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2.4 Priority Queues

- *APIs*
- *elementary implementations*
- *binary heaps*
- *heapsort*
- *event-driven simulation* ← see videos

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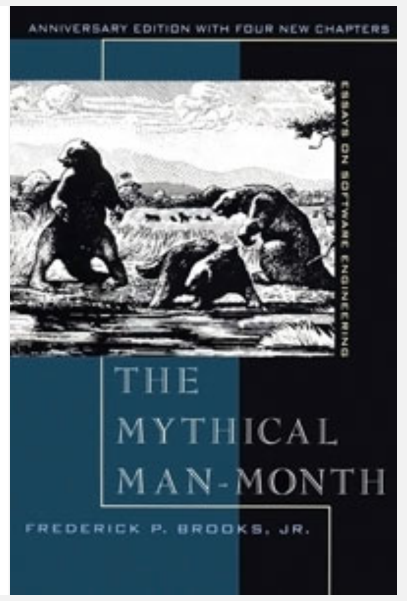
Collections

A collection is a data type that stores a group of items.

data type	core operations	data structure
stack	PUSH, POP	*linked list* *resizing array*
queue	ENQUEUE, DEQUEUE	
priority queue	INSERT, DELETE-MAX	*binary heap*
symbol table	PUT, GET, DELETE	*binary search tree* *hash table*
set	ADD, CONTAINS, DELETE	

> *" Show me your code and conceal your data structures, and I shall continue to be mystified. Show me your data structures, and I won't usually need your code; it'll be obvious."* — *Fred Brooks*



Priority queue

Collections. Insert and remove items. Which item to remove?

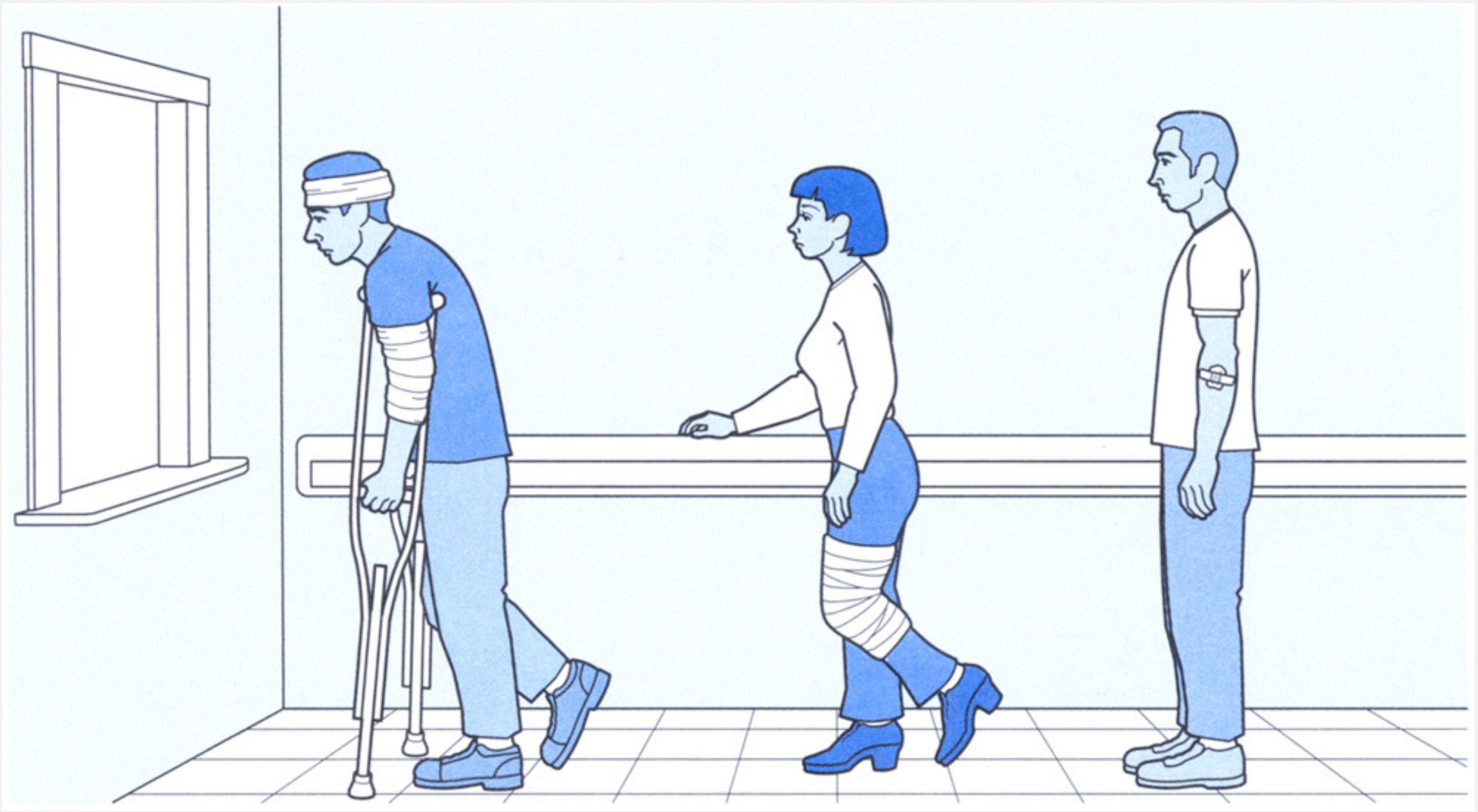
Stack. Remove the item most recently added.

Queue. Remove the item least recently added.

Randomized queue. Remove a random item.

Priority queue. Remove the largest (or smallest) item.

Generalizes: stack, queue, randomized queue.



triage in an emergency room
(priority = urgency of wound/illness)

operation	*argument*	*return value*
insert	P	
insert	Q	
insert	E	
remove max		Q
insert	X	
insert	A	
insert	M	
remove max		X
insert	P	
insert	L	
insert	E	
remove max		P

Max-oriented priority queue API

Requirement. Must insert keys of the same (generic) type; keys must be `Comparable`.

“bounded type parameter”

`public class MaxPQ<Key extends Comparable<Key>>`

	`MaxPQ()`	*create an empty priority queue*
`void`	`insert(Key v)`	*insert a key*
`Key`	`delMax()`	*return and remove a largest key*
`boolean`	`isEmpty()`	*is the priority queue empty?*
`Key`	`max()`	*return a largest key*
`int`	`size()`	*number of entries in the priority queue*

Note. Duplicate keys allowed; `delMax()` removes and returns any maximum key.

Min-oriented priority queue API

Analogous to `MaxPQ`.

```
public class MinPQ<Key extends Comparable<Key>>
```

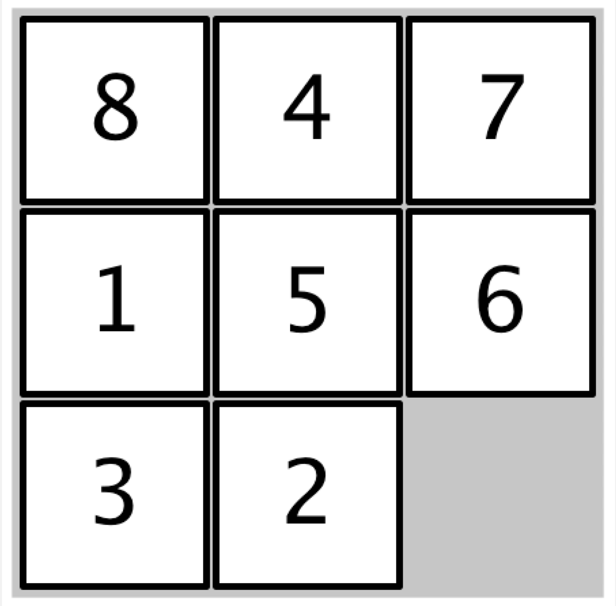
	`MinPQ()`	*create an empty priority queue*
`void`	`insert(Key v)`	*insert a key*
`Key`	`delMin()`	*return and remove a smallest key*
`boolean`	`isEmpty()`	*is the priority queue empty?*
`Key`	`min()`	*return a smallest key*
`int`	`size()`	*number of entries in the priority queue*

Warmup client. Sort a stream of integers from standard input.

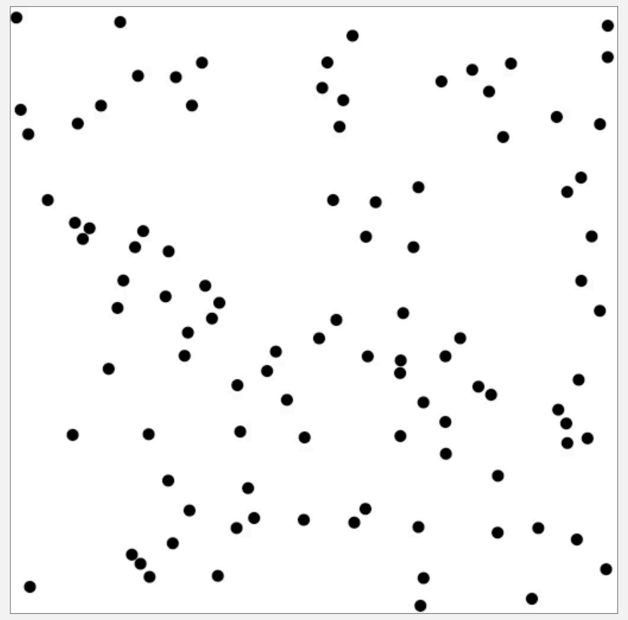
Priority queue: applications

- Event-driven simulation. [customers in a line, colliding particles]
- Discrete optimization. [bin packing, scheduling]
- Artificial intelligence. [A* search]
- Computer networks. [web cache]
- Data compression. [Huffman codes]
- Operating systems. [load balancing, interrupt handling]
- Graph searching. [Dijkstra's algorithm, Prim's algorithm]
- Number theory. [sum of powers]
- Spam filtering. [Bayesian spam filter]
- Statistics. [online median in data stream]

priority = length of best known path



priority = "distance" to goal board



priority = event time

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2.4 Priority Queues

- *APIs*
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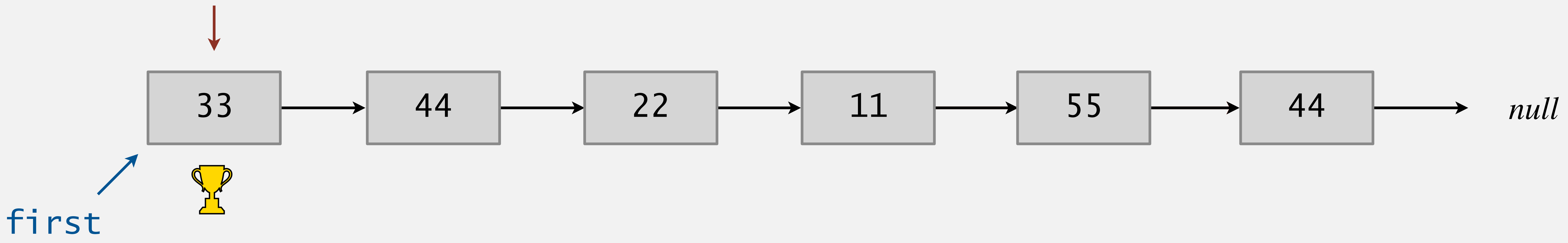
- *heapsort*
- *event-driven simulation*

Priority queue: elementary implementations

Unordered list. Store keys in a linked list.



Performance. INSERT takes $\Theta(1)$ time; DELETE-MAX takes $\Theta(n)$ time.

Priority queue: elementary implementations

Ordered array. Store keys in an array in ascending (or descending) order.

	0	1	2	3	4	5	6	7	8	9	10
a[]	11	22	33	44	44	55	99				
								n			

ordered array implementation of a MaxPQ

Priority queues: quiz 1

What are the worst-case running times for INSERT and DELETE-MAX, respectively, in a MaxPQ implemented with an ordered array?

ignore array resizing

A. $\Theta(1)$ and $\Theta(n)$

B. $\Theta(1)$ and $\Theta(\log n)$

C. $\Theta(\log n)$ and $\Theta(1)$

D. $\Theta(n)$ and $\Theta(1)$

0	1	2	3	4	5	6	7	8	9	10
11	22	33	44	44	55	99				
							n			

ordered array implementation of a MaxPQ

Priority queue: implementations cost summary

Elementary implementations. Either INSERT or DELETE-MAX takes $\Theta(n)$ time.

implementation	INSERT	DELETE–MAX	MAX
unordered list	1	n	n
ordered array	n	1	1
goal	$\log n$	$\log n$	$\log n$

order of growth of running time for priority queue with n items

Challenge. Implement both core operations efficiently.

Solution. "Somewhat-ordered" array.

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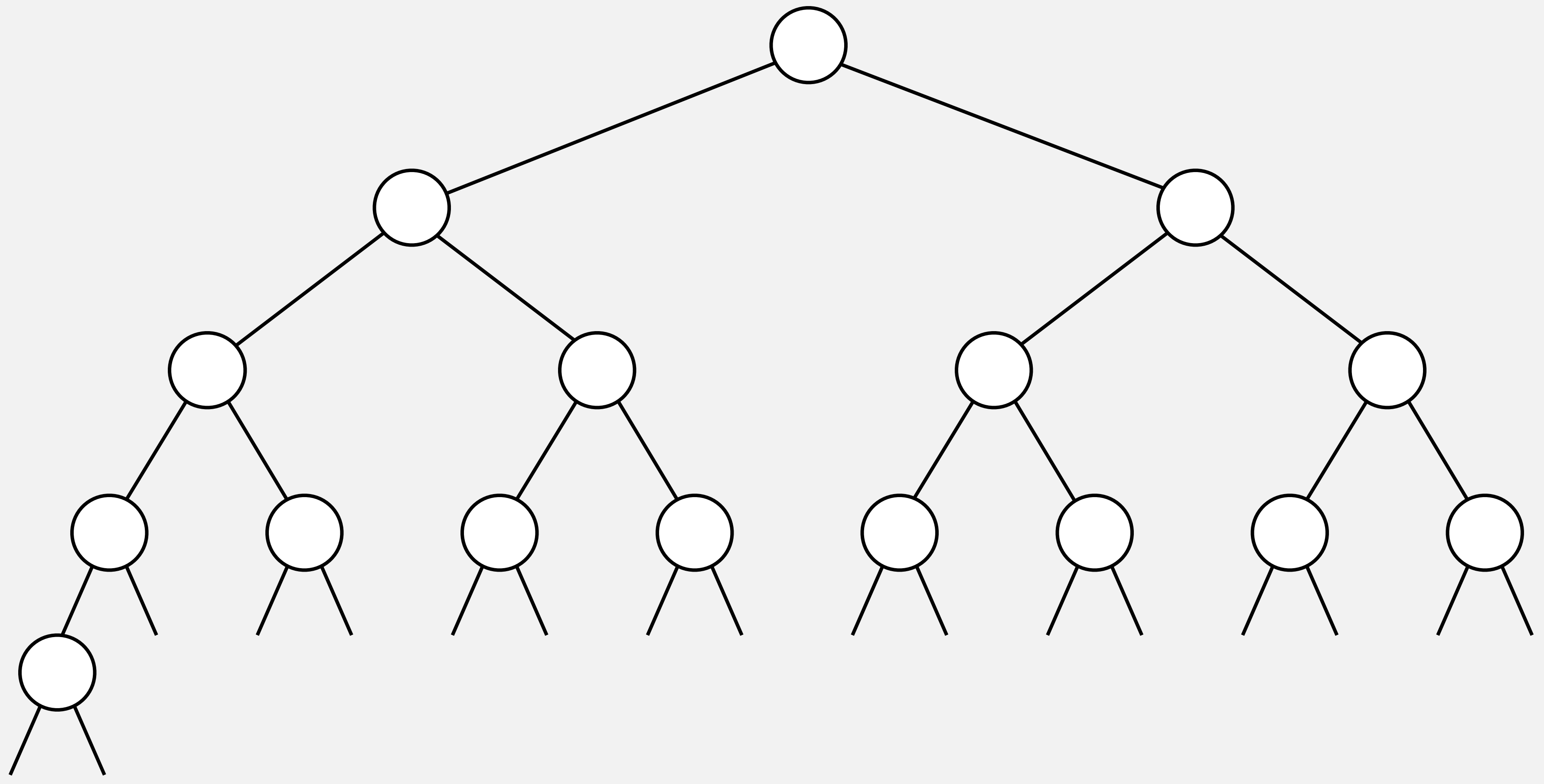
2.4 Priority Queues

- *API*
- *elementary implementations*
- ***binary heaps***
- *heapsort*
- *event-driven simulation*

Complete binary tree

Binary tree. Empty or node with links to two disjoint binary trees (left and right subtrees).

Complete tree. Every level (except possibly the last) is completely filled; the last level is filled from left to right.



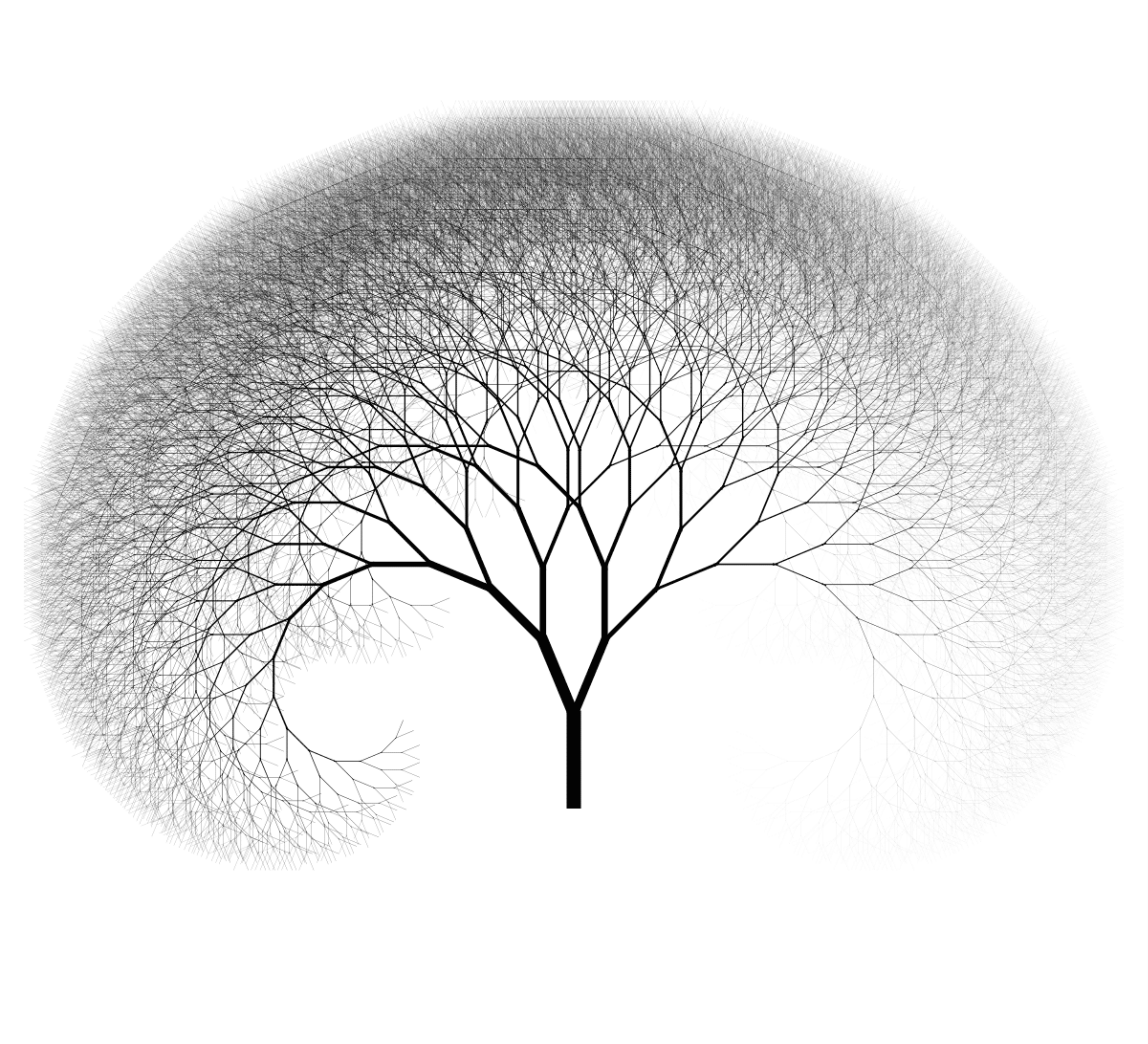
complete binary tree with n = 16 nodes (height = 4)

Property. Height of complete binary tree with n nodes is $\lfloor \log_2 n \rfloor$.

Pf. As you successively add nodes, height increases (by 1) only when n is a power of 2.

A complete binary tree in nature (of height 4)

A complete binary tree (of height 15)



Binary heap: representation

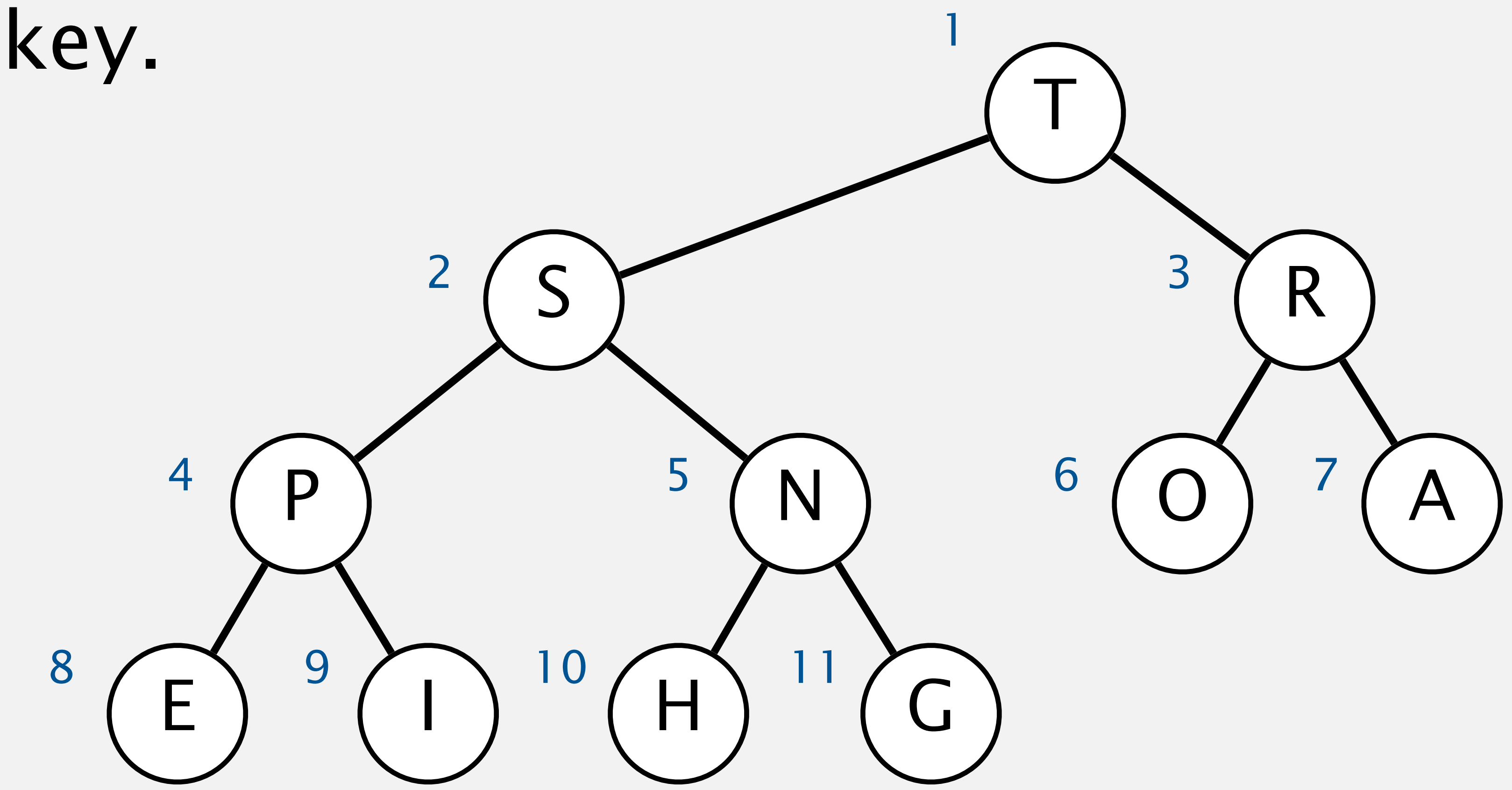
Binary heap. Array representation of a heap-ordered complete binary tree.

Heap-ordered tree.

- Keys in nodes.
- Child's key no larger than parent's key.

Array representation.

- Indices start at 1.
- Take nodes in level order.
- No explicit links!



	0	1	2	3	4	5	6	7	8	9	10	11
a[]	–	T	S	R	P	N	O	A	P	I	H	G

Priority queues: quiz 2

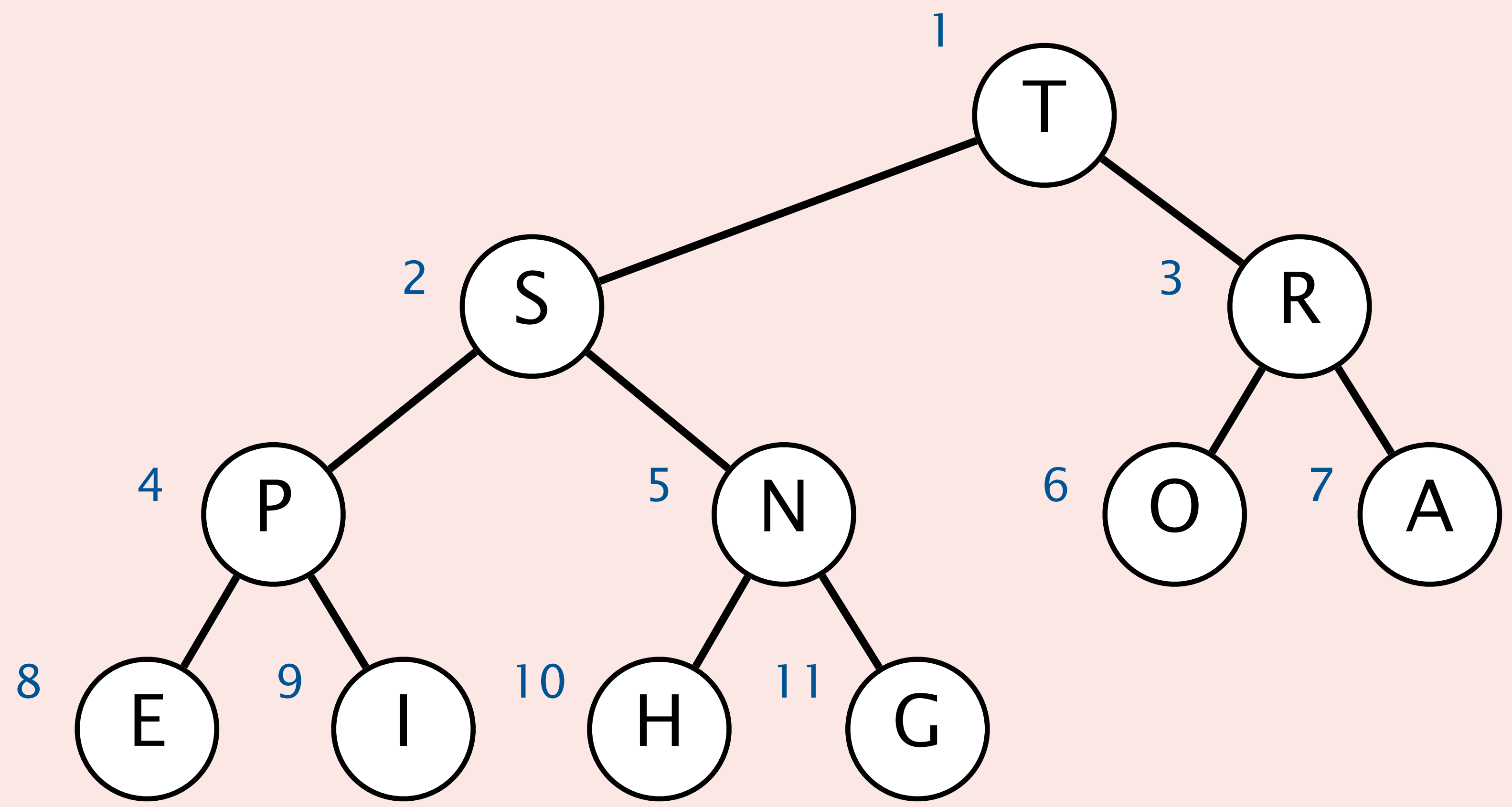
Consider the node at index k in a binary heap. Which Java expression produces the index of its parent?

A. `(k - 1) / 2`

B. `k / 2`

C. `(k + 1) / 2`

D. `2 * k`



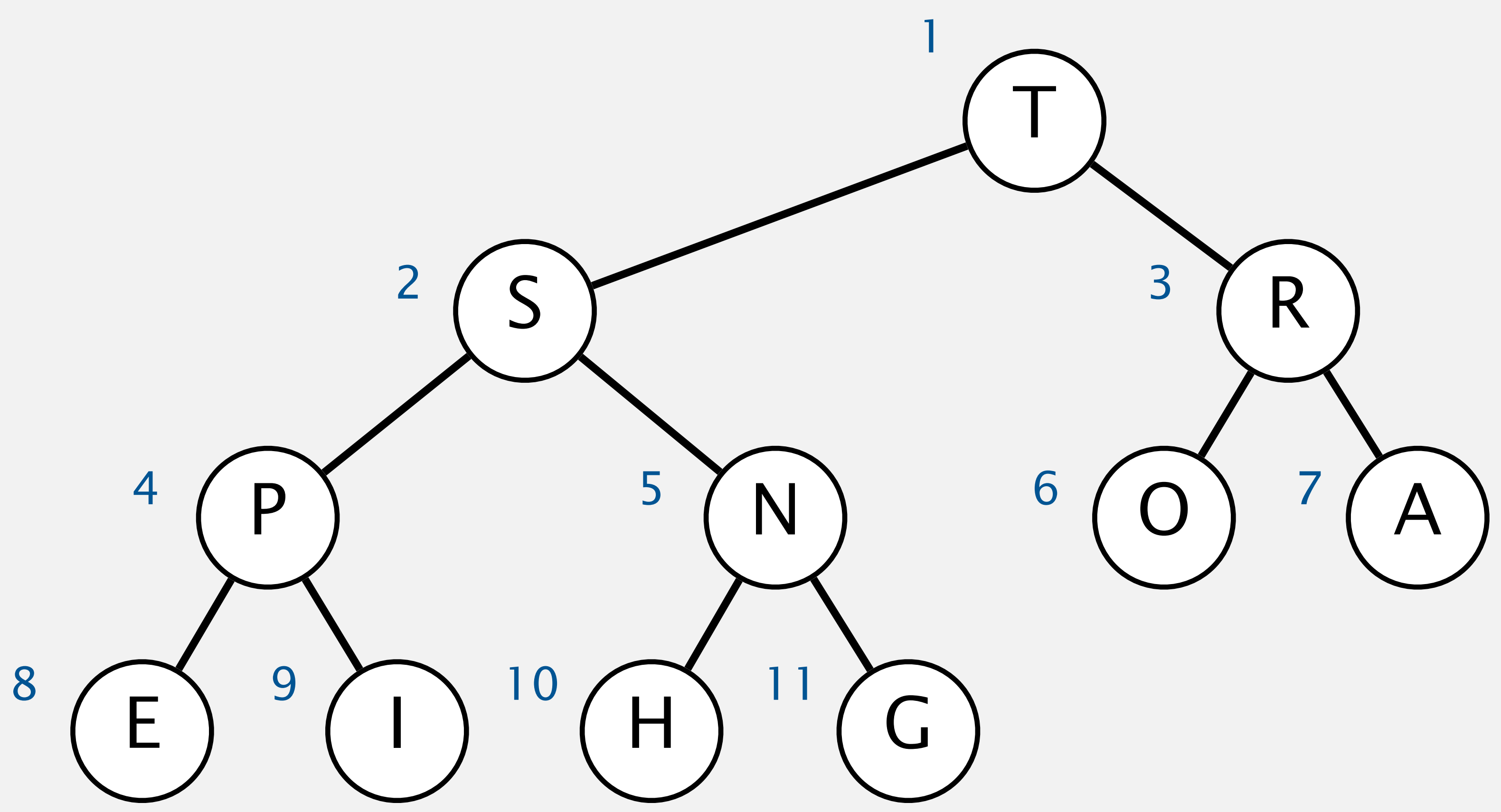
a[]	0	1	2	3	4	5	6	7	8	9	10	11
	–	T	S	R	P	N	O	A	E	I	H	G

Binary heap: properties

Proposition. Largest key is at index `1`, which is root of binary tree.

Proposition. Can use array indices to move up or down tree.

- Parent of key at index `k` is at index `k/2`.
- Children of key at index `k` are at indices `2*k` and `2*k + 1`.

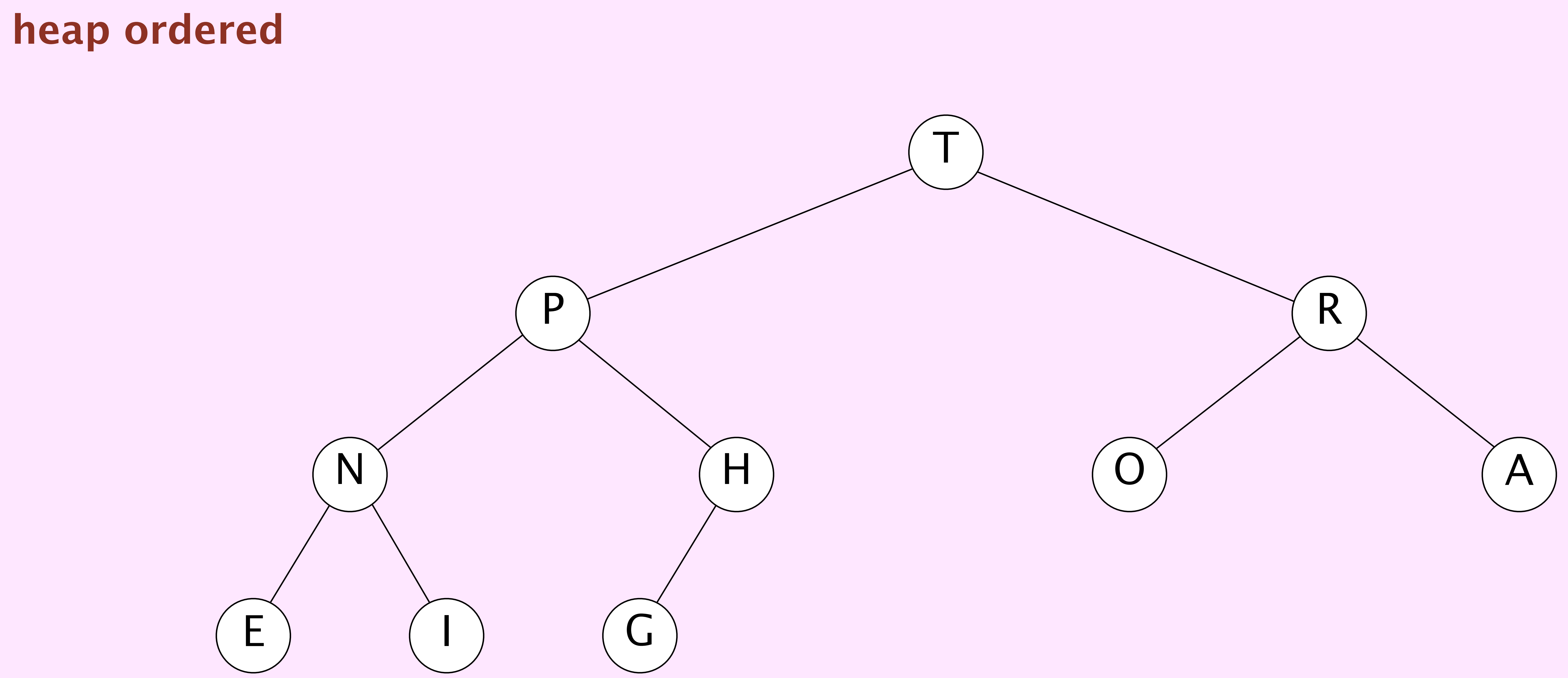


	0	1	2	3	4	5	6	7	8	9	10	11
a[]	–	T	S	R	P	N	O	A	P	I	H	G

Binary heap demo

Insert. Add node at end, then swim it up.

Remove the maximum. Exchange root with node at end, then sink it down.



T	P	R	N	H	O	A	E	I	G	

Binary heap: promotion

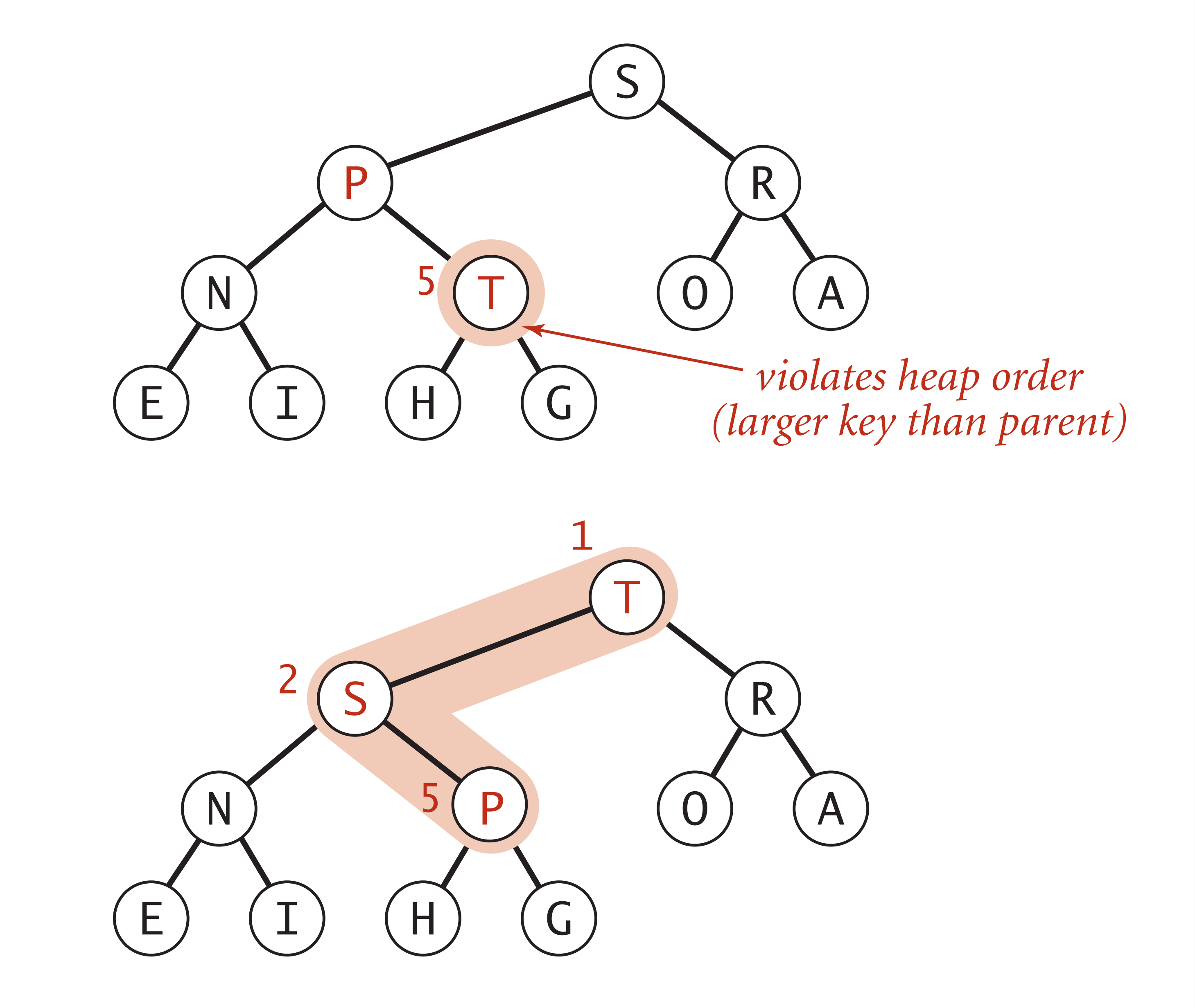
Scenario. Key in node becomes larger than key in parent's node.

To eliminate the violation:

- Exchange key in child node with key in parent node.
- Repeat until heap order restored.

```
private void swim(int k)
{
   while (k > 1 && less(k/2, k))
   {
      exch(k, k/2);
      k = k/2;
   }
}
```

parent of node at k is at k/2



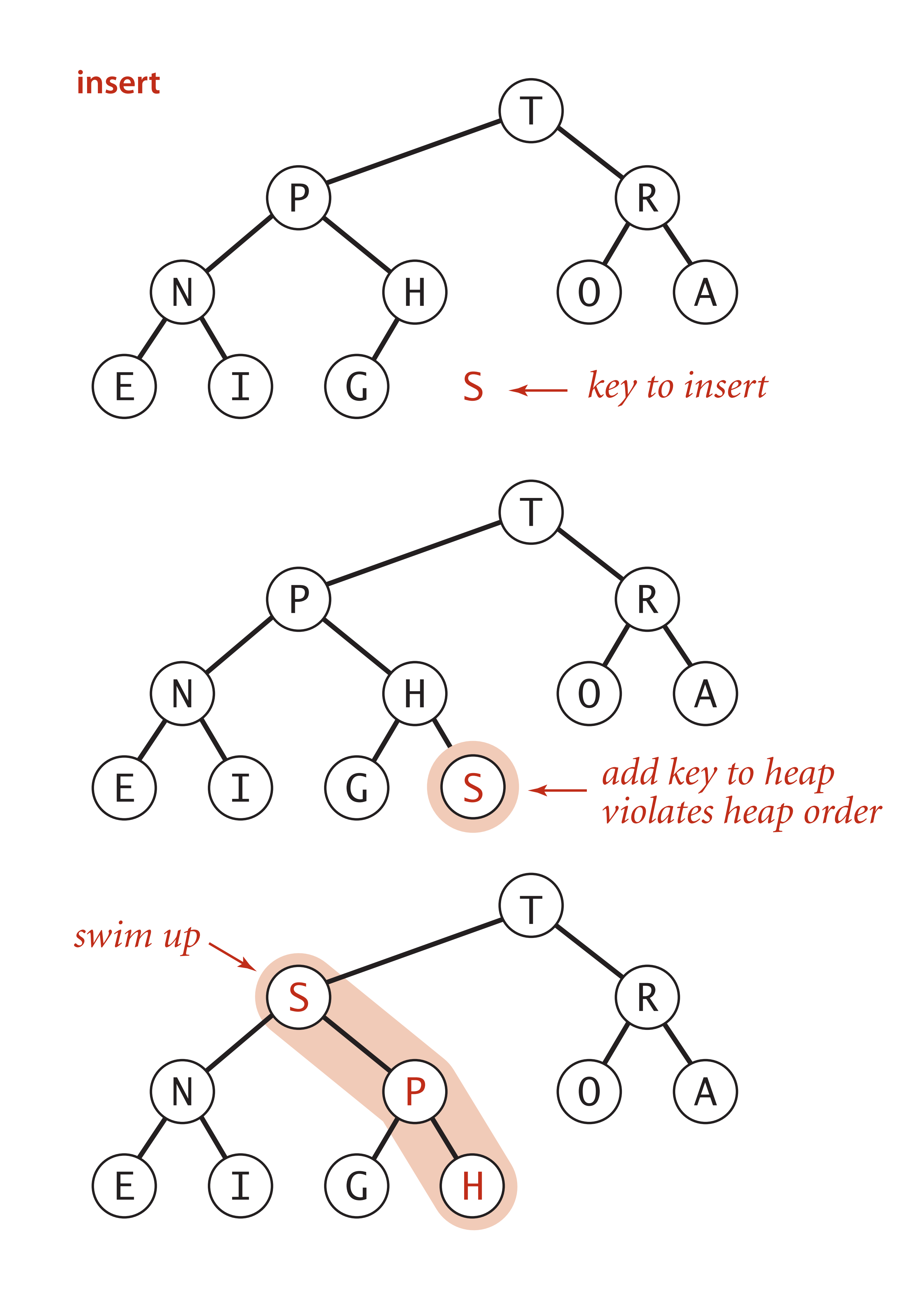
Peter principle. Node promoted to level of incompetence.

Binary heap: insertion

Insert. Add node at end in bottom level; then, swim it up.

Cost. At most $1 + \log_2 n$ compares.

```
public void insert(Key x)
{
   pq[++n] = x;
   swim(n);
}
```



Binary heap: demotion

Scenario. Key in node becomes smaller than one (or both) of keys in childrens' nodes.

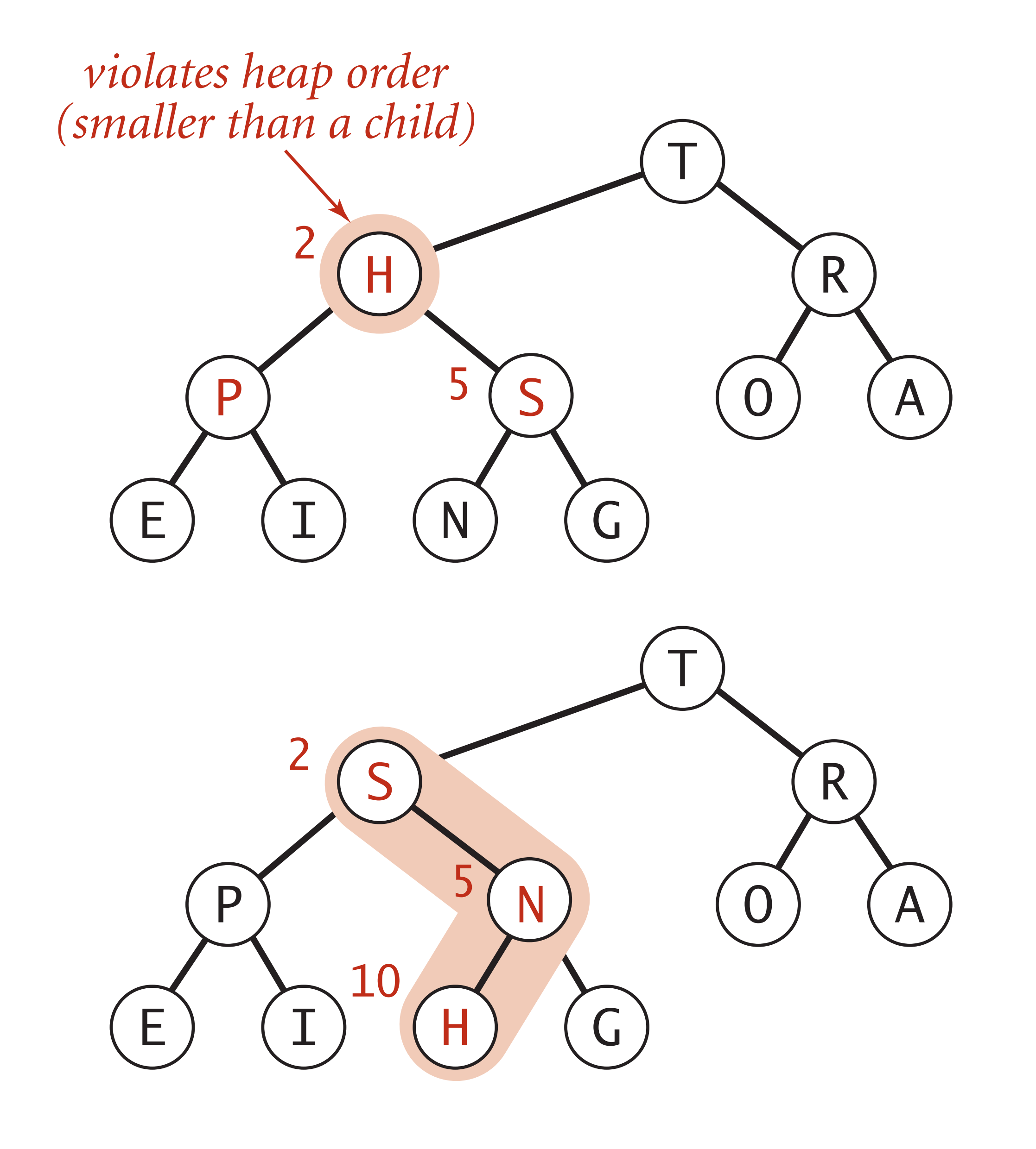
To eliminate the violation:

why not smaller child?

- Exchange key in parent node with key in larger child's node.
- Repeat until heap order restored.

```
private void sink(int k)
{
   while (2*k <= n)
   {
      int j = 2*k;
      if (j < n && less(j, j+1)) j++;
      if (!less(k, j)) break;
      exch(k, j);
      k = j;
   }
}
```

children of node at k
are at 2*k and 2*k+1



Power struggle. Better subordinate promoted.

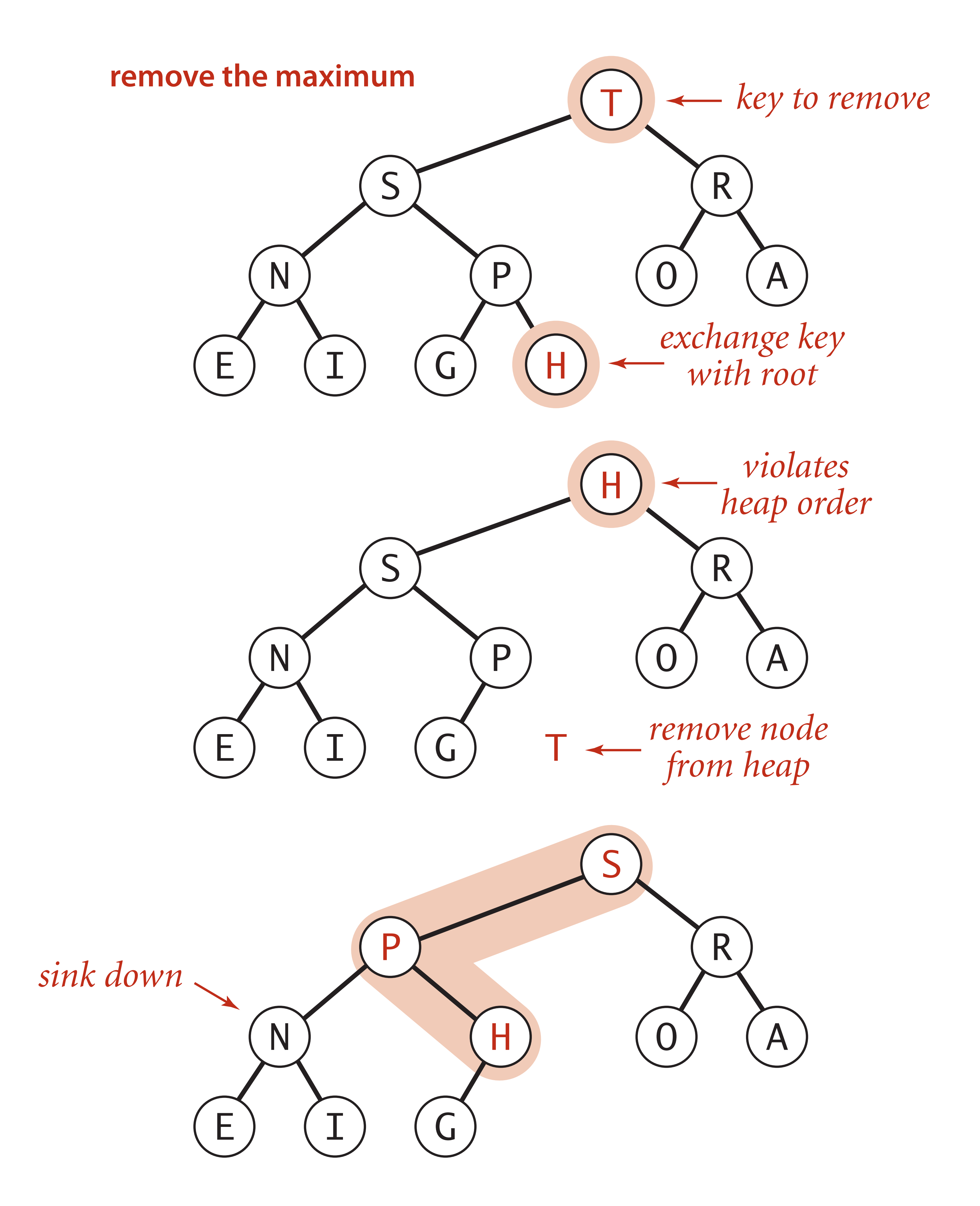
Binary heap: delete the maximum

Delete max. Exchange root with node at end; then, sink it down.

Cost. At most $2 \log_2 n$ compares.

```java
public Key delMax()
{
   Key max = pq[1];
   exch(1, n--);
   sink(1);
   pq[n+1] = null;
   return max;
}
```

prevent loitering



Binary heap: Java implementation

```
public class MaxPQ<Key extends Comparable<Key>>
{
   private Key[] a;
   private int n;

   public MaxPQ(int capacity)
   {  a = (Key[]) new Comparable[capacity+1];  }

   public boolean isEmpty()
   {  return n == 0;  }
   public void insert(Key key)  // see previous code
   public Key delMax()          // see previous code

   private void swim(int k)     // see previous code
   private void sink(int k)     // see previous code

   private boolean less(int i, int j)
   {  return a[i].compareTo(a[j]) < 0;  }
   private void exch(int i, int j)
   {  Key temp = a[i]; a[i] = a[j]; a[j] = temp;  }
}
```

fixed capacity (for simplicity)

PQ ops

heap helper functions

array helper functions

https://algs4.cs.princeton.edu/24pq/MaxPQ.java.html

Priority queue: implementations cost summary

Goal. Implement both INSERT and DELETE-MAX in $\Theta(\log n)$ time.

implementation	INSERT	DELETE-MAX	MAX
unordered list	1	n	n
ordered array	n	1	1
goal	$\boxed{\log n}$	$\boxed{\log n}$	1

order of growth of running time for priority queue with n items

Binary heap: considerations

Underflow and overflow.

- Underflow: throw exception if deleting from empty PQ.
- Overflow: add no-arg constructor and use resizing array.

leads to $O(\log n)$
amortized time per op
(how to make worst case?)

Minimum-oriented priority queue.

- Replace `less()` with `greater()`.
- Implement `greater()`.

Other operations.

- Remove an arbitrary item.
- Change the priority of an item.

can implement efficiently with `sink()` and `swim()`
[stay tuned for Prim/Dijkstra]

Immutability of keys.

- Assumption: client does not change keys while they're on the PQ.
- Best practice: use immutable keys.

immutable in Java: `String`, `Integer`, `Double`, ...

PRIORITY QUEUE WITH DELETE-RANDOM

Goal. Design an efficient data structure to support the following API:

- INSERT: insert a key.
- DELETE-MAX: return and remove a largest key.
- SAMPLE: return a random key.
- DELETE-RANDOM: return and remove a random key.

Midterm exam
Spring 2012

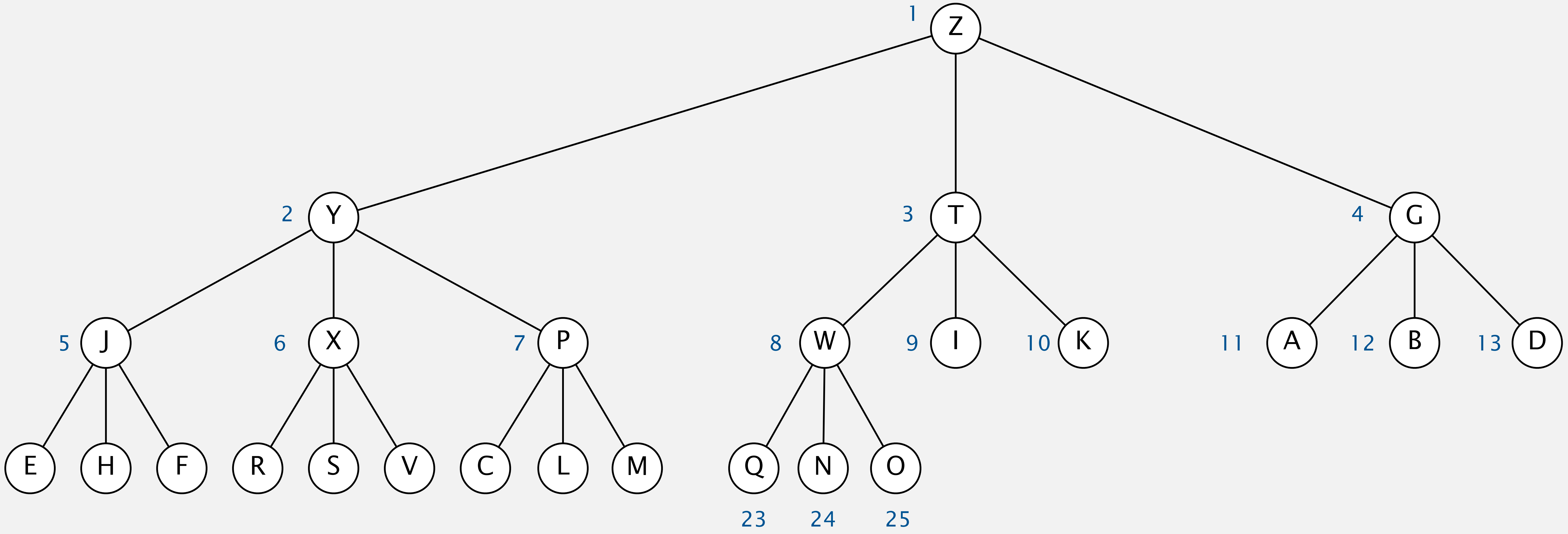
Multiway heaps

Multiway heaps.

- Complete d-way tree.
- Child's key no larger than parent's key.

Property. Height of complete d-way tree on n nodes is $\sim \log_d n$.

Property. Children of key at index k are at indices $3k - 1$, $3k$, and $3k + 1$.



3-way heap

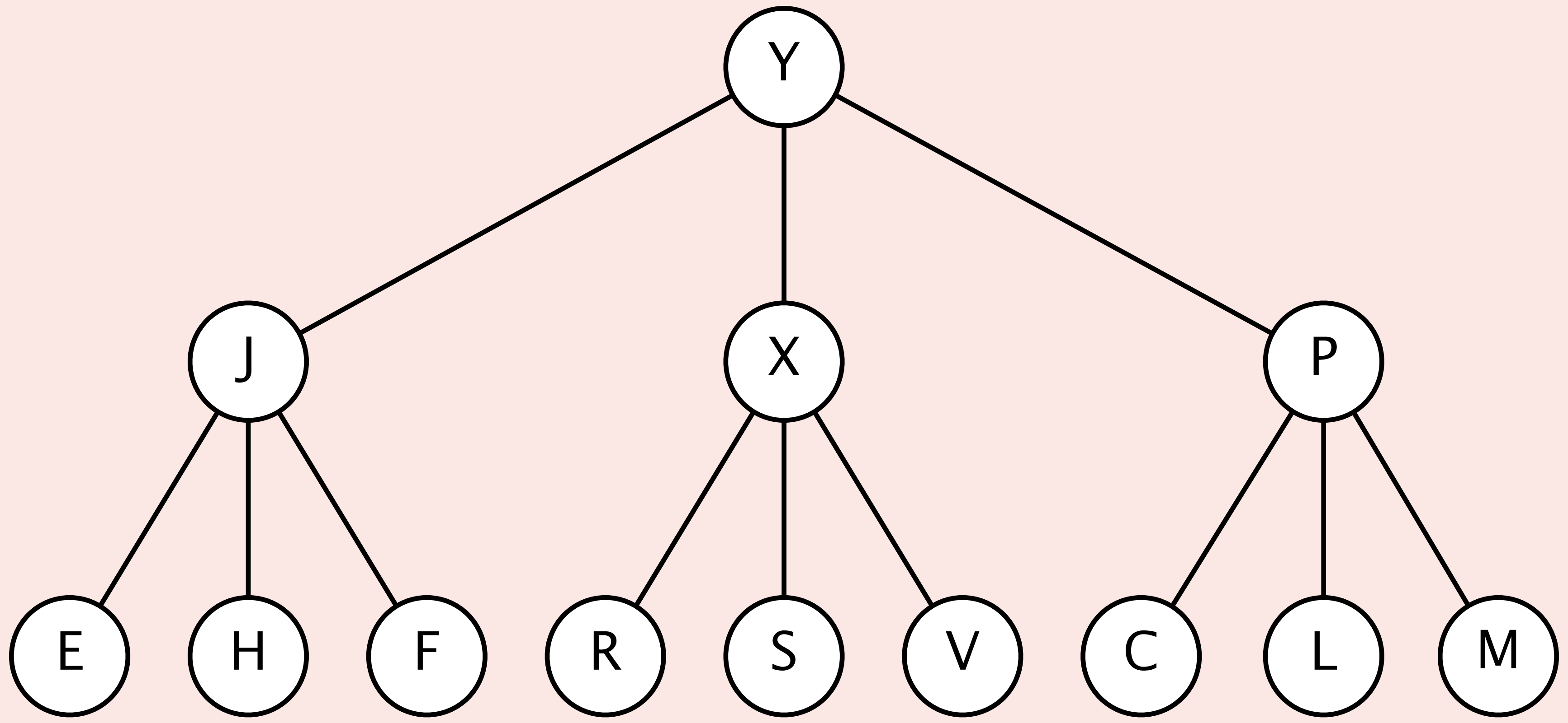
In the worst case, how many compares to INSERT and DELETE-MAX in a d-way heap as function of both n and d?

A. $\sim \log_d n$ and $\sim \log_d n$

B. $\sim \log_d n$ and $\sim d \log_d n$

C. $\sim d \log_d n$ and $\sim \log_d n$

D. $\sim d \log_d n$ and $\sim d \log_d n$



Priority queue: implementation cost summary

implementation	INSERT	DELETE-MAX	MAX	
unordered list	1	n	n	
ordered array	n	1	1	
binary heap	$\log n$	$\log n$	1	
d-ary heap	$\log_d n$	$d \log_d n$	1	← sweet spot: $d = 4$
Fibonacci	1	$\log n$	1	← see COS 423
impossible	1	1	1	← why impossible?

order-of-growth of running time for priority queue with n items

Algorithms

ROBERT SEDGEWICK | KEVIN WAYNE

https://algs4.cs.princeton.edu

2.4 PRIORITY QUEUES

- *APIs*
- *elementary implementations*
- *binary heaps*
- ***heapsort***
- *event-driven simulation*

Priority queues: quiz 5

What are the properties of this sorting algorithm?

```
public void sort(String[] a)
{
    int n = a.length;
    MinPQ<String> pq = new MinPQ<String>();

    for (int i = 0; i < n; i++)
        pq.insert(a[i]);

    for (int i = 0; i < n; i++)
        a[i] = pq.delMin();
}
```

A. $\Theta(n \log n)$ compares in the worst case.

B. In-place.

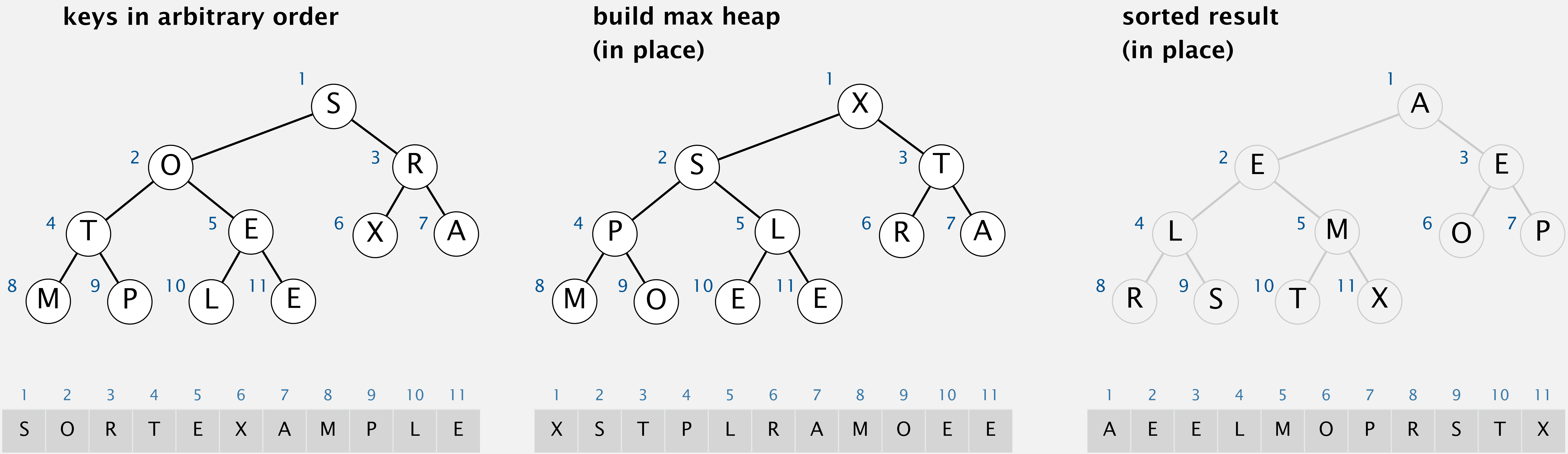
C. Stable.

D. *All of the above.*

Heapsort

Basic plan for in-place sort.

- View input array as a complete binary tree. ← we'll assume 1-indexed for now
- Heap construction: build a max-oriented heap with all n keys.
- Sortdown: repeatedly remove the maximum key.



Heapsort: top-down heap construction

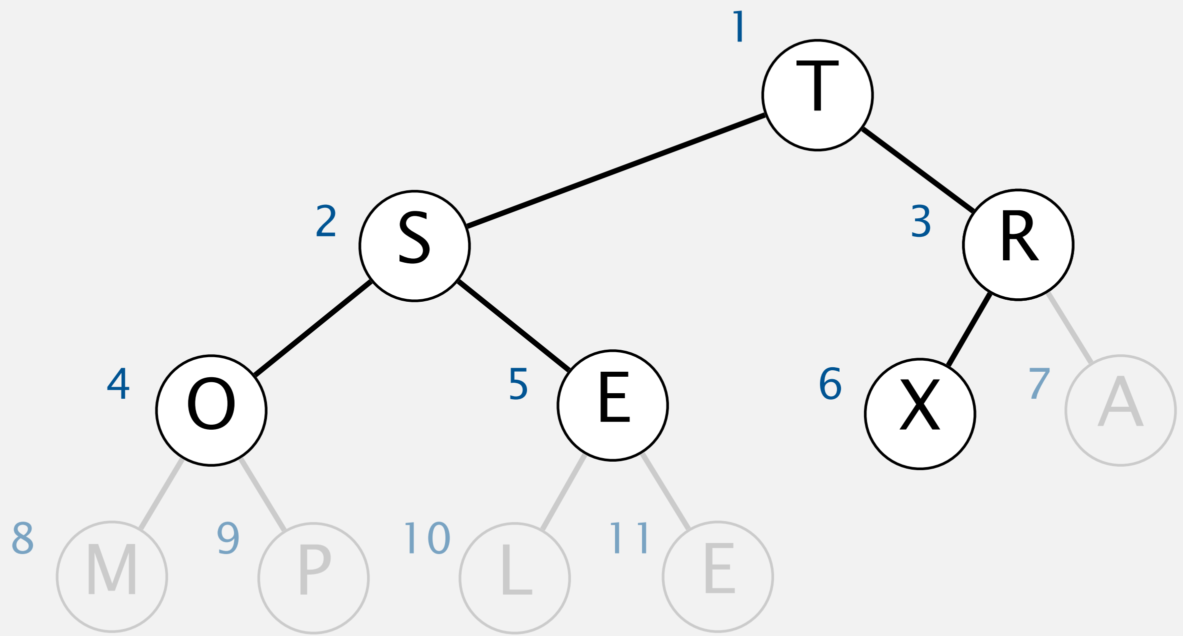
Top-down heap construction. Insert keys into a max heap, one at a time.

```
for (int k = 1; k <= n; k++)
   swim(a, k);
```

Invariants. After calling `swim(a, k)`,

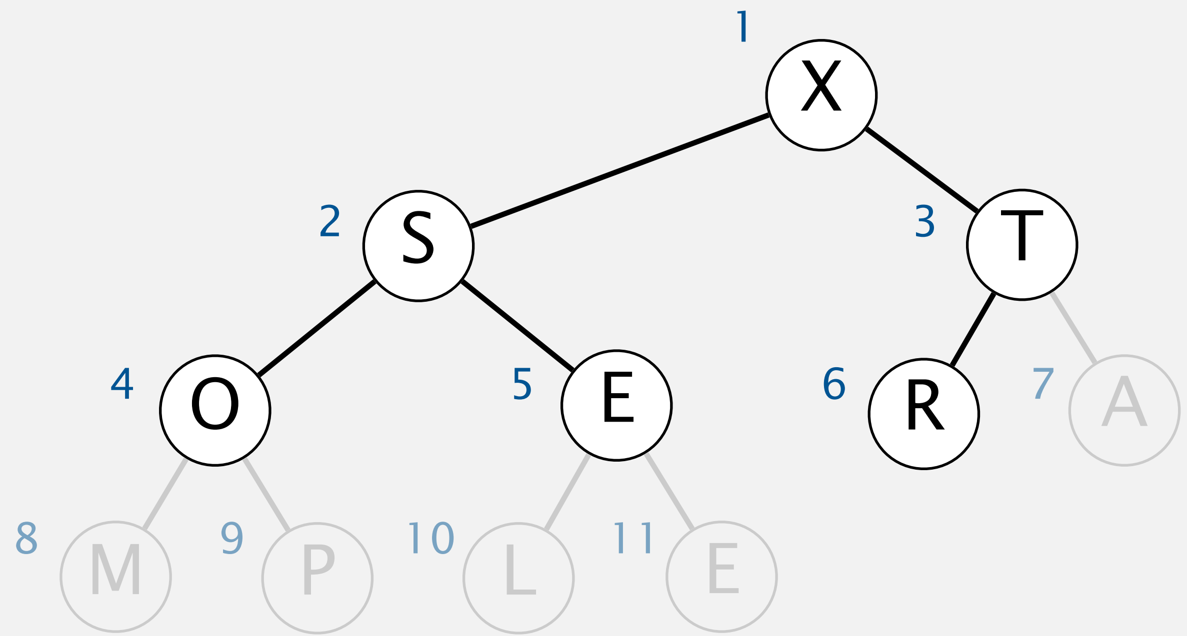
- `a[1..k]` is a max heap.
- `a[k+1..n]` are untouched.

before call to swim(a, k)
k = 6



1	2	3	4	5	6	7	8	9	10	11
T	S	R	O	E	X	A	M	P	L	E

after call to swim(a, k)
k = 6



1	2	3	4	5	6	7	8	9	10	11
X	S	T	O	E	R	A	M	P	L	E

Heapsort: sortdown

Second pass.

- Remove the maximum, one at a time.
- Leave in array (instead of nulling out).

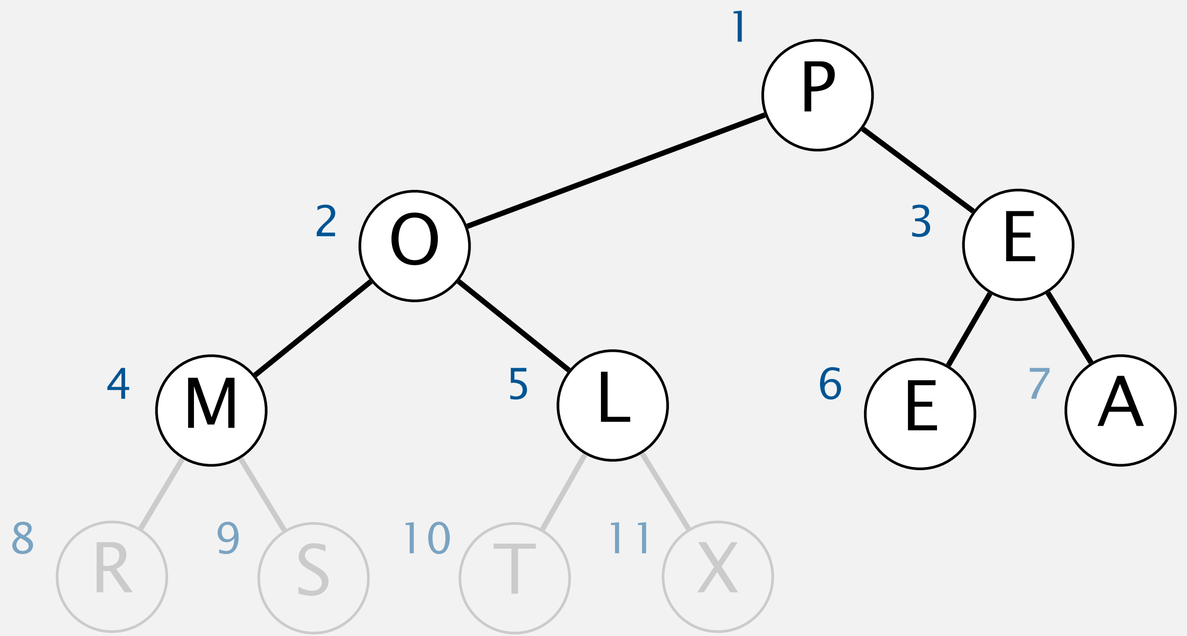
```
int k = n;
while (k > 1)
{
   exch(a, 1, k--);
   sink(a, 1, k);
}
```

delete-max
(but leave in array)

Invariants. After calling `sink(a, 1, k)`,

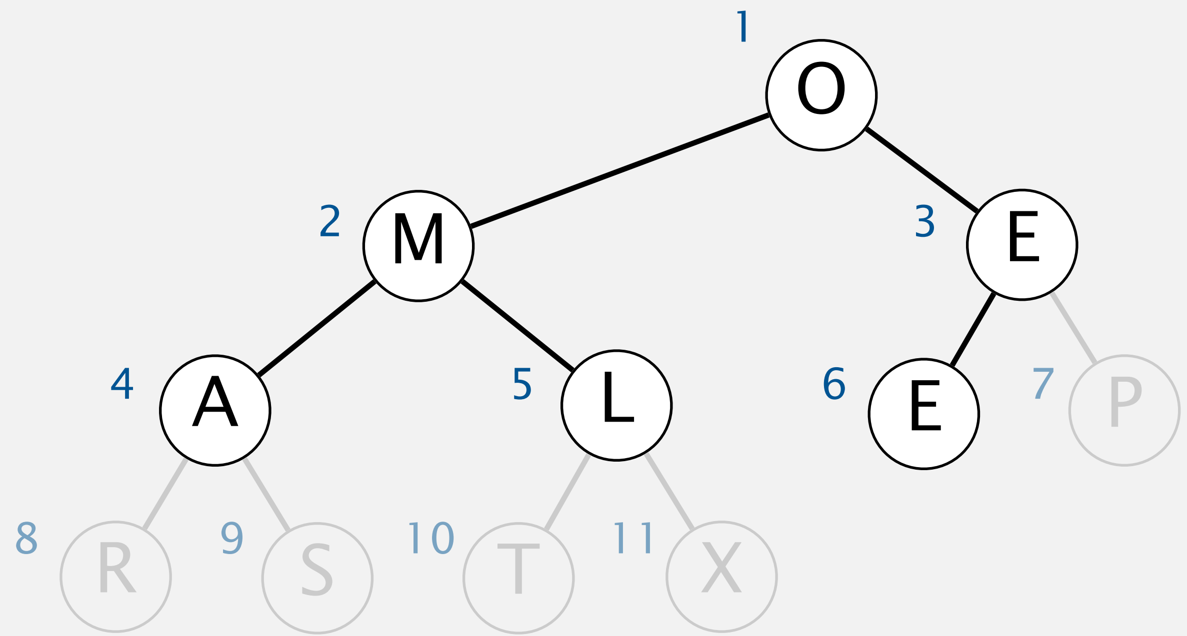
- `a[1..k-1]` is a max heap.
- `a[k..n]` are in final sorted order.

before call to sink(a, 1, k)
k = 7



1	2	3	4	5	6	7	8	9	10	11
P	O	E	M	L	E	A	R	S	T	X

after call to sink(a, 1, k)
k = 7



1	2	3	4	5	6	7	8	9	10	11
O	M	E	A	L	E	P	R	S	T	X

Heapsort: Java implementation

```java
public class HeapTopDown
{
   public static void sort(Comparable[] a)
   {
      // top-down heap construction
      int n = a.length;
      for (int k = 1; k <= n; k--)
         swim(a, k);

      // sortdown
      int k = n;
      while (k > 1)
      {
         exch(a, 1, k--);
         sink(a, 1, k);
      }
   }

   ...
}
```

https://algs4.cs.princeton.edu/24pq/HeapTopDown.java.html

```java
private static void sink(Comparable[] a, int k, int n)
{  /* as before */  }

private static void swim(Comparable[] a, int k)
{  /* as before */  }
```

but make static
(and pass arguments)

```java
private static boolean less(Comparable[] a, int i, int j)
{  /* as before */  }

private static void exch(Object[] a, int i, int j)
{  /* as before */  }
```

but convert from 1-based
indexing to 0-base indexing

Heapsort: mathematical analysis

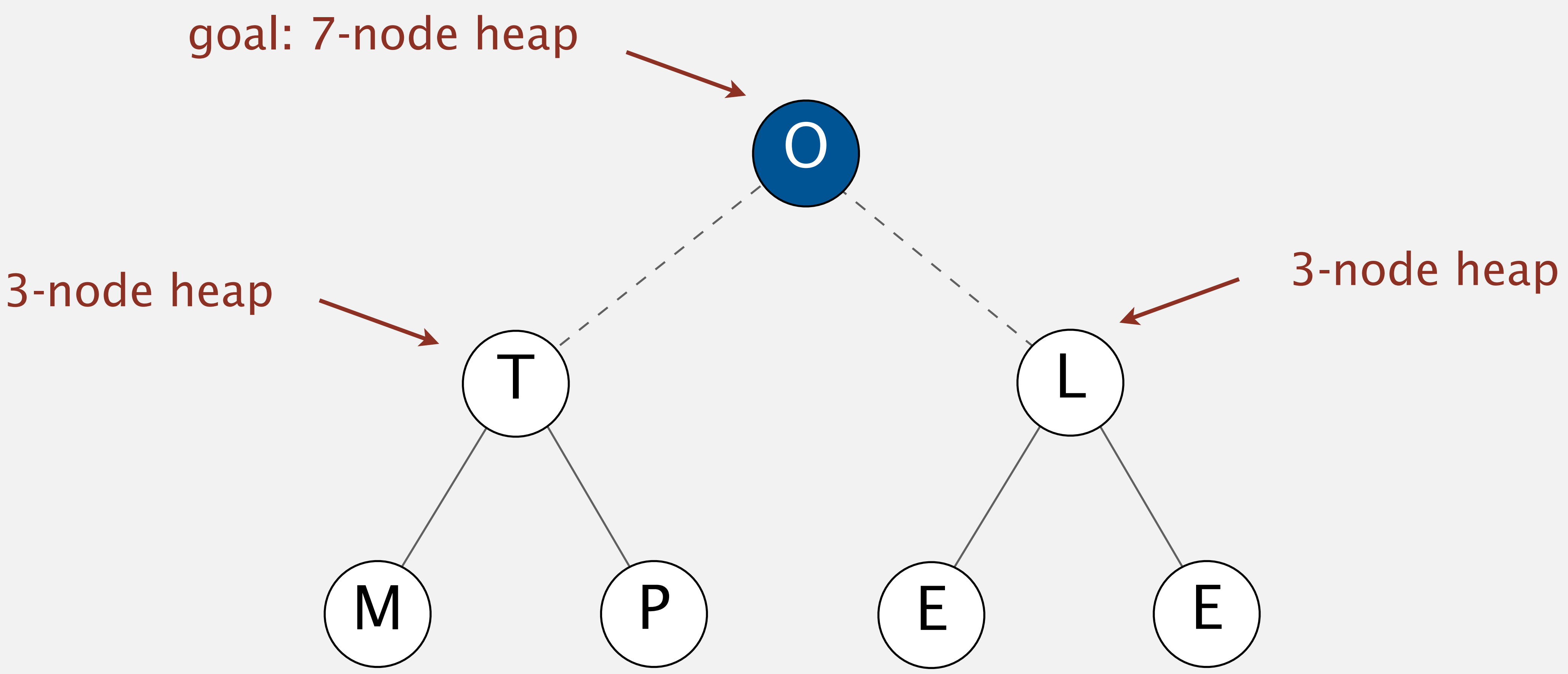
Proposition. Heapsort uses only $\Theta(1)$ extra space.

Proposition. Heapsort makes $\leq 3\, n \log_2 n$ compares (and $\leq 2\, n \log_2 n$ exchanges).

- Top-down heap construction: $\log_2 1 + \log_2 2 + \ldots + \log_2 n \;=\; \log_2 (n!) \;\sim\; n \log_2 n$ compares.
- Sortdown: $2\,(\log_2 1 + \log_2 2 + \ldots + \log_2 n) \;\sim\; 2n \log_2 n$ compares.

Bottom-up heap construction. [see book] Successively building larger heap from smaller ones.

Proposition. Makes $\leq 2\, n$ compares (and $\leq n$ exchanges).



Heapsort: context

Significance. In-place sorting algorithm with $\Theta(n \log n)$ worst-case.

- Mergesort: no, $\Theta(n)$ extra space. ← in-place merge possible, not practical
- Quicksort: no, $\Theta(n^2)$ time in worst case. ← $\Theta(n \log n)$ worst-case quicksort possible, not practical
- Heapsort: yes!

Bottom line. Heapsort is optimal for both time and space, but:

- Inner loop longer than quicksort's.
- Makes poor use of cache.
- Not stable.

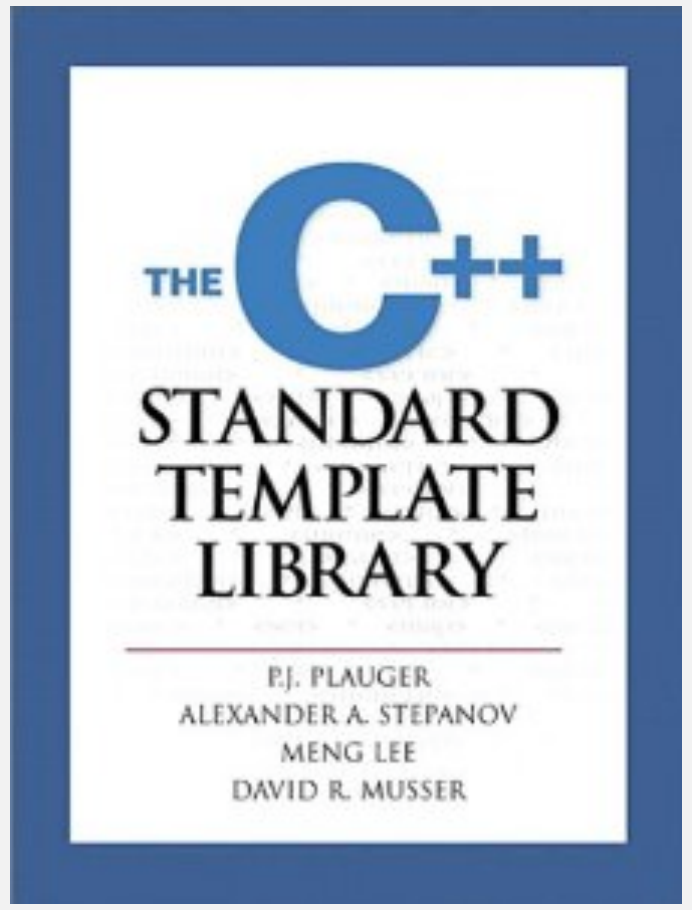
can be improved using advanced caching tricks

Introsort

Goal. As fast as quicksort in practice; $\Theta(n \log n)$ worst case; in place.

Introsort.

- Run quicksort.
- Cutoff to heapsort if function-call stack depth exceeds $2 \log_2 n$.
- Cutoff to insertion sort for $n \le 16$.



Microsoft® .NET

GO

In the wild. C++ STL, Microsoft .NET Framework, Go.

Sorting algorithms: summary

	inplace?	stable?	best	average	worst	remarks
selection	✔		½ n^2	½ n^2	½ n^2	n exchanges
insertion	✔	✔	n	¼ n^2	½ n^2	use for small n or partially ordered
merge		✔	½ $n \log_2 n$	$n \log_2 n$	$n \log_2 n$	$\Theta(n \log n)$ guarantee; stable
timsort		✔	n	$n \log_2 n$	$n \log_2 n$	improves mergesort when pre-existing order
quick	✔		$n \log_2 n$	$2 n \ln n$	½ n^2	$\Theta(n \log n)$ probabilistic guarantee; fastest in practice
3-way quick	✔		n	$2 n \ln n$	½ n^2	improves quicksort when duplicate keys
heap	✔		$3 n$	$2 n \log_2 n$	$2 n \log_2 n$	$\Theta(n \log n)$ guarantee; in-place
?	✔	✔	n	$n \log_2 n$	$n \log_2 n$	holy sorting grail

number of compares to sort an array of n elements