Project Report - Convex Optimization for Signal Processing and Communications

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Contents

1 Introduction

In this report, the formulation and the solution of the optimization problem proposed as lecture's project are presented. The problems were addressed using the `cvxpy 1.5` library [2], with all implementation and computations carried out in Python.

2 Problem's Formulation

2.1 Problem 1

The main goal of the project's problem is to find an optimal transmit covariance matrix $\mathbf{P}$ of a MIMO system so that its transmission rate is maximized under certain constraints. Hence, the problems has $\mathbf{P}$ as optimization variable and the channel capacity of the system as objective function.

The MIMO system has a channel given by $y = \mathbf{H}^H\mathbf{x} + \mathbf{n}$, with $\mathbf{H} \in \mathbb{C}^{T\times R}$, $\mathbf{x} \in \mathbb{C}^T$ and $\mathbf{n} \in \mathbb{C}^R$. Constant T denotes the number of transmitter antennas, whereas R the number of receiver antennas. The transmit covariance matrix $\mathbf{P}$ and the noise covariance matrix $\mathbf{R_n}$, when $\mathbb{E}[\mathbf{x}] = \mathbb{E}[\mathbf{n}] = 0$, are defined as:

$$\mathbf{P} = \mathbb{E}[\mathbf{x}\mathbf{x}^H], \quad \mathbf{P} \in \mathbb{C}^{T\times T} \tag{1}$$

$$\mathbf{R_n} = \mathbb{E}[\mathbf{n}\mathbf{n}^H], \quad \mathbf{R_n} \in \mathbb{C}^{R\times R} \tag{2}$$

Given the definition provided by Equation 1, the matrix $\mathbf{P}$ must therefore satisfy the conditions:

$$\mathbf{P} \succeq 0 \quad \text{and} \quad \mathbf{P} = \mathbf{P}^H \tag{3}$$

From 1, it is possible to note that the diagonal elements of $\mathbf{P}$ correspond to the transmitted power signal in each of the T transmitter antennas. Since limits are imposed to the signal power of each transmitter antenna, as well as a limit to the overall transmitted power, the variable $\mathbf{P}$ must also fulfill the conditions:

$$p_{ii} \leq L_i, \quad \forall i \in \{1, \dots, T\} \tag{4}$$

$$\mathrm{tr}(\mathbf{P}) \leq L' \tag{5}$$

In the equations above, L_i and L' denote the power limit to antenna i and the the limit to the overall transmitted power, respectively.

When $\mathbf{R_n}$ is not necessarily a diagonal matrix (assumption of "colored noise" proposed for the project), the equation for the channel capacity can be derived from its traditional form [4], which is defined for the white-noise case, by pre-multiplying the received signal $\mathbf{y}$ with $\mathbf{R_n}^{-\frac{1}{2}}$, the inverse square root of the noise covariance matrix. Details about such transformation are provided in the Appendices (Section 4). This procedure results in the following final equation for the channel capacity:

$$C = B \cdot \log\det\left(\mathbf{I} + \mathbf{R_n}^{-1/2}\mathbf{H}^H\mathbf{P}\mathbf{H}\mathbf{R_n}^{-1/2}\right) \tag{6}$$

Combining the objective function in Equation 6 with the constraints defined in Equations 3, 4 and 5, the optimization problem can be formulated as:

$$\begin{aligned} \underset{\mathbf{P}}{\text{maximize}} \quad & \log\det\left(\mathbf{I} + \mathbf{R_n}^{-1/2}\mathbf{H}^H\mathbf{P}\mathbf{H}\mathbf{R_n}^{-1/2}\right) \\ \text{subject to} \quad & \mathbf{P} \succeq 0, \quad \mathbf{P} = \mathbf{P}^H, \\ & p_{ii} \leq L_i, \quad i = 1, \dots, T \\ & \mathrm{tr}(\mathbf{P}) \leq L' \end{aligned}$$

In the formulation above, the objective function is concave, since it is an affine transformation of the function $f(\mathbf{X}) = \log\det X$, which is concave [1]. In addition, the constraints over the trace of $\mathbf{P}$ and the conditions over its diagonal elements are all affine. The optimization variable $\mathbf{P}$ is also defined in a convex set: set of the hermitian positive semi-definite matrices [1]. As a result, the formulated optimization problem above is convex.

2.2 Problem 2

Additionally, the problem was required to be solved with an additional constraint: the system's interference to another co-channel user, characterized by the channel vector h_c and the received symbol $y_c = \mathbf{h_c}^H\mathbf{x} + \mathbf{n_c}$, should be zero, i.e., $\mathbb{E}[\|y_c\|^2] = \sigma_c^2$. This new constraint can be interpreted as:

$$\mathbb{E}[\|y_c\|^2] = \sigma_c^2 \Rightarrow \mathbb{E}[y_c^H y_c] = \sigma^2 \Rightarrow \mathbb{E}[(\mathbf{x}^H\mathbf{h_c} + \mathbf{n_c}^H)(\mathbf{h_c}^H\mathbf{x} + \mathbf{n_c})] = \sigma_c^2$$

$$\mathbb{E}[\mathbf{x}^H\mathbf{h_c}\mathbf{h_c}^H\mathbf{x} + \mathbf{x}^H\mathbf{h_c}\mathbf{n_c} + \mathbf{n_c}^H\mathbf{h_c}^H\mathbf{x} + \mathbf{n_c}^H\mathbf{n_c}] = \sigma_c^2 \Rightarrow \mathbb{E}[\mathbf{x}^H\mathbf{h_c}\mathbf{h_c}^H\mathbf{x}] + \mathbb{E}[\mathbf{n_c}^H\mathbf{n_c}] = \sigma_c^2$$

$$\mathbb{E}[\mathbf{x}^H\mathbf{h_c}\mathbf{h_c}^H\mathbf{x}] + \sigma_c^2 = \sigma_c^2 \Rightarrow \mathbb{E}[\mathbf{x}^H\mathbf{h_c}\mathbf{h_c}^H\mathbf{x}] = 0$$

The expression $\mathbb{E}[\mathbf{x}^H\mathbf{h_c}\mathbf{h_c}^H\mathbf{x}] = 0$ can be reformulated in terms of the transmit covariance matrix $\mathbf{P}$ using the trace operator.

$$\mathbb{E}[\mathbf{x}^H\mathbf{h_c}\mathbf{h_c}^H\mathbf{x}] = \mathbb{E}[\mathrm{tr}(\mathbf{x}^H\mathbf{h_c}\mathbf{h_c}^H\mathbf{x})] = \mathbb{E}[\mathrm{tr}(\mathbf{h_c}\mathbf{h_c}^H\mathbf{x}\mathbf{x}^H) = 0$$

$$\mathbb{E}[\mathrm{tr}(\mathbf{h_c}\mathbf{h_c}^H\mathbf{x}\mathbf{x}^H) = 0 \Rightarrow \mathrm{tr}(\mathbf{h_c}\mathbf{h_c}^H\mathbb{E}[\mathbf{x}\mathbf{x}^H]) = 0 \Rightarrow \mathrm{tr}(\mathbf{h_c}\mathbf{h_c}^H\mathbf{P}) = 0$$

Hence, the additional constraint for the zero interference in the new co-channel user is:

$$\mathrm{tr}(\mathbf{h_c}\mathbf{h_c}^H\mathbf{P}) = 0 \tag{7}$$

The constraint in Equation 7 is affine in $\mathbf{P}$, because the trace operator is affine. Therefore, the problem remains convex if such constrained is added.

3 Simulation Results

3.1 Problem 1

For the problem's simulation, the input parameters were arbitrarily fixed/generated as follows:

1. $T = 6$ and $R = 4$;
2. $\mathbf{H}$: its entries were generated as complex random variables with zero mean and unit variance;

3. $\mathbf{R_n^{\frac{1}{2}}}$: hermitian positive semi-definite matrix generated from a normal distribution with zero mean and unit variance over its entries;

4. Total power constraint: $L' = 65$;

5. Power constraint per Transmitter: $[L_1, L_2, L_3, L_4, L_5, L_6] = [9, 18, 10, 8, 12, 16]$ (Total sum = 73).

With the execution of the solver SCS [3] offered by the python package `cvxpy`, it was possible to solve the problem with `optimal status` only when the number of iterations was increased from 10^5 (default value) to 10^6. Part of the output file obtained in the first execution of the solver is presented in the Listing 1.

Listing 1: Output of the simulation for the original problem

```
SIMULATION OUTPUT FILE
------------------------
INPUT PARAMETERS GENERATED
------------------------
Covariance Matrix Noise Rn =
[[3.233+0.j    2.31 +3.628j 1.95 -2.023j 0.12 +0.368j]
 [2.31 -3.628j 9.704+0.j    2.357-3.332j 2.855+1.571j]
 [1.95 +2.023j 2.357+3.332j 5.133+0.j    1.897+1.29j ]
 [0.12 -0.368j 2.855-1.571j 1.897-1.29j  1.99 +0.j   ]]
Channel Matrix  H =
[[-0.486+0.593j -0.598+0.658j -0.475+0.202j -0.009+0.626j]
 [-0.79 -0.533j  0.166+0.886j  1.174+0.363j  0.525-0.211j]
 [-0.136+0.345j -0.628-0.053j -0.528+0.8j    1.197+1.075j]
 [ 0.036+1.545j -0.45 -0.987j  0.135-1.021j  1.485-0.357j]
 [ 0.085+0.113j  0.436+0.62j   0.212+0.223j -0.249-1.43j ]
 [-0.808-0.217j -0.247+0.585j -0.148+0.163j  0.415+0.539j]]

--------------------
RESULTS
--------------------
Status: optimal
Num iterations: 1429225
Solve Time: 131.9184518
Solver name: SCS
Optimal value: 23.214224928245216
Num of Constraints: 8

P* =
[[ 9.    +0.j    -2.373-3.473j  2.746+5.303j -1.278-0.801j  0.591+3.39j   7.523-2.624j]
 [-2.373+3.473j 15.636+0.j    -2.624-5.58j  -1.858+4.622j  1.469+3.125j  5.333+1.519j]
 [ 2.746-5.303j -2.624+5.58j  10.   +0.j     1.304+1.983j -4.749+4.292j  2.13 -2.552j]
 [-1.278+0.801j -1.858-4.622j  1.304-1.983j  8.   +0.j    -3.688+3.548j -2.043-0.857j]
 [ 0.591-3.39j   1.469-3.125j -4.749-4.292j -3.688-3.548j 12.   +0.j    -0.251-5.904j]
 [ 7.523+2.624j  5.333-1.519j  2.13 +2.552j -2.043+0.857j -0.251+5.904j 10.364+0.j   ]]
Eigenvalues of P* = [-0.    -0.     0.    15.792 23.865 25.344]
Actual Total Power of the Optimal Solution: (65.00000082104829+0j)

--------------------
Lagrange Multipliers
--------------------
For total power: 0.03818879855590025
For the limit power of each TX:
[0.00420142082239887, 0.0, 0.006626014801552716, 0.04275083221556508, 0.00148208587881462, 0.0]
```

In total, 10 executions were performed, each one with new $\mathbf{H}$ and $\mathbf{R_n^{\frac{1}{2}}}$ randomly chosen. The average results among the 10 executions are presented in the Appendices (Section 4).

As it is possible to observe in Listing 1, the optimal matrix $\mathbf{P}^*$ satisfies all constraints. It is hermitian and positive semi-definite, as evidenced by all its non-negative eigenvalues. Moreover, the power allocated to each transmitter antenna, specified by the elements along the main diagonal of $\mathbf{P}^*$, meets the initial constraints in L_i (item 5). The total transmitter power was 65.0004, which is approximately equal to $L' = 65$ (item 4).

It is noteworthy that the optimal solution was found by reaching the bounds of the constraints regarding the total power and the lowest individual power limits, i.e L_1, L_3, L_4, L_5. Indeed, as it can be observed, the Lagrange multipliers associated to these constraints are positive, whereas the antennas with allocated power less than the established limit have Lagrange multipliers equal to

0 (constraints inactive). Such result over the power intuitive: the higher the desired channel capacity is, the higher is the necessary transmitted power. For this purpose, the solver maximizes power allocation until it reaches the most restrictive limits.

3.2 Problem 2

When the additional constraint stated in Equation 7 is incorporated and the problem is solved again with the same input parameters ($\mathbf{H}$ and $\mathbf{R}_\mathbf{n}^{\frac{1}{2}}$) , the simulation produces the output file partially depicted in the Listing 2. Note that now the number of constraints is 9, instead of 8.

Listing 2: Output of the simulation for the modified problem

```
SIMULATION OUTPUT FILE
------------------------
INPUT PARAMETERS GENERATED
------------------------
Covariance Matrix Noise Rn =
[[3.233+0.j    2.31 +3.628j 1.95 -2.023j 0.12 +0.368j]
 [2.31 -3.628j 9.704-0.j    2.357-3.332j 2.855+1.571j]
 [1.95 +2.023j 2.357+3.332j 5.133-0.j    1.897+1.29j ]
 [0.12 -0.368j 2.855-1.571j 1.897-1.29j  1.99 -0.j   ]]
Channel Matrix  H =
[[-0.486+0.593j -0.598+0.658j -0.475+0.202j -0.009+0.626j]
 [-0.79 -0.533j  0.166+0.886j  1.174+0.363j  0.525-0.211j]
 [-0.136+0.345j -0.628-0.053j -0.528+0.8j    1.197+1.075j]
 [ 0.036+1.545j -0.45 -0.987j  0.135-1.021j  1.485-0.357j]
 [ 0.085+0.113j  0.436+0.62j   0.212+0.223j -0.249-1.43j ]
 [-0.808-0.217j -0.247+0.585j -0.148+0.163j  0.415+0.539j]]
--------------------
RESULTS
--------------------
Status: optimal
Num iterations: 3210675
Solve Time: 290.0456827
Solver name: SCS
Optimal value: 22.84509444812582
Num of Constraints: 9
P* =
[[ 9.   +0.j     0.487-2.652j  1.466+4.601j -1.604+0.572j  0.542+5.812j  3.025-0.401j]
 [ 0.487+2.652j 18.   +0.j    -3.473-6.467j -2.015+5.292j  1.459+4.076j  7.325-4.86j ]
 [ 1.466-4.601j -3.473+6.467j 10.   +0.j     1.871+2.136j -3.833+3.436j  0.907+1.208j]
 [-1.604-0.572j -2.015-5.292j  1.871-2.136j  8.   +0.j    -2.411+2.588j -4.571-2.134j]
 [ 0.542-5.812j  1.459-4.076j -3.833-3.436j -2.411-2.588j 12.   +0.j     0.659-3.091j]
 [ 3.025+0.401j  7.325+4.86j   0.907-1.208j -4.571+2.134j  0.659+3.091j  6.179+0.j   ]]
Eigenvalues of P* = [-0.    -0.     0.    15.419 18.394 29.366]
Actual Total Power of the Optimal Solution: (63.178504463310595+0j)
Interference in new user with new solution: (7.302356108684904e-09+2.053218706166149e-14j)
Interference in new user with previous solution: (1605.32598882976-1.0658141036401503e-13j)
--------------------
Lagrange Multipliers
--------------------
For total power: 7.158127495889746e-19
For the limit power of each TX:
[0.0673043018782, 0.035324403246403, 0.0447811406921990, 0.084390637107537, 0.045738556080654, 0.0]
For the zero interference in new user:  0.6917069200017273
```

As can be observed, the new value of the objective function 22.85 is slightly lower than the one obtained in the previous case (i.e 23.21). Such result is intuitive. By adding a new constraint, the feasible region of the solution is reduced, which may lead to the exclusion of the optimal solution previously obtained. Therefore, the new optimal solution will lead to a new optimal objective value less or equal than the previous one.

Besides, the results in Listing 2 indicate that the newly found $\mathbf{P}^*$ resulted in an interference to the new co-channel user significantly close to 0, as desired: $7.30 \cdot 10^{-9} + j \cdot 2.05 \cdot 10^{-14}$. In contrast, the previous value for $\mathbf{P}^*$ would give an interference of $1605.33 - j \cdot 1.07 \cdot 10^{-13}$, as informed in the output file. Noteworthy is that, since this new constraint is of the type equality, the associated Lagrange multiplier is non-zero.

4 Appendices

4.1 Adaptation of the Equation for the Channel Capacity

Channel Capacity Calculation Using Noise Whitening

To calculate the channel capacity of a MIMO channel when the noise covariance matrix is not diagonal, it is necessary to apply a transformation to "whiten the noise". Such transformation can be achieved by pre-multiply the original channel model by $\mathbf{R_n}^{-1/2}$, as explained below.

1. **Original Channel Model**:
$$\mathbf{y} = \mathbf{H}^H \mathbf{x} + \mathbf{n}$$
2. **Whiten the Noise**: Transform the noise $\mathbf{n}$ to $\tilde{\mathbf{n}}$ such that $\tilde{\mathbf{n}}$ has has a covariance matrix $\mathbf{I}$. For that, define:
$$\tilde{\mathbf{n}} = \mathbf{R_n}^{-1/2}\mathbf{n}$$
3. **Transform the Channel Model**: Pre-multiply the original channel model by $\mathbf{R_n}^{-1/2}$:
$$\mathbf{R_n}^{-1/2}\mathbf{y} = \mathbf{R_n}^{-1/2}\mathbf{H}^H\mathbf{x} + \mathbf{R_n}^{-1/2}\mathbf{n}$$

Let $\tilde{\mathbf{y}} = \mathbf{R_n}^{-1/2}\mathbf{y}$ and $\tilde{\mathbf{H}} = \mathbf{R_n}^{-1/2}\mathbf{H}^H$, the model becomes:
$$\tilde{\mathbf{y}} = \tilde{\mathbf{H}}\mathbf{x} + \tilde{\mathbf{n}}$$

4. **Capacity Expression**: The capacity C of the transformed channel is the same as the original one. When the noise covariance matrix is $\mathbf{I}$, the capacity is [4]:
$$C = \log\det\left(\mathbf{I} + \tilde{\mathbf{H}}\mathbf{R_x}\tilde{\mathbf{H}}^H\right)$$
5. **Final Expression**: Substituting $\tilde{\mathbf{H}} = \mathbf{R_n}^{-1/2}\mathbf{H}^H$, we get:
$$C = \log\det\left(\mathbf{I} + \mathbf{R_n}^{-1/2}\mathbf{H}^H\mathbf{R_x}\mathbf{H}\mathbf{R_n}^{-1/2}\right)$$

4.1 Average Results

Listing 3: Output of the averaged results of the simulation for the original problem

```
------------------------
AVERAGE INPUT PARAMETERS
------------------------
Average Noise Covariance Matrix Rn =
[[22.859+0.j     0.037+0.403j -6.899-0.225j -7.477+0.041j]
 [ 0.037-0.403j 25.422+0.j    -0.279-0.37j   9.969+0.175j]
 [-6.899+0.225j -0.279+0.37j  19.73 +0.j     0.11 +0.143j]
 [-7.477-0.041j  9.969-0.175j  0.11 -0.143j 38.406+0.j    ]]
Average Channel Matrix H =
[[-0.296+0.579j  0.172-0.115j -0.06 -0.125j  0.095+0.106j]
 [ 0.069-0.011j  0.21 +0.06j  -0.245-0.161j -0.006-0.122j]
 [ 0.106+0.461j  0.402-0.275j -0.103+0.346j  0.248-0.126j]
 [-0.036-0.101j -0.375+0.087j  0.002+0.174j  0.044+0.287j]
 [-0.471+0.214j  0.211-0.334j -0.135-0.182j  0.01 +0.256j]
 [ 0.133-0.391j  0.054+0.571j -0.026+0.15j  -0.043+0.101j]]
------------------------
RESULTS
------------------------
Number of averaged iterations: 9
Optimal Value Average: 14.810346730644035
Average P* =
[[ 8.94 +0.j    -0.872-1.259j  1.419+0.23j  -1.56 +0.329j  0.31 +0.451j  1.187+0.178j]
 [-0.872+1.259j 14.319+0.j    -1.86 -0.075j  0.252-0.069j  1.086+0.247j -0.271-0.135j]
 [ 1.419-0.23j  -1.86 +0.075j  9.39 +0.j    -0.355-1.181j -2.641+0.154j  1.672-0.871j]
 [-1.56 -0.329j  0.252+0.069j -0.355+1.181j  7.658+0.j     0.929-0.293j  0.978+0.841j]
 [ 0.31 -0.451j  1.086-0.247j -2.641-0.154j  0.929+0.293j 11.524+0.j     0.831-0.139j]
 [ 1.187-0.178j -0.271+0.135j  1.672+0.871j  0.978-0.841j  0.831+0.139j 13.169+0.j    ]]
```

References

[1] Stephen Boyd and Lieven Vandenberghe. *Convex Optimization*. Cambridge University Press, Cambridge, UK, 2004.

[2] Steven Diamond, Akshay Agrawal, and Russell Murray. Welcome to cvxpy 1.5. `https://www.cvxpy.org/index.html`, 2024. Accessed: 2024-06-26.

[3] Brendan O'Donoghue. Scs 3.2.4 documentation — cvxgrp.org. `https://www.cvxgrp.org/scs/index.html`, 2021. [Accessed 26-06-2024].

[4] Dr.-Ing. Marius Pesavento. Lecture course: Information theory ii. Lecture slides, 2023. FG Nachrichtentechnische Systeme (NTS), Technische Universität Darmstadt.