509 Homework 1 Haocong Wang mw814

1. Let $C = \{x \mid f_i(x) \le 0\}$. Suppose $x_1, x_2 \in C$

$\Rightarrow f_i(x_1) \le 0,\ f_i(x_2) \le 0$

$f_i(x)$ is convex $\Rightarrow f_i(\alpha x_1 + (1-\alpha)x_2) \le \alpha f_i(x_1) + (1-\alpha) f_i(x_2) \le 0 \cdot \alpha + (1-\alpha)\cdot 0 = 0$ $(0 \le 1 \le \alpha)$

$\Rightarrow \alpha x_1 + (1-\alpha) x_2 \in C \Rightarrow C$ is convex

2. Suppose $\alpha x_1 + (1-\alpha) x_2 = x$ $(0 \le \alpha \le 1)$

$\because f(x)$ is convex $\therefore f(\alpha x_1 + (1-\alpha) x_2) \le \alpha f(x_1) + (1-\alpha) f(x_2) \le \alpha t + (1-\alpha) t = t$

$\Rightarrow \{(x, t) \mid f(x) \le t\}$ is convex

3. (1) $\vec{M}_1, \vec{M}_2$ are positive definite, so

for any non-zero vector $\vec{x} \in R^n$, we have

$\vec{x}' \vec{M}_1 \vec{x} > 0,\ \vec{x}' \vec{M}_2 \vec{x} > 0 \Rightarrow \vec{x}'(\vec{M}_1 + \vec{M}_2)\vec{x} = \vec{x}' \vec{M}_1 \vec{x} + \vec{x}' \vec{M}_2 \vec{x} > 0$

$\Rightarrow \vec{M}_1 + \vec{M}_2$ is positive definite

(2) $S^n_{++} = \{X \in R^{n\times n} \mid X = X^T,\ X > 0\}$

$\forall \alpha \in [0,1],\ \vec{A}, \vec{B} \in S^n_{++},\ \forall x \in R^n$, we have $x^T A x > 0,\ x^T B x > 0$

$\Rightarrow x^T(\alpha A + (1-\alpha) B) x = \alpha x^T A x + (1-\alpha) x^T B x > 0 \Rightarrow S^n_{++}$ is convex

4. Consider the function $f: R^3 \to S^2$ satisfying

$$f(x_1, x_2, x_3) = \begin{bmatrix} x_1 + x_2 & x_1 - 2x_2 \\ x_1 - 2x_3 & x_2 + 2x_3 \end{bmatrix}$$

$\Rightarrow f$ is linear and affine, so the set of interest can be written as

$\{(x_1, x_2, x_3) \mid f(x_1, x_2, x_3) \ge 0\} = f^{-1}(S^3_+)$

$\because S^3_+$ is convex and f is linear and affine $\therefore f^{-1}(S^3_+)$ is convex

5. Let $x = \begin{bmatrix} \alpha_1 x_1 \\ \alpha_2 x_2 \\ \vdots \\ \alpha_n x_n \end{bmatrix} \Rightarrow f(x) = x^T P x$

$\Rightarrow f'(x) = Px \Rightarrow f''(x) = P$

If $P \ge 0$, then $\nabla^2 f(x) \ge 0 \Rightarrow f(x)$ is convex (sufficient)

If $P < 0$, then $\nabla^2 f(x) < 0 \Rightarrow f(x)$ is nonconvex (necessary)

$\Rightarrow$ The condition is $P \ge 0$ or P is semidefinite positive