Homework 5

Convex Optimization

Due April 16

1. Consider the following non-smooth function on the real line:

$$f(x) = \max\left((x+1)^2, (x-3)^2\right)$$

Describe the subdifferential $\partial f(x)$ at every point $x \in \mathbb{R}$.

2. Prove or disprove: the subdifferential $\partial f(x)$ of a convex function is a convex set at every $x \in \mathbb{R}$.

3. Recall the subdifferential for the nuclear norm for an $n_1 \times n_2$ matrix $\boldsymbol{X}$.

$$\partial \left\|\boldsymbol{X}\right\|\star = \left\{\boldsymbol{U}\boldsymbol{V}^T + \boldsymbol{W} : \boldsymbol{U}\boldsymbol{W} = \boldsymbol{0}, \boldsymbol{W}\boldsymbol{V} = \boldsymbol{0}, \left\|\boldsymbol{W}\right\| \leq 1\right\}$$

In the expression above, $\boldsymbol{X}$ has rank r and its SVD is $\boldsymbol{X} = \boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^T$, where $\boldsymbol{U}$ is $n_1 \times r$, $\boldsymbol{\Sigma}$ is $r \times r$ and $\boldsymbol{V}$ is $n_2 \times r$. Recall that

$$\begin{aligned}
\boldsymbol{x}^+ &= \mathrm{prox}_{t\|cdot\|\star}(\boldsymbol{X}) \\
&= \arg\min_{\boldsymbol{Z}} \left(\left\|\boldsymbol{Z}\right\|\star + \frac{1}{2t}\left\|\boldsymbol{Z} - \boldsymbol{Z}\right\| F^2 \right)
\end{aligned}$$

if and only if

$$\boldsymbol{X} - \boldsymbol{X}^+ \in t\partial \left\|\boldsymbol{X}^+\right\|\star$$

Show that we can compute the prox operator above by singular value thresholding:

$$\boldsymbol{X}^+ = \boldsymbol{U}\boldsymbol{\Sigma}^+\boldsymbol{V}^T, where^+[i,i] = \max(\sigma_i t, 0).$$