table*, Constructive Aspects of the Fundamental Theorem of Algebra, B. Dejon and P. Henrici, eds., Wiley-Interscience, New York, 1969, pp. 131–150.

A. HURWITZ, *Über die Bedingungen, unter welchen ein Gleichung nur Wurzeln mit negativen reelen Teilen besitzt*, Math. Ann., 46 (1895), pp. 273–284.

THOMAS H. JEFFERSON, *Truncation error estimates for T-fractions*, SIAM J. Numer. Anal., 6 (1969), pp. 359–364.

WILLIAM B. JONES AND R. I. SNELL, *Truncation error bounds for continued fractions*, Ibid., 6 (1969), pp. 210–221.

WILLIAM B. JONES AND W. J. THRON, *A posteriori bounds for the truncation error of continued fractions*, Ibid., 8 (1971).

S. KARLIN, *Total Positivity*, vol. 1, Stanford University Press, Stanford, Calif., 1968.

JULIUS KÖNIG, *Über eine Eigenschaft der Potenzreihen*, Math. Ann., 23 (1884), pp. 447–449.

B. H. KUBLANOVSKAJA, *On some algorithms for the solution of the complete problem of proper values*, J. Comput. Math. and Math. Phys., 1 (1961), pp. 555–570.

C. LANCZOS, *An iteration method for the solution of the eigenvalue problem of linear differential and integral operators*, J. Res. Nat. Bur. Standards, 45 (1950), pp. 255–282.

H. J. MAEHLY, *Methods for fitting rational approximations*, J. Assoc. Comput. Mach., 7 (1960), pp. 150–162; 10 (1963), pp. 257–277.

ARNE MAGNUS, *Certain continued fractions associated with the Padé table*, Math. Z., 78 (1962), pp. 361–374.

———, *Expansion of power series into P-fractions*, Ibid., 80 (1962), pp. 209–216.

———, *On P-expansions of power series*, Det Kgl. Norske Videnskabers Selskabs Skrifter (Trondheim), 3 (1964), pp. 1–14.

A. MARKOFF, *On certain applications of algebraic continued fractions*, Thesis (in Russian), St. Petersburg, 1884.

——— *Deux démonstrations de la convergence de certaines fractions continues*, Acta Math., 19 (1895), pp. 93–104.

J. MEINGUET, *On the solubility of the Cauchy interpolation problem*, Univ. of Lancaster Symposium on Approximation Theory and Applications (Computation Centre Univ. Louvain, Belgium, 1969), Approximation Theory, A. Talbot, ed., Academic Press, New York, 1970.

GERHARD MERZ, *Padésche Näherungsbrüche und Iterationsverfahren höheren Ordnung*, Doctoral thesis, Technische Hochschule Clausthal, Clausthal, Germany, 1967.

R. DE MONTESSUS DE BALLORE, *Sur les fractions continues algébriques*, Bull. Soc. Math. France, 30 (1902), pp. 28–36.

R. NEVANLINNA, *Asymptotische Entwicklung beschränkter Funktionen und das Stieltjesschen Momenten problem*, Ann. Acad. Sci. Fenn. Ser. A, 18 (5) (1922), 52 pp.

J. M. ORTEGA, *On Sturm sequences for tridiagonal matrices*, J. Assoc. Comput. Mach., 7 (1960), pp. 260–263.

J. M. ORTEGA AND HENRY F. KAISER, *The LL^T and QR methods for symmetric tridiagonal matrices*, Comput. J., 6 (1963), pp. 99–101.

A. M. OSTROWSKI, *Über einen Satz von Herrn Hadamard*, Jber. Deutsch. Math.-Verein. 35 (1926), pp. 179–182.

———, *Recherches sur la méthode de Graeffe et les zéros des polynomes et les séries de Laurent*, Acta Math., 72 (1940), pp. 99–257.

———, *Solution of equations and systems of equations*, 2nd ed., Academic Press, New York, 1966.

H. PADÉ, *Sur la représentation approchée d'une fonction par des fractions rationelles*, Thesis, Ann. École Nor. (3), 9 (1892), pp. 1–93, supplement.

BERESFORD PARLETT, *The development and use of methods of LR type*, this Review, 6 (1964), pp. 275–295.

———, *Convergence of the QR algorithm*, Numer. Math., 7 (1965), pp. 187–193. Errata: 10 (1967), pp. 163–164.

———, *The LU and QR transformations*, Mathematical Methods for Digital Computers, vol. II, A. Ralston and H. S. Wilf, eds., John Wiley, New York, 1967, pp. 116–130.

———, *Singular and invariant matrices under the QR algorithm*, Math. Comp., 20 (1966), pp. 611–615.

———, *Canonical decomposition of Hessenberg matrices*, Ibid., 21 (1967), pp. 223–337.

———, *Global convergence of the basic QR algorithm on Hessenberg matrices*, Ibid., 22 (1968), pp. 803–817.

O. PERRON, *Die Lehre von den Kettenbrüchen*, vol. 2, B. G. Teubner, Stuttgart, 1957.

G. PÓLYA, *Graeffe's method for eigenvalues*, Numer. Math., 11 (1968), pp. 315–319.

HERMANUS VAN ROSSUM, *A theory of orthogonal polynomials based on the Padé table*, Van Gorcum, Assen, 1953, 76 pp.

E. J. ROUTH, *A Treatise on the Stability of a Given State of Motion*, Macmillan, London, 1877.

H. RUTISHAUSER, *Beiträge zur Kenntnis des Biorthogonalisierungs-Algorithmus von Lanczos*, Z. Angew. Math. Phys., 4 (1953), pp. 35–36.

———, *Der Quotienten-Differenzen-Algorithms*, Ibid., 5 (1954), pp. 233–251.

———, *Anwendungen des Quotienten-Differenzen Algorithms*, Ibid., 5 (1954), pp. 496–508.

———, *Ein infinitesimales Analogon zum Quotienten-Differenzen-Algorithmus*, Arch. Math., 5 (1954), pp. 132–137.

———, *Bestimmung der Eigenwerte und Eigenvektoren einer Matrix mit Hilfe des Quotienten-Differenzen-Algorithmus*, Z. Angew. Math. Phys., 6 (1955), pp. 387–401.

———, *Eine Formel von Wronski und ihre Bedeutung für den Quotienten-Differenzen-Algorithmus*, Ibid., 7 (1956), pp. 164–169.

———, *Der Quotienten-Differenzen-Algorithmus*, Mitt. Inst. Angew. Math. Zürich, 7 (1957), 74 pp.

———, *Beschleunigung der Konvergenz einer gewissen Klasse von Kettenbrüchen*, Z. Angew. Math. Mech., 38 (1958), pp. 187–190.

———, *Solution of eigenvalue problems with the LR-transformation*, National Bureau of Standards Applied Mathematics Series No. 49, U.S. Government Printing Office, Washington, D.C., 1958, pp. 47–81.

———, *Über eine Verallgemeinerung der Kettenbrüche*, Z. Angew. Math Mech., 38 (1958), pp. 278–279.

———, *Über eine kubisch konvergente Variante der LR-transformation*, Ibid., 40 (1960), pp. 49–54.

———, *On a modification of the QD-algorithm with Graeffe-type convergence*, Z. Angew. Math. Phys., 13 (1962), pp. 493–496.

———, *Stabile Sonderfälle des Quotienten-Differenzen Algorithmus*, Numer. Math., 5 (1963), pp. 95–112.

HANS-RUDOLF SCHWARZ, *Ein Verfahren zur Stabilitätsfrage bei Matrizen-Eigenwertproblemen*, Doctoral thesis, Eidgenössiche Technische Hochschule, Zurich, 1956, 30 pp.

I. J. SCHOENBERG, *On smoothing operations and their generating functions*, Bull. Amer. Math. Soc., 59 (1953), pp. 199–230.

———, *On the zeros of the generating functions of multiply positive sequences and functions*, Ann. of Math., 62 (1955), pp. 447–471.

D. SHANKS, *Nonlinear transformations of divergent and slowly convergent sequences*, J. Mathematical Phys., 34 (1955), pp. 1–42.

A. VAN DER SLUIS, *General orthogonal polynomials*, J. B. Wolters, Groningen, 1956. 50 pp.

G. W. STEWART, *On a companion operator for analytic functions*, Numer. Math., to appear.

E. STIEFEL, *Zur Interpolation von tabellienten Funktionen durch Exponentialsummen und zur Berechnung von Eigenwerten aus den Schwarzschen Konstanten*, Z. Angew. Math. Mech., 33 (1953), pp. 260–262.

T. J. STIELTJES, *Recherches sur les fractions continues*, Ann. Fac. Sci. Univ. Toulouse, 8 (1894), pp. 1–122; 9 (1895), pp. 1–47.

JOSEF STOER, *Über zwei Algorithmen zur Interpolation mit rationalen Funktionen*, Numer. Math., 3 (1961), pp. 285–304.

G. SZEGÖ, *Orthogonal Polynomials*, Colloquium Publications, vol. 23, Amer. Math. Soc., New York, 1958.

NICOLA TRUDI, *Teoria de' determinanti e loro applicazioni*, Libreria Scientifica e Industriale de B. Pellerano, Napoli, 1862.

H. S. WALL, *On the Padé approximants associated with the continued fraction and series of Stieltjes*, Trans. Amer. Math. Soc., 31 (1929), pp. 91–116.

———, *On the Padé approximants associated with a positive definite power series*, Ibid., 33 (1931), pp. 511–532.

———, *Continued fractions and totally monotone sequences*, Ibid., 48 (1940), pp. 165–184.

———, *Analytic Theory of Continued Fractions*, Van Nostrand, New York, 1948.

J. L. WALSH, *Padé approximants as limits of rational functions of best approximation*, J. Math. Mech., 13 (1964), pp. 305–312.

———, *On the convergence of sequences of rational functions*, SIAM J. Numer. Anal., 4 (1967), pp. 211–221.

J. H. WILKINSON, *Convergence of the LR, QR, and related algorithms*, Comput. J., 8 (1965), pp. 77–84.

———, *The QR algorithm for real symmetric matrices with multiple eigenvalues*, Ibid., 8 (1965), pp. 85–87.

———, *The Algebraic Eigenvalue Problem*, Oxford University Press, London, 1965.

———, *Global convergences of tridiagonal QR algorithm with origin shifts*, Linear Algebra and Appl., 1 (1968), pp. 409–420.

P. WYNN, *On a device for computing the $e_m(S_n)$ transformation*, Math. Tables Aids Comput., 10 (1956), pp. 91–96.

———, *On a Procrustean technique for the numerical transformation of slowly convergent series and sequences*, Proc. Cambridge Philos. Soc., 52 (1956), pp. 663–671.

———, *Converging factors for continued fractions*, Numer. Math., 1 (1959), pp. 272–320.

———, *On the propagation of error in certain non-linear algorithms*, Ibid., 1 (1959), pp. 142–149.

———, *Über einen Interpolations-Algorithmus und gewisse andere Formeln, die in der Theorie der Interpolation durch rationele Funktionen bestehen*, Ibid., 2 (1960), pp. 151–182.

———, *A note on a confluent form of the ε-algorithm*, Arch. Math., 11 (1960), pp. 237–240.

———, *Confluent forms of certain non-linear algorithms*, Ibid., 11 (1960), pp. 223–236.

———, *The rational approximation of functions which are formally defined by a power series expansion*, Math. Comp., 14 (1960), pp. 147–186.

———, *The epsilon algorithm and operational formulas of numerical analysis*, Ibid., 15 (1961), pp. 151–158.

———, *On repeated application of the ε-algorithm*, Chiffres, 4 (1961), pp. 19–22.

———, *Upon a second confluent form of the ε-algorithm*, Proc. Glasgow Math. Assoc., 5 (1962), pp. 160–165.

———, *A comparison technique for the numerical transformation of slowly convergent series based on the use of rational functions*, Numer. Math., 4 (1962), pp. 8–14.

———, *Note on a converging factor for a certain continued fraction*, Ibid., 5 (1963), pp. 332–351.

———, *Singular rules for certain non-linear algorithms*, BIT, 3 (1963), pp. 175–195.

———, *Partial differential equations associated with certain non-linear algorithms*, Z. Angew. Math. Phys., 15 (1964), pp. 273–289.

———, *On some recent developments in the theory and application of continued fractions*, SIAM J. Numer. Anal., 1 (1964), pp. 177–197.

———, *Upon systems of recursions which obtain among the quotients of the Padé table*, Numer. Math., 8 (1966), pp. 246–269.

———, *On the convergence and stability of the epsilon algorithm*, SIAM J. Numer. Anal., 3 (1966), pp. 91–122.

———, *Upon the Padé table derived from a Stieltjes series*, Ibid., 5 (1968), pp. 805–834.

———, *Upon the definition of an integral as the limit of a continued fraction*, Arch. Rational Mech. Anal., 28 (1968), pp. 83–148.