

# Fourier, Laplace and Z transformations

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September 15, 2018

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## 1 Convolution Product

For 2 functions  $x(t)$  and  $h(t)$  (usually denoting the impulse response of a LTI system), the **convolution product** is defined as:

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t - \tau)d\tau$$

And in case of time discrete convolution:

$$y[n] = x[n] * h[n] = \sum_{l=-\infty}^{\infty} x[l]h[n - l]$$

The product **commutes**, **associates** and **distributes**. By knowing the impulse response  $h(t)$  of any LTI system, providing an input  $x(t)$  to the system will produce an output  $y(t) = x(t) * h(t)$ . This can avoid solving a differential equation for this input.

## 2 Correlation

For energy limited signals, given by the condition:

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = E_x < \infty, \text{ or for discret signals } \sum_{l=-\infty}^{\infty} |x[l]|^2 = E_x < \infty$$

The correlation is calculated as

$$\varphi_{\mu\nu}(t) = \int_{-\infty}^{\infty} x_{\mu}(\tau)x_{\nu}(t + \tau)d\tau$$

If  $\mu = \nu$ , we speak of auto-correlation, while if  $\mu \neq \nu$ , we speak of the cross-correlation of the two signals. In case the signal is periodic, the integration happens on the interval of the signal, as follows:

$$\tilde{\varphi}_{\mu\nu}(t) = \frac{1}{T_0} \int_{T_0} x_{\mu}(\tau)x_{\nu}(t + \tau)d\tau$$

If the signals are discret, sum from 0 to N-1 and divide the sum by N.

## 3 Fourier Series

Any periodic signal, limited and continuously differentiable in the interval of length  $T_0$  can be written as a Fourier series:

$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}, \omega_0 = \frac{2\pi}{T_0}$$

$$c_k = \frac{1}{T_0} \int_{T_0} \tilde{x}(t)e^{-j\omega_0 t}dt$$

In case of real periodic signals  $\tilde{x}(t)$ ,  $c_{-k} = c_k^*$

### 3.1 Decomposition of the Fourier Series

The coefficients  $c_k$  can be rewritten as:

$$c_k = \begin{cases} \frac{a_k - jb_k}{2} & \text{pour } k \in N \\ a_0 & \text{pour } k = 0 \end{cases} \quad \text{avec } a_{-k} = a_k \text{ et } b_{-k} = b_k$$

Therefore the series can take the following form in the case of real periodic signals:

$$\boxed{\tilde{x}(t) = a_0 + \sum_{k=1}^{\infty} a_k \cos(k\omega_0 t) + \sum_{k=1}^{\infty} b_k \sin(k\omega_0 t)}$$

$$a_0 = \frac{1}{T_0} \int_{T_0} \tilde{x}(t) dt, \quad a_k = \frac{2}{T_0} \int_{T_0} \tilde{x}(t) \cos(k\omega_0 t) dt, \quad b_k = \frac{2}{T_0} \int_{T_0} \tilde{x}(t) \sin(k\omega_0 t) dt$$

It is important to see that for even functions  $\tilde{x}(t)$  all  $b_k = 0, \forall k \geq 1$ , while for uneven functions, all  $a_k = 0, \forall k \geq 0$ .

### 3.2 Parseval Theorem of the Fourier Series

$$\boxed{\frac{1}{T_0} \int_{T_0} |\tilde{x}(t)|^2 dt = \sum_{k=-\infty}^{\infty} |c_k|^2 = |c_0|^2 + 2 \sum_{k=1}^{\infty} |c_k|^2}$$

This result can be proved as follows:

$$\begin{aligned} \tilde{x}(t) &= \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} \\ \tilde{x}^*(t) &= \sum_{k'=-\infty}^{\infty} c_{k'}^* e^{-jk'\omega_0 t} \\ P &= \frac{1}{T_0} \int_{T_0} |\tilde{x}(t)|^2 dt = \frac{1}{T_0} \int_{T_0} \tilde{x}(t) \tilde{x}^*(t) dt = \frac{1}{T_0} \sum_{k=-\infty}^{\infty} \sum_{k'=-\infty}^{\infty} \underbrace{c_k c_{k'}^*}_{=|c_k|^2} \underbrace{\int_{T_0} e^{j(k-k')\omega_0 t} dt}_{T_0 \cdot \delta(k-k')} \end{aligned}$$

As  $k = k'$  because of the Dirac function  $\delta(k - k')$  that is equal to 0 if  $k \neq k'$ . We then find that:

$$P = \sum_{k=-\infty}^{\infty} |c_k|^2 \quad \square$$

The result of the integral ( $\int_{T_0} e^{j(k-k')\omega_0 t} dt$ ) will be discussed later on, when working on the Fourier transformation.

### 3.3 Time Shift of the Fourier Signal

A time shift of the signal will result in a phase shift of the coefficient  $c_k$  of the series.

$$\tilde{x}(t - \tau) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0(t-\tau)} = \sum_{k=-\infty}^{\infty} c_k e^{-jk\omega_0\tau} e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} c_k(\tau) e^{jk\omega_0 t}$$

Hence the phase shift is equal to  $-k\omega_0\tau$ .

### 3.4 Periodic Convolution

For two periodic signals  $\tilde{x}_1(t)$  and  $\tilde{x}_2(t)$ , we define on a period length (because of convergence issues) the *periodic convolution*:

$$\tilde{y}(t) = \tilde{x}_1(t) * \tilde{x}_2(t) = \int_{T_0} \tilde{x}_1(\tau) \tilde{x}_2(t - \tau) d\tau$$

The coefficients of the Fourier series follow the rule:

$$c_{yk} = T_0 \cdot c_{1k} \cdot c_{2k}$$

## 4 The Fourier Transformation FT

The advantage of the Fourier transformation is to enable us to transform not periodical signals. The Fourier series can only work for periodic signals! Note here that both transformations can be multiplied by a factor  $\frac{1}{\sqrt{2\pi}}$ , depending on the convention.

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega$$

The condition for the existence of the transformation is that  $x(t)$  is differentiable on a finite number of places and that it is absolutely integrable:

$$\int_{-\infty}^{\infty} |x(t)| dt < \infty$$

### 4.1 Parseval Theorem for the Fourier Transformation

$$E_x = \int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 d\omega$$

Of course this does not converge for periodic signals. The Parseval Theorem for the Fourier series has to be used instead in case of periodic signals. For the sake of generality, the theorem is given for two signals:

$$E_{\mu\nu} = \int_{-\infty}^{\infty} x_{\mu}^*(t) x_{\nu}(t) dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} X_{\mu}^*(\omega) X_{\nu}(\omega) d\omega$$

### 4.2 Example of the Fourier Transformation

Let us prove the following result:

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{j\omega_0 t} dt = \delta(\omega_0)$$

Let us set  $x(t) = \delta(t - t')$ :

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t} dt = \int_{-\infty}^{\infty} \delta(t - t')e^{-j\omega t} dt \quad \text{with} \quad \delta(t - t')e^{-j\omega t} \neq 0 \text{ for } t = t'$$

which yields,

$$X(\omega) = e^{-j\omega t'}, \quad x(t) = \delta(t - t')$$

By applying the inverse transform,

$$x(t) = \delta(t - t') = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega)e^{j\omega t} d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{j\omega(t-t')} d\omega$$

which proves

$$\delta(t - t') = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{j\omega(t-t')} d\omega \quad \square$$

### 4.3 Relation between the transformation and the series

From the relations we saw until now:

$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} \quad c_k = \frac{1}{T_0} \int_{T_0} \tilde{x}(t) e^{-j\omega_0 t} dt$$

and

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega)e^{j\omega t} d\omega \quad X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t} dt$$

it comes that

$$c_k = \frac{1}{T_0} X(k\omega_0) \quad \Leftrightarrow \quad T_0 c_k = X(k\omega_0)$$

This means that  $X(\omega)$  envelops the coefficients  $c_k$  with a factor  $T_0$ . The  $c_k$  are separated by a distance  $\omega_0 = \frac{2\pi}{T_0}$ .

### 4.4 Linear Differential Equations with constant coefficients

A differential equation of order N has the form:

$$\sum_{l=0}^N a_l \frac{d^l}{dt^l} y(t) = \sum_{m=0}^M b_m \frac{d^m}{dt^m} x(t)$$

And by knowing that:

$$\frac{d^n}{dt^n} x(t) \longleftrightarrow (j\omega)^n X(\omega)$$

The equation can be written after transformation as:

$$Y(\omega) \sum_{l=0}^N (j\omega)^l a_l = X(\omega) \sum_{m=0}^M (j\omega)^m b_m$$

Hence the **transfer function**  $H(\omega)$ :

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{\sum_{m=0}^M (j\omega)^m b_m}{\sum_{l=0}^N (j\omega)^l a_l}$$

The phase and magnitude of the transfer function are usually represented in logarithmic scales, typically with a *Bode-Diagram*.

## 5 The Time Discrete Fourier Transformation TDFT

Time is now discrete so the normalisations are necessary, with the sampling period  $T_s$ :

$$n = \frac{t}{T_s}$$

$$\Omega = \omega T_s = 2\pi f \cdot \frac{1}{f_s}$$

This is due to the fact that  $\phi(t) = e^{j\omega t} = e^{j\omega n T_s} = e^{j\Omega n}$ . The transformations are then given as follows:

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega n}$$

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\Omega) e^{j\Omega n} d\Omega$$

In order for the transformation to exist, the sequence has to be *stable*, hence:

$$|X(\Omega)| = \sum_{n=-\infty}^{\infty} |x[n]| < \infty, \quad \forall \Omega$$

It is important to see that the spectrum is **periodical**. It can be expressed as:

$$X(\Omega) = X(\Omega + 2\pi), \quad \forall \Omega$$

### 5.1 Infinite Sum of Complex Exponentials

With the following series in mind, with  $|a| < 1$ :

$$\sum_{k=0}^N a^k = \frac{1 - a^{N+1}}{1 - a} \quad \longleftrightarrow \quad \sum_{k=0}^{\infty} (ae^x)^k = \frac{1}{1 - ae^x}$$

We have:

$$\begin{aligned} \sum_{k=-\infty}^{\infty} (ae^{jx})^k &= \sum_{k=0}^{\infty} (ae^{jx})^k + \sum_{k=0}^{\infty} (ae^{-jx})^k - \underbrace{(ae^{jx})^0}_{*)} \\ &= \frac{1}{1 - ae^{jx}} + \frac{1}{1 - ae^{-jx}} - 1 = \frac{2 - a(e^{jx} + e^{-jx})}{1 - a(e^{jx} + e^{-jx}) + a^2} - 1 \underset{\lim_{a \rightarrow 1}}{=} 1 - 1 = 0 \end{aligned}$$

\*) If not subtracted here, it would be present two times in the sums otherwise.

If we take the definition of the series with  $a = 1$  and let  $e^x = e^{\pm j2\pi n} = 1$ , then the series diverges towards infinity. For any other value of  $e^x$ , the series is equal to zero. This is equivalent to a series of Dirac impulses!

As a final result, by letting  $a = 1$  and  $x = \Omega$  in the previous equations:

$$\sum_{n=-\infty}^{\infty} e^{-j\Omega n} = 2\pi \sum_{r=-\infty}^{\infty} \delta(\Omega - 2\pi r)$$

This result may come handy when calculating a TDFT.

## 5.2 Parseval Theorem for the TDFT

The *Energy Signal* is given as:

$$E_x = \sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi} \int_{2\pi} |X(\Omega)|^2 d\Omega$$

$|X(\Omega)|^2$  describes the Spectrum of the Energy Density. The Energy Signal of two overlapping signals is given as:

$$E_{xy} = \sum_{n=-\infty}^{\infty} x[n]y^*[n] = \frac{1}{2\pi} \int_{2\pi} X(\Omega)Y^*(\Omega) d\Omega$$

## 5.3 Transfer Function for LTI System

$$y[n] = x[n] * h[n] = \sum_{l=-\infty}^{\infty} x[l]h[n-l] = \sum_{l=-\infty}^{\infty} h[l]x[n-l]$$

Therefore, we can express the transfer function as:

$$Y(\Omega) = H(\Omega)X(\Omega) \Leftrightarrow \boxed{H(\Omega) = \frac{Y(\Omega)}{x(\Omega)}}$$

With the **stability** condition:

$$\boxed{\sum_{n=-\infty}^{\infty} |h[n]| < \infty}$$

## 5.4 Linear Differential Equations with constant coefficients

A differential equation is given as:

$$\sum_{l=0}^N a_l y[n-l] = \sum_{m=0}^M b_m x[n-m]$$

After transformation:

$$Y(\Omega) \sum_{l=0}^N a_l e^{-j\Omega l} = X(\Omega) \sum_{m=0}^M b_m e^{-j\Omega m}$$

Hence

$$H(\Omega) = \frac{Y(\Omega)}{X(\Omega)} = \frac{\sum_{m=0}^M b_m e^{-j\Omega m}}{\sum_{l=0}^N a_l e^{-j\Omega l}}$$

- *First Order*: The denominator only has values for  $l = 0, 1$ .
- *Second Order*: The denominator only has values for  $l = 0, 1, 2$ .
- *Higher Order*: They can be built from First and Second order systems.

## 6 Signal Sampling and Reconstruction

A time continuous signal  $x(t)$  is modulated by a sampling function  $s(t)$  at a rate  $\omega_s$ .

$$s(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s) \leftrightarrow S(\omega) = \frac{2\pi}{T_s} \sum_{r=-\infty}^{\infty} \delta(\omega - r\omega_s)$$

Therefore we have the sampled signal  $x_s(t)$ :

$$x_s(t) = x(t)s(t) = \sum_{n=-\infty}^{\infty} x(nT_s)\delta(t - nT_s)$$

$$X_s(\omega) = \frac{1}{2\pi} X(\omega) * S(\omega) = \frac{1}{T_s} \sum_{r=-\infty}^{\infty} X(\omega - r\omega_s)$$

Which proves, the spectrum of a time discrete signal is **periodical!**

### 6.1 The Shannon Sampling Theorem

With  $\omega_s$  the sampling rate and  $\omega_g$  the maximal rate of the signal:

$$\omega_s \geq 2\omega_g \longleftrightarrow T_s \leq \frac{\pi}{\omega_g}$$

This means that  $X(\omega) = 0$  for  $|\omega| > \omega_g$ . This means the width of the Spectrum has to be equal or smaller than the sampling rate. Otherwise, there is some overlapping and information is lost.

### 6.2 Signal Reconstruction

By applying a low pass to the signal given as

$$H_r(\omega) = T_s$$

for  $|\omega| \leq \omega_r$ . A common choice for  $\omega_r$  is  $\frac{\pi}{T_s}$ .

$$h_r(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} H_r(\omega) e^{j\omega t} d\omega = \frac{1}{2\pi} \int_{-\omega_r}^{\omega_r} T_s e^{j\omega t} d\omega = \frac{1}{2\pi} \int_{-\frac{\pi}{T_s}}^{\frac{\pi}{T_s}} T_s e^{j\omega t} d\omega = \frac{T_s}{\pi t} \sin(\omega_r t) = \text{si}\left(\frac{\pi t}{T_s}\right)$$

Hence

$$x_r(t) = \int_{-\infty}^{\infty} x_s(\tau) h_r(t-\tau) d\tau = \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} x(nT_s) \delta(\tau - nT_s) h_r(t-\tau) d\tau = \sum_{n=-\infty}^{\infty} x(nT_s) h_r(t - nT_s)$$

$$x_r(t) = \frac{\omega_r T_s}{\pi} \sum_{n=-\infty}^{\infty} x(nT_s) \text{si}\left(\omega_r(t - nT_s)\right)$$

The Reconstruction of a sampled signal is done by the summation of a multitude of si-functions!

### 6.3 Sampling in Frequency Domain

The function

$$P(\omega) = \sum_{n=-\infty}^{\infty} \delta(\omega - k\omega_p) \leftrightarrow p(t) = \frac{T_p}{2\pi} \sum_{r=-\infty}^{\infty} \delta(t - rT_p)$$

is used for the sampling of the signal. The signal in time domain of a sampled spectrum is **periodical** with a period  $T_p$ :

$$x_p(t) = x(t) * p(t) = \frac{T_p}{2\pi} \sum_{r=-\infty}^{\infty} x(t - rT_p)$$

#### 6.3.1 Sampling Theorem in Frequency Domain

To avoid an overlap of the time-signals:

$$\boxed{T_p \geq 2T_g}$$

as the time-periodical signal is  $2T_g$  wide,  $T_p$  must guarantee that each chunk of signal has enough room to avoid overlapping.

## 7 The Laplace Transformation LT

The Laplace transformation allows some better convergence properties than the Fourier transform. By setting:

$$s = \sigma + j\omega$$

The Laplace transform is given as:

$$\boxed{X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt}$$

$$\boxed{x(t) = \frac{1}{2\pi j} \int_{\sigma-j\infty}^{\sigma+j\infty} X(s)e^{st}ds = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\sigma + j\omega)e^{(\sigma+j\omega)t}d\omega} \quad ^1$$

Note: To solve the inverse transform, one can use the *Residue Theorem* as follows:

$$\frac{1}{2\pi j} \oint_C X(s)e^{st}ds = \sum_{k=1}^K \text{Res}\{X(s), s_k\}$$

Considering that

$$X(s) = X(\sigma + j\omega) = \int_{-\infty}^{\infty} (x(t)e^{-\sigma t})e^{-j\omega t}dt$$

the LT can be interpreted as a FT:

$$L(x(t)) = F(x(t)e^{-\sigma t})$$

<sup>1</sup>The presence of  $j$  here is due to the change of variables from the FT:  $s = \sigma + j\omega$  which with partial derivation gives  $ds = jd\omega \rightarrow d\omega = \frac{ds}{j}$

- $\sigma > 0$ : damping
- $\sigma < 0$ : amplification

In order for the LT to exist and to **converge**, with  $\sigma_0$  providing the limit for the convergence domain:

$$\int_{-\infty}^{\infty} |x(t)|e^{-\sigma_0 t} dt < \infty$$

**IMPORTANT:** If the *Imaginary Axis*  $\in$  *Convergence Domain* then the FT converges also!

## 7.1 The Unilateral Laplace Transformation

This is useful to study the response of systems to a stimulation.

$$X_+(s) = \int_0^{\infty} x(t)e^{-st} dt$$

$$x(t) = \frac{1}{2\pi j} \int_{\sigma-\infty}^{\sigma+\infty} X_+(s)e^{st} ds$$

## 7.2 Linear Differential Equations using the LT

From the differential equation:

$$\sum_{l=0}^N a_l \frac{d^l}{dt^l} y(t) = \sum_{m=0}^M b_m \frac{d^m}{dt^m} x(t)$$

The transfer function of the system can be expressed as:

$$H(s) = \frac{Y(s)}{X(s)} = \frac{\sum_{m=0}^M b_m s^m}{\sum_{l=0}^N a_l s^l}$$

## 7.3 Convergence of the Laplace Transformation

From the condition for the convergence, which provides a value for  $\sigma_0$ :

$$\int_{-\infty}^{\infty} |x(t)|e^{-\sigma_0 t} dt < \infty$$

The Convergence Domain (CD) has following properties:

1. The CD is parallel to the imaginary axis in the s-plane.
2. If the Laplace Transform is rational, then there is no Pole within the CD.
3. If  $x(t)$  is **finite in time**, and there is at least one value of  $s$  that lets the LT converge, then the whole s-plane is a CD.
4. If  $x(t)$  is a **right hand signal**, then all values of  $s$  with  $Re(s) > \sigma_0$  are in the CD too.
5. If  $x(t)$  is a **left hand signal**, then all values of  $s$  with  $Re(s) < \sigma_0$  are in the CD too.
6. If  $x(t)$  is **infinite in time**, then points 4 and 5 apply at the same time.

## 8 The z-Transformation ZT

This transformation is a generalisation of the TDFT. It offers a stronger convergence, is usually simpler to handle with analytic problems and the complex writing is introduced by time discrete signals. The bilateral ZT is given as:

$$z = re^{j\omega}$$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

$$x[n] = \frac{1}{2\pi j} \oint_C X(z)z^{n-1}dz$$

$C$ : Integration way along a closed circuit anti-clockwise within the convergence domain.  $X(z)$  can also be unilateral:

$$X_+(z) = \sum_{n=0}^{\infty} x[n]z^{-n}$$

### 8.1 Convergence of the z-Transformation

The condition for the convergence is:

$$\sum_{n=-\infty}^{\infty} |x[n]r^{-n}| < \infty$$

The convergence properties come from the properties of the Laurent series.

1. The Convergence Domain (CD) is **a ring or a disc around the origin** of the z-plane!  
 $0 \leq R_1 < |z| < R_2 \leq \infty$
2. The TDFT converges only if the CD of the z-Transform contains the unit circle.
3. The CD contains no Pole.
4. If  $x[n]$  is a sequence of finite length, then the whole z-plane is a CD. Excepted maybe for  $z = 0$  and/or  $|z| = \infty$ .
5. If  $x[n]$  is a right hand signal, the CD is the outer part of the circle containing the furthest Pole.
6. If  $x[n]$  is a left hand signal, the CD is the inner part of the circle containing the closest Pole.
7. If  $x[n]$  has no limit for  $n$  (left and right), the CD is a ring free of any Pole, containing the circle  $|z| = r_0$ .
8. The CD is a one piece domain and has no detached sections.

## 8.2 Comparison to the TDFT

For  $z = e^{j\Omega}$  with  $|z| = 1$ , the z-transformation becomes:

$$(ZT) \quad X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} = \sum_{n=-\infty}^{\infty} x[n](e^{j\Omega})^{-n} = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n} = X(\Omega) \quad (\text{TDFT})$$

For  $z = re^{j\Omega}$  with  $|z| = r$ , the ZT can be expressed as a TDFT:

$$X(z) = X(re^{j\Omega}) = \sum_{n=-\infty}^{\infty} (x[n]r^{-n})e^{-j\Omega n}$$

As said above, the TDFT converges only if the CD of the z-Transform contains the unit circle.

## 8.3 Residue Theorem

The computation of the inverse ZT being given as:

$$x[n] = \frac{1}{2\pi j} \oint_C X(z)z^{n-1}dz$$

The complex integral on a closed path can be computed using the **Residue Theorem**:

$$\oint_C f(z)dz = 2\pi j \sum_{k=1}^K \text{Res}\{f(z), z_k\}$$

with  $K$  the number of poles contained in the integration path  $C$ . For each pole  $z_k$ , the residue is calculated as:

$$f(z) = \frac{A(z)}{(z - z_k)^m}$$

$$\text{Res}\{f(z), z_k\} = \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} A(z)|_{z=z_k}$$

with  $k \in [1, K]$ .

In our case  $f(z) = X(z) \cdot z^{n-1}$ .

To calculate Integrals of the form:

$$\int_{-\infty}^{\infty} f(x)e^{i\omega x}dx = \int_{-\infty}^{\infty} \frac{p(x)}{q(x)} e^{i\omega x}dx$$

with  $f(x) = \frac{p(x)}{q(x)}$  where  $q$  and  $p$  are *polynomial* functions so that  $q(x) \neq 0, \forall x \in R$  and  $\deg(q) - \deg(p) \geq 2$ . Then only the residues in the upper part of the complex plane must be included when solving the integral. If the exponential is negative or for negative values of  $\omega$  ( $\omega \leq 0$ ), only the poles of the lower part of the complex plain must be included.

## 8.4 Transition from the LT to the ZT

There are many ways of numerically representing a function. Depending on how a function is represented, the relation between the LT and ZT will change. This transition is important for the implementation of filters, controllers, etc... that were derived in the Laplace domain and that need to be numerically implemented.

### 8.4.1 Integral Substitution

An integration from 0 to  $t_k$  is given by:

$$y(t_k) = \underbrace{\int_0^{t_{k-1}} x(t) dt}_{=y_{k-1}} + \underbrace{\int_{t_{k-1}}^{t_k} x(t) dt}_{=\Delta y_k}$$

**Rectangle Approximation** We express  $\Delta y_k$  as

$$\Delta y_k = x_k h$$

Therefore

$$\begin{aligned} y_k &= y_{k-1} + x_k h \\ Y(z) &= Y(z)z^{-1} + X(z)h \end{aligned}$$

hence

$$\frac{Y(z)}{X(z)} = \frac{h}{1 - z^{-1}} = \frac{hz}{z - 1} \quad \equiv \quad \frac{1}{s} = \frac{Y(s)}{X(s)}$$

Which means we can substitute  $s$  by

$$s \hat{=} \frac{z - 1}{hz}$$

**Trapeze Approximation** Also called *Tustin* or *Bilinear Transform*. We express this time  $\Delta y_k$  as

$$\Delta y_k = \frac{h}{2}(x_{k-1} + x_k)$$

Therefore

$$\begin{aligned} y_k &= y_{k-1} + \frac{h}{2}(x_{k-1} + x_k) \\ Y(z) &= Y(z)z^{-1} + \frac{h}{2}(X(z)z^{-1} + X(z)) \end{aligned}$$

hence

$$\frac{Y(z)}{X(z)} = \frac{h z + 1}{2 z - 1} \quad \equiv \quad \frac{1}{s} = \frac{Y(s)}{X(s)}$$

Which means we can substitute  $s$  by

$$s \hat{=} \frac{2 z - 1}{h z + 1} \quad \Leftrightarrow \quad z \hat{=} \frac{1 + \frac{h}{2}s}{1 - \frac{h}{2}s}$$

## 8.5 Pre-warping of the Bilinear Transform

With the previous result:

$$s \hat{=} \frac{2 z - 1}{h z + 1} \quad \Leftrightarrow \quad z \hat{=} \frac{1 + \frac{h}{2}s}{1 - \frac{h}{2}s}$$

the behaviour of the digital transfer function will differ from the continuous one when the frequency reaches the Nyquist frequency. This is due to the warping of the s-domain into the z-domain.

Considering the frequency only, we can write:

$$z = e^{j\omega_d T} = \frac{1 + j\frac{\omega_a}{2}T}{1 - j\frac{\omega_a}{2}T}$$

which shows that  $\omega_a \neq \omega_d$  (approx. linear for low frequencies):

$$\omega_d = \frac{1}{jT} \ln \left( \frac{1 + j\frac{\omega_a}{2}T}{1 - j\frac{\omega_a}{2}T} \right)$$

Therefore the frequency warping can be expressed as follows:

$$\begin{aligned} H_d(z) &= H_d(e^{j\omega_d T}) = H_a \left( \frac{2}{T} \frac{e^{j\omega_d T} - 1}{e^{j\omega_d T} + 1} \right) \\ &= H_a \left( \frac{2}{T} \cdot \frac{e^{j\omega_d T/2}(e^{j\omega_d T/2} - e^{-j\omega_d T/2})}{e^{j\omega_d T/2}(e^{j\omega_d T/2} + e^{-j\omega_d T/2})} \right) \\ &= H_a \left( j \frac{2}{T} \cdot \frac{\sin(\omega_d T/2)}{\cos(\omega_d T/2)} \right) \\ &= H_a \left( j \frac{2}{T} \cdot \tan(\omega_d T/2) \right) \end{aligned}$$

This shows that every point on the unit circle in the discrete-time filter z-plane,  $z = e^{j\omega T}$  is mapped to a point on the  $j\omega$  axis of the s-plane,  $s = j\omega_a$ . Hence the discrete-time to continuous-time frequency mapping of the bilinear transform:

$$\boxed{\omega_a = \frac{2}{T} \tan \left( \omega_d \frac{T}{2} \right) \Leftrightarrow \omega_b = \frac{2}{T} \tan^{-1} \left( \omega_a \frac{T}{2} \right)}$$

This means the *discrete-time filter* behaves at frequency  $\omega_d$  the same way the *continuous-time filter* behaves at frequency  $\omega_a = \frac{2}{T} \tan \left( \omega_d \frac{T}{2} \right)$ .

Finally, given a specific frequency  $\omega_0$ , we can pre-warp this frequency when applying the Bilinear Transform as follows:

$$\boxed{s \hat{=} \frac{\omega_0}{\tan \left( \frac{\omega_0 T}{2} \right)} \frac{z - 1}{z + 1}}$$

Note that for  $\omega_0 \rightarrow 0$ ,  $s \hat{=} \frac{2}{h} \frac{z-1}{z+1}$ .

## 8.6 Matched Z-transform method

Also called *Zero-Pole Matching Method* is a pretty straight forward method. It is applicable with a few steps:

1. Map the poles and zeros of the s-plane to the z-plane with the transformation  $z = e^{sT}$ . This provides the corresponding new transfer function in the z-domain.
2. Add zeros at infinity, which corresponds to multiplying the previously obtained transfer function by  $(z + 1)^n$  in order to have the same degree between the numerator and denominator of the transfer function.
3. Retrieve zeros to make the function strictly proper ( $\deg(\text{numerator}) < \deg(\text{denominator})$ ).

4. Calculate the DC-gain of both discrete-time and continuous time transfer functions and multiply the discrete-time transfer function by a gain  $k_d$  such as  $\text{DC-gain}(G(s)) = \text{DC-gain}(G(z)) \cdot k_d$ .

The DC-gain is obtained by setting  $s = 0$  and  $z = 1$  respectively.

## 8.7 ZOH Approach (Zero Order Hold)

To apply a discrete signal to a continuous plant, a DA conversion is required. The most commonly used approach is the *Zero Order Hold* method, where a discrete value will be hold until a second command value is emitted. This signal form will affect the discrete system and has to be accounted for. The signal sequence  $v[k]$  goes through following blocks, until the output  $Y(s)$ :

$$v[k] \longrightarrow \text{ZOH} \longrightarrow G(s) \longrightarrow Y(s) \longrightarrow \text{AD} \longrightarrow y[k]$$

We can express the ZOH as a series of steps, as the discrete values are hold:

$$\text{ZOH} = v(t) = \sum_{k=0}^{\infty} v[k]_{\text{hold}} = \sum_{k=0}^{\infty} v[k] \left( u(t - kT) - u(t - (k+1)T) \right)$$

Hence

$$V(s) = \sum_{k=0}^{\infty} v[k] \left( \frac{1}{s} e^{-kTs} - \frac{1}{s} e^{-(k+1)Ts} \right) = \frac{1 - e^{-Ts}}{s} \sum_{k=0}^{\infty} v[k] e^{-kTs}$$

Therefore

$$\begin{aligned} Y(s) &= V(s) \cdot G(s) \\ &= \frac{1 - e^{-Ts}}{s} \sum_{k=0}^{\infty} v[k] e^{-kTs} \cdot G(s) \\ z &= e^{sT} \Rightarrow \frac{1 - z^{-1}}{s} \sum_{k=0}^{\infty} v[k] z^{-k} \cdot G(s) \\ Y(s) &= (1 - z^{-1}) \frac{G(s)}{s} V(z) \end{aligned}$$

Finally

$$H(z) = \frac{Y(z)}{V(z)} = (1 - z^{-1}) \cdot \mathcal{Z} \left( \mathcal{L}^{-1} \left( \frac{G(s)}{s} \right) \right)_{t=kT}$$

## 9 Lists of Properties

### 9.1 Continuous Fourier transformation

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt \quad x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega$$

	$x(t)$	$X(\omega)$
Symmetrie	$x^*(t)$	$X^*(-\omega)$
	$x^*(-t)$	$X^*(\omega)$
Linearität	$ax_1(t) + bx_2(t)$	$aX_1(\omega) + bX_2(\omega)$
Verschiebung (Zeit)	$x(t - \tau)$	$e^{-j\omega\tau}X(\omega)$
(Frequenz)	$e^{j\omega_0 t}x(t)$	$X(\omega - \omega_0)$
Maßstabänderung	$x(at)$	$\frac{1}{ a }X\left(\frac{\omega}{a}\right)$
Differentiation (Zeit)	$\frac{d^n}{dt^n}x(t)$	$(j\omega)^n X(\omega)$
(Frequenz)	$\frac{t^n}{j^n}x(t)$	$\frac{d^n}{d\omega^n}X(\omega)$
Integration (Zeit)	$\int_{-\infty}^t x(\tau)d\tau$	$\frac{1}{j\omega}X(\omega) + \pi X(0)\delta(\omega)$
(Frequenz)	$\frac{j}{t}x(t) + \pi x(0)\delta(t)$	$\int_{-\infty}^{\omega} X(\Omega)d\Omega$
Faltung	$x_1(t) * x_2(t)$	$X_1(\omega)X_2(\omega)$
Modulation	$x_1(t)x_2(t)$	$\frac{1}{2\pi}X_1(\omega) * X_2(\omega)$
	$x(t)\cos(\omega_0 t)$	$\frac{1}{2}(X(\omega + \omega_0) + X(\omega - \omega_0))$
Korrelation	$\int_{-\infty}^{\infty} x_1(\tau)x_2(t + \tau)d\tau$	$X_1^*(\omega)X_2(\omega)$

Figure 1: A list of properties of the Fourier transformation

$x(t)$	$X(\omega)$
$u(t - a) - u(t - b); \quad a < b$	$\frac{e^{-j\omega a} - e^{-j\omega b}}{j\omega}$
$u(t + a) - u(t - a); \quad a > 0$	$2a \frac{\sin(a\omega)}{a\omega} = 2a \text{si}(a\omega)$
$\frac{a}{\pi} \text{ si}(at)$	$u(\omega + a) - u(\omega - a); \quad a > 0$
$e^{-bt}u(t); \quad b > 0$	$\frac{1}{b + j\omega}$
$e^{-b t }; \quad b > 0$	$\frac{2b}{b^2 + \omega^2}$
$\frac{1}{b^2 + t^2}$	$\frac{\pi}{b} e^{-b \omega }; \quad b > 0$
$\frac{1}{\sqrt{2\pi}} e^{\frac{-t^2}{2}}$	$e^{\frac{-\omega^2}{2}}$
1	$2\pi\delta(\omega)$
$\delta(t)$	1
$\int_{-\infty}^{\infty} e^{j\omega_0 t} dt = 2\pi \delta(\omega - \omega_0) \leftrightarrow e^{j\omega_0 t}$	$2\pi\delta(\omega - \omega_0) \longleftrightarrow \int_{-\infty}^{\infty} \delta(\omega - \omega_0) e^{j\omega_0 t} dt$
$\cos(\omega_0 t)$	$\pi(\delta(\omega + \omega_0) + \delta(\omega - \omega_0))$
$\sin(\omega_0 t)$	$j\pi(\delta(\omega + \omega_0) - \delta(\omega - \omega_0))$
$\sum_{n=-\infty}^{\infty} \delta(t - nT)$	$\frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_0); \quad \omega_0 = \frac{2\pi}{T}$
$\delta(t - T)$	$e^{-j\omega T}$
$u(t)$	$\frac{1}{j\omega} + \pi\delta(\omega)$
$\sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$	$2\pi \sum_{k=-\infty}^{\infty} c_k \delta(\omega - k\omega_0)$

Figure 2: A list of known Fourier transformations

## 9.2 Time Discrete Fourier transformation

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n} \quad x[n] = \frac{1}{2\pi} \int_{2\pi} X(\Omega)e^{j\Omega n}d\Omega$$

	$x[n]$	$\mathcal{F}$	$X(\Omega)$
Symmetrie	$x[-n]$		$X(-\Omega) = X^*(\Omega)$
	$x^*[n]$		$X^*(-\Omega) = X(\Omega)$ wenn $x[n]$ = reell
Linearität	$ax_1[n] + bx_2[n]$		$aX_1(\Omega) + bX_2(\Omega)$
Verschiebung (Zeit)	$x[n - n_0]$		$e^{-j\Omega n_0} X(\Omega)$
(Frequenz)	$e^{j\Omega_0 n} x[n]$		$X(\Omega - \Omega_0)$
Maßstabsänderung	$x\left[\frac{n}{a}\right]$		$X(a\Omega)$
	mit $\left(\frac{n}{a}\right) \in \mathbb{Z}$		
Differenz	$x[n] - x[n - 1]$		$(1 - e^{-j\Omega})X(\Omega)$
Superposition	$\sum_{m=-\infty}^n x[m]$		$\frac{1}{1 - e^{-j\Omega}} X(\Omega) + \pi X(0) \sum_{r=-\infty}^{\infty} \delta(\Omega - 2\pi r)$
Differentiation (Frequenz)	$j^{-k} n^k x[n]$		$\frac{d^k}{d\Omega^k} X(\Omega)$
Faltung	$x_1[n] * x_2[n] =$		$X_1(\Omega)X_2(\Omega)$
	$\sum_{l=-\infty}^{\infty} x_1[l]x_2[n-l]$	<i>↓ Faltung</i>	
Modulation	$x_1[n]x_2[n]$		$\frac{1}{2\pi} \int_{2\pi} X_1(\Omega')X_2(\Omega - \Omega')d\Omega'$
Korrelation	$\sum_{l=-\infty}^{\infty} x_1[l]x_2[n+l]$		$X_1^*(\Omega)X_2(\Omega)$

Figure 3: A list of properties of the Time Discrete Fourier transformation

Sequenz: $x[n]$	ZDFT: $X(\Omega)$
$\delta[n]$	1
$\delta[n - n_0]$	$e^{-j\Omega n_0}$
$\sum_{k=-\infty}^{\infty} \delta[n - kN]$	$\frac{2\pi}{N} \sum_{k=-\infty}^{\infty} \delta\left(\Omega - \frac{2\pi k}{N}\right)$
$x[n] = 1$	$2\pi \sum_{r=-\infty}^{\infty} \delta(\Omega - 2\pi r)$
$u[n]$	$\frac{1}{1 - e^{-j\Omega}} + \pi \sum_{r=-\infty}^{\infty} \delta(\Omega - 2\pi r)$
$a^n u[n] \quad  a  < 1$	$\frac{1}{1 - ae^{-j\Omega}}$
$(n+1)a^n u[n] \quad  a  < 1$	$\frac{1}{(1 - ae^{-j\Omega})^2}$
$\frac{(n+q-1)!}{n!(q-1)!} a^n u[n] \quad  a  < 1$	$\frac{1}{(1 - ae^{-j\Omega})^q}$
$e^{j\Omega_0 n}$	$2\pi \sum_{r=-\infty}^{\infty} \delta(\Omega - \Omega_0 - 2\pi r)$
$\cos[\Omega_0 n + \phi]$	$\pi \sum_{r=-\infty}^{\infty} [e^{-j\phi} \delta(\Omega + \Omega_0 - 2\pi r) + e^{j\phi} \delta(\Omega - \Omega_0 - 2\pi r)]$
$\sin[\Omega_0 n + \phi]$	$j\pi \sum_{r=-\infty}^{\infty} [e^{-j\phi} \delta(\Omega + \Omega_0 - 2\pi r) - e^{j\phi} \delta(\Omega - \Omega_0 - 2\pi r)]$
$\frac{\sin \Omega_c n}{\pi n} = \frac{\Omega_c}{\pi} \text{si}[\Omega_c n]$	$X(\Omega) \begin{cases} 1 & \text{für }  \Omega  < \Omega_c \\ 0 & \text{für } \Omega_c <  \Omega  \leq \pi \end{cases}$
$x[n] = \begin{cases} 1 & \text{für } 0 \leq n \leq M \\ 0 & \text{sonst} \end{cases}$	$\frac{\sin\left(\Omega \frac{M+1}{2}\right)}{\sin\left(\frac{\Omega}{2}\right)} e^{-j\frac{\Omega M}{2}}$

Figure 4: A list of known Time Discrete Fourier transformations

### 9.3 Laplace transformation

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt \quad x(t) = \frac{1}{2\pi j} \int_{\sigma-\infty}^{\sigma+\infty} X(s)e^{st}ds$$

	$x(t)$	$X(s)$	$R$
Linearität	$ax_1(t) + bx_2(t)$	$aX_1(s) + bX_2(s)$	$R_1 \cap R_2$
Zeitverschiebung	$x(t - t_0)$	$e^{-st_0} X(s)$	$R$
Frequenzverschiebung	$e^{s_0 t} x(t)$	$X(s - s_0)$	$R + \text{Re}(s_0)$
Maßstabsänderung	$x(at)$	$\frac{1}{ a } X\left(\frac{s}{a}\right)$	$R \cdot a$
Faltung	$x_1(t) * x_2(t)$	$X_1(s)X_2(s)$	$R_1 \cap R_2$
Differentiation	$\frac{d}{dt} x(t)$	$sX(s)$	$R$
	$-t x(t)$	$\frac{d}{ds} X(s)$	$R$
Integration	$\int_{-\infty}^t x(\tau)d\tau$	$\frac{1}{s} X(s)$	$R \cap \{\text{Re}(s) > 0\}$
Impulsübertragung	$y(t) = x(t) * h(t)$	$Y(s) = X(s)H(s)$	

Figure 5: A list of properties of the Laplace transformation

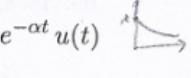
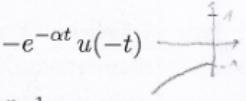
$x(t)$	$X(s)$	Bereich $R$
$\delta(t)$	1	alle s
$u(t)$	$\frac{1}{s}$	$\text{Re}(s) > 0$
$-u(-t)$	$\frac{1}{s}$	$\text{Re}(s) < 0$
$\frac{t^{n-1}}{(n-1)!} u(t)$	$\frac{1}{s^n}$	$\text{Re}(s) > 0$
$-\frac{t^{n-1}}{(n-1)!} u(-t)$	$\frac{1}{s^n}$	$\text{Re}(s) < 0$
$e^{-\alpha t} u(t)$ 	$\frac{1}{s + \alpha}$	$\text{Re}(s) > -\alpha$
$-e^{-\alpha t} u(-t)$ 	$\frac{1}{s + \alpha}$	$\text{Re}(s) < -\alpha$
$\frac{t^{n-1}}{(n-1)!} e^{-\alpha t} u(t)$	$\frac{1}{(s + \alpha)^n}$	$\text{Re}(s) > -\alpha$
$-\frac{t^{n-1}}{(n-1)!} e^{-\alpha t} u(-t)$	$\frac{1}{(s + \alpha)^n}$	$\text{Re}(s) < -\alpha$
$\delta(t - T)$	$e^{-sT}$	alle s
$\cos(\omega_0 t) u(t)$	$\frac{s}{s^2 + \omega_0^2}$	$\text{Re}(s) > 0$
$\sin(\omega_0 t) u(t)$	$\frac{\omega_0}{s^2 + \omega_0^2}$	$\text{Re}(s) > 0$
$e^{-\alpha t} \cos(\omega_0 t) u(t)$	$\frac{s + \alpha}{(s + \alpha)^2 + \omega_0^2}$	$\text{Re}(s) > -\alpha$
$e^{-\alpha t} \sin(\omega_0 t) u(t)$	$\frac{\omega_0}{(s + \alpha)^2 + \omega_0^2}$	$\text{Re}(s) > -\alpha$

Figure 6: A list of known Laplace transformations

## 9.4 Z-transformation

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} \quad x[n] = \frac{1}{2\pi j} \oint_C X(z)z^{n-1}dz$$

	$x[n]$	$X(z)$	$R$
Linearität	$ax_1[n] + bx_2[n]$	$aX_1(z) + bX_2(z)$	mindestens $R_1 \cap R_2$
Verschiebung auf der n-Achse	$x[n - n_0]$	$z^{-n_0}X(z)$	$R_x$ , evtl. zuzüglich oder abzüglich $z = 0, z = \infty$
Verschiebung im z-Bereich	$a^n x[n]$	$X\left(\frac{z}{a}\right)$	$ a R_x$
	$e^{j\Omega_0 n} x[n]$	$X(e^{-j\Omega_0}z)$	$R_x$
Differentiation nach $z$	$n x[n]$	$-z \frac{d}{dz} X(z)$	$R_x$ , evtl. zuzüglich oder abzüglich $z = 0, z = \infty$
Faltung	$x_1[n] * x_2[n]$	$X_1(z)X_2(z)$	enthält $R_1 \cap R_2$
Modulation	$x_1[n]x_2[n]$	$\frac{1}{2\pi j} \oint_C X_1(\xi)X_2\left(\frac{z}{\xi}\right) \xi^{-1} d\xi$	enthält $R_1 \cap R_2$
Superposition	$\sum_{m=-\infty}^n x[m]$	$\frac{1}{1-z^{-1}}X(z)$	mindestens $R_x \cap  z  > 1$
Symmetrie	$x^*[n]$	$X^*(z^*)$	$R_x$
	$x[-n]$	$X(z^{-1})$	$R_x^{-1}$
	$\text{Re}\{x[n]\}$	$\frac{1}{2}(X(z) + X^*(z^*))$	enthält $R_x$
	$\text{Im}\{x[n]\}$	$\frac{1}{2j}(X(z) - X^*(z^*))$	enthält $R_x$
Anfangswert	wenn $x[n] = 0$ für $n < 0$	dann $X(0) = \lim_{z \rightarrow \infty} X(z)$	$R_x$

Figure 7: A list of properties of the z-transformation

$x[n]$	$X(z)$	$R$
$\delta[n]$	1	alle $z$
$\delta[n - m]$	$z^{-m}$	alle $z$ außer: $z = 0$ für $m > 0$ $z = \infty$ für $m < 0$
$u[n]$	$\frac{1}{1 - z^{-1}}$	$ z  > 1$
$-u[-n - 1]$	$\frac{1}{1 - z^{-1}}$	$ z  < 1$
$a^n u[n]$	$\frac{1}{1 - az^{-1}}$	$ z  >  a $
$-a^n u[-n - 1]$	$\frac{1}{1 - az^{-1}}$	$ z  <  a $
$na^n u[n]$	$\frac{az^{-1}}{(1 - az^{-1})^2}$	$ z  >  a $
$-na^n u[-n - 1]$	$\frac{az^{-1}}{(1 - az^{-1})^2}$	$ z  <  a $
$\cos[\Omega_0 n] u[n]$	$\frac{1 - (\cos \Omega_0)z^{-1}}{1 - 2(\cos \Omega_0)z^{-1} + z^{-2}}$	$ z  > 1$
$\sin[\Omega_0 n] u[n]$	$\frac{(\sin \Omega_0)z^{-1}}{1 - 2(\cos \Omega_0)z^{-1} + z^{-2}}$	$ z  > 1$
$r^n \cos[\Omega_0 n] u[n]$	$\frac{1 - r(\cos \Omega_0)z^{-1}}{1 - 2r(\cos \Omega_0)z^{-1} + r^2 z^{-2}}$	$ z  > r$
$r^n \sin[\Omega_0 n] u[n]$	$\frac{r(\sin \Omega_0)z^{-1}}{1 - 2r(\cos \Omega_0)z^{-1} + r^2 z^{-2}}$	$ z  > r$
$x[n] = \begin{cases} a^n & 0 \leq n \leq N-1 \\ 0 & \text{sonst} \end{cases}$	$\frac{1 - a^N z^{-N}}{1 - az^{-1}}$	$ z  > 0$

Figure 8: A list of known z-transformations

## 10 Complement

### 10.1 Solving Differential Equations

Tableau I : équations différentielles du 1er ordre :  $y' = \phi(x, y)$ 

Type d'équation	Méthode(s) de résolution
I.1 séparable: $y' = f(x)g(y)$	separation et intégration $\rightarrow \int \frac{dy}{g(y)} = \int f(x) dx$
I.2 homogène (de degré 0) en $x$ et $y$ exemple: $y' = \sqrt{1 - \frac{x^2}{xy+y^2}}$	on pose $y = ux \implies y' = u'x + u \rightarrow$ équation séparable $\rightarrow$ I.1
I.3 $y' = \phi\left(\frac{ax+by+c}{dx+ey+f}\right)$	on pose $\begin{cases} x = \xi + \alpha \\ y = u + \beta \end{cases}$ avec $\begin{cases} a\alpha + b\beta + c = 0 \\ d\alpha + e\beta + f = 0 \end{cases} \rightarrow$ équation homogène en $\xi$ et $u \rightarrow$ I.2
I.4 linéaire $y' + a(x)y = b(x)$	solution : $y_h(x) = Cw(x)$ avec $w(x) = e^{-A(x)}$ et $A(x)$ une primitive de $a(x)$
	A : Méthode des essais B : Var. des cstes : $y_P(x) = c(x)w(x)$ avec $c(x) = \int \frac{b(x)}{w(x)} dx$
I.5 Equation de Bernoulli $y' = f(x)y + g(x)y^n \quad n \neq 1$	on pose $u = y^{1-n} \implies u' = (1-n)y' \cdot y^{-n} \rightarrow$ équation linéaire en $u$ : $u' + (n-1)f(x)u = (1-n)g(x) \rightarrow$ I.4
I.6 Équation de Riccati $y' = g(x)y^2 + f(x)y + h(x)$	on pose $y = -\frac{1}{g(x)} \frac{u'}{u} \rightarrow$ équation linéaire homogène du 2ème ordre : $u'' - \left(\frac{g'}{g} + f\right)u' + ghu = 0 \rightarrow$ tableau II

Figure 9: Methods for solving first order differential equations.

Tableau II : équations différentielles du 2ème ordre :  $y'' = \phi(x, y; y')$ 

Type d'équation	Méthode(s) de résolution
II.1 $\phi$ ne contient pas de $y$	on pose $u = y' \Rightarrow u' = y'' \rightarrow$ équation du 1er ordre en $u \rightarrow$ tableau I
II.2 autonome; $\phi$ ne contient pas de $x$	on pose $\begin{cases} y' = z(y) \\ \Rightarrow y'' = z \cdot z' \end{cases} \rightarrow$ éq. lin. du 1er ordre en $z(y) \rightarrow$ I.4. Lorsque $z = z(y)$ est trouvé, on résout l'équa. diff. du 1er ordre: $y' = z(y)$
homogène	pas de solution générale. Mais si $y_1(x)$ est trouvé on cherche $y_2(x)$ sous la forme $y_2(x) = u(x)y_1(x) \rightarrow$ éq. en $u', u'' \rightarrow$ éq. en $v', v$ du 1er ordre.
II.3 homogène à coefficients constants: $y'' + py' + qy = 0$ $p, q \in \mathbb{R}$	Polynôme caractéristique : $P(\lambda) = \lambda^2 + p\lambda + q = 0$ . 3 cas possibles: (1) $\lambda_1 \neq \lambda_2 \in \mathbb{R} \Rightarrow y_1 = e^{\lambda_1 x}, y_2 = e^{\lambda_2 x}$ (2) $\lambda_1 = \lambda_2 = \lambda \in \mathbb{R} \Rightarrow y_1 = e^{\lambda x}, y_2 = x \cdot e^{\lambda x}$ (3) $\lambda_{1,2} = \mu \pm \omega_0 i \in \mathbb{C} \Rightarrow y_1 = e^{\mu x} \cos(\omega_0 x), y_2 = e^{\mu x} \sin(\omega_0 x)$
éq. d'Euler homogène: $x^2y'' + \alpha xy' + \beta y = 0$ $\alpha, \beta \in \mathbb{R}$	Polynôme caractéristique : $r^2 + (\alpha - 1)r + \beta = 0$ . 3 cas possibles: (1) $r_1 \neq r_2 \in \mathbb{R} \Rightarrow y_1 = x^{r_1}, y_2 = x^{r_2}$ (2) $r_1 = r_2 = r \in \mathbb{R} \Rightarrow y_1 = x^r, y_2 = x^r \ln x$ (3) $r_{1,2} = a \pm bi \in \mathbb{C} \Rightarrow y_1 = x^a \cos(b \ln x), y_2 = x^a \sin(b \ln x)$
sol. part. de l'éq. inhomogène	A : Méthode des essais B : Méthode de var. des constantes : $y_P(x) = \varphi_1(x)y_1(x) + \varphi_2(x)y_2(x)$ $W = \det \begin{pmatrix} y_1 & y_2 \\ y'_1 & y'_2 \end{pmatrix}, \begin{pmatrix} \varphi'_1 \\ \varphi'_2 \end{pmatrix} = \frac{1}{W} \begin{pmatrix} -b(x) \cdot y_2 \\ b(x) \cdot y_1 \end{pmatrix} \xrightarrow{f} \varphi_1, \varphi_2$

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Figure 10: Methods for solving second order differential equations.