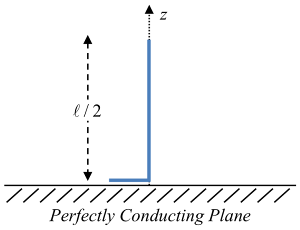
Problem 5

Derive the *far-field vector potential* $\vec{A}$, *far-field electric field* $\vec{E}$, *total radiated power* P_{rad}, and the *radiation resistance* R_r of a monopole above a perfectly conducting plane, as shown. The length of this antenna is $\ell/2=\lambda/4$ and assume its current distribution is $I=I_0e^{j\omega t}(1-2z/\ell)$ for $0\le z\le \ell/2$.



Answers: $\vec{A}=\hat{a}_z\frac{?}{4\pi r}\left(\frac{?}{0.25\pi\cos\theta}\right)^2$ (*Weber / m*), $\vec{E}=\ldots$, $P_{rad}\approx 1?.32\,I_0^2$ (*Watts*), *and ....*

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$$\vec{I}=\hat{a}_z I=\hat{a}_z I_0 e^{j\omega t}\left(1-\frac{2z}{\ell}\right)\ (A)$$

$$\eta=377\ (\Omega)$$

$$\mu=\mu_0 \quad C=3\times10^8\ (m/s)$$

$$\vec{A}=\int\mu\vec{I}\,\frac{e^{-jkR}}{4\pi R}dz\simeq\int\mu\vec{I}\,\frac{e^{-jk(r-z\cos\theta)}}{4\pi r}dz=\int\mu\,\hat{a}_z I_0 e^{j\omega t}\left(1-\frac{2z}{\ell}\right)\frac{e^{-jk(r-z\cos\theta)}}{4\pi r}dz$$

$$=\hat{a}_z\mu I_0\frac{e^{j(\omega t-kr)}}{4\pi r}\int\left(1-\frac{2z}{\ell}\right)e^{jkz\cos\theta}dz$$

$$=\hat{a}_z\mu I_0\frac{e^{j(\omega t-kr)}}{4\pi r}\left(\int_0^{\ell/2}\left(1-\frac{2z}{\ell}\right)e^{jkz\cos\theta}dz+\int_{-\ell/2}^{0}\left(1+\frac{2z}{\ell}\right)e^{jkz\cos\theta}dz\right)$$

$$\int_0^{\ell/2}\left(1-\frac{2z}{\ell}\right)e^{jkz\cos\theta}dz=\frac{1}{jk\cos\theta}e^{jkz\cos\theta}\Big|_0^{\ell/2}-\frac{2}{\ell}\int_0^{\ell/2}ze^{jkz\cos\theta}dz$$

$$=\frac{1}{jk\cos\theta}\left(e^{jk\frac{\ell}{2}\cos\theta}-1\right)-\frac{2}{\ell}\left(\frac{\ell/2}{jk\cos\theta}e^{jk\frac{\ell}{2}\cos\theta}-\frac{1}{(jk\cos\theta)^2}e^{jk\frac{\ell}{2}\cos\theta}+\frac{1}{(jk\cos\theta)^2}\right)$$

$$\begin{array}{ll} z & e^{az} \\ & {\scriptstyle +} \\ 1 & \frac{1}{a}e^{az} \\ & {\scriptstyle -} \\ 0 & \frac{1}{a^2}e^{az} \end{array}$$

$$\frac{z}{a}e^{az}\Big|_0^{\ell/2}=\frac{\ell/2}{jk\cos\theta}e^{jk\frac{\ell}{2}\cos\theta}$$

$$-\frac{1}{a^2}e^{az}\Big|_0^{\ell/2}=-\frac{1}{(jk\cos\theta)^2}e^{jk\frac{\ell}{2}\cos\theta}+\frac{1}{(jk\cos\theta)^2}$$

$$\int_{-\ell/2}^{0}\left(1+\frac{2z}{\ell}\right)e^{jkz\cos\theta}dz=\frac{1}{jk\cos\theta}e^{jkz\cos\theta}\Big|_{-\ell/2}^{0}+\frac{2}{\ell}\int_{-\ell/2}^{0}ze^{jkz\cos\theta}dz$$

$$=\frac{1}{jk\cos\theta}\left(1-e^{jk\frac{-\ell}{2}\cos\theta}\right)+\frac{2}{\ell}\left[\frac{\ell/2}{jk\cos\theta}e^{jk\frac{-\ell}{2}\cos\theta}+\frac{-1}{(jk\cos\theta)^2}+\frac{e^{jk\frac{-\ell}{2}\cos\theta}}{(jk\cos\theta)^2}\right]$$

$$\begin{array}{ll} z & e^{az} \\ & {\scriptstyle +} \\ 1 & \frac{1}{a}e^{az} \\ & {\scriptstyle -} \\ 0 & \frac{1}{a^2}e^{az} \end{array}$$

$$\frac{z}{a}e^{az}\Big|_{-\ell/2}^{0}=-\frac{-\ell/2}{a}e^{a(\frac{-\ell}{2})}=\frac{\ell/2}{a}e^{a(\frac{-\ell}{2})}$$

$$\frac{-1}{a^2}e^{az}\Big|_{-\ell/2}^{0}=\frac{-1}{a^2}-\left(\frac{-1}{a^2}e^{a(\frac{-\ell}{2})}\right)=\frac{-1}{a^2}+\frac{1}{a^2}e^{a(\frac{-\ell}{2})}$$

$$\Rightarrow \int_0^{l/2}\left(1-\frac{2z}{l}\right)e^{jkz\cos\theta}\,dz+\int_{-l/2}^{0}\left(1+\frac{2z}{l}\right)e^{jkz\cos\theta}\,dz$$

$$=\frac{2j\sin(k\frac{l}{2}\cos\theta)}{jk\cos\theta}+\frac{1}{l}\left[\frac{\frac{l}{2}(-2j)\sin(k\frac{l}{2}\cos\theta)}{jk\cos\theta}-\frac{2}{(jk\cos\theta)^2}+\frac{2\cos(k\frac{l}{2}\cos\theta)}{(jk\cos\theta)^2}\right]$$

$$=\frac{2\sin(k\frac{l}{2}\cos\theta)}{k\cos\theta}+\frac{-2\sin(k\frac{l}{2}\cos\theta)}{k\cos\theta}+\frac{4/l}{+(k\cos\theta)^2}+\frac{4/l\,\cos(k\frac{l}{2}\cos\theta)}{-(k\cos\theta)^2}$$

$$=\frac{4/l\,(1-\cos(k\frac{l}{2}\cos\theta))}{(k\cos\theta)^2}=\frac{(4-4\cos(\frac{\pi}{2}\cos\theta))\,l}{(\pi\cos\theta)^2}$$

$$\Rightarrow \vec{A}=\hat{a}_z\mu I_0\frac{e^{j(\omega t-kr)}}{4\pi r}\cdot\frac{(4-4\cos(\frac{\pi}{2}\cos\theta))\,l}{(\pi\cos\theta)^2}\quad(\mathrm{Wb/m})\ ※$$

$$\because \hat{a}_z=\cos\theta\,\hat{a}_r-\sin\theta\,\hat{a}_\theta\quad\therefore \vec{A}_t=(\vec{A}\cdot\hat{a}_\theta)\hat{a}_\theta=\hat{a}_\theta(-\sin\theta)\mu I_0\frac{e^{j(\omega t-kr)}}{4\pi r}\cdot\frac{(4-4\cos(\frac{\pi}{2}\cos\theta))\,l}{(\pi\cos\theta)^2}$$

$\because \vec{A}_t\propto\frac{e^{-jkr}}{r}\vec{R}_A(\theta,\phi)$ for far field

$$\therefore \vec{E}=-j\omega\vec{A}_t=\hat{a}_\theta\, j\omega\sin\theta\,\mu I_0\frac{e^{j(\omega t-kr)}}{4\pi r}\cdot\frac{(4-4\cos(\frac{\pi}{2}\cos\theta))\,l}{(\pi\cos\theta)^2}\quad(\mathrm{V/m})\ ※$$

$$\Rightarrow \vec{H}=\frac{\nabla\times\vec{E}}{-j\omega\mu}=\frac{-jk\,\hat{a}_r\times\vec{E}}{-j\omega\mu}=\frac{\hat{a}_r\times\vec{E}}{\eta}$$

$$=\hat{a}_\phi\frac{1}{\eta}j\omega\sin\theta\,\mu I_0\frac{e^{j(\omega t-kr)}}{4\pi r}\cdot\frac{(4-4\cos(\frac{\pi}{2}\cos\theta))\,l}{(\pi\cos\theta)^2}\quad(\mathrm{A/m})\ ※$$

$P_{rad}=\frac{1}{2}\int\vec{E}\times\vec{H}^*\cdot d\vec{s}$ $\quad\because$ half-sphere radiation $\therefore 0\le\phi\le2\pi,\ 0\le\theta\le\frac{\pi}{2}$

$$=\frac{1}{2}\int_0^{2\pi}\int_0^{\frac{\pi}{2}}\hat{a}_r\frac{\omega^2}{\eta}\sin^2\theta\,\mu^2I_0^2\cdot\frac{1}{(4\pi r)^2}\cdot\frac{(4-4\cos(\frac{\pi}{2}\cos\theta))^2\,l^2}{(\pi\cos\theta)^4}\,r^2\sin\theta\,d\theta\,d\phi\cdot\hat{a}_r$$

$$=\frac{\omega^2\mu^2I_0^2\,l^2}{2\eta\cdot16\pi^2\cdot\pi^4}\cdot2\pi\cdot\int_0^{\pi/2}\frac{\sin^3\theta\,(4-4\cos(\frac{\pi}{2}\cos\theta))^2}{\cos^4\theta}\,d\theta$$

$$\because \omega=v_p k=c\cdot\frac{2\pi}{\lambda}\quad\therefore \omega^2l^2=\left(c\cdot\frac{2\pi}{\lambda}\cdot\frac{\lambda}{2}\right)^2=c^2\pi^2\Rightarrow\frac{\cancel{\omega^2}\mu^2I_0^2\cancel{l^2}^{\,c^2\cancel{\pi^2}}}{\cancel{2}\eta\cdot16\cancel{\pi^2}\cdot\pi^{\cancel{4}3}}\cdot\cancel{2\pi}=\frac{c^2\mu^2I_0^2}{\eta\cdot16\pi^3}$$

$$\Rightarrow P_{rad}=\frac{c^2\mu^2I_0^2}{\eta\cdot16\pi^3}\cdot14.9355=11.3494\,I_0^2\ (\mathrm{W})\ ※$$

$\because P_{rad} = \frac{1}{2} R_{rad} I_0^2 \quad \therefore R_{rad} = \frac{2P_{rad}}{I_0^2} = 22.6987(\Omega)$ #