Avaliação de Álgebra Linear

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Nome e matrícula:

Leia atentamente cada item antes de começar a resolver. É preciso colocar detalhes que expliquem matematicamente sua resposta. Cada item correto vale 1.5 pontos.

Problema: Sejam $\mathcal{A} = \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\} = \{v_1, v_2, v_3\}$, $k = \begin{bmatrix} 4 \\ 0 \\ 1 \\ 2 \end{bmatrix}$ e $T : \mathbb{R}^3 \to \mathbb{R}^4$ a transformação linear definida como

$$T(u) = xv_1 + yv_2 + zv_3 = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & 1 \\ 0 & -1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \qquad u = \begin{bmatrix} x \\ y \\ z \end{bmatrix},$$

e seja $T^* : \mathbb{R}^4 \to \mathbb{R}^3$ a transformação linear dada por transposta

$$T^*(v) = \begin{bmatrix} 1 & -1 & 0 & 1 \\ 1 & 1 & -1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} q \\ r \\ s \\ t \end{bmatrix}, \qquad v = \begin{bmatrix} q \\ r \\ s \\ t \end{bmatrix}.$$

a) Determine os subespaços vetoriais $U = T(\mathbb{R}^3)$, a imagem de T, e W, o núcleo de T^*.
Determine quatro elementos em $a_i \in U$ e cinco elementos $b_j \in W$ e verifique que o produto interno $\langle a_i, b_j \rangle = a_i \cdot b_j = 0$; ou seja, o núcleo de T^* é ortogonal à imagem de T.
Verifique também que $U + W = \{u + w, \mid u \in U, w \in W\} = \mathbb{R}^4$ e $U \cap W = \emptyset$.

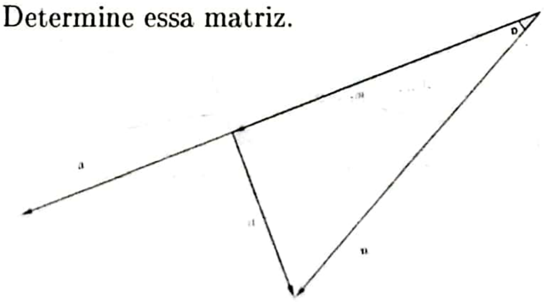
b) Determine

$$k_1 = Proj_W k,$$

calcule $k_2 = k - k_1$ e verifique que $k_1 \perp k_2$;

c) Verifique que $T(u) = k$ não tem solução, ou seja $k \notin U$. Resolva a equação $T(u) = k_2$, por eliminação ou escalonamento.

d) Determine uma base ortogonal $\mathcal{B}$ para U, tal que a matriz de mudança da base $\mathcal{A}$ para a base $\mathcal{B}$ seja triangular. Determine essa matriz.



Mantenha a calma e boa prova!

Aluno: Lucas Carrijo Ferrari

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a) $\boxed{Im(T) = Cob\left(\begin{bmatrix}1\\-1\\0\\1\end{bmatrix}, \begin{bmatrix}1\\1\\-1\\0\end{bmatrix}, \begin{bmatrix}1\\1\\1\\1\end{bmatrix}\right) = U}$

$x - y - w$

$Nucleo(T^t) = (1,-1,0,1) \cdot (q,r,s,t) = 0 \rightarrow q - r + t = 0$

$(1,1,-1,0) \cdot (q,r,s,t) = 0 \quad q + r - s = 0$

$(1,1,1,1) \cdot (q,r,s,t) = 0 \quad q + r + s + t = 0$

$x \quad y \quad z \quad w$

$q = r - t \rightarrow \begin{cases} r - t + r - s = 0 \\ r - t + r + s + t = 0 \end{cases} \longrightarrow \left(\frac{3}{2}s, -\frac{1}{2}s, s, -2s\right)$

$s\left(\frac{3}{2}, -\frac{1}{2}, 1, -2\right)$

$\boxed{Nucleo(T^t) = \left\{\left[\frac{3}{2}, -\frac{1}{2}, 1, -2\right]\right\} = W}$

$a_1 = \begin{bmatrix}1\\-1\\0\\1\end{bmatrix} + \begin{bmatrix}1\\1\\-1\\0\end{bmatrix} + \begin{bmatrix}1\\1\\1\\1\end{bmatrix} = \begin{bmatrix}3\\1\\0\\2\end{bmatrix}$

$a_2 = \begin{bmatrix}2\\-2\\0\\2\end{bmatrix} + \begin{bmatrix}1\\1\\-1\\0\end{bmatrix} + \begin{bmatrix}1\\1\\1\\1\end{bmatrix} = \begin{bmatrix}4\\0\\0\\3\end{bmatrix}$

$a_3 = \begin{bmatrix}1\\-1\\0\\1\end{bmatrix} + \begin{bmatrix}2\\2\\-2\\0\end{bmatrix} + \begin{bmatrix}1\\1\\1\\1\end{bmatrix} = \begin{bmatrix}4\\2\\-1\\2\end{bmatrix}$

$a_4 = \begin{bmatrix}1\\-1\\0\\1\end{bmatrix} + \begin{bmatrix}1\\1\\-1\\0\end{bmatrix} + \begin{bmatrix}2\\2\\2\\2\end{bmatrix} = \begin{bmatrix}4\\2\\1\\3\end{bmatrix}$

$b_1 = \left[\frac{3}{2}, -\frac{1}{2}, 1, -2\right]$

$b_2 = 2 \cdot \left[\frac{3}{2}, -\frac{1}{2}, 1, -2\right] = [3, -1, 2, -4]$

$b_3 = 4 \cdot \left[\frac{3}{2}, -\frac{1}{2}, 1, -2\right] = [6, -2, 4, -8]$

$b_4 = 6 \cdot \left[\frac{3}{2}, -\frac{1}{2}, 1, -2\right] = [9, -3, 6, -12]$

$b_5 = 8 \cdot \left[\frac{3}{2}, -\frac{1}{2}, 1, -2\right] = [12, -4, 8, -16]$

$\langle a_2, b_3 \rangle = \begin{bmatrix}4\\0\\0\\3\end{bmatrix} \cdot \begin{bmatrix}6\\-2\\4\\-8\end{bmatrix} = 24 - 24 = 0$

$a_1 + b_2 = \begin{bmatrix}6\\0\\2\\-2\end{bmatrix}$ $\mathbb{R}^4$

Cofundo!

$U \cap W$ a sinal, não há combinações de U que gerem essa soma, nem um escalar p/ W

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10,5

b) $h_1 = proj_w h_2 = \dfrac{\langle w, h_2 \rangle}{\langle w, w \rangle} w$

$= \dfrac{\langle [\frac{3}{2}, \frac{-1}{2}, 1, -2] [4, 0, 1, 2] \rangle}{\langle [\frac{3}{2}, \frac{-1}{2}, 1, -2], [\frac{3}{2}, \frac{-1}{2}, 1, -2] \rangle} w$

$= \dfrac{6 + 0 + 1 - 4}{\frac{9}{4} + \frac{1}{4} + 1 + 4} [\frac{3}{2}, \frac{-1}{2}, 1, -2]$

$= \dfrac{3}{\frac{5}{2} + 5} w$

$= \dfrac{3}{\frac{15}{2}} w$

$= \cancel{3} \cdot \dfrac{2}{\cancel{15}_{5}} w$

$= \frac{2}{5} [\frac{3}{2}, \frac{-1}{2}, 1, -2]$

$= [\frac{6}{10}, \frac{-2}{10}, \frac{2}{5}, \frac{-4}{5}] \rightarrow [\frac{3}{5}, \frac{-1}{5}, \frac{2}{5}, \frac{-4}{5}]$

Não precisa!

$\boxed{h_1 = [0{,}6 \ , \ -0{,}2 \ , \ 0{,}4 \ , -0{,}8]}$

$h_2 = h - h_1$

$h_2 = [4, 0, 1, 2] - [\frac{3}{5}, \frac{1}{5}, \frac{2}{5}, \frac{-4}{5}]$

$\boxed{h_2 = [3{,}4 \ , \ 0{,}2 \ , \ 0{,}6 \ , \ 2{,}8]}$

$h_1 \perp h_2 \rightarrow \langle h_1, h_2 \rangle = 0$

$\langle [0{,}6 \ , -0{,}2 \ , 0{,}4 \ , -0{,}8] \ , [3{,}4 \ , 0{,}2 \ , 0{,}6 \ , 2{,}8] \rangle = 0$

$2{,}04 + -0{,}04 + 0{,}24 - 2{,}24$

$2 - 2$

0

1,5

$T(u) = h_2$

c) $x\begin{bmatrix}1\\-1\\0\\1\end{bmatrix} + y\begin{bmatrix}1\\1\\-1\\0\end{bmatrix} + z\begin{bmatrix}1\\1\\1\\1\end{bmatrix} = \begin{bmatrix}4\\0\\1\\2\end{bmatrix}$

$$\left[\begin{array}{ccc|c}1&1&1&4\\-1&1&1&0\\0&-1&1&1\\1&0&1&2\end{array}\right] \rightarrow \left[\begin{array}{ccc|c}1&2&0&3\\-1&2&0&-1\\0&-1&1&1\\1&1&0&1\end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|c}1&2&0&3\\0&4&0&2\\0&-1&1&1\\0&-1&0&-2\end{array}\right] \rightarrow \left[\begin{array}{ccc|c}1&2&0&3\\0&1&0&\frac{1}{2}\\0&-1&1&1\\0&1&0&2\end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|c}1&0&0&2\\0&1&0&\frac{1}{2}\\0&0&1&\frac{3}{2}\\0&0&0&\frac{3}{2}\end{array}\right] \rightarrow \boxed{0 = \frac{3}{2} \;?}$$

impossível! , $K \notin U$

$T(u) = h_2$

$x\begin{bmatrix}1\\-1\\0\\1\end{bmatrix} + y\begin{bmatrix}1\\1\\-1\\0\end{bmatrix} + z\begin{bmatrix}1\\1\\1\\1\end{bmatrix} = \begin{bmatrix}3{,}4\\0{,}2\\0{,}6\\2{,}8\end{bmatrix}$

x10 tudo

$$\left[\begin{array}{ccc|c}10&10&10&34\\-10&10&10&2\\0&-10&10&6\\10&0&10&28\end{array}\right] \rightarrow \left[\begin{array}{ccc|c}1&1&1&\frac{17}{5}\\-1&1&1&\frac{1}{5}\\0&-1&1&\frac{3}{5}\\1&0&1&\frac{14}{5}\end{array}\right] \rightarrow \left[\begin{array}{ccc|c}1&2&0&14/5\\-1&2&0&-2/5\\0&-1&1&3/5\\1&1&0&11/5\end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|c}1&2&0&14/5\\0&1&0&3/5\\0&-1&1&3/5\\0&0&0&0\end{array}\right] \rightarrow \left[\begin{array}{ccc|c}1&0&0&\frac{8}{5}\\0&1&0&3/5\\0&0&1&6/5\end{array}\right]$$

$$\boxed{(x, y, z) = \left(\frac{8}{5}, \frac{3}{5}, \frac{6}{5}\right)}$$

1,5

(cálculos com calculadora científica permitida na prova e no rascunho)

d) $U \to v_1 = [1,-1,0,1]$ $\quad A = \{v_1, v_2, v_3\}$

$v_2 = [1,1,-1,0]$

$v_3 = [1,1,1,1]$

$q_1 = v_1 = [1,-1,0,1]$

$$q_2 = v_2 - proj_{q_1} v_2 = [1,1,-1,0] - \frac{\langle [1,-1,0,1], [1,1,-1,0] \rangle}{\langle [1,-1,0,1], [1,-1,0,1] \rangle} q_1 =$$

$$= [1,1,-1,0] - 0$$

$$= [1,1,-1,0]$$

$$q_3 = v_3 - proj_{q_1} v_3 - proj_{q_2} v_3 = [1,1,1,1] - \frac{\langle q_1, v_3 \rangle}{\langle q_1, q_1 \rangle} q_1 - \frac{\langle q_2, v_3 \rangle}{\langle q_2, q_2 \rangle} q_2$$

$$= [1,1,1,1] - \frac{\langle [1,-1,0,1], [1,1,1,1] \rangle}{\langle [1,-1,0,1], [1,-1,0,1] \rangle} q_1 - \frac{\langle [1,1,-1,0], [1,1,1,1] \rangle}{\langle [1,1,-1,0], [1,1,-1,0] \rangle} q_2$$

$$= [1,1,1,1] - \frac{1-1+1}{1+1+1} q_1 - \frac{1+1-1}{1+1+1} q_2$$

$$= [1,1,1,1] - \frac{1}{3} q_1 - \frac{1}{3} q_2$$

$$= [1,1,1,1] - \frac{1}{3}(q_1 + q_2)$$

$$= [1,1,1,1] - \frac{1}{3}([1,-1,0,1] + [1,1,-1,0])$$

$$= [1,1,1,1] - \frac{1}{3}([2,0,-1,1])$$

$$= [1,1,1,1] + [-\tfrac{2}{3}, 0, \tfrac{1}{3}, -\tfrac{1}{3}]$$

$$= [\tfrac{1}{3}, 1, \tfrac{4}{3}, \tfrac{2}{3}]$$

1, 0

$q_1 = [1,-1,0,1]$

$q_2 = [1,1,-1,0]$

$q_3 = [\frac{1}{3}, 1, \frac{4}{3}, \frac{2}{3}]$

$$\boxed{\text{matriz de mudança de base} = \begin{bmatrix} 1 & 1 & \frac{1}{3} \\ -1 & 1 & 1 \\ 0 & -1 & \frac{4}{3} \\ 1 & 0 & \frac{2}{3} \end{bmatrix}}$$