Homework 3 Haocong Wang mw814

Q1. Suppose $\min t$ s.t. $(Ax+b)^T F(x)^{-1} (Ax+b) \le t$

$F(x) > 0$

$\Rightarrow \begin{bmatrix} t & (Ax+b)^T \\ Ax+b & F(x) \end{bmatrix} \succeq 0 \Rightarrow \min t$ s.t. $\begin{bmatrix} t & (Ax+b)^T \\ Ax+b & F(x) \end{bmatrix} \succeq 0$

Q2. We have $L(x, \lambda) = x^T W(\lambda) x - \lambda^T$

$W(\lambda) = W_0 + \lambda^T W_1$, $x^T W_1 x - 1 \le 0$

$\Rightarrow$ dual function $g(\lambda) = \inf_x L(x-\lambda)$

$= \inf_x [x^T(W_0 + \lambda^T W_1)x - \lambda^T]$

$= \begin{cases} -\lambda & W_0 + \lambda^T W_1 \succeq 0 \\ -\infty & \text{otherwise} \end{cases}$

Q3. Primal Dual

$\max\ 0^T x \Rightarrow \min\ b^T x$ s.t. $A^T \lambda = 0$

s.t. $Ax \le b$ $\lambda \ge 0$

If the given set is nonempty, we have $\exists \lambda \in R^m$, $b^T \lambda < 0$

$\Rightarrow$ no optimal solution for dual problem

$\Rightarrow$ no feasible solution for primal problem $\Rightarrow \{x \mid x \in R^n, Ax \le b\}$ is empty

Q4. For P_1, we can set $P_1^*(a)$: $\min\ a^T x$ s.t. $Ax \le b$

$L_1(x, \lambda_1) = a^T x + \lambda_1^T (Ax - b) = (a^T + \lambda_1^T A) x - \lambda^T b$

$g_1(\lambda) = \inf_x L_1(x, \lambda_1) = \begin{cases} -b^T \lambda, & A^T \lambda_1 + a = 0 \\ -\infty, & \text{otherwise} \end{cases}$

$\Rightarrow$ Dual problem: $\max\ -b^T \lambda_1$ s.t. $A^T x_1 + a = 0$ $\lambda \ge 0$

Also, for P_2, we have $P_2^*(a)$: $\max\ a^T x$ $\Rightarrow$ Dual: $\min\ d^T \lambda_2$

s.t. $Cx \le d$ s.t. $C^T \lambda_2 - a = 0$, $\lambda \ge 0$

For γ, suppose $p_2^*(a) < \gamma < p_1^*(a)$.

Assume $\|a\|_1 \le 1$ since this is homogeneous

$\Rightarrow \max\ P_1^*(a) - P_2^*(a)$ s.t. $\|a\|_1 \le 1$

Since $P_1^*(a)$, $P_2^*(a)$ satisfy strong duality, we have

$$\begin{aligned} \max\ & -b^T\lambda_1 - d^T\lambda_2 \\ \text{s.t. } & A^T\lambda_1 + a = 0 \\ & C^T\lambda_2 - a = 0 \\ & \lambda_1 \geq 0,\ \lambda_2 \geq 0,\ \|a\|_1 \leq 1 \end{aligned}$$

in which, we minimize with respect to λ_1, λ_2 and a