Quaternions Theory

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1 Quaternion Manipulations

A quaternion is given as

$$q = (w, x, y, z) \longleftrightarrow w + xi + yj + zk$$

where w is the *scalar* and (x, y, z) is the *vector*. The following rule generates all properties of the quaternions:

$$\boxed{i^2 = j^2 = k^2 = ijk = -1}$$

which implies following multiplication properties:

$\cdot \vec{}$	i	j	k
i	-1	k	-j
j	-k	-1	i
k	j	-i	-1

1.1 Addition

$$q_1 + q_2 = (a, b, c, d) + (e, f, g, h) = (a + e, b + f, c + g, d + h)$$

1.2 Multiplication

$$q_1 = a + bi + cj + dk$$
$$q_2 = e + fi + gj + hk$$

The multiplication is then:

$$\begin{aligned} q_1 q_2 = ae + afi + agj + ahk + \\ bei + bfi^2 + bgij + bhik + \\ cej + cfji + cgj^2 + chjk + \\ dek + dfki + dgkj + dhk^2 \end{aligned}$$

Which results in:

$$\boxed{\begin{aligned} q_1 q_2 = (ae - bf - cg - dh, \\ af + be + ch - dg, \\ ag - bh + ce + df, \\ ah + bg - cf + de) \end{aligned}}$$

Note that $q_1 q_2 \neq q_2 q_1$.

1.2.1 Quaternions seen as Matrices

We can rearrange the multiplication like this:

$$\begin{aligned} q_1 q_2 = (a, b, c, d)(e, f, g, h) = (ae - bf - cg - dh, \\ be + af - dg + ch, \\ ce + df + ag - bh, \\ de - cf + bg + ah) \end{aligned}$$

Which we rewrite as:

$$q_1 q_2 = \begin{bmatrix} a & -b & -c & -d \\ b & a & -d & c \\ c & d & a & -b \\ d & -c & b & a \end{bmatrix} \cdot \begin{bmatrix} e \\ f \\ g \\ h \end{bmatrix}$$

This can be applied to:

$$1 = (1, 0, 0, 0) \longrightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = 1$$

$$i = (0, 1, 0, 0) \longrightarrow \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} = i$$

$$j = (0, 0, 1, 0) \longrightarrow \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} = j$$

$$k = (0, 0, 0, 1) \longrightarrow \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} = k$$

1.3 Extracting the Dot and Cross Products

Let us consider the multiplication with the following notation:

$$q_1 q_2 = (w_1, \vec{v_1})(w_2, \vec{v_2})$$

If we compare it to the first notation in section 1.2.1, we observe that:

$$\boxed{q_1 q_2 = (w_1 w_2 - \vec{v_1} \cdot \vec{v_2}, w_2 \vec{v_1} + w_1 \vec{v_2} + \vec{v_1} \wedge \vec{v_2})}$$

Let us build the following quaternions:

$$v_1 v_2 = (0, \vec{v}_1)(0, \vec{v}_2) = (-\vec{v}_1 \cdot \vec{v}_2, \vec{v}_1 \wedge \vec{v}_2)$$

We can see that:

$$\begin{aligned} v_1 v_2 &= (-\vec{v}_1 \cdot \vec{v}_2, \vec{v}_1 \wedge \vec{v}_2) \\ v_2 v_1 &= (-\vec{v}_1 \cdot \vec{v}_2, -\vec{v}_1 \wedge \vec{v}_2) \end{aligned}$$

1.3.1 Dot Product

$$v_1 v_2 + v_2 v_1 = (-2\vec{v}_1 \cdot \vec{v}_2, \vec{0})$$

hence

$$\boxed{\vec{v}_1 \cdot \vec{v}_2 = -\frac{1}{2}(v_1 v_2 + v_2 v_1)}$$

1.3.2 Cross Product

$$v_1 v_2 - v_2 v_1 = (0, 2\,\vec{v}_1 \wedge \vec{v}_2)$$

hence

$$\boxed{\vec{v}_1 \wedge \vec{v}_2 = \frac{1}{2}(v_1 v_2 - v_2 v_1)}$$

2 Quaternion Exponentials

From

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + ...$$

we can plug $q = (0, \vec{v})$:

$$\begin{aligned} e^q &= 1 + q + \frac{q^2}{2!} + \frac{q^3}{3!} + \frac{q^4}{4!} + ... \\ &= 1 + \vec{v} - \frac{\|\vec{v}\|^2}{2!} - \frac{\|\vec{v}\|^2 \vec{v}}{3!} + \frac{\|\vec{v}\|^4}{4!} + \frac{\|\vec{v}\|^4 \vec{v}}{5!} - \frac{\|\vec{v}\|^6}{6!} - \frac{\|\vec{v}\|^6 \vec{v}}{7!} + ... \end{aligned}$$

The even powers are pure scalar and odd powers are pure vectors. By searching for a function whose Taylor series equals the scalar part and another that corresponds to the vector part, we end up finding that:

$$e^q = e^{ai+bj+ck} = \cos(\|\vec{v}\|) + \frac{\sin(\|\vec{v}\|)}{\|\vec{v}\|}(ai + bj + ck)$$

Generalizing for quaternions with non-zero scalar part:

$$\boxed{e^q = e^{a+bi+cj+dk} = e^a \left(\cos(\|\vec{v}\|) + \frac{\sin(\|\vec{v}\|)}{\|\vec{v}\|}(bi + cj + dk) \right)}$$

Note therefore that $e^{i+j} \neq e^i e^j$.

3 3D Rotations

3.1 Special Case

In the first case, we will consider the rotation of a vector $\vec{v}$ perpendicular to a normal rotation vector $\hat{n}$ ($\|\hat{n}\| = 1$). This case will help construct any rotation afterwards. We build as well another vector $\hat{n} \wedge \vec{v}$ ($\|\hat{n} \wedge \vec{v}\| = \|\vec{v}\|$) perpendicular to $\hat{n}$, lying in the same plane as $\vec{v}$.

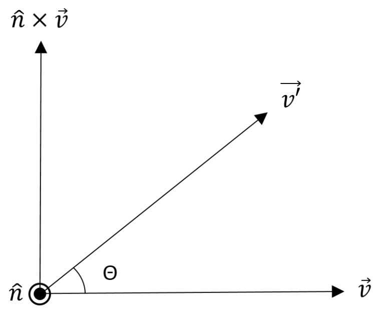

Figure 1: Perpendicular Rotation

Therefore, we can express $\vec{v}\,'$ as referred in figure 1:

$$\boxed{\vec{v}\,' = \cos(\theta)\vec{v} + \sin(\theta)(\hat{n} \wedge \vec{v})}$$

Let us write:

$$v = (0, \vec{v}),\ v' = (0, \vec{v}\,'),\ n = (0, \hat{n})$$
$$nv = (-\hat{n} \cdot \vec{v}, \hat{n} \wedge \vec{v}) = \hat{n} \wedge \vec{v}$$
$$n^2 = -1$$

Therefore, in quaternion notation, the equation becomes:

$$v' = \cos(\theta)v + \sin(\theta)nv = (\cos(\theta) + \sin(\theta)n)\, v$$

by rewriting this result as a quaternion exponential with the vector part equal to $\theta\hat{n}$:

$$\boxed{v' = e^{\theta n}v = ve^{-\theta n}}$$

So we can extract an important result for **normal** vectors $\hat{n}$, respectively quaternions $n = (0, \hat{n})$:

$$\boxed{e^{\theta n} = \cos(\theta) + \sin(\theta)\hat{n} = \cos(\theta) + \sin(\theta)(n_x i + n_y j + n_z k)}$$

3.2 General Case

If $\vec{v}$ is no longer perpendicular to $\hat{n}$, $\vec{v}$ can be decomposed into:

$$\vec{v} = \vec{v}_{\|} + \vec{v}_{\perp}$$

where after a rotation, the only change is:

$$\vec{v}\,' = \vec{v}_{\|} + \vec{v}_{\perp}\,'$$

3.2.1 3D Rotation with Vectors

Therefore, from the special case:

$$\vec{v}\,' = \vec{v}_{\|} + \cos(\theta)\vec{v}_{\perp} + \sin(\theta)(\hat{n} \wedge \vec{v}_{\perp})$$

with

$$\hat{n} \wedge \vec{v} = \hat{n} \wedge (\vec{v}_{\|} + \vec{v}_{\perp}) = \hat{n} \wedge \vec{v}_{\perp}$$
$$\vec{v}_{\perp} = \vec{v} - \vec{v}_{\|}$$

we can rewrite it as a function of $\vec{v}$ and $\vec{v}_{\|}$:

$$\vec{v}\,' = (1 - \cos(\theta))\vec{v}_{\|} + \cos(\theta)\vec{v} + sin(\theta)(\hat{n} \wedge \vec{v})$$

With $\vec{v}_{\|} = (\vec{v}\cdot\hat{n})\hat{n}$, we obtain the *Rodrigez Formula* for rotating any vector around any axis:

$$\boxed{\vec{v}\,' = (1 - \cos(\theta))(\vec{v} \cdot \hat{n})\hat{n} + \cos(\theta)\vec{v} + \sin(\theta)(\hat{n} \wedge \vec{v})}$$

3.2.2 3D Rotations with Quaternions

By taking the notation:

$$v = (0, \vec{v})$$

From the special case, we can rewrite the general case as:

$$v' = v_{\parallel} + v'_{\perp} = v_{\parallel} + e^{\theta n} v_{\perp}$$

We need to observe the following properties (confirmed by applying the multiplication rule):

$$e^{\theta n} v_{\perp} = v_{\perp} e^{-\theta n}$$
$$e^{\theta n} v_{\parallel} = v_{\parallel} e^{\theta n}$$

Keeping these properties in mind, we can write:

$$\begin{aligned}
v' &= v_{\parallel} + e^{\theta n} v_{\perp} \\
&= e^{\frac{\theta}{2} n} e^{-\frac{\theta}{2} n} v_{\parallel} + e^{\frac{\theta}{2} n} e^{\frac{\theta}{2} n} v_{\perp} \\
&= e^{\frac{\theta}{2} n} v_{\parallel} e^{-\frac{\theta}{2} n} + e^{\frac{\theta}{2} n} v_{\perp} e^{-\frac{\theta}{2} n} \\
&= e^{\frac{\theta}{2} n} (v_{\parallel} + v_{\perp}) e^{-\frac{\theta}{2} n} \\
&= e^{\frac{\theta}{2} n} v e^{-\frac{\theta}{2} n}
\end{aligned}$$

The compact form for rotation with quaternions is then given by:

$$\boxed{v' = e^{\frac{\theta}{2} n} v e^{-\frac{\theta}{2} n}}$$

$$\begin{aligned}
q = e^{\frac{\theta}{2} n} \quad &= \cos\left(\frac{\theta}{2}\right) + \sin\left(\frac{\theta}{2}\right)(n_x i + n_y j + n_z k) \\
q^* = e^{-\frac{\theta}{2} n} \quad &= \cos\left(\frac{\theta}{2}\right) - \sin\left(\frac{\theta}{2}\right)(n_x i + n_y j + n_z k)
\end{aligned}$$

$$\boxed{v' = q v q^*}$$

4 Summary

$$\boxed{i^2 = j^2 = k^2 = ijk = -1}$$

$\cdot$ ↱	i	j	k
i	-1	k	-j
j	-k	-1	i
k	j	-i	-1

$$\boxed{q_1 q_2 = (w_1 w_2 - \vec{v_1} \cdot \vec{v_2}, w_2 \vec{v_1} + w_1 \vec{v_2} + \vec{v_1} \wedge \vec{v_2})}$$

$$\boxed{\vec{v}_1 \cdot \vec{v}_2 = -\frac{1}{2}(v_1 v_2 + v_2 v_1)}$$

$$\boxed{\vec{v}_1 \wedge \vec{v}_2 = \frac{1}{2}(v_1 v_2 - v_2 v_1)}$$

$$\boxed{e^q = e^{a+bi+cj+dk} = e^a \left(\cos(\|\vec{v}\|) + \frac{\sin(\|\vec{v}\|)}{\|\vec{v}\|}(bi + cj + dk) \right)}$$

$$\boxed{e^{\theta n} = \cos(\theta) + \sin(\theta)\hat{n} = \cos(\theta) + \sin(\theta)(n_x i + n_y j + n_z k)}$$

$$\boxed{\vec{v}\,' = (1 - \cos(\theta))(\vec{v} \cdot \hat{n})\hat{n} + \cos(\theta)\vec{v} + \sin(\theta)(\hat{n} \wedge \vec{v})}$$

$$\boxed{v' = e^{\frac{\theta}{2} n} v e^{-\frac{\theta}{2} n}}$$

$$\boxed{v' = q v q^*}$$