1. $\min (Ax+b)^T F(x)^{-1} (Ax+b)$

$\text{s.t. } F(x) > 0 \quad \text{where } F(x) = F_0 + x_1 F_1 + \cdots + x_n F_n$

$\Rightarrow \min \ t$

$\text{s.t } (Ax+b)^T F(x)^{-1} (Ax+b) \le t$

$F(x) > 0$

$\Rightarrow t \geq (Ax+b)^T F(x)^{-1} (Ax+b)$

$F(x) > 0$

$$\Rightarrow \begin{bmatrix} t & (Ax+b)^T \\ (Ax+b) & F(x) \end{bmatrix} \succeq 0$$

$\Rightarrow \min \ t$

$$\text{s.t } \begin{bmatrix} t & (Ax+b)^T \\ Ax+b & F(x) \end{bmatrix} \succeq 0$$

2. $\min x^T W_0 x$

$\text{s.t. } x^T W_1 x \le 1 \iff x^T W_1 x - 1 \le 0.$

$L(x, \lambda) = x^T W(\lambda) x - \lambda^T$

$W(\lambda) = W_0 + \lambda^T W_1$

$\Rightarrow$ dual function $g(\lambda) = \inf_x L(x, \lambda)$

$$= \inf_x \left[x^T (W_0 + \lambda^T W_1) x - \lambda^T \right]$$

$$= \begin{cases} -\lambda & W_0 + \lambda^T W_1 \succeq 0 \\ -\infty & \text{otherwise.} \end{cases}$$

3. Primal problem

$$\max\ 0^T x \quad \text{s.t.}\ Ax \preceq b$$

$\Rightarrow$ Dual problem

$$\min\ b^T \lambda \quad \text{s.t.}\ A^T\lambda = 0,\ \lambda \succeq 0.$$

if the set $\{\lambda \mid \lambda \in R^m, \lambda \succeq 0, A^T\lambda = 0, b^T\lambda < 0\}$ is not empty

which means that $\exists \lambda \in R^m$, $b^T\lambda < 0$.

$\Rightarrow$ no optimal solution for dual problem

$\Rightarrow$ primal problem don't have feasible solution

$\Rightarrow$ $\{x \mid x \in R^n, Ax \preceq b\}$ is empty

4. For P_1, we can set a optimize problem like.

$$p_1^*(a): \min \; a^T x \quad \text{s.t.} \; Ax \preceq b$$

$$L_1(x, \lambda_1) = a^T x + \lambda_1^T (Ax - b) = (a^T + \lambda_1^T A) x - \lambda^T b$$

$$\Rightarrow g_1(\lambda) = \inf_x L_1(x, \lambda_1) = \begin{cases} -b^T \lambda_1 & A^T \lambda_1 + a = 0 \\ -\infty & \text{otherwise.} \end{cases}$$

$$\Rightarrow \text{Dual}: \max \; -b^T \lambda_1 \quad \text{s.t.} \; A^T \lambda_1 + a = 0 \quad \lambda \succeq 0$$

Similarly, we can set

$$p_2^*(a) \quad \max \; a^T x \quad \text{s.t.} \; Cx \preceq d \quad \Rightarrow \text{Dual}: \min \; d^T \lambda_2 \quad \text{s.t.} \; C^T \lambda_2 - a = 0, \lambda \succeq 0$$

to find γ, we let $p_2^*(a) < \gamma < p_1^*(a)$., which is to find $p_1^*(a) - p_2^*(a)$. Because this is homogeneous, it may can't be bounded unless we assume $\|a\|_1 \leq 1$

So we form the problem to $\max \; p_1^*(a) - p_2^*(b)$ s.t. $\|a\|_1 \leq 1$

Because $p_1^*(a), p_2^*(a)$ represent LP problem which satisfies strong duality, we can form the LP

$$\begin{aligned} \max \quad & -b^T \lambda_1 - d^T \lambda_2 \\ \text{s.t.} \quad & A^T \lambda_1 + a = 0 \\ & C^T \lambda_2 - a = 0 \\ & \lambda_1 \succeq 0 \quad \lambda_2 \succeq 0 \\ & \|a\|_1 \leq 1 \end{aligned}$$

Where we are minimizing with respect to λ_1, λ_2 and a