Chapter 4
Utility

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Utility Representation

- A function $u : X \rightarrow R$ is a utility function representing preference relation $\succcurlyeq$ $if\ \forall x, y \in X, x \succcurlyeq y \Leftrightarrow u(x) \geq u(y)$
- $u(x)\ is\ not\ unique$: $f'(.) > 0 \Longrightarrow$
$v(x) = f\big(u(x)\big)\ is\ also\ utility\ function.$
- Ordinal property (preference relation of u) and Cardinal property (numerical value of u)

Utility Representation

- X finite and $\succcurlyeq$ rational $\Longrightarrow$ can be represented by a utility function
- X Infinite but Countable and $\succcurlyeq$ rational $\Longrightarrow$ can be represented by a utility function
- If X is separable and $\succcurlyeq$ is Rational and Continuous $\Leftrightarrow$ $\succcurlyeq$ can be represented by a Continuous utility function

Rationality and utility representation

- A Preference can be represented by a Utility Function $\Longrightarrow$ The preference must be Rational (why?)
- The other way around is not true. Counter example: Lexicographic

Utility Functions

- Utility is an ordinal (i.e. ordering) concept. [序数效用]
- *E.g.* if U(x) = 6 and U(y) = 2 then bundle x is strictly preferred to bundle y. But x is not preferred three times as much as is y.

Utility Functions & Indiff. Curves

- Consider the bundles (4,1), (2,3) and (2,2).
- Suppose $(2,3) \succ (4,1) \sim (2,2)$.
- Assign to these bundles any numbers that preserve the preference ordering;
  e.g. $U(2,3) = 6 > U(4,1) = U(2,2) = 4$.
- Call these numbers utility levels.

Utility Functions & Indiff. Curves

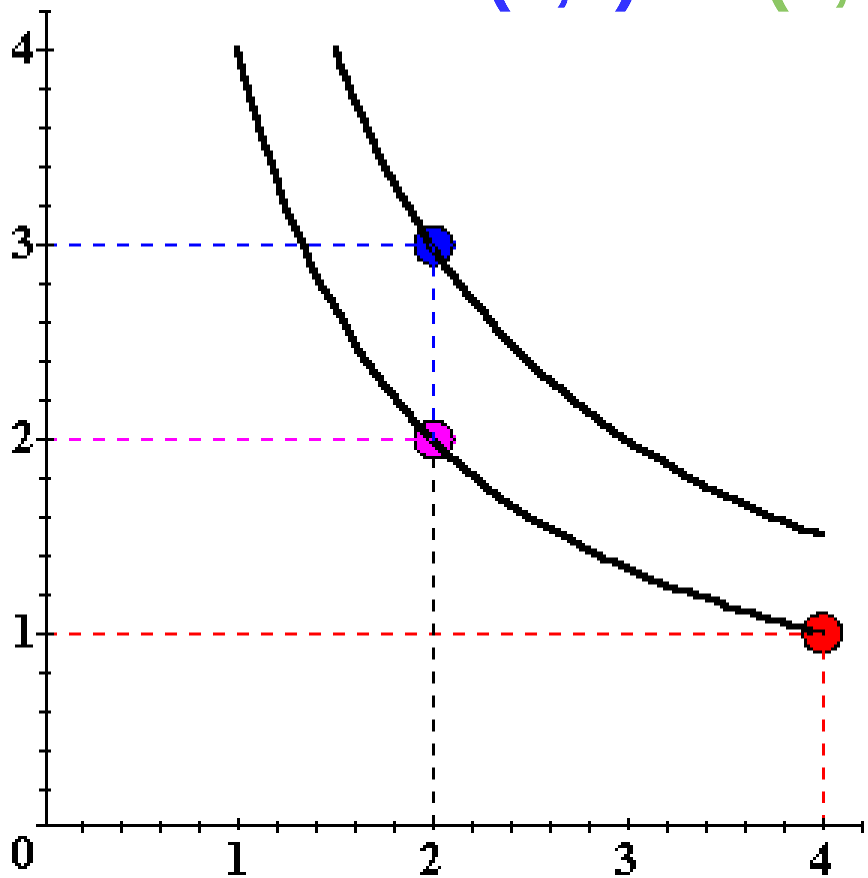
- An indifference curve contains equally preferred bundles.
- Equal preference $\Rightarrow$ same utility level.
- Therefore, all bundles in an indifference curve have the same utility level.

Utility Functions & Indiff. Curves

- So the bundles (4,1) and (2,2) are in the indiff. curve with utility level $U \equiv 4$
- But the bundle (2,3) is in the indiff. curve with utility level $U \equiv 6$.
- On an indifference curve diagram, this preference information looks as follows:

Utility Functions & Indiff. Curves

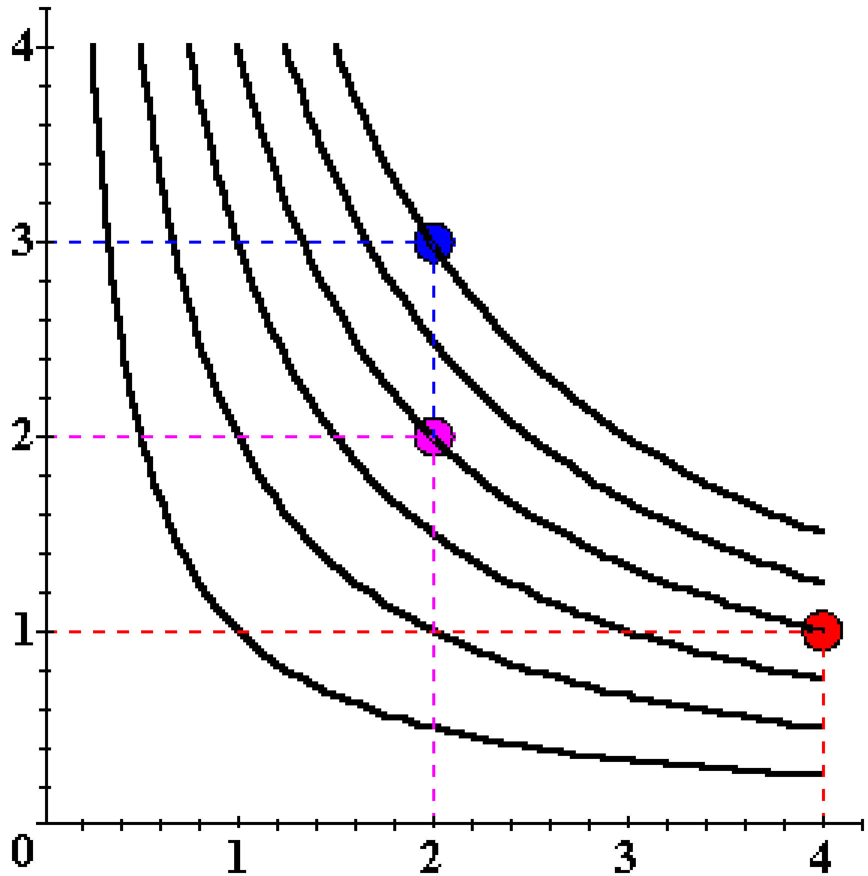
(2,3) (2,2) (4,1)



Utility Functions & Indiff. Curves

- Comparing more bundles will create a larger collection of all indifference curves and a better description of the consumer's preferences.

Utility Functions & Indiff. Curves



Utility Functions

- There is no unique utility function representation of a preference relation.
- Suppose $U(x_1,x_2) = x_1x_2$ represents a preference relation.
- Again consider the bundles (4,1), (2,3) and (2,2).

Utility Functions

- $U(x_1,x_2) = x_1x_2$, so

  $U(2,3) = 6 > U(4,1) = U(2,2) = 4$;

  that is, $(2,3) \succ (4,1) \sim (2,2)$

Utility Functions

- $U(x_1,x_2) = x_1x_2 \Longrightarrow (2,3) \succ (4,1) \sim (2,2)$
- Define $V = U^2$.
- Then $V(x_1,x_2) = x_1^2x_2^2$ and
  $V(2,3) = 36 > V(4,1) = V(2,2) = 16$
  so again
  $(2,3) \succ (4,1) \sim (2,2)$
- V preserves the same order as U and so represents the same preferences.

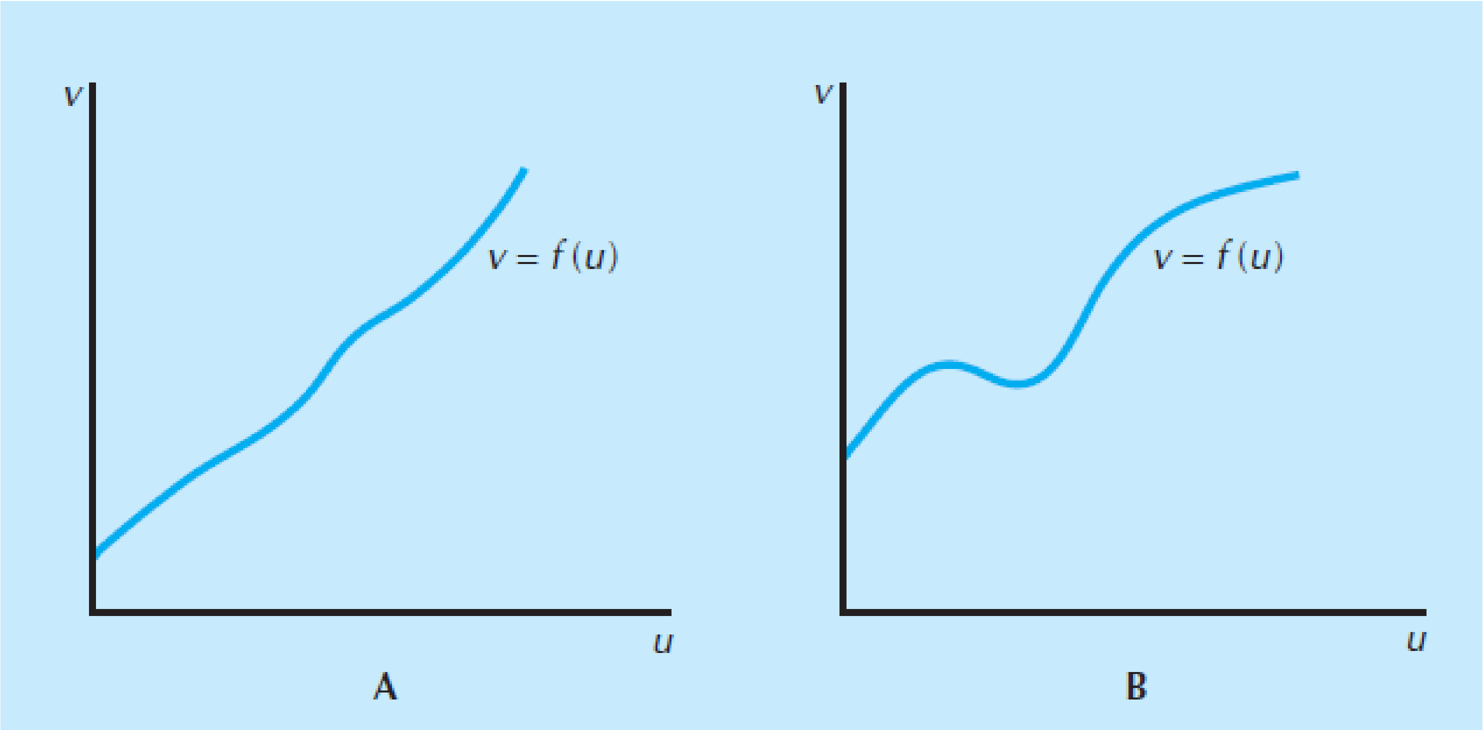
Utility Functions

- $U(x_1,x_2) = x_1x_2$ $\Rightarrow$ $(2,3) \succ (4,1) \sim (2,2)$
- Define $W = 2U + 10$.
- Then $W(x_1,x_2) = 2x_1x_2+10$ so $W(2,3) = 22 > W(4,1) = W(2,2) = 18$. Again, $(2,3) \succ (4,1) \sim (2,2)$
- W preserves the same order as U and V and so represents the same preferences.

Utility Functions: Monotonic Transformation

- If U is a utility function that represents a preference relation $\succcurlyeq$ and
  - f is a strictly increasing function,
- then V = f(U) is also a utility function representing $\succcurlyeq$.

Monotonic Transformation



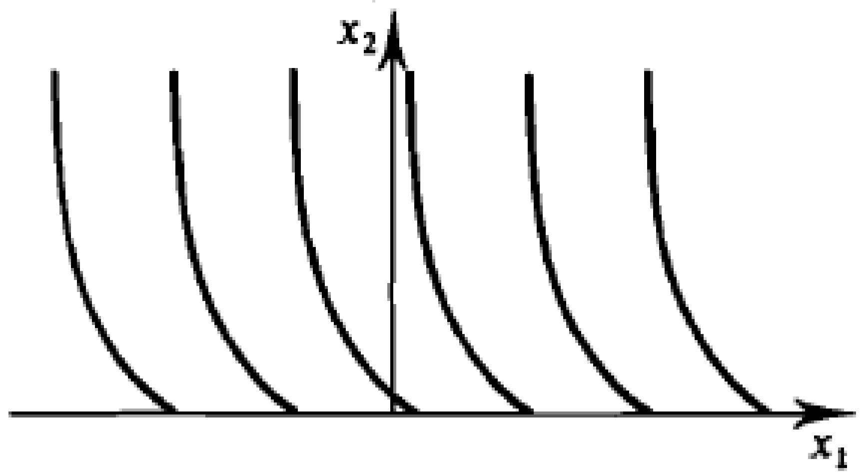
Some Other Utility Functions and Their Indifference Curves

- Perfect substitute
  - $V(x_1,x_2) = x_1 + x_2$.
- Perfect complement
  - $W(x_1,x_2) = \min\{x_1,x_2\}$
- Quasi-linear
  - $U(x_1,x_2) = f(x_1) + x_2$
- Cobb-Douglas Utility Function
  - $U(x_1,x_2) = x_1^a x_2^b$
- What do the indifference curves for these utility functions look like?

Quasilinear

- Quasilinear with respect to commodity 1 (numeraire commodity): $X = (-\infty, +\infty) \times R_+^{L-1}$
  - (1) all indifference sets are parallel displacements of each other along the axis of commodity 1; $x \sim y \Rightarrow (x + ae_1) \sim (y + ae_1)\ \forall a \in R$, where $e_1 = (1,0, \dots, 0)$
  - (2) commodity 1 is desirable $(x + ae_1) \succ x, \forall x\ \forall a$
- A continuous preference is quasilinear wrt commodity 1 $\Leftrightarrow$ it admits a utility function of the form $u(x) = x_1 + v(x_2, x_3, \dots, x_L)$

Quasilinear



Cobb-Douglas Utility Function

- Any utility function of the form

$$U(x_1,x_2) = x_1^{a} x_2^{b}$$

  with a > 0 and b > 0 is called a Cobb-Douglas utility function.

- *E.g.* $U(x_1,x_2) = x_1^{1/2} x_2^{1/2}$ $(a = b = 1/2)$
  $V(x_1,x_2) = x_1 x_2^{3}$ $(a = 1, b = 3)$

Marginal Utilities

- Marginal means "incremental".
- The marginal utility of commodity i is the rate-of-change of total utility as the quantity of commodity 1 consumed changes; *i.e.*

$$MU_1 = \frac{\Delta U}{\Delta x_1} = \frac{u(x_1 + \Delta x_1, x_2) - u(x_1, x_2)}{\Delta x_1},$$

MRS and Utility Function

$$U(x_1, x_2) = \bar{U}$$

$$d\bar{U} = dU(x_1, x_2) = \frac{\partial U}{\partial x_1} dx_1 + \frac{\partial U}{\partial x_2} dx_2 = 0$$

$$MRS = -\frac{dx_2}{dx_1}\Big|_{u=\bar{u}} = \frac{\partial U}{\partial x_1} / \frac{\partial U}{\partial x_2}$$

Monotonic Transformations & Marginal Rates-of-Substitution

- Applying a monotonic transformation to a utility function representing a preference relation simply creates another utility function representing the same preference relation.
- What happens to marginal rates-of-substitution when a monotonic transformation is applied?

Positive monotonic transformation

$$V(x_1, x_2) = f[u(x_1, x_2)], \quad f'(u) > 0$$

$$MRS = \frac{\partial v / \partial x_1}{\partial v / \partial x_2} = \frac{f'(u)}{f'(u)} \frac{\partial u / \partial x_1}{\partial u / \partial x_2} = \frac{\partial u / \partial x_1}{\partial u / \partial x_2}$$

Excise

- Write down the MU and MRS of the following utility functions
- $U(x_1, x_2) = {x_1}^{a} \; {x_2}^{1-a}$
- $U(x_1,x_2) = f(x_1) + x_2$
- $U(x_1, x_2) = [a x_1^{\rho} + \; (1-a) x_2^{\rho}]^{\frac{1}{\rho}}$