Fourier, Laplace and Z transformations

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Contents

1 Convolution Product

For 2 functions $x(t)$ and $h(t)$ (usually denoting the impulse response of a LTI system), the **convolution product** is defined as:

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

And in case of time discrete convolution:

$$y[n] = x[n] * h[n] = \sum_{l=-\infty}^{\infty} x[l]h[n-l]$$

The product **commutes**, **associates** and **distributes**. By knowing the impulse response $h(t)$ of any LTI system, providing an input $x(t)$ to the system will produce an output $y(t) = x(t) * h(t)$. This can avoid solving a differential equation for this input.

2 Correlation

For energy limited signals, given by the condition:

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = E_x < \infty, \text{ or for discret signals } \sum_{l=-\infty}^{\infty} |x[l]|^2 = E_x < \infty$$

The correlation is calculated as

$$\varphi_{\mu\nu}(t) = \int_{-\infty}^{\infty} x_\mu(\tau)x_\nu(t+\tau)d\tau$$

If $\mu = \nu$, we speak of auto-correlation, while if $\mu \neq \nu$, we speak of the cross-correlation of the two signals. In case the signal is periodic, the integration happens on the interval of the signal, as follows:

$$\tilde{\varphi}_{\mu\nu}(t) = \frac{1}{T_0}\int_{T_0} x_\mu(\tau)x_\nu(t+\tau)d\tau$$

If the signals are discret, sum from 0 to N-1 and divide the sum by N.

3 Fourier Series

Any periodic signal, limited and continuously differentiable in the interval of length T_0 can be written as a Fourier series:

$$\boxed{\tilde{x}(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}}\ , \ \omega_0 = \frac{2\pi}{T_0}$$

$$\boxed{c_k = \frac{1}{T_0}\int_{T_0} \tilde{x}(t)e^{-j\omega_0 t}dt}$$

In case of real periodic signals $\tilde{x}(t)$, $\boxed{c_{-k} = c_k^*}$

3.1 Decomposition of the Fourier Series

The coefficients c_k can be rewritten as:

$$c_k = \begin{cases} \frac{a_k - jb_k}{2} & \text{pour } k \in N \\ a_0 & \text{pour } k = 0 \end{cases} \quad \text{avec } a_{-k} = a_k \text{ et } b_{-k} = b_k$$

Therefore the series can take the following form in the case of real periodic signals:

$$\boxed{\tilde{x}(t) = a_0 + \sum_{k=1}^{\infty} a_k \cos(k\omega_0 t) + \sum_{k=1}^{\infty} b_k \sin(k\omega_0 t)}$$

$$a_0 = \frac{1}{T_0}\int_{T_0} \tilde{x}(t)dt \ , \ a_k = \frac{2}{T_0}\int_{T_0} \tilde{x}(t)\cos(k\omega_0 t)dt \ , \ b_k = \frac{2}{T_0}\int_{T_0} \tilde{x}(t)\sin(k\omega_0 t)dt$$

It is important to see that for even functions $\tilde{x}(t)$ all $b_k = 0, \forall k \geq 1$, while for uneven functions, all $a_k = 0, \forall k \geq 0$.

3.2 Parseval Theorem of the Fourier Series

$$\boxed{\frac{1}{T_0}\int_{T_0} |\tilde{x}(t)|^2 dt = \sum_{k=-\infty}^{\infty} |c_k|^2 = |c_0|^2 + 2\sum_{k=1}^{\infty} |c_k|^2}$$

This result can be proved as follows:

$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$$

$$\tilde{x}^*(t) = \sum_{k'=-\infty}^{\infty} c_{k'}^* e^{-jk'\omega_0 t}$$

$$P = \frac{1}{T_0}\int_{T_0} |\tilde{x}(t)|^2 dt = \frac{1}{T_0}\int_{T_0} \tilde{x}(t)\tilde{x}^*(t)dt = \frac{1}{T_0}\sum_{k=-\infty}^{\infty}\sum_{k'=-\infty}^{\infty} \underbrace{c_k c_{k'}^*}_{=|c_k^2|} \underbrace{\int_{T_0} e^{j(k-k')\omega_0 t}dt}_{T_0 \cdot \delta(k-k')}$$

As $k = k'$ because of the Dirac function $\delta(k-k')$ that is equal to 0 if $k \neq k'$. We then find that:

$$P = \sum_{k=-\infty}^{\infty} |c_k|^2 \qquad \square$$

The result of the integral ($\int_{T_0} e^{j(k-k')\omega_0 t}dt$) will be discussed later on, when working on the Fourier transformation.

3.3 Time Shift of the Fourier Signal

A time shift of the signal will result in a phase shift of the coefficient c_k of the series.

$$\tilde{x}(t-\tau) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0(t-\tau)} = \sum_{k=-\infty}^{\infty} c_k e^{-jk\omega_0\tau} e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} c_k(\tau) e^{jk\omega_0 t}$$

Hence the phase shift is equal to $-k\omega_0\tau$.

3.4 Periodic Convolution

For two periodic signals $\tilde{x}_1(t)$ and $\tilde{x}_2(t)$, we define on a period length (because of convergence issues) the *periodic convolution*:

$$\tilde{y}(t) = \tilde{x}_1(t) * \tilde{x}_2(t) = \int_{T_0} \tilde{x}_1(\tau)\tilde{x}_2(t-\tau)d\tau$$

The the coefficients of the Fourier series follow the rule:

$$\boxed{c_{yk} = T_0 \cdot c_{1k} \cdot c_{2k}}$$

4 The Fourier Transformation FT

The advantage of the Fourier transformation is to enable us to transform not periodical signals. The Fourier series can only work for periodic signals!

$$\boxed{X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt}$$

$$\boxed{x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega}$$

The condition for the existence of the transformation is that $x(t)$ is differentiable on a finite number of places and that it is absolutely integrable:

$$\int_{-\infty}^{\infty} |x(t)|dt < \infty$$

4.1 Parseval Theorem for the Fourier Transformation

$$\boxed{E_x = \int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|^2 d\omega}$$

Of course this does not converge for periodic signals. The Parseval Theorem for the Fourier series has to be used instead in case of periodic signals. For the sake of generality, the theorem is given for two signals:

$$E_{\mu\nu} = \int_{-\infty}^{\infty} x_\mu^*(t)x_\nu(t)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} X_\mu^*(\omega)X_\nu(\omega)d\omega$$

4.2 Example of the Fourier Transformation

Let us prove the following result:

$$\boxed{\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{j\omega_0 t}dt = \delta(\omega_0)}$$

Let us set $x(t) = \delta(t-t')$:

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt = \int_{-\infty}^{\infty} \delta(t-t')e^{-j\omega t}dt \qquad \text{with} \qquad \delta(t-t')e^{-j\omega t} \neq 0 \text{ for } t = t'$$

which yields,

$$X(\omega) = e^{-j\omega t'}, \qquad x(t) = \delta(t - t')$$

By applying the inverse transform,

$$x(t) = \delta(t - t') = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{j\omega(t-t')}d\omega$$

which proves

$$\delta(t - t') = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{j\omega(t-t')}d\omega \qquad \square$$

4.3 Relation between the transformation and the series

From the relations we saw until now:

$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} \qquad c_k = \frac{1}{T_0}\int_{T_0} \tilde{x}(t)e^{-j\omega_0 t}dt$$

and

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega \qquad X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

it comes that

$$\boxed{c_k = \frac{1}{T_0}X(k\omega_0) \qquad \Leftrightarrow \qquad T_0 c_k = X(k\omega_0)}$$

This means that $X(\omega)$ envelops the coefficients c_k with a factor T_0. The c_k are separated by a distance $\omega_0 = \frac{2\pi}{T_0}$.

4.4 Linear Differential Equations with constant coefficients

A differential equation of order N has the form:

$$\sum_{l=0}^{N} a_l \frac{d^l}{dt^l}y(t) = \sum_{m=0}^{M} b_m \frac{d^m}{dt^m}x(t)$$

And by knowing that:

$$\frac{d^n}{dt^n}x(t) \quad \longleftrightarrow \quad (j\omega)^n X(\omega)$$

The equation can be written after transformation as:

$$Y(\omega)\sum_{l=0}^{N}(j\omega)^l a_l = X(\omega)\sum_{m=0}^{M}(j\omega)^m b_m$$

Hence the **transfer function** $H(\omega)$:

$$\boxed{H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{\sum_{m=0}^{M}(j\omega)^m b_m}{\sum_{l=0}^{N}(j\omega)^l a_l}}$$

The phase and magnitude of the transfer function are usually represented in logarithmic scales, typically with a *Bode-Diagram*.

5 The Time Discrete Fourier Transformation TDFT

Time is now discrete so the normalisations are necessary, with the sampling period T_s:

$$n = \frac{t}{T_s}$$

$$\Omega = \omega T_s = 2\pi f \cdot \frac{1}{f_s}$$

This is due to the fact that $\phi(t) = e^{j\omega t} = e^{j\omega n T_s} = e^{j\Omega n}$. The transformations are then given as follows:

$$\boxed{X(\Omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega n}}$$

$$\boxed{x[n] = \frac{1}{2\pi} \int_{2\pi} X(\Omega) e^{j\Omega n} d\Omega}$$

In order for the transformation to exist, the sequence has to be *stable*, hence:

$$|X(\Omega)| = \sum_{n=-\infty}^{\infty} |x[n]| < \infty, \quad \forall\, \Omega$$

It is important to see that the spectrum is **periodical**. It can be expressed as:

$$\boxed{X(\Omega) = X(\Omega + 2\pi), \quad \forall\, \Omega}$$

5.1 Infinite Sum of Complex Exponentials

With the following series in mind, with $|a| < 1$:

$$\sum_{k=0}^{N} a^k = \frac{1 - a^{N+1}}{1 - a} \qquad \longleftrightarrow \qquad \sum_{k=0}^{\infty} (ae^x)^k = \frac{1}{1 - ae^x}$$

We have:

$$\sum_{k=-\infty}^{\infty} (ae^{jx})^k = \sum_{k=0}^{\infty} (ae^{jx})^k + \sum_{k=0}^{\infty} (ae^{-jx})^k \underbrace{-(ae^{jx})^0}_{*)}$$

$$= \frac{1}{1 - ae^{jx}} + \frac{1}{1 - ae^{-jx}} - 1 = \frac{2 - a(e^{jx} + e^{-jx})}{1 - a(e^{jx} + e^{-jx}) + a^2} - 1 \underbrace{=}_{\lim_{a\to 1}} 1 - 1 = 0$$

*) If not subtracted here, it would be present two times in the sums otherwise.

If we take the definition of the series with $a = 1$ and let $e^x = e^{\pm j2\pi n} = 1$, then the series diverges towards infinity. For any other value of e^x, the series is equal to zero. This is equivalent to a series of Dirac impulses!

As a final result, by letting $a = 1$ and $x = \Omega$ in the previous equations:

$$\boxed{\sum_{n=-\infty}^{\infty} e^{-j\Omega n} = 2\pi \sum_{r=-\infty}^{\infty} \delta(\Omega - 2\pi r)}$$

This result may come handy when calculating a TDFT.

5.2 Parseval Theorem for the TDFT

The *Energy Signal* is given as:

$$E_x = \sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi} \int_{2\pi} |X(\Omega)|^2 d\Omega$$

$|X(\Omega)|^2$ describes the Spectrum of the Energie Density. The Energy Signal of two overlapping signals is given as:

$$E_{xy} = \sum_{n=-\infty}^{\infty} x[n] y^*[n] = \frac{1}{2\pi} \int_{2\pi} X(\Omega) Y^*(\Omega) d\Omega$$

5.3 Transfer Function for LTI System

$$y[n] = x[n] * h[n] = \sum_{l=-\infty}^{\infty} x[l] h[n-l] = \sum_{l=-\infty}^{\infty} h[l] x[n-l]$$

Therefore, we can express the transfer function as:

$$Y(\Omega) = H(\Omega) X(\Omega) \leftrightarrow \boxed{H(\Omega) = \frac{Y(\Omega)}{x(\Omega)}}$$

With the **stability** condition:

$$\boxed{\sum_{n=-\infty}^{\infty} |h[n]| < \infty}$$

5.4 Linear Differential Equations with constant coefficients

A differential equation is given as:

$$\sum_{l=0}^{N} a_l y[n-l] = \sum_{m=0}^{M} b_m x[n-m]$$

After transformation:

$$Y(\Omega) \sum_{l=0}^{N} a_l e^{-j\Omega l} = X(\Omega) \sum_{m=0}^{M} b_m e^{-j\Omega m}$$

Hence

$$H(\Omega) = \frac{Y(\Omega)}{X(\Omega)} = \frac{\sum_{m=0}^{M} b_m e^{-j\Omega m}}{\sum_{l=0}^{N} a_l e^{-j\Omega l}}$$

- *First Order*: The denominator only has values for $l = 0, 1$.
- *Second Order*: The denominator only has values for $l = 0, 1, 2$.
- *Higher Order*: They can be built from First and Second order systems.

6 Signal Sampling and Reconstruction

A time continuous signal $x(t)$ is modulated by a sampling function $s(t)$ at a rate ω_s.

$$s(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s) \quad \leftrightarrow \quad S(\omega) = \frac{2\pi}{T_s} \sum_{r=-\infty}^{\infty} \delta(\omega - r\omega_s)$$

Therefore we have the sampled signal $x_s(t)$:

$$x_s(t) = x(t)s(t) = \sum_{n=-\infty}^{\infty} x(nT_s)\delta(t - nT_s)$$

$$\boxed{X_s(\omega) = \frac{1}{2\pi} X(\omega) * S(\omega) = \frac{1}{T_s} \sum_{r=-\infty}^{\infty} X(\omega - r\omega_s)}$$

Which proves, the spectrum of a time discrete signal is **periodical**!

6.1 The Shannon Sampling Theorem

With ω_s the sampling rate and ω_g the maximal rate of the signal:

$$\boxed{\omega_s \geq 2\omega_g \longleftrightarrow T_s \leq \frac{\pi}{\omega_g}}$$

This means that $X(\omega) = 0$ for $|\omega| > \omega_g$. This means the width of the Spectrum has to be equal or smaller than the sampling rate. Otherwise, there is some overlapping and information is lost.

6.2 Signal Reconstruction

By applying a low pass to the signal given as

$$H_r(\omega) = T_s$$

for $|\omega| \leq \omega_r$. A common choice for ω_r is $\frac{\pi}{T_s}$.

$$h_r(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} H_r(\omega) e^{j\omega t} d\omega = \frac{1}{2\pi} \int_{-\omega_r}^{\omega_r} T_s e^{j\omega t} d\omega = \frac{1}{2\pi} \int_{-\frac{\pi}{T_s}}^{\frac{\pi}{T_s}} T_s e^{j\omega t} d\omega = \frac{T_s}{\pi t} \sin(\omega_r t) = \mathrm{si}(\frac{\pi t}{T_s})$$

Hence

$$x_r(t) = \int_{-\infty}^{\infty} x_s(\tau) h_r(t-\tau) d\tau = \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} x(nT_s)\delta(\tau - nT_s) h_r(t-\tau) d\tau = \sum_{n=-\infty}^{\infty} x(nT_s) h_r(t - nT_s)$$

$$\boxed{x_r(t) = \frac{\omega_r T_s}{\pi} \sum_{n=-\infty}^{\infty} x(nT_s)\mathrm{si}(\omega_r(t - nT_s))}$$

The Reconstruction of a sampled signal is done by the summation of a multitude of si-functions!

6.3 Sampling in Frequency Domain

The function

$$P(\omega) = \sum_{n=-\infty}^{\infty} \delta(\omega - k\omega_p) \quad \leftrightarrow \quad p(t) = \frac{T_p}{2\pi} \sum_{r=-\infty}^{\infty} \delta(t - rT_p)$$

is used for the sampling of the signal. The signal in time domain of a sampled spectrum is **periodical** with a period T_p:

$$x_p(t) = x(t) * p(t) = \frac{T_p}{2\pi} \sum_{r=-\infty}^{\infty} x(t - rT_p)$$

6.3.1 Sampling Theorem in Frequency Domain

To avoid an overlap of the time-signals:

$$\boxed{T_p \geq 2T_g}$$

as the time-periodical signal is $2T_g$ wide, T_p must guaranty that each chunk of signal has enough room to avoid overlapping.

7 The Laplace Transformation LT

The Laplace transformation allows some better convergence properties than the Fourier transform. By setting:

$$s = \sigma + j\omega$$

The Laplace transform is given as:

$$\boxed{X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt}$$

$$\boxed{x(t) = \frac{1}{2\pi j} \int_{\sigma-\infty}^{\sigma+\infty} X(s) e^{st} ds} \quad [1]$$

Considering that

$$X(s) = X(\sigma + j\omega) = \int_{-\infty}^{\infty} (x(t) e^{-\sigma t}) e^{-j\omega t} dt$$

the LT can be interpreted as a FT:

$$L(x(t)) = F(x(t) e^{-\sigma t})$$

- $\sigma > 0$: damping
- $\sigma < 0$: amplification

In order for the LT to exist and to **converge**, with σ_0 providing the limit for the convergence domain:

$$\int_{-\infty}^{\infty} |x(t)| e^{-\sigma_0 t} < \infty$$

IMPORTANT: If $\Im m\{s\} \in$ *Convergence Domain* then the FT converges also!

[1] The presence of j here is due to the change of variables from the FT: $s = \sigma + j\omega$ which with partial derivation gives $ds = jd\omega \longrightarrow d\omega = \frac{ds}{j}$

7.1 The Unilateral Laplace Transformation

This is useful to study the response of systems to a stimulation.

$$X_+(s) = \int_0^\infty x(t)e^{-st}dt$$

$$x(t) = \frac{1}{2\pi j}\int_{\sigma-\infty}^{\sigma+\infty} X_+(s)e^{st}ds$$

7.2 Linear Differential Equations using the LT

From the differential equation:

$$\sum_{l=0}^{N} a_l \frac{d^l}{dt^l} y(t) = \sum_{m=0}^{M} b_m \frac{d^m}{dt^m} x(t)$$

The transfer function of the system can be expressed as:

$$\boxed{H(s) = \frac{Y(s)}{X(s)} = \frac{\sum_{m=0}^{M} b_m s^m}{\sum_{l=0}^{N} a_l s^l}}$$

7.3 Convergence of the Laplace Transformation

From the condition for the convergence, which provides a value for σ_0:

$$\int_{-\infty}^{\infty} |x(t)|e^{-\sigma_0 t} < \infty$$

The Convergence Domain (CD) has following properties:

1. The CD is parallel to the imaginary axis in the s-plane.
2. If the Laplace Transform is rational, then there is no Pole within the CD.
3. If $x(t)$ is **finite in time**, and there is at least one value of s that lets the LT converge, then the whole s-plane is a CD.
4. If $x(t)$ is a **right hand signal**, then all values of s with $Re(s) > \sigma_0$ are in the CD too.
5. If $x(t)$ is a **left hand signal**, then all values of s with $Re(s) < \sigma_0$ are in the CD too.
6. If $x(t)$ is **infinite in time**, then points 4 and 5 apply at the same time.

8 The z-Transformation ZT

This transformation is a generalisation of the TDFT. It offers a stronger convergence, is usually simpler to handle with analytic problems and the complex writing is introduced by time discrete signals. The bilateral ZT is given as:

$$z = re^{j\omega}$$

$$\boxed{X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}}$$

$$\boxed{x[n] = \frac{1}{2\pi j} \oint_C X(z)z^{n-1}dz}$$

C: Integration way along a closed circuit anti-clockwise within the convergence domain. $X(z)$ can also be unilateral:

$$X_+(z) = \sum_{n=0}^{\infty} x[n]z^{-n}$$

8.1 Convergence of the z-Transformation

The condition for the convergence is:

$$\sum_{n=-\infty}^{\infty} |x[n]r^{-n}| < \infty$$

The convergence properties come from the properties of the Laurent series.

1. The Convergence Domain (CD) is **a ring or a disc around the origin** of the z-plane! $0 \leq R_1 < |z| < R_2 \leq \infty$
2. The TDFT converges only if the CD of the z-Transform contains the unit circle.
3. The CD contains no Pole.
4. If $x[n]$ is a sequence of finite length, then the whole z-plane is a CD. Excepted maybe for $z = 0$ and/or $|z| = \infty$.
5. If $x[n]$ is a right hand signal, the CD is the outer part of the circle containing the furthest Pole.
6. If $x[n]$ is a left hand signal, the CD is the inner part of the circle containing the closest Pole.
7. If $x[n]$ has no limit for n (left and right), the CD is a ring free of any Pole, containing the circle $|z| = r_0$.
8. The CD is a one piece domain and has no detached sections.

8.2 Comparison to the TDFT

For $z = e^{j\Omega}$ with $|z| = 1$, the z-transformation becomes:

$$\text{(ZT)} \quad X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} = \sum_{n=-\infty}^{\infty} x[n](e^{j\Omega})^{-n} = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n} = X(\Omega) \quad \text{(TDFT)}$$

For $z = re^{j\Omega}$ with $|z| = r$, the ZT can expressed as a TDFT:

$$X(z) = X(re^{j\Omega}) = \sum_{n=-\infty}^{\infty} (x[n]r^{-n})e^{-j\Omega n}$$

As said above, the TDFT converges only if the CD of the z-Transform contains the unit circle.

8.3 Residue Theorem

The computation of the inverse ZT being given as:

$$x[n] = \frac{1}{2\pi j} \oint_C X(z)z^{n-1}dz$$

The complex integral on a closed path can be computed using the **Residue Theorem**:

$$\boxed{\oint_C f(z)dz = 2\pi j \sum_{k=1}^{K} Res\{f(z), z_k\}}$$

with K the number of poles contained in the integration path C. For each pole z_k, the residue is calculated as:

$$f(z) = \frac{A(z)}{(z - z_k)^m}$$

$$\boxed{Res\{f(z), z_k\} = \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} A(z)|_{z=z_k}}$$

with $k \in [1, K]$.

In our case $f(z) = X(z) \cdot z^{n-1}$.

8.4 Transition from the LT to the ZT

There are many ways of numerically representing a function. Depending on how a function is represented, the relation between the LT and ZT will change. This transition is important for the implementation of filters, controllers, etc... that where derived in the Laplace domain and that need to be numerically implemented.

8.4.1 Integral Substitution

An integration from 0 to t_k is given by:

$$y(t_k) = \underbrace{\int_0^{t_{k-1}} x(t)\, dt}_{=y_{k-1}} + \underbrace{\int_{t_{k-1}}^{t_k} x(t)\, dt}_{=\Delta y_k}$$

Rectangle Approximation We express Δy_k as

$$\Delta y_k = x_k h$$

Therefore

$$y_k = y_{k-1} + x_k h$$
$$Y(z) = Y(z)z^{-1} + X(z)h$$

hence

$$\frac{Y(z)}{X(z)} = \frac{h}{1 - z^{-1}} = \frac{hz}{z-1} \quad \equiv \quad \frac{1}{s} = \frac{Y(s)}{X(s)}$$

Which means we can substitute s by

$$\boxed{s \mathrel{\hat{=}} \frac{z-1}{hz}}$$

Trapeze Approximation We express this time Δy_k as

$$\Delta y_k = \frac{h}{2}(x_{k-1} + x_k)$$

Therefore

$$y_k = y_{k-1} + \frac{h}{2}(x_{k-1} + x_k)$$
$$Y(z) = Y(z)z^{-1} + \frac{h}{2}(X(z)z^{-1} + X(z))$$

hence

$$\frac{Y(z)}{X(z)} = \frac{h}{2}\frac{z+1}{z-1} \quad \equiv \quad \frac{1}{s} = \frac{Y(s)}{X(s)}$$

Which means we can substitute s by

$$\boxed{s \mathrel{\hat{=}} \frac{2}{h}\frac{z-1}{z+1}}$$

9 Lists of Properties

9.1 Continuous Fourier transformation

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt \qquad x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega$$

	$x(t)$	$X(\omega)$
Symmetrie	$x^*(t)$	$X^*(-\omega)$
	$x^*(-t)$	$X^*(\omega)$
Linearität	$ax_1(t) + bx_2(t)$	$aX_1(\omega) + bX_2(\omega)$
Verschiebung (Zeit)	$x(t-\tau)$	$e^{-j\omega\tau}X(\omega)$
(Frequenz)	$e^{j\omega_0 t}x(t)$	$X(\omega-\omega_0)$
Maßstabänderung	$x(at)$	$\frac{1}{\lvert a\rvert}X\left(\frac{\omega}{a}\right)$
Differentiation (Zeit)	$\frac{d^n}{dt^n}x(t)$	$(j\omega)^n X(\omega)$
(Frequenz)	$\frac{t^n}{j^n}x(t)$	$\frac{d^n}{d\omega^n}X(\omega)$
Integration (Zeit)	$\int_{-\infty}^{t} x(\tau)d\tau$	$\frac{1}{j\omega}X(\omega) + \pi X(0)\delta(\omega)$
(Frequenz)	$\frac{j}{t}x(t) + \pi x(0)\delta(t)$	$\int_{-\infty}^{\omega} X(\Omega)d\Omega$
Faltung	$x_1(t) * x_2(t)$	$X_1(\omega)X_2(\omega)$
Modulation	$x_1(t)x_2(t)$	$\frac{1}{2\pi}X_1(\omega) * X_2(\omega)$
	$x(t)\cos(\omega_0 t)$	$\frac{1}{2}\Big(X(\omega+\omega_0) + X(\omega-\omega_0)\Big)$
Korrelation	$\int_{-\infty}^{\infty} x_1(\tau)x_2(t+\tau)d\tau$	$X_1^*(\omega)X_2(\omega)$

Figure 1: A list of properties of the Fourier transformation

$x(t)$	$X(\omega)$
$u(t-a)-u(t-b)\ ;\quad a<b$	$\dfrac{e^{-j\omega a}-e^{-j\omega b}}{j\omega}$
$u(t+a)-u(t-a)\ ;\quad a>0$	$2a\,\dfrac{\sin(a\omega)}{a\omega}=2a\,\mathrm{si}(a\omega)$
$\dfrac{a}{\pi}\,\mathrm{si}(at)$	$u(\omega+a)-u(\omega-a);\quad a>0$
$e^{-bt}u(t)\ ;\quad b>0$	$\dfrac{1}{b+j\omega}$
$e^{-b\lvert t\rvert}\ ;\quad b>0$	$\dfrac{2b}{b^2+\omega^2}$
$\dfrac{1}{b^2+t^2}$	$\dfrac{\pi}{b}e^{-b\lvert\omega\rvert}\ ;\quad b>0$
$\dfrac{1}{\sqrt{2\pi}}e^{\frac{-t^2}{2}}$	$e^{\frac{-\omega^2}{2}}$
1	$2\pi\delta(\omega)$
$\delta(t)$	1
$-j\omega t$ $dt = 2\pi\delta(\omega-\omega_0) \longleftrightarrow e^{j\omega_0 t}$ $\omega_0)t$	$2\pi\delta(\omega-\omega_0) \xleftrightarrow{2\pi} \int_{-\infty}^{\infty}\delta(\omega-\omega_0)e^{j\omega t}$
$\cos(\omega_0 t)$	$\pi\Big(\delta(\omega+\omega_0)+\delta(\omega-\omega_0)\Big)$
$\sin(\omega_0 t)$	$j\pi\Big(\delta(\omega+\omega_0)-\delta(\omega-\omega_0)\Big)$
$\sum_{n=-\infty}^{\infty}\delta(t-nT)$	$\dfrac{2\pi}{T}\sum_{k=-\infty}^{\infty}\delta(\omega-k\omega_0);\quad \omega_0=\dfrac{2\pi}{T}$
$\delta(t-T)$	$e^{-j\omega T}$
$u(t)$	$\dfrac{1}{j\omega}+\pi\delta(\omega)$
$\sum_{k=-\infty}^{\infty}c_k e^{jk\omega_0 t}$	$2\pi\sum_{k=-\infty}^{\infty}c_k\delta(\omega-k\omega_0)$

Figure 2: A list of known Fourier transformations

9.2 Time Discrete Fourier transformation

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n} \qquad x[n] = \frac{1}{2\pi}\int_{2\pi} X(\Omega)e^{j\Omega n}d\Omega$$

	$x(t)$	$X(\omega)$
Symmetrie	$x^*(t)$	$X^*(-\omega)$
	$x^*(-t)$	$X^*(\omega)$
Linearität	$ax_1(t) + bx_2(t)$	$aX_1(\omega) + bX_2(\omega)$
Verschiebung (Zeit)	$x(t-\tau)$	$e^{-j\omega\tau}X(\omega)$
(Frequenz)	$e^{j\omega_0 t}x(t)$	$X(\omega-\omega_0)$
Maßstabänderung	$x(at)$	$\frac{1}{\|a\|}X\left(\frac{\omega}{a}\right)$
Differentiation (Zeit)	$\frac{d^n}{dt^n}x(t)$	$(j\omega)^n X(\omega)$
(Frequenz)	$\frac{t^n}{j^n}x(t)$	$\frac{d^n}{d\omega^n}X(\omega)$
Integration (Zeit)	$\int_{-\infty}^{t} x(\tau)d\tau$	$\frac{1}{j\omega}X(\omega) + \pi X(0)\delta(\omega)$
(Frequenz)	$\frac{j}{t}x(t) + \pi x(0)\delta(t)$	$\int_{-\infty}^{\omega} X(\Omega)d\Omega$
Faltung	$x_1(t) * x_2(t)$	$X_1(\omega)X_2(\omega)$
Modulation	$x_1(t)x_2(t)$	$\frac{1}{2\pi}X_1(\omega) * X_2(\omega)$
	$x(t)\cos(\omega_0 t)$	$\frac{1}{2}\Big(X(\omega+\omega_0) + X(\omega-\omega_0)\Big)$
Korrelation	$\int_{-\infty}^{\infty} x_1(\tau)x_2(t+\tau)d\tau$	$X_1^*(\omega)X_2(\omega)$

Figure 3: A list of properties of the Time Discrete Fourier transformation

$x(t)$	$X(\omega)$
$u(t-a)-u(t-b)\ ;\quad a<b$	$\dfrac{e^{-j\omega a}-e^{-j\omega b}}{j\omega}$
$u(t+a)-u(t-a)\ ;\quad a>0$	$2a\,\dfrac{\sin(a\omega)}{a\omega}=2a\,\mathrm{si}(a\omega)$
$\dfrac{a}{\pi}\,\mathrm{si}(at)$	$u(\omega+a)-u(\omega-a);\quad a>0$
$e^{-bt}u(t)\ ;\quad b>0$	$\dfrac{1}{b+j\omega}$
$e^{-b\|t\|}\ ;\quad b>0$	$\dfrac{2b}{b^2+\omega^2}$
$\dfrac{1}{b^2+t^2}$	$\dfrac{\pi}{b}e^{-b\|\omega\|}\ ;\quad b>0$
$\dfrac{1}{\sqrt{2\pi}}e^{\frac{-t^2}{2}}$	$e^{\frac{-\omega^2}{2}}$
1	$2\pi\delta(\omega)$
$\delta(t)$	1
$dt = 2\pi\,\delta(\omega-\omega_0) \longleftrightarrow e^{j\omega_0 t}$	$2\pi\delta(\omega-\omega_0) \longleftrightarrow \int_{-\infty}^{\infty}\delta(\omega-\omega_0)\,e^{j\omega t}$
$\cos(\omega_0 t)$	$\pi\Big(\delta(\omega+\omega_0)+\delta(\omega-\omega_0)\Big)$
$\sin(\omega_0 t)$	$j\pi\Big(\delta(\omega+\omega_0)-\delta(\omega-\omega_0)\Big)$
$\sum_{n=-\infty}^{\infty}\delta(t-nT)$	$\dfrac{2\pi}{T}\sum_{k=-\infty}^{\infty}\delta(\omega-k\omega_0);\quad \omega_0=\dfrac{2\pi}{T}$
$\delta(t-T)$	$e^{-j\omega T}$
$u(t)$	$\dfrac{1}{j\omega}+\pi\delta(\omega)$
$\sum_{k=-\infty}^{\infty}c_k e^{jk\omega_0 t}$	$2\pi\sum_{k=-\infty}^{\infty}c_k\delta(\omega-k\omega_0)$

Figure 4: A list of known Time Discrete Fourier transformations

9.3 Laplace transformation

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt \qquad x(t) = \frac{1}{2\pi j}\int_{\sigma-\infty}^{\sigma+\infty} X(s)e^{st}ds$$

	$x(t)$	$X(\omega)$
Symmetrie	$x^*(t)$	$X^*(-\omega)$
	$x^*(-t)$	$X^*(\omega)$
Linearität	$ax_1(t) + bx_2(t)$	$aX_1(\omega) + bX_2(\omega)$
Verschiebung (Zeit)	$x(t-\tau)$	$e^{-j\omega\tau}X(\omega)$
(Frequenz)	$e^{j\omega_0 t}x(t)$	$X(\omega - \omega_0)$
Maßstabänderung	$x(at)$	$\frac{1}{\lvert a\rvert}X\left(\frac{\omega}{a}\right)$
Differentiation (Zeit)	$\frac{d^n}{dt^n}x(t)$	$(j\omega)^n X(\omega)$
(Frequenz)	$\frac{t^n}{j^n}x(t)$	$\frac{d^n}{d\omega^n}X(\omega)$
Integration (Zeit)	$\int_{-\infty}^{t} x(\tau)d\tau$	$\frac{1}{j\omega}X(\omega) + \pi X(0)\delta(\omega)$
(Frequenz)	$\frac{j}{t}x(t) + \pi x(0)\delta(t)$	$\int_{-\infty}^{\omega} X(\Omega)d\Omega$
Faltung	$x_1(t) * x_2(t)$	$X_1(\omega)X_2(\omega)$
Modulation	$x_1(t)x_2(t)$	$\frac{1}{2\pi}X_1(\omega) * X_2(\omega)$
	$x(t)\cos(\omega_0 t)$	$\frac{1}{2}\Big(X(\omega+\omega_0) + X(\omega-\omega_0)\Big)$
Korrelation	$\int_{-\infty}^{\infty} x_1(\tau)x_2(t+\tau)d\tau$	$X_1^*(\omega)X_2(\omega)$

Figure 5: A list of properties of the Laplace transformation

$x(t)$	$X(\omega)$
$u(t-a) - u(t-b)\ ; \quad a < b$	$\dfrac{e^{-j\omega a} - e^{-j\omega b}}{j\omega}$
$u(t+a) - u(t-a)\ ; \quad a > 0$	$2a\,\dfrac{\sin(a\omega)}{a\omega} = 2a\,\mathrm{si}(a\omega)$
$\dfrac{a}{\pi}\,\mathrm{si}(at)$	$u(\omega+a) - u(\omega-a); \quad a > 0$
$e^{-bt}u(t)\ ; \quad b > 0$	$\dfrac{1}{b+j\omega}$
$e^{-b\lvert t\rvert}\ ; \quad b > 0$	$\dfrac{2b}{b^2+\omega^2}$
$\dfrac{1}{b^2+t^2}$	$\dfrac{\pi}{b}e^{-b\lvert\omega\rvert}\ ; \quad b > 0$
$\dfrac{1}{\sqrt{2\pi}}e^{\frac{-t^2}{2}}$	$e^{\frac{-\omega^2}{2}}$
1	$2\pi\delta(\omega)$
$\delta(t)$	1
$-j\omega t$ $dt = 2\pi\,\delta(\omega-\omega_0) \longleftrightarrow e^{j\omega_0 t}$ $\omega_0)t$	$2\pi\delta(\omega-\omega_0) \xleftrightarrow{\frac{1}{2\pi}} \int_{-\infty}^{\infty}\delta(\omega-\omega_0)\,e^{j\omega t}$
$\cos(\omega_0 t)$	$\pi\Big(\delta(\omega+\omega_0) + \delta(\omega-\omega_0)\Big)$
$\sin(\omega_0 t)$	$j\pi\Big(\delta(\omega+\omega_0) - \delta(\omega-\omega_0)\Big)$
$\sum_{n=-\infty}^{\infty}\delta(t-nT)$	$\dfrac{2\pi}{T}\sum_{k=-\infty}^{\infty}\delta(\omega-k\omega_0); \quad \omega_0 = \dfrac{2\pi}{T}$
$\delta(t-T)$	$e^{-j\omega T}$
$u(t)$	$\dfrac{1}{j\omega} + \pi\delta(\omega)$
$\sum_{k=-\infty}^{\infty}c_k e^{jk\omega_0 t}$	$2\pi\sum_{k=-\infty}^{\infty}c_k\delta(\omega-k\omega_0)$

Figure 6: A list of known Laplace transformations

9.4 Z-transformation

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} \qquad x[n] = \frac{1}{2\pi j}\oint_C X(z)z^{n-1}dz$$

	$x(t)$	$X(\omega)$
Symmetrie	$x^*(t)$	$X^*(-\omega)$
	$x^*(-t)$	$X^*(\omega)$
Linearität	$ax_1(t) + bx_2(t)$	$aX_1(\omega) + bX_2(\omega)$
Verschiebung (Zeit)	$x(t-\tau)$	$e^{-j\omega\tau}X(\omega)$
(Frequenz)	$e^{j\omega_0 t}x(t)$	$X(\omega-\omega_0)$
Maßstabänderung	$x(at)$	$\frac{1}{\lvert a\rvert}X\left(\frac{\omega}{a}\right)$
Differentiation (Zeit)	$\frac{d^n}{dt^n}x(t)$	$(j\omega)^n X(\omega)$
(Frequenz)	$\frac{t^n}{j^n}x(t)$	$\frac{d^n}{d\omega^n}X(\omega)$
Integration (Zeit)	$\int_{-\infty}^{t} x(\tau)d\tau$	$\frac{1}{j\omega}X(\omega) + \pi X(0)\delta(\omega)$
(Frequenz)	$\frac{j}{t}x(t) + \pi x(0)\delta(t)$	$\int_{-\infty}^{\omega} X(\Omega)d\Omega$
Faltung	$x_1(t) * x_2(t)$	$X_1(\omega)X_2(\omega)$
Modulation	$x_1(t)x_2(t)$	$\frac{1}{2\pi}X_1(\omega) * X_2(\omega)$
	$x(t)\cos(\omega_0 t)$	$\frac{1}{2}\Big(X(\omega+\omega_0) + X(\omega-\omega_0)\Big)$
Korrelation	$\int_{-\infty}^{\infty} x_1(\tau)x_2(t+\tau)d\tau$	$X_1^*(\omega)X_2(\omega)$

Figure 7: A list of properties of the z-transformation

$x(t)$	$X(\omega)$
$u(t-a)-u(t-b)\ ;\quad a<b$	$\dfrac{e^{-j\omega a}-e^{-j\omega b}}{j\omega}$
$u(t+a)-u(t-a)\ ;\quad a>0$	$2a\,\dfrac{\sin(a\omega)}{a\omega}=2a\,\mathrm{si}(a\omega)$
$\dfrac{a}{\pi}\,\mathrm{si}(at)$	$u(\omega+a)-u(\omega-a);\quad a>0$
$e^{-bt}u(t)\ ;\quad b>0$	$\dfrac{1}{b+j\omega}$
$e^{-b\|t\|}\ ;\quad b>0$	$\dfrac{2b}{b^2+\omega^2}$
$\dfrac{1}{b^2+t^2}$	$\dfrac{\pi}{b}e^{-b\|\omega\|}\ ;\quad b>0$
$\dfrac{1}{\sqrt{2\pi}}e^{\frac{-t^2}{2}}$	$e^{\frac{-\omega^2}{2}}$
1	$2\pi\delta(\omega)$
$\delta(t)$	1
$-j\omega t$ $dt = 2\pi\,\delta(\omega-\omega_0) \longleftrightarrow e^{j\omega_0 t}$ $\omega_0)t$	$2\pi\delta(\omega-\omega_0) \longleftrightarrow \int_{-\infty}^{\infty}\delta(\omega-\omega_0)\,e^{j\omega t}$
$\cos(\omega_0 t)$	$\pi\Big(\delta(\omega+\omega_0)+\delta(\omega-\omega_0)\Big)$
$\sin(\omega_0 t)$	$j\pi\Big(\delta(\omega+\omega_0)-\delta(\omega-\omega_0)\Big)$
$\sum_{n=-\infty}^{\infty}\delta(t-nT)$	$\dfrac{2\pi}{T}\sum_{k=-\infty}^{\infty}\delta(\omega-k\omega_0);\quad \omega_0=\dfrac{2\pi}{T}$
$\delta(t-T)$	$e^{-j\omega T}$
$u(t)$	$\dfrac{1}{j\omega}+\pi\delta(\omega)$
$\sum_{k=-\infty}^{\infty}c_k e^{jk\omega_0 t}$	$2\pi\sum_{k=-\infty}^{\infty}c_k\delta(\omega-k\omega_0)$

Figure 8: A list of known z-transformations