Sparse Recovery and Dictionary Learning from Nonlinear Compressive Measurements

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Abstract

Sparse coding and dictionary learning are popular techniques for linear inverse problems such as denoising or inpainting. However in many cases, the measurement process is nonlinear, for example for clipped, quantized or 1-bit measurements. These problems have often been addressed by solving constrained sparse coding problems, which can be difficult to solve, and assuming that the sparsifying dictionary is known and fixed. Here we propose a simple and unified framework to deal with nonlinear measurements. We propose a cost function that minimizes the distance to a convex feasibility set, which models our knowledge about the nonlinear measurement. This provides an unconstrained, convex, and differentiable cost function that is simple to optimize, and generalizes the linear least squares cost commonly used in sparse coding. We then propose proximal based sparse coding and dictionary learning algorithms, that are able to learn directly from nonlinearly corrupted signals. We show how the proposed framework and algorithms can be applied to clipped, quantized and 1-bit data.

Index Terms

Sparse coding, dictionary learning, nonlinear measurements, saturation, quantization, 1-bit sensing

I. Introduction

SPARSE decomposition and dictionary learning are popular techniques for linear inverse problems in signal processing, such as denoising [1], [2], inpainting [3], [4] or super-resolution [5], [6]. Sparse coding aims at finding a sparse set of coefficients $\boldsymbol{\alpha} \in \mathbb{R}^M$ that accurately represents a signal $\mathbf{x} \in \mathbb{R}^N$ from a fixed overcomplete dictionary $\mathbf{D} \in \mathbb{R}^{N\times M}$, and is often formulated as:

$$\min_{\boldsymbol{\alpha}} \|\mathbf{x} - \mathbf{D}\,\boldsymbol{\alpha}\,\|_2^2 \quad + \lambda\Psi(\boldsymbol{\alpha}), \tag{1}$$

where $\Psi(\cdot)$ is a sparsity inducing regularizer, such as the ℓ_0 pseudo-norm or the ℓ_1-norm. Dictionary learning on the other hand, jointly learns the dictionary $\mathbf{D}$ and sparse coefficients $\boldsymbol{\alpha}_t$ from a set of training signals $\{\mathbf{x}_t\}_{t=1\ldots T}$:

$$\min_{\mathbf{D},\boldsymbol{\alpha}_t} \sum_{t=1}^{T} \left[\|\mathbf{x}_t - \mathbf{D}\,\boldsymbol{\alpha}_t\,\|_2^2 \quad + \lambda\Psi(\boldsymbol{\alpha}_t)\right]. \tag{2}$$

However, the observed signals are often distorted or measured in a nonlinear way:

$$\mathbf{y} = f(\mathbf{x}), \tag{3}$$

where f is a nonlinear measurement function, and $\mathbf{x}$ is the original (unknown) clean signal. Examples of nonlinear distortions include *clipping* (or saturation) and *quantization*. Clipping is often due to dynamic range limitations in acquisition systems, when a signal reaches a maximum allowed amplitude, and the waveform is truncated above that threshold [7]–[14]. Quantization is a common process in analog-to-digital conversion that maps a signal from a continuous input space to a (finite) discrete space [15]. More recently, 1-bit compression has attracted a lot of interest, as an extreme quantization scheme where

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samples are coded using only one bit per sample [16], i.e. only measuring the signs of the signal. Clipping and quantization are non-linear, non-smooth, and *compressive* measurements, i.e. the measurement map is non-invertible. For these reasons, the recovery of clipped/quantized signals is a challenging problem.

Recovering a signal from clipped or quantized measurements can be treated as linear inverse problems, by simply ignoring the nonlinearities, i.e. treating clipped samples as missing [4], and quantization error as additive noise [17]. Similarly, 1-bit signals can be tackled by using the sign measurements directly as an input [16], [18]. However using a formulation that is *consistent* with the measurement process, i.e. that takes into account our knowledge about the nonlinear measurement function, has been shown to greatly improve the reconstruction [4], [7]–[14], [16], [18]–[24]. Specially tailored cost functions, constraints, or regularizers have independently been proposed to deal with clipped [4], [7]–[14], quantized [19]–[23] or 1-bit [16], [18], [24] measurements. These formulations often involve solving constrained sparse coding problems, which can be difficult and computationally expensive to solve, since they involve computing expensive non-orthogonal projections at each iteration [13], [23].

Reconstruction methods proposed in the literature assume that the signal is sparse in some orthogonal basis [16], [18]–[22], [22]–[25], or in a fixed dictionary [4], [9], [10], [13]. However in some cases, the sparsifying dictionary might not be known in advance. Even when a good sparsifying dictionary is known, it has been shown in a different range of applications that adapting (or *learning*) the dictionary from the observed data greatly improves the reconstruction compared to a fixed dictionary [1], [2], [26], [27]. To the best of our knowledge, dictionary learning for saturated and quantized measurements has not been addressed in the literature.

A. Limitations, and motivations

Several algorithms have been proposed in the literature to address saturated, quantized and 1-bit measurements, however we review here some of their limitations:

- Although sparse recovery from linear measurements is now a well studied topic, recovery from clipped, quantized and 1-bit measurements have been treated independently, using specially-tailored algorithms for each problem.
- Strong assumptions on the sparsifying dictionary are often made in the literature, such as orthogonality [16], [18]–[22], [22]–[25], tightness [13], [28] or analysis dictionaries [13], [28].
- Algorithms in the literature often assume that the sparsifying dictionary is known and fixed. Dictionary learning algorithms from 1-bit measurements have recently been proposed in [29], [30]. However, dictionary learning from saturated and general quantized measurements have not been addressed in the literature.

B. Contributions, and main results

Our contributions are as follows:

- We propose a unifying framework for signal recovery from nonlinear measurements such as clipping, quantization and 1-bit measurements, i.e. addressing these three problems in a unified fashion rather than individually.
- We show how these problems can be formulated as minimizing the distance to a convex feasibility set, which models our assumption about the nonlinear measurement process. In particular, the proposed cost generalizes the linear least squares commonly used in sparse coding, as well as several cost functions proposed independently for declipping and 1-bit recovery.
- Using properties of projection operators over convex sets, we show that the proposed cost function is continuous, convex and differentiable with Lipschitz gradient. Our main result uses Danskin's Min-Max theorem [31], that allows us to derive a closed-form gradient for the proposed cost.
- We propose proximal based consistent sparse coding, and dictionary learning algorithms, for non-linear measurements. We show that this algorithm can be applied to clipped, quantized and 1-bit measurements (preliminary results for clipped measurements were already presented in [32]).

We show experimentally that the proposed framework performs well on signal reconstruction from saturated, quantized and 1-bit measurements. For these three problems, the main experimental results are as follows:

- The proposed framework for consistent sparse coding and dictionary learning, performs better than using classical sparse coding and dictionary learning (i.e. discarding the nonlinearities and treating declipping and de-quantization as linear inverse problems).
- The proposed consistent dictionary learning outperforms consistent sparse coding with a fixed dictionary.

C. Organization of paper

The paper is organised as follows: in Section II, we briefly review sparse recovery and and dictionary learning from linear measurements, and some strategies proposed to deal with clipped, quantized and 1-bit measurements. In Section III we propose a unifying cost function for nonlinear measurements. We present the assumptions made on the measurement function f, and show some properties of the proposed cost function that make it simple to optimize. In Section IV we propose a sparse coding, and a dictionary learning algorithm for nonlinear measurements. Applications of the proposed framework, and links to previous work are presented in Section V. The performance of the proposed algorithm is presented in Section VI, before the conclusion is drawn.

D. Notation

In this paper, bold lowercase letters $\mathbf{x}$ denote vectors and bold uppercase letters $\mathbf{X}$ denote matrices. The i-th element of a vector $\mathbf{x}$ is noted x_i. The identity matrix is noted $\mathbf{I}$. The p-norm of a vector $\mathbf{x}$ is $\|\mathbf{x}\|_p = (\sum x_i^p)^{1/p}$. The ℓ_0 pseudo-norm (i.e. the number of non-zero elements) of $\mathbf{x}$ is noted $\|\mathbf{x}\|_0$. For a matrix $\mathbf{X}$, $\|\mathbf{X}\|_2$ denotes the matrix 2-norm, i.e. the largest singular value of $\mathbf{X}$. We denote $(\mathbf{x})_+ = \max(\mathbf{0}, \mathbf{x})$ (where $\max$ is the element-wise maximum), and $(\mathbf{x})_- = -(-\mathbf{x})_+$. The floor (i.e. closest lower integer) of a vector is noted $\lfloor \mathbf{x} \rfloor$ (applied element-wise). The sign (positive or negative) of each element of $\mathbf{x}$ is noted $\mathrm{sign}(\mathbf{x})$. The element-wise multiplication is $\odot$. For a set $\mathcal{C}$, $\mathrm{cl}(\mathcal{C})$ is the closure of $\mathcal{C}$, and $\mathbb{1}_{\mathcal{C}}(\cdot)$ is the indicator function of that set, i.e. $\mathbb{1}_{\mathcal{C}}(\mathbf{x}) = 0$ when $\mathbf{x} \in \mathcal{C}$, $+\infty$ otherwise.

II. Background

In this paper, we denote observed vectors as $\mathbf{y} \in \mathbb{R}^L$, with $\mathbf{y} = f(\mathbf{x})$, where $\mathbf{x} \in \mathbb{R}^N$ is the original un-observed clean signal, and f is a measurement or distortion function. We further assume that the signal $\mathbf{x}$ can be sparsely represented by some *overcomplete* dictionary $\mathbf{D} \in \mathbb{R}^{N \times M}$ of M atoms ($N < M$), i.e $\mathbf{x} = \mathbf{D}\boldsymbol{\alpha}$ where $\boldsymbol{\alpha} \in \mathbb{R}^M$ is a sparse activation vector. In this section we review the different types of linear and nonlinear measurement functions f, and the associated problem formulations appearing in the literature.

A. Sparse coding from linear measurements

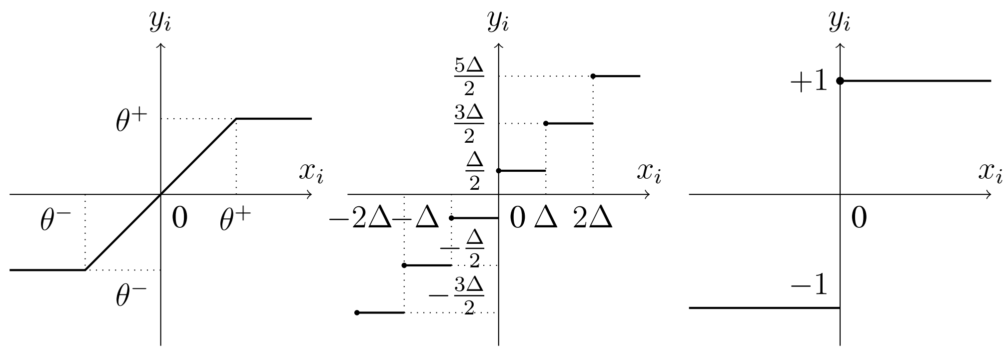
A widely studied case is when the measurement function is linear, i.e. $f(\mathbf{x}) = \mathbf{M}\mathbf{x}$ with $\mathbf{M} \in \mathbb{R}^{L \times N}$. The corresponding sparse coding model is problem formulated as:

$$\min_{\boldsymbol{\alpha}} \|\mathbf{y} - \mathbf{M}\mathbf{D}\boldsymbol{\alpha}\|_2^2 + \lambda\Psi(\boldsymbol{\alpha}). \tag{4}$$

For example, $\mathbf{M} = \mathbf{I}$ (the identity matrix) leads to a denoising problem [1], [2]. When $\mathbf{M}$ is a diagonal binary matrix, (4) corresponds to an inpainting problem [3], [4]. The linear least squares $\|\mathbf{y} - \mathbf{M}\mathbf{D}\boldsymbol{\alpha}\|_2^2$ is a convex and smooth data-fidelity term, with a closed form gradient, which allows one to derive efficient gradient-based algorithms such as Iterative Hard Thresholding (IHT) [33] or fast proximal descent algorithms [34]–[36].

B. *Sparse coding from nonlinear measurements*

Often in signal acquisition and processing, signals are measured or distorted in a nonlinear way, such as in clipped, quantized and 1-bit measurements. These three problems can be approximated as linear inverse problems by ignoring the nonlinearities. However the main focus in the literature has been to design specific methods that are *consistent* with the measurement process, i.e. fully models the nonlinear measurements, instead of discarding the nonlinearities. In the following we review different nonlinear measurement functions, and consistent formulations proposed in the literature.



(a) Clipping (b) Quantization (c) 1-bit compression

Fig. 1: Visualization of different nonlinear measurement functions f (output $y_i = f(x_i)$ versus input x_i).

1) Clipped measurements: We consider the case of hard clipping, where each sample x_i is clipped as:

$$f(x_i) = \begin{cases} \theta^+ & \text{if} \quad x_i \geq \theta^+ \\ \theta^- & \text{if} \quad x_i \leq \theta^- \\ x_i & \text{otherwise} \end{cases} \tag{5}$$

where $\theta^+ > \theta^-$ are positive and negative clipping thresholds respectively (see Figure 1a). This can be written in vector form as:

$$f(\mathbf{x}) = \mathbf{M^r}\,\mathbf{x} + \theta^+\,\mathbf{M^{c+}}\,\mathbf{1} + \theta^-\,\mathbf{M^{c-}}\,\mathbf{1}, \tag{6}$$

where $\mathbf{1}$ is the all-ones vector in $\mathbb{R}^N$, and $\mathbf{M^r}, \mathbf{M^{c+}}$ and $\mathbf{M^{c-}}$ are diagonal binary sensing matrices, that define the *reliable* (i.e. unclipped), positive and negative clipped samples respectively (e.g., $[\mathbf{M^{c+}}]_{i,i} = 1$ if $x_i \geq \theta^+$, or 0 otherwise), and such that $\mathbf{M^r} + \mathbf{M^{c+}} + \mathbf{M^{c-}} = \mathbf{I}$.

Declipping can be treated as a linear inverse problem by discarding the clipped samples, treating declipping as a linear *inpainting* problem, i.e. solving (4) with $\mathbf{M} = \mathbf{M^r}$ [4]. However, the reconstruction can be improved by adding extra knowledge about the clipping process. Indeed, we know that the clipped samples should have an amplitude that is higher than the clipping threshold. This extra information can be enforced by constraining clipped samples to be above their clipping level [9]:

$$\begin{aligned} &\min_{\boldsymbol{\alpha}} \|\mathbf{y} - \mathbf{M^r}\,\mathbf{D}\,\boldsymbol{\alpha}\|_2^2 + \lambda\Psi(\boldsymbol{\alpha}) \\ &\text{s.t.} \begin{cases} \mathbf{M^{c+}}\,\mathbf{D}\,\boldsymbol{\alpha} \succeq \theta^+\,\mathbf{M^{c+}}\,\mathbf{1} \\ \mathbf{M^{c-}}\,\mathbf{D}\,\boldsymbol{\alpha} \preceq \theta^-\,\mathbf{M^{c-}}\,\mathbf{1} \end{cases} \end{aligned} \tag{7}$$

This can be re-formulated as:

$$\min_{\boldsymbol{\alpha}} \Psi(\boldsymbol{\alpha}) + \mathbb{1}_{\mathcal{C}(\mathbf{y})}(\mathbf{D}\,\boldsymbol{\alpha}), \tag{8}$$

where:

$$\begin{aligned} \mathcal{C}(\mathbf{y}) \triangleq \{\mathbf{x} |\, \mathbf{M^r}\mathbf{y} = \mathbf{M^r}\,\mathbf{x}, \mathbf{M^{c+}}\,\mathbf{x} \succeq \theta^+\,\mathbf{M^{c+}}\,\mathbf{1}, \\ \mathbf{M^{c-}}\,\mathbf{x} \preceq \theta^-\,\mathbf{M^{c-}}\,\mathbf{1}\} \end{aligned} \tag{9}$$

is the *clipping consistency* set. Eqn. (8) is a constrained, non-smooth and possibly non-convex sparse decomposition problem, which can be difficult to solve. An alternating direction method of multipliers (ADMM) [37] based declipping algorithm was proposed in [13]. However, this algorithm proves to be computationally expensive, since it involves computing non-orthogonal projections at each iteration. Convex ℓ_1-based constrained formulations were also proposed in [7], [11], [14] and solved using general purpose optimization toolboxes [38] which can also be slow. A soft consistency metric was used in [8], [10], [12]:

$$\begin{aligned}\min_{\boldsymbol{\alpha}} \| \mathbf{M}^{\mathbf{r}}(\mathbf{y} - \mathbf{D}\,\boldsymbol{\alpha})\|_2^2 + \| \mathbf{M}^{\mathbf{c}+}(\theta^+\mathbf{1} - \mathbf{D}\,\boldsymbol{\alpha})_+\|_2^2 \\ + \| \mathbf{M}^{\mathbf{c}\text{-}}(\theta^-\mathbf{1} - \mathbf{D}\,\boldsymbol{\alpha})_-\|_2^2 + \lambda\Psi(\boldsymbol{\alpha}),\end{aligned} \tag{10}$$

where the clipped samples are penalized with a 1-sided square loss. The cost (10) is convex and smooth with a closed-form gradient, so methods based on iterative hard thresholding [10], [33] or proximal algorithms [12], [36] can be directly applied.

2) Quantized measurements: Quantization maps a continuous input space onto a finite discrete set of codewords $\mathcal{Y} = \{y_1, ..., y_p\}$. A quantization map f is defined by a set of quantization levels $\mathcal{R}_i = [l_i, u_i)$ and the relation $x \in \mathcal{R}_i \Leftrightarrow f(x) = y_i$, i.e. samples that fall into $\mathcal{R}_i$ are quantized as y_i. For example in the case of a uniform mid-riser quantizer, $\mathcal{R}_i = [\Delta i, \Delta(i+1))$, and the quantization function can be written as:

$$f(\mathbf{x}) = \Delta \left\lfloor \frac{\mathbf{x}}{\Delta} \right\rfloor + \frac{\Delta}{2}, \tag{11}$$

where $\Delta > 0$ is the size of each quantization region. The quantization map in the case of a mid-riser quantizer is presented in Figure 1b.

De-quantization can be treated as a simple linear inverse problem by considering quantization error as additive noise, and using a linear sparse model (4) [17]. However it has been shown that using a more accurate model of the quantization process improves the reconstruction. Bayesian approaches [20], ℓ_p data-fidelity terms [21], or piecewise linear cost functions [22] were proposed in the literature to enforce quantization consistency. Constrained formulations were proposed in [19], [23] in order to enforce consistency:

$$\min_{\boldsymbol{\alpha}} \Psi(\boldsymbol{\alpha}) + \mathbb{1}_{\mathcal{R}}(\mathbf{D}\,\boldsymbol{\alpha}) \tag{12}$$

where $\mathcal{R} = \mathcal{R}_1 \times ... \times \mathcal{R}_N$ and $\mathcal{R}_i$ is the quantization region associated with the i-th sample y_i. Eqn. (12) is a constrained sparse coding problem, which was solved using a Douglas-Rachford algorithm [35] implemented in [39]. However similarly as in the constrained declipping scenario (7), solving (12) with an overcomplete dictionary requires computing non-orthogonal projections at each iteration, which can be computationally expensive.

3) 1-bit measurements: 1-bit measurement can be seen as an extreme quantization using only one bit per sample, or similarly an extreme saturation where the clipping level tends to zero:

$$f(\mathbf{x}) = \operatorname{sign}(\mathbf{x}). \tag{13}$$

One-bit signal reconstruction is thus a problem of reconstructing a signal from its signs. Since only the signs of the signal are known, the signal can only be reconstructed up to an amplitude factor [16]. Boufounos [16], [18] proposed to enforce measurement consistency by solving the following problem:

$$\min_{\boldsymbol{\alpha}} \|\big(\mathbf{y} \odot (\mathbf{D}\,\boldsymbol{\alpha})\big)_-\|_2^2 + \lambda\Psi(\boldsymbol{\alpha}) \tag{14}$$

The data-fidelity term $\|\big(\mathbf{y} \odot (\mathbf{D}\,\boldsymbol{\alpha})\big)_-\|_2^2$ forces the reconstructed samples to have the same sign as the measurement y_i. The penalty can be seen as a one sided squared loss, which is convex and smooth. A variant of (14) that uses an ℓ_1 cost instead of an ℓ_2 was later proposed in [24], although the ℓ_1 norm leads to a non-smooth cost function.

C. Dictionary learning

Dictionary learning from clean or noisy measurements is often formulated as [27]:

$$\min_{\mathbf{D}\in\mathcal{D},\boldsymbol{\alpha}_t} \sum_{t=1}^{T} \left[\|\mathbf{y}_t - \mathbf{D}\,\boldsymbol{\alpha}_t\|_2^2 + \lambda\Psi(\boldsymbol{\alpha}_t)\right] \tag{15}$$

where $\{\mathbf{y}_t\}_{1\ldots T}$ is a collection of T signals in $\mathbb{R}^N$, and $\boldsymbol{\alpha}_t$ are the corresponding sparse activation vectors. The dictionary $\mathbf{D}$ is often constrained to be in $\mathcal{D} = \{\mathbf{D} \in \mathbb{R}^{N\times M} | \forall i, \|\mathbf{d}_i\|_2 \leq 1\}$ in order to avoid scaling ambiguity [27]. Many algorithms have been proposed in the literature to solve (15), such as MOD [40], K-SVD [1] or stochastic gradient descent [41], [42]. In the case of inpainting a weighted K-SVD has been proposed in [2] in order to handle missing values.

Dictionary learning for 1-bit data have recently been addressed in [29], [30]. To our knowledge, dictionary learning from saturated and quantized measurements, however, has not been addressed in the literature. Since dictionary learning involves updating sparse coefficients and dictionary for many iterations, over large datasets, one needs to define a formulation that is simple and computationally tractable. In the next section, we show how sparse coding and dictionary learning from nonlinear measurements can be tackled using the same, unified framework, that leads to a simple optimization problem.

III. A UNIFYING FRAMEWORK FOR NONLINEAR SIGNAL RECONSTRUCTION

Let $f : \mathcal{X} \mapsto f(\mathcal{X}) = \mathcal{Y}$ be an arbitrary - and possibly nonlinear - measurement function from a clean input space $\mathcal{X}$ to a measurement space $\mathcal{Y}$. For a measured signal $\mathbf{y} \in \mathcal{Y}$, we propose a cost function (or data-fidelity term) defined for all $\mathbf{x} \in \mathcal{X}$ as:

$$\mathcal{L}_f(\mathbf{x}, \mathbf{y}) = \frac{1}{2}\,\mathrm{d}(\mathbf{x}, f^{-1}\{\mathbf{y}\})^2 \tag{16}$$

where $f^{-1}\{\mathbf{y}\}$ is the *pre-image* of $\{\mathbf{y}\}$ under the measurement map f:

$$f^{-1}\{\mathbf{y}\} \triangleq \{\mathbf{x} \in \mathcal{X} | f(\mathbf{x}) = \mathbf{y}\}, \tag{17}$$

and $\mathrm{d}(\mathbf{x}, \mathcal{C})$ is the distance between $\mathbf{x}$ and the set $\mathcal{C}$, defined as:

$$\mathrm{d}(\mathbf{x}, \mathcal{C}) \triangleq \inf_{\mathbf{z}\in\mathcal{C}} \|\mathbf{x} - \mathbf{z}\|. \tag{18}$$

The set $f^{-1}\{\mathbf{y}\}$ can be seen as a *feasibility set*, i.e. the set of all possible input signals $\mathbf{x} \in \mathcal{X}$ that could have generated $\mathbf{y}$ when measured through f. The cost (16) thus measures how "close" a signal $\mathbf{x}$ is to the feasibility set associated with the measurement $\mathbf{y}$. The proposed cost is thus *consistent* with the measurement function f since it fully takes into account f. However unlike constrained formulations (7) or (12), here measurement-consistency is enforced in a simple unconstrained way.

Without any assumptions on the feasibility sets $f^{-1}\{\mathbf{y}\}$ and the norm $\|\cdot\|$, $\mathbf{x} \mapsto \mathcal{L}_f(\mathbf{x}, \mathbf{y})$ is in general non-convex and non-smooth, and therefore difficult to optimize. However we here show that when $f^{-1}\{\mathbf{y}\}$ is convex (which is the case in many inverse problems such as denoising, inpainting, declipping and de-quantization as will be shown in Section V), the proposed cost (16) exhibits convenient properties such as convexity, and differentiability with Lipschitz gradient.

In the following we assume $\mathcal{X} = \mathbb{R}^N$, $\|\cdot\| = \|\cdot\|_2$ (the Euclidean distance), and for all $\mathbf{y} \in f^{-1}\{\mathbf{y}\}$, $f^{-1}\{\mathbf{y}\}$ is a convex set. Note that for a set $\mathcal{C}$, $\mathrm{d}(\mathbf{x}, \mathcal{C}) = \mathrm{d}(\mathbf{x}, \mathrm{cl}(\mathcal{C}))$, so we can assume without loss of generality that $f^{-1}\{\mathbf{y}\}$ is closed. Note also that since $\mathcal{Y} = f(\mathcal{X})$, $f^{-1}\{\mathbf{y}\}$ is non-empty for all $\mathbf{y} \in \mathcal{Y}$.

A. Properties of the proposed cost function

In the rest of this section we consider a fixed $\mathbf{y} \in \mathcal{Y}$, and review the properties of $\mathbf{x} \mapsto \mathcal{L}_f(\mathbf{x}, \mathbf{y})$. We first recall the following theorem [43, Prop. B.11] which will be useful in rest of the paper:

Theorem 1 (Projection Theorem [43, Prop. B.11])**.** *Let $\mathcal{C}$ be a closed convex set in $\mathbb{R}^N$. Then, the following hold:*

a) For every $\mathbf{x} \in \mathbb{R}^N$, there exists a unique $\mathbf{z}^* \in \mathcal{C}$ such that $\mathbf{z}^*$ minimizes $\|\mathbf{x} - \mathbf{z}\|_2$ over all $\mathbf{z} \in \mathcal{C}$. $\mathbf{z}^*$ is called the projection *of $\mathbf{x}$ onto $\mathcal{C}$ and is noted $\Pi_{\mathcal{C}}(\mathbf{x})$. In other words:*

$$\Pi_{\mathcal{C}}(\mathbf{x}) \triangleq \underset{\mathbf{z} \in \mathcal{C}}{\operatorname{argmin}} \|\mathbf{x} - \mathbf{z}\|_2. \tag{19}$$

b) For $\mathbf{x} \in \mathbb{R}^N$, $\mathbf{z}^* = \Pi_{\mathcal{C}}(\mathbf{x})$ if and only if:

$$(\mathbf{z} - \mathbf{z}^*)^T (\mathbf{x} - \mathbf{z}^*) \leq 0 \quad \forall \mathbf{z} \in \mathcal{C}. \tag{20}$$

c) $\mathbf{x} \mapsto \Pi_{\mathcal{C}}(\mathbf{x})$ is continuous and non-expansive, i.e:

$$\|\Pi_{\mathcal{C}}(\mathbf{x}_1) - \Pi_{\mathcal{C}}(\mathbf{x}_2)\|_2 \leq \|\mathbf{x}_1 - \mathbf{x}_2\|_2 \quad \forall \mathbf{x}_1, \mathbf{x}_2 \in \mathbb{R}^N. \tag{21}$$

The Projection Theorem (Theorem 1) thus ensures existence and uniqueness of a minimizer of $\|\mathbf{x}-\mathbf{z}\|_2$ in $f^{-1}\{\mathbf{y}\}$. In particular, the infinimum is a minimum and $\mathcal{L}_f(\mathbf{x}, \mathbf{y})$ can be redefined as:

$$\mathcal{L}_f(\mathbf{x}, \mathbf{y}) = \frac{1}{2} \|\mathbf{x} - \Pi_{f^{-1}\{\mathbf{y}\}}(\mathbf{x})\|_2^2. \tag{22}$$

In particular, Theorem 1c ensures that $\mathcal{L}_f(\cdot, \mathbf{y})$ is a continuous function (as a composition of continuous functions). We now show some properties of $\mathcal{L}_f(\cdot, \mathbf{y})$, which make it suitable for a range of optimization algorithms. We define $\phi(\mathbf{x}, \mathbf{z}) = \frac{1}{2}\|\mathbf{x} - \mathbf{z}\|_2^2$ such that $\mathcal{L}_f(\mathbf{x}, \mathbf{y}) = \min_{\mathbf{z} \in f^{-1}\{\mathbf{y}\}} \phi(\mathbf{x}, \mathbf{z})$. Note that ϕ is convex in $(\mathbf{x}, \mathbf{z})$.

Proposition 1. *$\mathcal{L}_f(\cdot, \mathbf{y})$ is convex.*

Proof. $\mathcal{L}_f(\cdot, \mathbf{y})$ is a minimum of a family of convex functions ϕ over a non-empty convex set, so by [44, Section 3.2.5], $\mathcal{L}_f(\cdot, \mathbf{y})$ is convex. □

We then recall a theorem in optimization from [45, Section 4.1], originally due to Danskin [31]:

Theorem 2 (Danskin's Min-Max Theorem [31])**.** *Let $\mathcal{C}$ be a compact*[1] *set, and $g(\mathbf{x}) = \min_{\mathbf{z} \in \mathcal{C}} \phi(\mathbf{x}, \mathbf{z})$. Suppose that for each $\mathbf{z} \in \mathbb{R}^N$, $\phi(\cdot, \mathbf{z})$ is differentiable with gradient $\nabla_{\mathbf{x}}\phi(\mathbf{x}, \mathbf{z})$, and $\phi(\mathbf{x}, \mathbf{z})$ and $\nabla_{\mathbf{x}}\phi(\mathbf{x}, \mathbf{z})$ are continuous on $\mathbb{R}^N \times \mathbb{R}^N$. Define $\mathcal{Z}(\mathbf{x}) = \operatorname{argmin}_{\mathbf{z} \in \mathcal{C}} \phi(\mathbf{x}, \mathbf{z})$. Then g is directionally differentiable with directional derivative:*

$$\nabla g(\mathbf{x}; \mathbf{h}) = \min_{\mathbf{z} \in \mathcal{Z}(\mathbf{x})} \nabla_{\mathbf{x}}\phi(\mathbf{x}, \mathbf{z})^T \mathbf{h} \quad \forall \mathbf{h}, \tag{23}$$

where $\nabla g(\mathbf{x}; \mathbf{h})$ is the directional derivative in the direction $\mathbf{h}$. In particular, when $\mathcal{Z}(\mathbf{x})$ is unique $\mathcal{Z}(\mathbf{x}) = \{\mathbf{z}^*\}$, g is differentiable with gradient:

$$\nabla g(\mathbf{x}) = \nabla_{\mathbf{x}}\phi(\mathbf{x}, \mathbf{z}^*). \tag{24}$$

In other words, Danskin's Min-Max theorem says that if the minimum over a family of continuous and continuously differentiable functions is attained at a unique point $\mathbf{z}^*$, then the gradient of the minimum over this family of functions can be computed by simply evaluating that gradient at the argmin $\mathbf{z}^*$. A corollary is the following proposition:

Proposition 2. *$\mathcal{L}_f(\cdot, \mathbf{y})$ is differentiable, with gradient:*

$$\nabla_{\mathbf{x}}\mathcal{L}_f(\mathbf{x}, \mathbf{y}) = \mathbf{x} - \Pi_{f^{-1}\{\mathbf{y}\}}(\mathbf{x}). \tag{25}$$

[1] Note that compactness is only required to ensure existence of a minimum, according to Weierstrass' theorem.

Proof. $\mathcal{L}_f(\mathbf{x},\mathbf{y}) = \min_{\mathbf{z}\in f^{-1}\{\mathbf{y}\}} \phi(\mathbf{x},\mathbf{z})$ with $\phi(\mathbf{x},\mathbf{z}) = \frac{1}{2}\|\mathbf{x}-\mathbf{z}\|_2^2$. For all $\mathbf{z} \in \mathbb{R}^N$, $\phi(\cdot,\mathbf{z})$ is differentiable with gradient $\nabla_{\mathbf{x}}\phi(\mathbf{x},\mathbf{z}) = \mathbf{x}-\mathbf{z}$. Furthermore, $\phi(\mathbf{x},\mathbf{z})$ and $\nabla_{\mathbf{x}}\phi(\mathbf{x},\mathbf{z})$ are continuous in $(\mathbf{x},\mathbf{z})$, and $\mathcal{Z}(\mathbf{x}) = \operatorname{argmin}_{\mathbf{z}\in\mathcal{C}} \phi(\mathbf{x},\mathbf{z})$ is uniquely defined as $\mathcal{Z}(\mathbf{x}) = \{\Pi_{f^{-1}\{\mathbf{y}\}}(\mathbf{x})\}$ by the Projection Theorem (Theorem 1a). Using Danskin's Min-Max theorem, we can then conclude that:

$$\begin{aligned}\nabla_{\mathbf{x}}\mathcal{L}_f(\mathbf{x},\mathbf{y}) =& \nabla_{\mathbf{x}}\phi(\mathbf{x},\Pi_{f^{-1}\{\mathbf{y}\}}(\mathbf{x}))\\ =& \mathbf{x}-\Pi_{f^{-1}\{\mathbf{y}\}}(\mathbf{x}).\end{aligned} \tag{26}$$

□

Danskin's theorem thus provides a simple closed form solution for the gradient $\nabla_{\mathbf{x}}\mathcal{L}_f(\mathbf{x},\mathbf{y})$, which is computed as the difference between the vector $\mathbf{x}$ and its projection on $f^{-1}\{\mathbf{y}\}$. We further show a useful property of the gradient:

Proposition 3. *The gradient $\nabla_{\mathbf{x}}\mathcal{L}_f(\mathbf{x},\mathbf{y})$ is Lipschitz continuous, with a Lipschitz constant $L = 1$, i.e:*

$$\forall \mathbf{x}_1,\mathbf{x}_2, \|\nabla_{\mathbf{x}}\mathcal{L}_f(\mathbf{x}_1,\mathbf{y}) - \nabla_{\mathbf{x}}\mathcal{L}_f(\mathbf{x}_2,\mathbf{y})\|_2^2 \le \|\mathbf{x}_1-\mathbf{x}_2\|_2^2 \tag{27}$$

Proof. Let $\mathbf{x}_1,\mathbf{x}_2 \in \mathbb{R}^N$. By Theorem 1b, we have:

$$\begin{aligned}&(\Pi_{f^{-1}\{\mathbf{y}\}}(\mathbf{x}_2) - \Pi_{f^{-1}\{\mathbf{y}\}}(\mathbf{x}_1))^T(\mathbf{x}_1 - \Pi_{f^{-1}\{\mathbf{y}\}}(\mathbf{x}_1)) \le 0\\ &(\Pi_{f^{-1}\{\mathbf{y}\}}(\mathbf{x}_1) - \Pi_{f^{-1}\{\mathbf{y}\}}(\mathbf{x}_2))^T(\mathbf{x}_2 - \Pi_{f^{-1}\{\mathbf{y}\}}(\mathbf{x}_2)) \le 0.\end{aligned} \tag{28}$$

Subtracting and rearranging these two equations gives:

$$\begin{aligned}&\|\Pi_{f^{-1}\{\mathbf{y}\}}(\mathbf{x}_1) - \Pi_{f^{-1}\{\mathbf{y}\}}(\mathbf{x}_2)\|_2^2\\ &\qquad\le(\Pi_{f^{-1}\{\mathbf{y}\}}(\mathbf{x}_1) - \Pi_{f^{-1}\{\mathbf{y}\}}(\mathbf{x}_2))^T(\mathbf{x}_1-\mathbf{x}_2).\end{aligned} \tag{29}$$

We can then show that:

$$\begin{aligned}&\|\nabla_{\mathbf{x}}\mathcal{L}_f(\mathbf{x}_1,\mathbf{y}) - \nabla_{\mathbf{x}}\mathcal{L}_f(\mathbf{x}_2,\mathbf{y})\|_2^2\\ &\quad= \|\mathbf{x}_1 - \Pi_{f^{-1}\{\mathbf{y}\}}(\mathbf{x}_1) - (\mathbf{x}_2 - \Pi_{f^{-1}\{\mathbf{y}\}}(\mathbf{x}_2))\|_2^2\\ &\quad= \|\mathbf{x}_1-\mathbf{x}_2\|_2^2 + \|\Pi_{f^{-1}\{\mathbf{y}\}}(\mathbf{x}_1) - \Pi_{f^{-1}\{\mathbf{y}\}}(\mathbf{x}_2)\|_2^2\\ &\qquad\quad - 2(\mathbf{x}_1-\mathbf{x}_2)^T(\Pi_{f^{-1}\{\mathbf{y}\}}(\mathbf{x}_1) - \Pi_{f^{-1}\{\mathbf{y}\}}(\mathbf{x}_2))\\ &\quad\le \|\mathbf{x}_1-\mathbf{x}_2\|_2^2 - \|\Pi_{f^{-1}\{\mathbf{y}\}}(\mathbf{x}_1) - \Pi_{f^{-1}\{\mathbf{y}\}}(\mathbf{x}_2)\|_2^2\\ &\quad\le \|\mathbf{x}_1-\mathbf{x}_2\|_2^2\end{aligned} \tag{30}$$

where we have used (29) in the third line. □

B. Summary and Remarks

To summarize, we have proposed a cost function $\mathcal{L}_f(\mathbf{x},\mathbf{y}) = \frac{1}{2}\,\mathrm{d}(\mathbf{x}, f^{-1}\{\mathbf{y}\}))^2$ that enforces measurement-consistency, and is:

- unconstrained
- continuous
- convex
- differentiable, with Lipschitz continuous gradient: $\nabla_{\mathbf{x}}\mathcal{L}_f(\mathbf{x},\mathbf{y}) = \mathbf{x} - \Pi_{f^{-1}\{\mathbf{y}\}}(\mathbf{x})$.

Convexity makes this cost function very attractive since convex optimization, and convexity theory can be applied [44], [46]. Convexity theory has been well studied and attracted a lot of interest in signal processing and machine learning. Differentiability, and availability of a closed-form gradient, makes this cost suitable for gradient-based optimization algorithms. Moreover, Lipschitz continuity of the gradient provides convergence guarantees in gradient descent algorithms [43], and proximal based algorithms [35], [36]. The proposed cost enforces consistency in an *unconstrained* way, which makes it easier to optimize than constrained formulations such as (7) or (12).

In addition, when f is the identity map ($f(\mathbf{x}) = \mathbf{x}$), we have $f^{-1}\{\mathbf{y}\} = \mathbf{y}$ and $\mathcal{L}_f(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|_2^2$. The proposed cost thus generalizes the least squares cost commonly used in signal processing and machine learning. We will show in Section V how the proposed cost also generalizes several cost functions already proposed in the literature for inpainting, declipping and 1-bit signals, and how it can be applied to quantized measurements. The proposed cost thus provides a unifying framework to tackle all these problems.

To demonstrate how the proposed consistency metric (16) can easily be optimized along with sparsity-inducing regularizers, in the next section we propose simple proximal-based algorithms for sparse decomposition and dictionary learning.

IV. Proposed consistent sparse coding and dictionary learning algorithms

Let $\{\mathbf{y}_t\}_{1\ldots T}$ be a collection of T signals measured through a known measurement function f. Consistent dictionary learning can be formulated using the proposed cost as:

$$\min_{\mathbf{D} \in \mathcal{D}, \boldsymbol{\alpha}_t} \sum_{t=1}^{T} \Big(\mathcal{L}_f(\mathbf{D}\,\boldsymbol{\alpha}_t, \mathbf{y}_t) + \lambda \Psi(\boldsymbol{\alpha}_t) \Big) \tag{31}$$

Jointly minimizing $\mathbf{D}$ and $\{\boldsymbol{\alpha}_t\}_{t=1,\ldots,T}$ in (31) is a non-convex problem. For this reason, dictionary learning algorithms typically alternate between a sparse coding, and a dictionary update step [27], [40], [42], [47]. In the following we first propose a sparse coding algorithm when the dictionary is fixed, and then a dictionary learning algorithm.

A. Sparse coding algorithm

The sparse coding problem, for an observation $\mathbf{y}$ and a fixed dictionary $\mathbf{D}$, can be formulated as:

$$\min_{\boldsymbol{\alpha}} \mathcal{L}_f(\mathbf{D}\,\boldsymbol{\alpha}, \mathbf{y}) + \lambda \Psi(\boldsymbol{\alpha}). \tag{32}$$

The sparse coding (32) is thus a problem of minimizing a smooth and convex cost, plus a non-smooth regularizer. This can be commonly solved using proximal gradient descent algorithms [35], [36], which have already been successfully applied to sparse coding problems [34]. A proximal descent algorithm alternates between a gradient descent step, and a proximal minimization using the proximal operator of Ψ:

$$\mathrm{prox}_{\lambda\Psi}(\boldsymbol{\alpha}) \triangleq \operatorname*{argmin}_{\mathbf{u}} \|\mathbf{u} - \boldsymbol{\alpha}\,\|_2^2 + \lambda \Psi(\boldsymbol{\alpha}) \tag{33}$$

The proposed sparse coding algorithm is presented in Algorithm 1. When $\Psi(\boldsymbol{\alpha}) = \|\,\boldsymbol{\alpha}\,\|_1$, the proximal minimization can be computed in closed form, and is equal to the soft-thresholding operator [36]. Algorithm 1 is thus similar to the well known Iterative Shrinkage/Thresholding Algorithm (ISTA) [34], [36]. We here present the algorithm in its simplest form, but note that the algorithm can easily be accelerated as presented in [34]. When $\Psi(\boldsymbol{\alpha}) = \|\,\boldsymbol{\alpha}\,\|_0$, the proximal operator is not properly defined since Ψ is non-convex. However, (33) can still be computed as the hard-thresholding operator [33], in which case Algorithm 1 is similar to the Iterative Hard Thresholding (IHT) algorithm [33].

Algorithm 1 Proposed consistent sparse coding algorithm

Require: $\mathbf{y}, \mathbf{D}, \boldsymbol{\alpha}^0$
  initialize: $\boldsymbol{\alpha} \leftarrow \boldsymbol{\alpha}^0$
  while stopping criterion not reached **do**
    // Gradient descent step:
    $\boldsymbol{\alpha} \leftarrow \boldsymbol{\alpha} + \mu_1 \mathbf{D}^T (\Pi_{f^{-1}\{\mathbf{y}\}}(\mathbf{D}\,\boldsymbol{\alpha}) - \mathbf{D}\,\boldsymbol{\alpha})$
    // Proximal thresholding:
    $\boldsymbol{\alpha} \leftarrow \mathrm{prox}_{\lambda\Psi}(\boldsymbol{\alpha})$
  end while
  return $\hat{\alpha}$

B. Algorithm for dictionary learning

Once the sparse codes $\{\boldsymbol{\alpha}_t\}_{1...T}$ have been updated, the dictionary update step can be formulated as:

$$\min_{\mathbf{D}\in\mathcal{D}} \sum_{t=1}^{T} \mathcal{L}_f(\mathbf{D}\,\boldsymbol{\alpha}_t, \mathbf{y}_t), \tag{34}$$

which is a problem of minimizing a convex and smooth function over a convex set. This can be classically solved using a projected gradient descent algorithm [43]. A projected gradient descent typically alternates between a gradient descent step, and a projection step $\Pi_{\mathcal{D}}$, which here simply re-normalizes each column $\mathbf{d}_i$ of $\mathbf{D}$ as $\mathbf{d}_i \leftarrow \mathbf{d}_i / \max(\|\mathbf{d}_i\|_2, 1)$. The proposed dictionary update step is thus similar to classical projected gradient descent approaches already proposed for dictionary learning [41], [48]–[50].

The proposed dictionary learning algorithm is presented in Algorithm 2, where μ_2 is a parameter that controls the step size and can be either fixed or estimated at each iteration. The algorithm can be stopped when a maximum number of iterations has been reached, or when the cost reaches a desired error level ϵ.

Algorithm 2 Proposed consistent dictionary learning algorithm

Require: $\{\mathbf{y}_t\}_{1...T}$, $\mathbf{D}^0$, $\{\boldsymbol{\alpha}_t^0\}_{1...T}$
  initialize: $\mathbf{D}^{(0)} \leftarrow \mathbf{D}^0$, $\boldsymbol{\alpha}_t^{(0)} \leftarrow \boldsymbol{\alpha}_t^0$, $i \leftarrow 0$
  while stopping criterion not reached **do**
    $i \leftarrow i + 1$
    // Sparse coding step:
    for $t = 1...T$ **do**
      Initialize $\boldsymbol{\alpha}_t \leftarrow \boldsymbol{\alpha}_t^{(i-1)}$.
      Update $\boldsymbol{\alpha}_t^{(i)}$ using Algorithm 1 with $\mathbf{D} = \mathbf{D}^{(i-1)}$.
    end for
    // Dictionary update step:
    Initialize $\mathbf{D} \leftarrow \mathbf{D}^{(i-1)}$
    while not converged **do**
      $\mathbf{D} \leftarrow \mathbf{D} + \mu_2 \sum_t (\Pi_{f^{-1}\{\mathbf{y}_t\}}(\mathbf{D}\,\boldsymbol{\alpha}_t^{(i)}) - \mathbf{D}\,\boldsymbol{\alpha}_t^{(i)})\,\boldsymbol{\alpha}_t^{(i)T}$
      $\mathbf{D} \leftarrow \Pi_{\mathcal{D}}(\mathbf{D})$
    end while
    $\mathbf{D}^{(i)} \leftarrow \mathbf{D}$
  end while
  return $\hat{\mathbf{D}}$, $\{\hat{\boldsymbol{\alpha}}_t\}_{1...T}$

V. Applications and link with previous work

In this section we show how the proposed framework can be applied to linear inverse problems such as denoising and declipping, and nonlinear inverse problems such as declipping, de-quantization and 1-bit recovery. As an analogy with classical sparse coding algorithms, we define $\mathbf{r} \triangleq \Pi_{f^{-1}\{\mathbf{y}\}}(\mathbf{D}\,\boldsymbol{\alpha}) - \mathbf{D}\,\boldsymbol{\alpha}$ as the *residual* vector, i.e. measuring the (signed) error between $\mathbf{D}\,\boldsymbol{\alpha}$ and its projection on $f^{-1}\{\mathbf{y}\}$. Note that $\mathcal{L}_f(\mathbf{D}\,\boldsymbol{\alpha}, \mathbf{y}) = \|\mathbf{r}\|_2^2$. Algorithm 1 and 2 involve the computation of residual vectors at each iteration. It this section we show how these residuals can be computed in closed form, often involving simple elementwise maximum. Note that in the examples below, f is applied elementwise, i.e. (with a slight abuse of notations) $f(\mathbf{x}) = [f(x_1), ..., f(x_N)]$, and $f^{-1}\{\mathbf{y}\} = f^{-1}\{y_1\} \times ... \times f^{-1}\{y_N\}$.

A. Denoising/inpainting

When $\mathbf{y} = \mathbf{M}\mathbf{x}$ with $\mathbf{M}$ a diagonal binary matrix, i.e. in the denoising or inpainting case, the feasibility set is defined as:

$$\begin{aligned} f^{-1}\{\mathbf{y}\} &\triangleq \{\mathbf{x} | \mathbf{M}\mathbf{x} = \mathbf{y}\} \\ &= \{\mathbf{x} | x_i = y_i \quad \text{when} \quad [\mathbf{M}]_{i,i} = 1\} \end{aligned} \tag{35}$$

The projection can thus be computed for each sample x_i as:

$$\Pi_{f^{-1}\{\mathbf{y}\}}(x_i) = \begin{cases} y_i & \text{if} \quad [\mathbf{M}]_{i,i} = 1 \\ x_i & \text{otherwise,} \end{cases} \tag{36}$$

or in vector form as:

$$\Pi_{f^{-1}\{\mathbf{y}\}}(\mathbf{x}) = \mathbf{y} + (\mathbf{I} - \mathbf{M})\mathbf{x} \tag{37}$$

When $\mathbf{x} = \mathbf{D}\,\boldsymbol{\alpha}$, the proposed cost is then written as:

$$\begin{aligned} \mathcal{L}_f(\mathbf{D}\,\boldsymbol{\alpha}, \mathbf{y}) &= \|\Pi_{f^{-1}\{\mathbf{y}\}}(\mathbf{D}\,\boldsymbol{\alpha}) - \mathbf{D}\,\boldsymbol{\alpha}\,\|_2^2 \\ &= \|\mathbf{y} - \mathbf{M}\mathbf{D}\,\boldsymbol{\alpha}\,\|_2^2 \end{aligned} \tag{38}$$

We thus retrieve the linear least squares (4) commonly used in sparse coding and dictionary learning. In particular for $\mathbf{M} = \mathbf{I}$, $f^{-1}\{\mathbf{y}\} = \{\mathbf{y}\}$ and $\mathcal{L}_f(\mathbf{D}\,\boldsymbol{\alpha}, \mathbf{y}) = \|\mathbf{y} - \mathbf{D}\,\boldsymbol{\alpha}\,\|_2^2$. In this case Algorithm 1 becomes a classical linear sparse coding algorithm, like IHT (in the case of ℓ_0) or ISTA (in the case of ℓ_1), and Algorithm 2 becomes a classical linear dictionary learning algorithm.

B. Saturated/clipped measurements

In the case of saturated signals, using the notations of Section II, the feasibility set can be defined in closed form as:

$$\begin{aligned} f^{-1}\{\mathbf{y}\} = \{\mathbf{x} |\, \mathbf{M^r}\mathbf{y} = \mathbf{M^r}\,\mathbf{x}, \mathbf{M^{c+}}\,\mathbf{x} \succeq \mathbf{M^{c+}}\,\mathbf{y}, \\ \mathbf{M^{c\text{-}}}\,\mathbf{x} \preceq \mathbf{M^{c\text{-}}}\,\mathbf{y}\} \end{aligned} \tag{39}$$

which is a convex set. The projection can be computed as:

$$\begin{aligned} \Pi_{f^{-1}\{\mathbf{y}\}}(\mathbf{D}\,\boldsymbol{\alpha}) = \mathbf{M^r}\,\mathbf{y} + \mathbf{M^{c+}} \max(\mathbf{y}, \mathbf{D}\,\boldsymbol{\alpha}) \\ + \mathbf{M^{c\text{-}}} \min(\mathbf{y}, \mathbf{D}\,\boldsymbol{\alpha}), \end{aligned} \tag{40}$$

and the residual as:

$$\mathbf{r} = \mathbf{M^r}(\mathbf{y} - \mathbf{D}\,\boldsymbol{\alpha}) + \mathbf{M^{c+}}(\mathbf{y} - \mathbf{D}\,\boldsymbol{\alpha})_+ + \mathbf{M^{c\text{-}}}(\mathbf{y} - \mathbf{D}\,\boldsymbol{\alpha})_-. \tag{41}$$

This shows that the proposed cost can be written in closed form as:

$$\begin{aligned} \mathcal{L}_f(\mathbf{D}\,\boldsymbol{\alpha}, \mathbf{y}) = \|\,\mathbf{M^r}(\mathbf{y} - \mathbf{D}\,\boldsymbol{\alpha})\|_2^2 \\ + \|\,\mathbf{M^{c+}}(\mathbf{y} - \mathbf{D}\,\boldsymbol{\alpha})_+\|_2^2 + \|\,\mathbf{M^{c\text{-}}}(\mathbf{y} - \mathbf{D}\,\boldsymbol{\alpha})_-\|_2^2. \end{aligned} \tag{42}$$

The proposed cost thus generalizes the soft consistency metric proposed in [8], [10], [12] for declipping. When $\Psi(\boldsymbol{\alpha}) = \|\,\boldsymbol{\alpha}\,\|_0$, Algorithm 1 is thus equivalent to the consistent IHT declipping algorithm proposed in [10]. When $\Psi(\boldsymbol{\alpha}) = \|\,\boldsymbol{\alpha}\,\|_1$, Algorithm 1 is similar to the ISTA-type declipping algorithms proposed in [12]. We applied Algorithm 2 to declipping in [32].

C. Quantized measurements

We consider a general quantizer defined by quantization levels y_i and quantization sets $f^{-1}y_i = [l_i, u_i)$ for each sample i. As commented earlier and discussed in [19], [20], one can assume $f^{-1}(y_i) = [l_i, u_i]$ (the closure of $[l_i, u_i)$) without affecting the cost function. The projection operator for each sample x_i can be computed as:

$$\Pi_{f^{-1}\{\mathbf{y}\}}(x_i) = \begin{cases} u_i & \text{if} \quad x_i \geq u_i \\ l_i & \text{if} \quad x_i \leq l_i \\ x_i & \text{otherwise} \end{cases} \tag{43}$$

and the residual:

$$\Pi_{f^{-1}\{\mathbf{y}\}}(x_i) - x_i = \begin{cases} u_i - x_i & \text{if} \quad u_i - x_i \leq 0 \\ l_i - x_i & \text{if} \quad l_i - x_i \geq 0 \\ 0 & \text{otherwise.} \end{cases} \tag{44}$$

This can be simplified as:

$$\Pi_{f^{-1}\{\mathbf{y}\}}(x_i) - x_i = (u_i - x_i)_- + (l_i - x_i)_+ \tag{45}$$

In vector form (with $\mathbf{l} = [l_1, ..., l_N]$ and $\mathbf{u} = [u_1, ..., u_N]$) the cost can be written as:

$$\mathcal{L}_f(\mathbf{x}, \mathbf{y}) = \|(\mathbf{l} - \mathbf{x})_-\|_2^2 + \|(\mathbf{u} - \mathbf{x})_+\|_2^2, \tag{46}$$

In particular, we have with $\mathbf{x} = \mathbf{D}\,\boldsymbol{\alpha}$:

$$\mathcal{L}_f(\mathbf{D}\,\boldsymbol{\alpha}, \mathbf{y}) = \|(\mathbf{l} - \mathbf{D}\,\boldsymbol{\alpha})_-\|_2^2 + \|(\mathbf{u} - \mathbf{D}\,\boldsymbol{\alpha})_+\|_2^2. \tag{47}$$

For example in the case of uniform mid-riser quantizer:

$$\mathcal{L}_f(\mathbf{D}\,\boldsymbol{\alpha}, \mathbf{y}) = \|(\mathbf{y} - \frac{\Delta}{2} - \mathbf{D}\,\boldsymbol{\alpha})_-\|_2^2 + \|(\mathbf{y} + \frac{\Delta}{2} - \mathbf{D}\,\boldsymbol{\alpha})_+\|_2^2. \tag{48}$$

To the best of our knowledge the cost (47) for quantized measurements has not been proposed in the literature before. Eqn. (47) is an unconstrained, convex and smooth cost function that is consistent with the quantized measurements since it takes into account the quantization levels $[l_i, u_i]$. It is therefore simpler to optimize than constraint based formulations (12) proposed in [19], [23], or non-smooth formulations [22].

D. 1-bit sensing

In the case of 1-bit measurements, the projection operator can easily be computed for each sample x_i as:

$$\Pi_{f^{-1}\{\mathbf{y}\}}(x_i) = \begin{cases} x_i & \text{if} \quad \operatorname{sign}(x_i) = y_i \\ 0 & \text{otherwise,} \end{cases} \tag{49}$$

hence:

$$\Pi_{f^{-1}\{\mathbf{y}\}}(x_i) - x_i = \begin{cases} 0 & \text{if} \quad \operatorname{sign}(x_i) = y_i \\ -x_i & \text{otherwise} \end{cases} \tag{50}$$

and it can be easily verified that:

$$\mathcal{L}_f(\mathbf{x}, \mathbf{y}) = \|\big(\mathbf{y} \odot \mathbf{x}\big)_-\|_2^2. \tag{51}$$

In particular,

$$\mathcal{L}_f(\mathbf{D}\,\boldsymbol{\alpha}, \mathbf{y}) = \|\big(\mathbf{y} \odot (\mathbf{D}\,\boldsymbol{\alpha})\big)_-\|_2^2, \tag{52}$$

which shows that the proposed cost is equivalent to the cost (14) proposed for 1-bit signals [16], [18].

E. Summary

We have shown how the proposed framework can be applied to different problems such as declipping, de-quantization and 1-bit signal recovery. In these three problems, the projection operator can be computed in a simple and closed form solution, involving simple elementwise maxima, which leads to efficient implementations of the proposed consistent sparse coding and dictionary algorithms. The advantages of the proposed framework can be summarized as follows:

- The proposed cost generalizes several soft consistency metrics proposed in the literature for declipping, and 1-bit sensing. It also generalizes the classical linear least squares commonly used in denoising and inpainting.
- The proposed cost enforces consistency with the measurement function.
- The proposed cost follows some of the properties of the linear least squares, namely convexity and smoothness, which makes it suitable for gradient descent based algorithms. Moreover, the formulation is unconstrained, unlike constrained formulations that can be difficult to solve. In particular, solving the proposed consistent dictionary learning problem (31) is as simple as solving classical dictionary learning (15).
- The proposed framework can potentially be applied to a wide range of nonlinear functions.

In the next section we evaluate the performance of the proposed framework and algorithms on declipping, de-quantization and 1-bit reconstruction tasks.

VI. Evaluation

In this section, we evaluate the performance of the proposed framework. The goal here is not to compare the best algorithm to solve (31) or (32), but to demonstrate how the proposed framework can be used to reconstruct signal or learn dictionaries from nonlinear measurements. Other algorithms, based on greedy pursuits [51] or fixed point continuation [52] have been proposed, e.g., for 1-bit signals. Adaptive sparsity [10], [13], structured sparsity priors [12], or extra information about the signal energy [16], [18], [24] have also been proposed to improve the reconstruction. An extensive comparative study of all algorithms proposed in the literature is out of the scope of this paper. The goal here is to evaluate the proposed framework, compared with classical linear sparse coding and dictionary learning (15). We also compare the proposed consistent dictionary learning to consistent sparse coding with fixed dictionaries. Although many algorithms are available to solve (15), for a fair comparison we use the same gradient-descent based algorithm in all experiments, i.e. using Algorithm 1 and Algorithm 2 with $f^{-1}\{\mathbf{y}\} = \{\mathbf{y}\}$ as representative of "classical" linear sparse coding and dictionary learning (Algorithm 1 thus corresponds to IHT [33] in the case of ℓ_0 or ISTA in the case of ℓ_1 [34], and Algorithm 2 correspond to a classical gradient-descent based dictionary learning algorithm).

A. Implementation

We first discuss some implementation details.

1) Choice of parameters: The proposed algorithms can be iterated until a satisfactory error ϵ is reached, or when a maximum number of iterations is attained. In the following experiments, the sparse coding algorithm (with fixed dictionary) was computed for 50 iterations, which usually ensures convergence of the objective function. The dictionary learning algorithm was computed by alternating 50 iterations of sparse coding and dictionary update step, with 20 inner iterations for each update step. We use $\mu_1 = 1/L_1$ and $\mu_2 = 1/L_2$ where L_1 and L_2 are the Lipschitz constants of the gradients in Algorithm 1 and 2 respectively, which can be computed as $L_1 = \|\mathbf{D}\|_2^2$ and $L_2 = \|\mathbf{A}\|_2^2$ (where $\mathbf{A} \triangleq [\boldsymbol{\alpha}_1, ..., \boldsymbol{\alpha}_T]$) using Proposition 3. This ensures the convergence of the sparse coding step (Algorithm 1) in the case of ℓ_1 [35], [36]. Although there is no such guarantee in the case of ℓ_0 since ℓ_0 is non-convex, we observe good convergence behaviour in practice.

2) Initialization: The sparse coefficients $\boldsymbol{\alpha}_t$ can be initialized as the zero vector in the case of linear, saturated or quantized measurements. However in the case of 1-bit measurements, the gradient $\nabla_{\boldsymbol{\alpha}}\mathcal{L}_f(\mathbf{D}\,\boldsymbol{\alpha}^0, \mathbf{y})$ is zero when $\boldsymbol{\alpha}^0 = 0$, so we instead initialize the algorithm with a random vector whose elements follow a zero-mean Gaussian distribution.

For the dictionary learning algorithm, each sparse coding and dictionary update step is then initialized using a *warm start* strategy, i.e. using the estimate from the previous iteration.

3) Evaluation metrics: The quality of the reconstructed signal is commonly evaluated using the Signal-to-Noise Ratio (SNR): $\mathrm{SNR}(\hat{\mathbf{x}}, \mathbf{x}) = 20 \log \frac{\|\mathbf{x}\|_2}{\|\mathbf{x}-\hat{\mathbf{x}}\|_2}$ where $\hat{\mathbf{x}}$ is the estimated signal, and $\mathbf{x}$ is the reference clean signal. However in the case of 1-bit measurements, since the signal can only be recovered up to an amplitude factor, we define the *"angular" SNR* (already used in [22]) to evaluate the result:

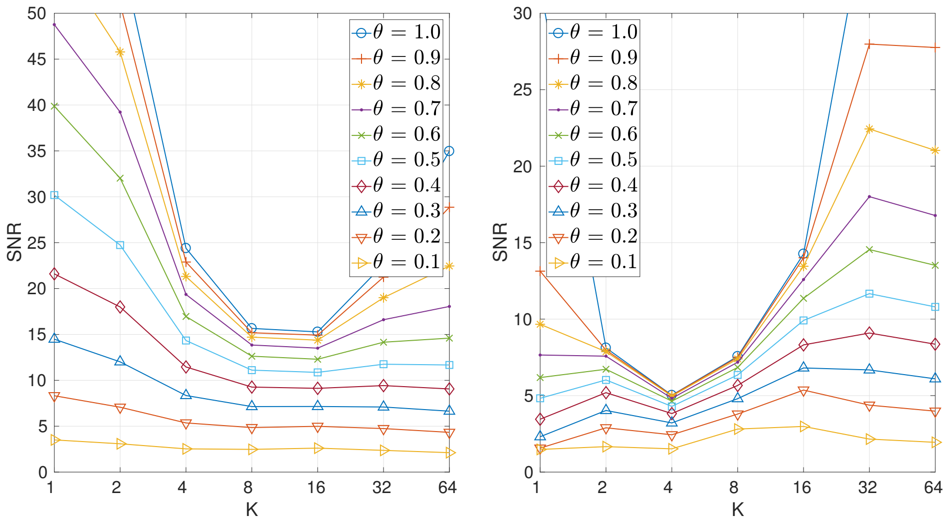
$$\mathrm{SNR}_{\text{angular}}(\hat{\mathbf{x}}, \mathbf{x}) \triangleq 20 \log \frac{\|\mathbf{x}\|_2}{\|\mathbf{x} - \frac{\|\mathbf{x}\|_2}{\|\hat{\mathbf{x}}\|_2}\hat{\mathbf{x}}\|_2}. \tag{53}$$

B. Experiments with synthetic data

We first evaluate the performance of the proposed algorithms with synthetic data. We generate a random dictionary $\mathbf{D} \in \mathbb{R}^{32\times 64}$, with i.i.d. normally distributed entries, and normalize each column to be of unit ℓ_2-norm. We then generate $T = 2000$ K-sparse activation vectors $\boldsymbol{\alpha}_t \in \mathbb{R}^{64}$ using i.i.d normal distribution for the sparse support and coefficients. We normalize the resulting dictionary-sparse vectors $\mathbf{x}_t = \mathbf{D}\,\boldsymbol{\alpha}_t$ to unit ℓ_∞ norm, and then artificially clip or quantize the signals $\mathbf{x}_t$ as $\mathbf{y}_t = f(\mathbf{x}_t)$. We consider clipping with different level θ. For quantization, we consider a uniform mid-rise quantizer that quantizes the input space $[-1, 1]$ using N_b bits, i.e. using 2^{N_b} quantization levels of size $\Delta = 2/2^{N_b}$.

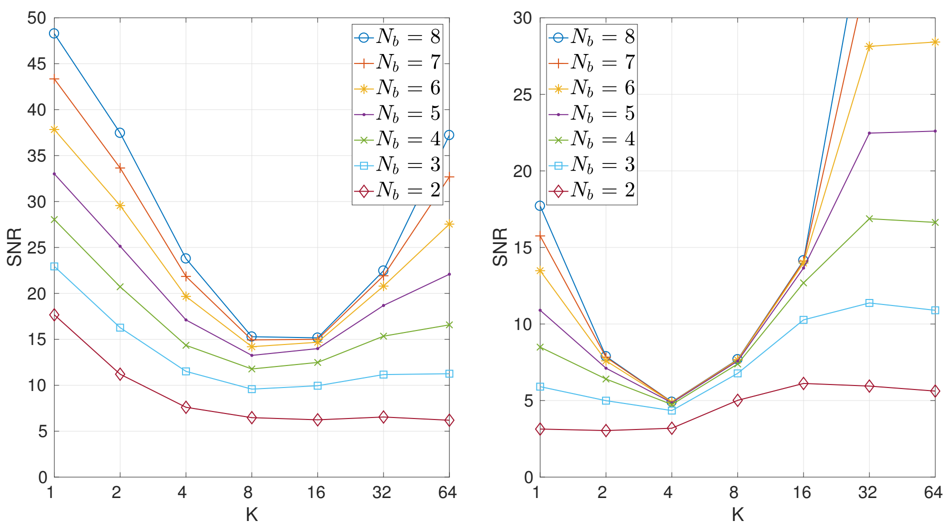
We consider two scenarios: first, we assume that the dictionary used to generate the data $\mathbf{D}$ is known, and recover the input signals $\mathbf{x}_t$ from the nonlinear measurements $\mathbf{y}_t$ using the proposed consistent sparse coding algorithm (Algorithm 1) with the known dictionary $\mathbf{D}$. Then we consider a *blind* scenario where the dictionary is unknown, and recover the input signals $\mathbf{x}_t$ using the proposed consistent dictionary learning algorithm (Algorithm 2). The algorithm is initialized with 50 different random dictionaries and we present the average result. In both experiments we assume the sparsity level K of the input signals is known, and thus use $\Psi(\boldsymbol{\alpha}) = \mathbb{1}(\|\,\boldsymbol{\alpha}\,\|_0 \leq K)$.

Figure 2 shows the declipping performance as a function of the sparsity level K of the input signals, for different clipping threshold θ. We include the case $\theta = 1$, which thus corresponds to a clean signal and can be seen as an upper-bound on the reconstruction performance when the signal is clean. Figure 2(a) shows that the proposed consistent sparse coding (with fixed dictionary) algorithm performs better when the input signal is highly sparse, i.e. $K = 1$ or $K = 2$. When $K = 1$, the algorithm manages to recover the signal with up to 14dB even the signal is highly clipped ($\theta = 0.3$). In the blind dictionary setting (with unknown dictionary), the proposed algorithm works better when $K = 32$, and sometimes even outperforms the sparse coding with a known dictionary. This shows that the proposed dictionary learning algorithm is able to recover clipped sparse signals even when the sparsifying dictionary is unknown. Figure 3 shows the de-quantization performance as a function of the sparsity level K of the input signals, and for different number of quantization bits N_b. Similarly as for the declipping case, Figure 3(a) shows that the proposed sparse coding works best for highly sparse signals. When $K = 1$, signals are reconstructed with a quality above 17dB, even from only 2 bits (i.e. 4 quantization levels). Figure 3(b) also shows how blind dictionary learning can outperform sparse coding (with a known dictionary) when $K = 32$. The results for 1-bit data can be found in Figure 4. When $K = 1$ and the dictionary is known, the signals can be reconstructed with a quality of 33dB (in angular SNR) from their signs. The blind dictionary learning setting fails to reconstruct the signals when the sparsity level is low ($K \leq 4$), however it performs as good as with a known dictionary when $K \geq 8$. Note that in practice (with real data), the dictionary learning algorithm can be initialized with a known sparsifying dictionary, as will be seen in the next experiment.



(a) Known dictionary (b) Unknown dictionary

Fig. 2: Signal recovery from clipped measurements with: (a) known dictionary and (b) unknown dictionary



(a) Known dictionary (b) Unknown dictionary

Fig. 3: Signal recovery from quantized measurements with: (a) known dictionary and (b) unknown dictionary

TABLE I: Performance for 1-bit signals

Angular SNR (dB)	Classical sparse coding (DCT)	Classical dictionary learning	Consistent sparse coding (DCT) (Algorithm 1)	Consistent dictionary learning (Algorithm 2)
Female speech	5.69	5.50	5.91	**6.12**
Male speech	4.67	4.47	4.50	**4.70**

TABLE II: Performance for 1-bit signals, for different sparsity parameters

Angular SNR (dB)	ℓ_0 $(K = 16)$	ℓ_0 $(K = 32)$	ℓ_0 $(K = 64)$	ℓ_1 $(\lambda = 10^{-3})$	ℓ_1 $(\lambda = 10^{-2})$	ℓ_1 $(\lambda = 5.10^{-2})$
Consistent sparse coding (DCT)	**5.54**	5.51	5.00	2.49	2.83	4.63
Consistent dictionary learning	**5.88**	5.78	5.09	3.27	0.19	0.00

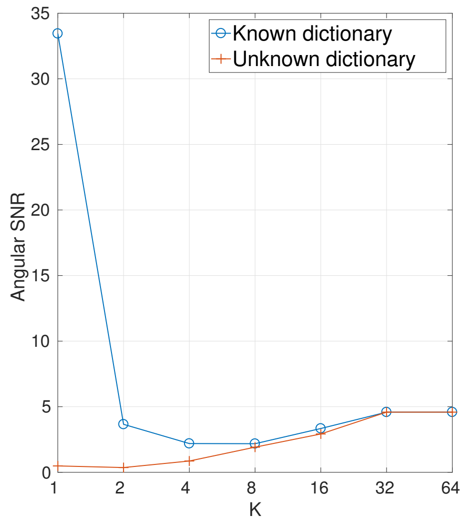


Fig. 4: Performance of proposed framework on 1-bit data

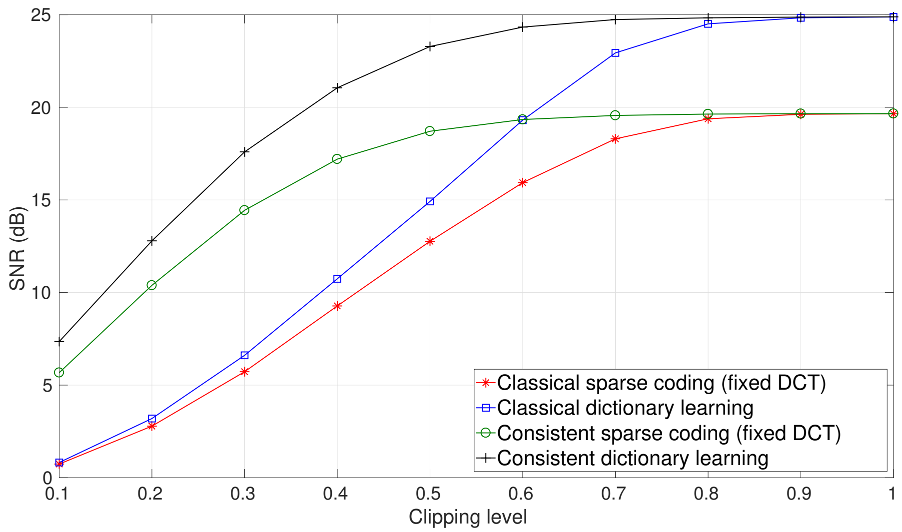
C. *Experiments with real data*

The following experiments are computed on audio signals, which are known to be sparse or approximately sparse [53]. The dataset consists of 10 female and male speech signals, taken from the SISEC dataset [54]. Each signal is 10s long, sampled at 16kHz, with 16 bits per sample. Each signal is normalized to unit ℓ_∞, and then processed using overlapping time frames of size $N = 256$, with 75% overlap, for a total of approximately $T = 2500$ frames per signal. All sparse coding experiments were computed using a fixed DCT dictionary of $M = 512$ atoms, and dictionary learning algorithms were initialized using the same DCT dictionary.

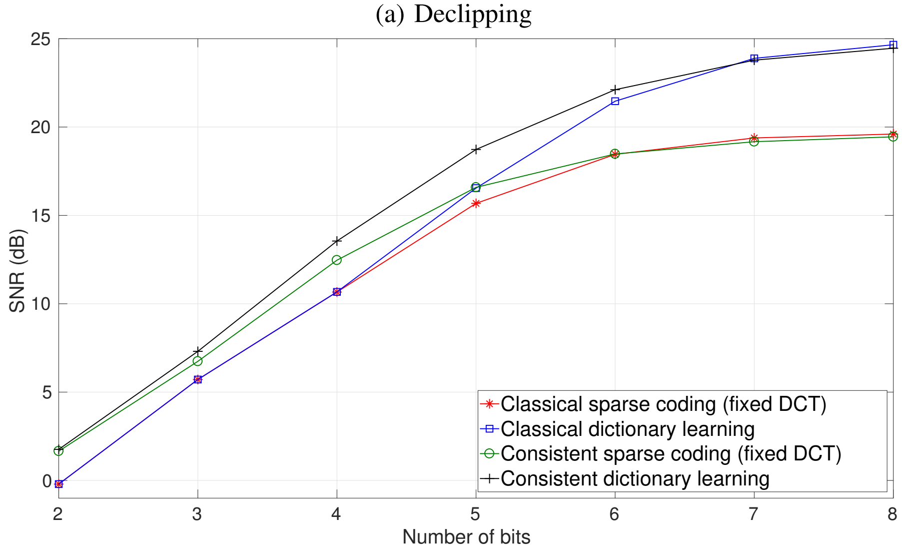
1) Influence of measurement consistency, and of dictionary learning: In the first experiment, we compare the performance of the proposed framework for consistent sparse coding and dictionary learning, compared to "classical" linear sparse coding and dictionary learning. In the case of declipping, the "classical" approach is to discard the clipped samples and treat the clipped signal as a signal with missing data [4]. In the quantization case, the "classical" approach is to treat the quantized signal as a noisy signal with variance $\frac{\Delta^2}{12}$ [18]. For 1-bit signals, we simply used the sign measurements directly as an input [18]. All experiments are computed with an ℓ_0 constraint and $K = 32$. In all three cases classical sparse coding is computed using IHT [33], and classical dictionary learning alternates between IHT to update the sparse coefficients and gradient descent to update the dictionary, similar to Algorithm 2.

Figure 5(a) shows the declipping performance, for different clipping levels ranging from $\theta = 0.1$ (highly clipped) to $\theta = 1$ (unclipped). Note that since $\Psi(\boldsymbol{\alpha}) = \|\,\boldsymbol{\alpha}\,\|_0$, the classical linear sparse coding simply corresponds to the IHT algorithm [33] trained on the unclipped data, and the consistent sparse coding corresponds to the "consistent IHT" proposed in [10]. Figure 5(a) demonstrates several things: First, using measurement consistency greatly improves the reconstruction. Consistent sparse coding shows an improvement of up to 8dB compared to classical sparse coding. Consistent dictionary learning shows up to 10dB improvement compared to classical dictionary learning. This improvement is greater when the signals are highly distorted ($\theta \leq 0.5$). As expected, the two frameworks give equivalent results when $\theta = 1$, which proves that the proposed consistent framework simply generalizes classical sparse coding. Second, we can see that consistent dictionary learning greatly improves the reconstruction performance compared to consistent sparse coding with a fixed DCT dictionary. This shows that the learned dictionaries generalize well to the unclipped samples. In particular, the proposed consistent dictionary learning algorithm outperforms all the other methods. Finally, it is interesting to point out that when the signals are highly clipped ($\theta = 0.1$), classical dictionary learning does not improve compared to classical sparse coding with DCT. This is probably due to a lack of data to learn from, since most of the data is clipped and discarded. Our consistent dictionary learning algorithm on the other hand, makes use of the clipped data, and is able to learn and improve the performance by 1.7dB.

Figure 5(b) shows the results for quantization, from highly quantized ($N_b = 2$ bits) to lightly quantized ($N_b = 8$). Similarly, we can see that using measurement consistency improves the performance when the signals are heavily quantized $N_b \leq 5$. As expected, the consistent and classical framework are equivalent, when the signals are lightly distorted ($N_b = 8$). Dictionary learning improves the performance (compared to using a fixed dictionary), and the proposed consistent dictionary learning algorithm outperforms the other methods.



(a) Declipping



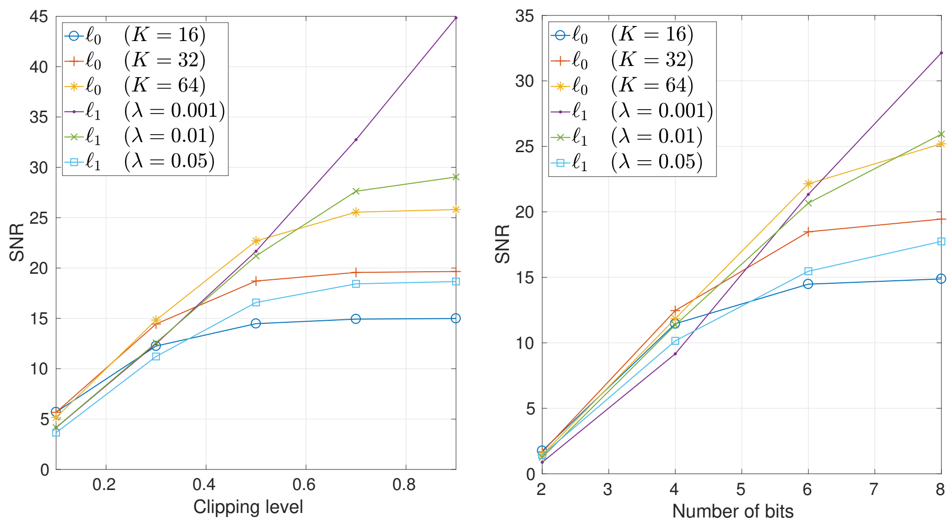
(b) De-quantization

Fig. 5: Comparison of the proposed consistent sparse coding and dictionary learning algorithms, compared to classical sparse coding and dictionary learning.

The results for 1-bit data are shown in Table I. Note that here the consistent sparse coding is equivalent to minimizing the cost (14) proposed in [16], [18], using and IHT-like algorithm. The results for sparse coding and consistent sparse coding are comparable (note that here for simplicity we don't enforce the signal to be on the unit circle, unlike in [16], [18]). Classical dictionary learning here performs worse than classical sparse coding with a fixed dictionary, presumably because the reconstructed signal overfits the ± 1 sign measurements. This shows that classical dictionary learning does not perform well with 1-bit data.

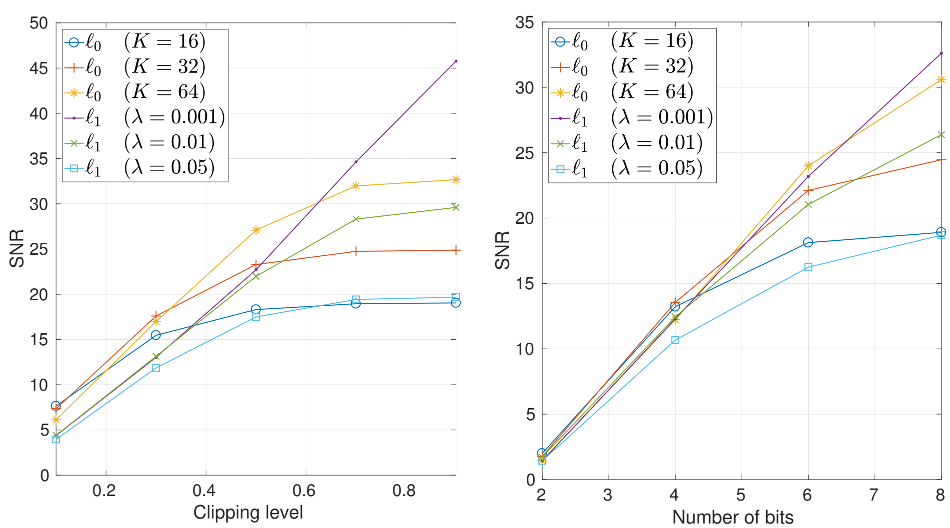
The proposed consistent dictionary learning however, outperforms consistent sparse coding and classical dictionary learning.

2) Influence of the sparsity regularizer: To show how the proposed algorithms can be used with different sparsity regularizers, we show the results for different sparsity inducing regularizers (ℓ_1 and ℓ_0) and different sparsity parameters. The results are exposed in Figure 6 for consistent sparse coding with fixed DCT, and Figure 7 for consistent dictionary learning. These experiments show that using an ℓ_0 regularizer is better when the signal is highly corrupted (low clipping level or low number of bits), but an ℓ_1 regularizer is better when the signal becomes lightly corrupted. The same conclusion can be reached in the case of 1-bit sensing (Table II).



(a) Declipping (b) De-quantization

Fig. 6: Comparison of consistent sparse coding algorithm (with fixed DCT) for different sparsity regularizers



(a) Declipping (b) De-quantization

Fig. 7: Comparison of consistent dictionary learning algorithm for different sparsity regularizers

VII. Conclusion and future work

We have presented a unified framework for signal reconstruction from nonlinear measurements. We proposed a cost function that takes into account our knowledge about the measurement process, by minimizing the distance to the pre-image of the received signal. When the pre-image is convex, we have shown that the proposed cost is a convex and smooth cost function with a Lipschitz gradient, which makes it ideal for proximal based algorithms. The proposed cost generalizes the classical linear least squares commonly used in signal processing, and classical sparse coding algorithms such as IHT or ISTA. We have shown how the proposed framework can be applied to different nonlinear measurement functions commonly found in signal processing, such as clipping, quantization and 1-bit signals. In the case of clipping and 1-bit, the proposed cost generalizes several cost functions already proposed in the literature. In the case of quantization, the proposed cost leads to a formulation that is conveniently unconstrained, convex and smooth. Experimentally, we showed that using measurement consistency greatly improves the reconstruction, compared to ignoring the nonlinearities and using classical sparse coding and dictionary learning. We also showed that consistent dictionary learning outperforms consistent sparse coding with fixed dictionaries. We have thus proposed a framework that can tackle declipping, dequantization and 1-bit signal reconstruction in a single unified way. But the proposed framework can also potentially be used for a wide range of nonlinear measurements, such as zero-crossing distortion [55], or color image quantization [56].

Future work will investigate how the proposed framework can be used in the case of non-convex sets. A non-convex set would mean that the minimizer in (18) is no longer unique, and the proposed cost is no longer convex. However this could potentially be applied to a wide range of problems, such as phase retrieval [57] and unlimited sampling [58].

We have proposed here a simple proximal gradient descent algorithm. However in the case of 1-bit signals for example, other type of algorithms have been proposed, such as greedy algorithms [51] or fixed point continuation [52]. Future work will also extend classical sparse coding algorithm (such as greedy pursuits) to the proposed framework, and consider an analytical and experimental study of different algorithms to solve (31) and (32).

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