蘇品瑄 E14082181

工數 2nd 期中

1、(a) True

(b) True

(c) $(1-x^2)y''-2xy'+n(n+1)y=0$, $n=\frac{1}{2}$

$$y(x)=a_0\left[1-\frac{n(n+1)}{2!}x^2+\frac{n(n-2)(n+1)(n+3)}{4!}x^4+\dots\right]+a_1\left[x-\frac{(n-1)(n+2)}{3!}x^3+\frac{(n-1)(n-3)(n+2)(n+4)}{5!}\dots+\dots\right]$$

$$=a_0\left(1-\frac{3}{8}x^2-\frac{21}{128}x^4+\dots\right)+a_1\left(x+\frac{5}{24}x^3+\frac{15}{128}x^5+\dots\right)\ \#$$

(d) Bessel: $x^2y''+xy'-(\alpha^2x^2+\nu^2)y=0 \Rightarrow y(x)=C_1I_\nu(\alpha x)+C_2K_\nu(\alpha x)$

$\alpha=\sqrt{15}$, $\nu=\frac{4}{5}$ $\Rightarrow y(x)=C_1I_{\frac{4}{5}}(\sqrt{15}x)+C_2K_{\frac{4}{5}}(\sqrt{15}x)$ #

(e)
$$\int_{-1}^{1}e^{-x}\left(x^2e^x-\frac{1}{3}e^x\right)dx=\int_{-1}^{1}\left(x^2-\frac{1}{3}\right)dx=\left[\frac{x^3}{3}-\frac{x}{3}\right]_{-1}^{1}=\frac{1}{3}-\frac{1}{3}+\frac{1}{3}-\frac{1}{3}=0$$

True

(f) True

3、 $\lambda=0$, $y=C_1+\cancel{C_2}x$, $y'=\cancel{C_2}$ $\xrightarrow{BC's}$ $C_2=0$, $C_1+0=0$ $\Rightarrow y=0$ X

$y'(0)=0$

$y(L)+y'(L)=0$

$\lambda<0$, $y=C_1e^{\sqrt{-\lambda}x}+C_2e^{-\sqrt{-\lambda}x}$

$y'=C_1\sqrt{-\lambda}e^{\sqrt{-\lambda}x}-C_2\sqrt{-\lambda}e^{-\sqrt{-\lambda}x}$

$\xrightarrow{BC's}$ $\begin{cases}C_1\sqrt{-\lambda}-C_2\sqrt{-\lambda}=0\\ C_1e^{\sqrt{-\lambda}L}+C_2e^{-\sqrt{-\lambda}L}+C_1\sqrt{-\lambda}e^{\sqrt{-\lambda}L}-C_2\sqrt{-\lambda}e^{-\sqrt{-\lambda}L}\end{cases}$

$=C_1e^{\sqrt{-\lambda}L}(1+\sqrt{-\lambda})+C_2e^{-\sqrt{-\lambda}L}(1-\sqrt{-\lambda})=0$

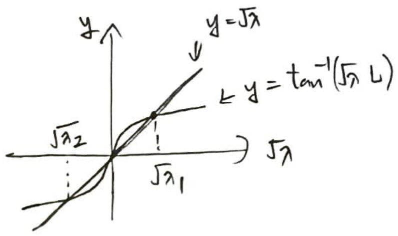
$\Rightarrow C_1=0, C_2=0 \Rightarrow y=0$

$\lambda>0$, $y=C_1\cos(\sqrt{\lambda}x)+\cancel{C_2}\sin(\sqrt{\lambda}x)$

$y'=-C_1\sqrt{\lambda}\sin(\sqrt{\lambda}x)+\cancel{C_2}\sqrt{\lambda}\cos(\sqrt{\lambda}x)$

$\xrightarrow{BC's}$ $\begin{cases}C_2=0\\ C_1\cos(\sqrt{\lambda}L)-C_1\sqrt{\lambda}\sin(\sqrt{\lambda}L)=C_1(\cos\sqrt{\lambda}L-\sqrt{\lambda}\sin\sqrt{\lambda}L)=0\end{cases}$ $C_1\neq0$

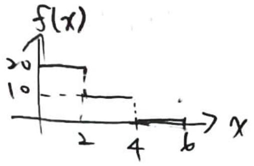
$\Rightarrow \sqrt{\lambda}=\frac{\cos(\sqrt{\lambda}L)}{\sin(\sqrt{\lambda}L)}=\tan^{-1}(\sqrt{\lambda}L)$ $\Rightarrow \sqrt{\lambda_1}$, $\sqrt{\lambda_2}$



$\Rightarrow y=C_1\cos(\sqrt{\lambda_1}x)$

or $C_1\cos(\sqrt{\lambda_2}x)$

2、



$$\boxed{\ddot{x} + x = f(t)}$$

$20\times 2 \qquad 10\times 2$

$$a_0 = \frac{1}{6}\int_0^6 f(x)\,dx = \frac{1}{6}\left[\int_0^2 (20)\,dx + \int_2^4 (10)\,dx\right] = 10$$

$$a_n = \frac{2}{6}\int_0^6 f(x)\cos\frac{n\pi x}{6}\,dx = \frac{1}{3}\left[\int_0^2 20\cos\left(\frac{n\pi}{6}x\right)dx + \int_2^4 10\cos\left(\frac{n\pi}{6}x\right)dx\right]$$

$$= \frac{20}{3}\left[\frac{\sin\left(\frac{n\pi}{6}x\right)}{\frac{n\pi}{6}}\right]_0^2 + \frac{10}{3}\left[\frac{\sin\left(\frac{n\pi}{6}x\right)}{\frac{n\pi}{6}}\right]_2^4 = \frac{120}{3n\pi}\left[\sin\frac{n\pi}{3} - 0\right] + \frac{60}{3n\pi}\left[\sin\frac{2}{3}n\pi - \sin\frac{n\pi}{3}\right]$$

$$= \frac{20}{n\pi}\left(\sin\frac{n\pi}{3} + \sin\frac{2n\pi}{3}\right)$$

$$b_n = \frac{2}{6}\int_0^6 f(x)\sin\frac{n\pi x}{6}\,dx = \frac{1}{3}\left[\int_0^2 20\sin\left(\frac{n\pi}{6}x\right)dx + \int_2^4 10\sin\left(\frac{n\pi}{6}x\right)dx\right]$$

$$= \frac{20}{3}\left[\frac{-\cos\left(\frac{n\pi}{6}x\right)}{\frac{n\pi}{6}}\right]_0^2 + \frac{10}{3}\left[\frac{-\cos\left(\frac{n\pi}{6}x\right)}{\frac{n\pi}{6}}\right]_2^4 = -\frac{40}{n\pi}\left[\cos\frac{n\pi}{3} - 1\right] - \frac{20}{n\pi}\left[\cos\frac{2n\pi}{3} - \cos\frac{n\pi}{3}\right]$$

$$= -\frac{20}{n\pi}\left(\cos\frac{n\pi}{3} + \cos\frac{2n\pi}{3} - 2\right)$$

$$F(t) = 10 + \sum_{n=1}^{\infty}\frac{20}{n\pi}\left[\left(\sin\frac{n\pi}{3} + \sin\frac{2n\pi}{3}\right)\cos\frac{n\pi t}{6} - \left(\cos\frac{n\pi}{3} + \cos\frac{2n\pi}{3} - 2\right)\sin\frac{n\pi t}{6}\right]$$

$$x_{p0} = 10$$

$$x_{pn} = A_n\cos\left(\frac{n\pi t}{6}\right) + B_n\sin\left(\frac{n\pi t}{6}\right) = \frac{1-n^2}{(1-n^2)^2 + 0n^2}\cos\left(\frac{n\pi t}{6}\right) + \frac{0n}{(1-n^2)^2 + 0n^2}\sin(nt) = \frac{1}{1-n^2}\cos\left(\frac{n\pi t}{6}\right)$$

$$x_{steady}(t) = 10 + \sum_{n=1}^{\infty}\frac{20}{n\pi}\left[\left(\sin\frac{n\pi}{3} + \sin\frac{2n\pi}{3}\right)\frac{1}{1-n^2}\cos\left(\frac{n\pi t}{6}\right) - \left(\cos\frac{n\pi}{3} + \cos\frac{2n\pi}{3} - 2\right)\frac{1}{1-n^2}\cos\left(\frac{n\pi t}{6}\right)\right] \#$$

4、 $\lambda = n(n+1)$ $(n = 0, 1, 2, \ldots$ or $\lambda = 0, 2, 6, \ldots)$

$\lambda = 0,\ y = P_0(x) = 1$

$\lambda = 2,\ y = P_1(x) = x$

$\lambda = 3,\ y = P_2(x) = \frac{1}{2}(3x^2 - 1)$