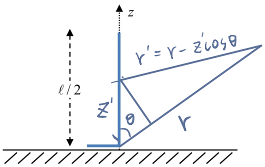
電磁作業III 謝宋翰 E14084078.

Problem 5

Derive the *far-field vector potential* $\vec{A}$, *far-field electric field* $\vec{E}$, *total radiated power* P_{rad}, and the *radiation resistance* R_r of a monopole above a perfectly conducting plane, as shown. The length of this antenna is $\ell/2 = \lambda/4$ and assume its current distribution is $I = I_0 e^{j\omega t}(1-2z/\ell)$ for $0 \le z \le \ell/2$.



Perfectly Conducting Plane

Answers: $\vec{A} = \hat{a}_z \frac{?}{4\pi r}\left(\frac{?}{0.25\pi\cos\theta}\right)^2$ (*Weber / m*), $\vec{E} = \ldots$, $P_{rad} \approx 1?.32 I_0^2$ (*Watts*), *and* ....

$$\begin{cases} I = I_0 \cdot e^{j\omega t}\left(1-\frac{2z}{\ell}\right) & \text{for } 0 \le z \le \frac{\ell}{2} \\ I = I_0 \cdot e^{j\omega t}\left(1+\frac{2z}{\ell}\right) & \text{for } -\frac{\ell}{2} \le z \le 0. \end{cases}$$

$$\vec{A} = \int \mu \vec{I} \cdot \frac{e^{-jkr'}}{4\pi r'} dz' \Rightarrow \int \mu \vec{I} \cdot \frac{e^{-jk(r-z'\cos\theta)}}{4\pi r} dz' = \frac{\mu_0 I_0}{4\pi r} e^{j\omega t}\left[\int_0^{\frac{\ell}{2}} \left(1-\frac{2}{\ell}z'\right) e^{-jk(r-z'\cos\theta)} dz' + \int_{-\frac{\ell}{2}}^{0} \left(1-\frac{2}{\ell}z'\right) e^{-jk(r-z'\cos\theta)} dz'\right].$$

解: $\frac{\mu_0 I_0}{4\pi r} e^{j\omega t} e^{-jkr}\left[\int_0^{\ell/2} \left(1-\frac{2}{\ell}z'\right) e^{+jkz'\cos\theta} dz' + \int_{-\frac{\ell}{2}}^{0} \left(1-\frac{2}{\ell}z'\right) e^{jkz'\cos\theta} dz'\right].$

① $\int_0^{\ell/2} \left(1-\frac{2}{\ell}z'\right) e^{jkz'\cos\theta} dz'$ // 左微右積

$1-\frac{2}{\ell}z'$	+	$e^{jkz'\cos\theta}$
$-\frac{2}{\ell}$	−	$\frac{1}{jk\cos\theta} e^{jkz'\cos\theta}$
0		$\frac{-1}{k^2\cos^2\theta} e^{jkz'\cos\theta}$

$$\Rightarrow \left(1-\frac{2}{\ell}z'\right)\frac{1}{jk\cos\theta} e^{jkz'\cos\theta} - \frac{2}{\ell}\cdot\frac{1}{k^2\cos^2\theta} e^{jkz'\cos\theta} \Big|_0^{\frac{\ell}{2}}$$

$$= \left(1-\frac{2}{\ell}\cdot\frac{\ell}{2}\right) \cdots - \frac{2}{\ell}\cdot\frac{1}{k^2\cos^2\theta} e^{jk\frac{\ell}{2}\cos\theta} - \frac{1}{jk\cos\theta} + \frac{2}{\ell}\cdot\frac{1}{k^2\cos^2\theta}$$

② $\int_{-\ell/2}^{0} \left(1+\frac{2}{\ell}z'\right) e^{jkz'\cos\theta} dz'$ // 左微右積

$1+\frac{2}{\ell}z'$	+	$e^{jkz'\cos\theta}$
$+\frac{2}{\ell}$	−	$\frac{1}{jk\cos\theta} e^{jkz'\cos\theta}$
0		$\frac{-1}{k^2\cos^2\theta} e^{jkz'\cos\theta}$

$$\Rightarrow \left(1+\frac{2}{\ell}z'\right)\frac{1}{jk\cos\theta} e^{jkz'\cos\theta} + \frac{2}{\ell}\cdot\frac{1}{k^2\cos^2\theta} e^{jkz'\cos\theta} \Big|_{-\frac{\ell}{2}}^{0}$$

$$= \left(1-\frac{2}{\ell}\cdot\frac{\ell}{2}\right) \cdots - \frac{2}{\ell}\cdot\frac{1}{k^2\cos^2\theta} e^{jk\frac{\ell}{2}\cos\theta} + \frac{1}{jk\cos\theta} + \frac{2}{\ell}\cdot\frac{1}{k^2\cos^2\theta}$$

① + ② $\Rightarrow \frac{4}{\ell k^2\cos^2\theta}\left[1-\left(e^{jk\frac{\ell}{2}\cos\theta} + e^{-jk\frac{\ell}{2}\cos\theta}\right)\right] = \frac{4}{\ell k^2\cos^2\theta}\left[1-\cos\left(\frac{\pi}{2}\cos\theta\right)\right]$ by Euler's formula

$$\therefore \vec{A} = \frac{\mu_0 I_0}{4\pi r} e^{j\omega t} e^{-jkr} \frac{4\ell}{0.25\pi^2\cos^2\theta}\cdot 2\sin^2\left[\frac{\pi}{4}\cos\theta\right]$$

$$= \frac{\mu_0 I_0}{4\pi r} e^{j(\omega t-kr)} \left(\frac{\sin\left(\frac{\pi}{4}\cos\theta\right)}{0.25\pi\cos\theta}\right)^2 \cdot 2\ell \ \vec{a}_z \ \text{Wb/m} \#$$

$$\vec{E} = -j\omega \vec{A_\theta}$$

$$\vec{A_\theta} = A(-\sin\theta)\,\vec{a_\theta}$$

$$= \frac{\mu_0 I_0}{4\pi r} e^{j(\omega t - kr)} \left(\frac{\sin(\frac{\pi}{4}\cos\theta)}{0.25\pi\cos\theta}\right)^2 \cdot 2l\,(-\sin\theta)$$

$$\Rightarrow \vec{E} = j\omega \frac{\mu_0 I_0}{4\pi r} e^{j(\omega t - kr)} \left(\frac{\sin(\frac{\pi}{4}\cos\theta)}{0.25\pi\cos\theta}\right)^2 \cdot 2l\sin\theta \quad V/m \ \#.$$

$$P_{rad} = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{2\pi} U(\theta,\phi)\sin\theta\, d\theta\, d\phi$$

$$= 2\pi \cdot \eta_0 \cdot \frac{k^2 \cdot I_0^2}{2\cdot 32\pi^2} (2l)^2 \int_0^{\pi} \left(\frac{\sin(\frac{\pi}{4}\cos\theta)}{0.25\pi\cos\theta}\right)^2 \cdot \sin\theta\, d\theta$$

$$= \frac{120\pi \cdot \frac{\pi^2}{4} I_0 \cdot 4 \cdot 2\pi}{32\pi^2} \cdot 0.612$$

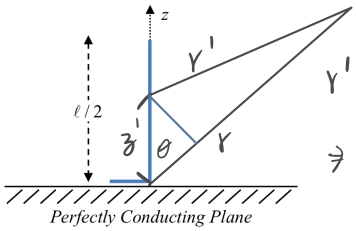
$$= 11.37\, I_0^2 \quad W \ \#.$$

$$R_{rad} = 2P_{rad}/I_0^2 = 22.93\ \Omega \ \#.$$

电磁 H.W p.5   E14086622 採三晴.

Problem 5

Derive the *far-field vector potential* $\bar{A}$, *far-field electric field* $\bar{E}$, *total radiated power* P_{rad}, and the *radiation resistance* R_r of a monopole above a perfectly conducting plane, as shown. The length of this antenna is $\ell/2 = \lambda/4$ and assume its current distribution is $I = I_0 e^{j\omega t}(1-2z/\ell)$ for $0 \le z \le \ell/2$.



$r' = r - z'\cos\theta$

$\ni \frac{1}{r'} = \frac{1}{r}$

Answers: $\bar{A} = \hat{a}_z \frac{?}{4\pi r}\left(\frac{?}{0.25\pi\cos\theta}\right)^2$ (*Weber / m*), $\bar{E} = \ldots$, $P_{rad} \approx 1?.32 I_0^2$ (*Watts*), and ....

$\frac{\ell}{2} = \frac{\lambda}{4}$

$I = I_0 e^{j\omega t}(1 - \frac{2}{\ell}z) \quad 0 \le z \le \frac{\ell}{2}$

$\vec{A} = \int \frac{\mu_0}{4\pi} \frac{I d\ell}{r'} e^{-jkr'} \vec{a_z}$

$= \frac{\mu_0}{4\pi r} \int I_0 e^{j\omega t}(1-\frac{2}{\ell}z')dz' \cdot e^{-jkr'}$

$= \frac{\mu_0 I_0}{4\pi r} e^{j\omega t}\left[\int_0^{\frac{\ell}{2}}(1-\frac{2}{\ell}z')e^{-jk(r-z'\cos\theta)}dz' + \int_{-\frac{\ell}{2}}^{0}(1+\frac{2}{\ell}z')e^{-jk(r-z'\cos\theta)}dz'\right]$

$= \frac{\mu_0 I_0}{4\pi r} e^{j\omega t} e^{-jkr}\left[\int_0^{\frac{\ell}{2}}(1-\frac{2}{\ell}z')e^{jkz'\cos\theta}dz' + \int_{-\frac{\ell}{2}}^{0}(1+\frac{2}{\ell}z')e^{jkz'\cos\theta}dz'\right]$

$\left[(1-\frac{2}{\ell}z')\frac{e^{jkz'\cos\theta}}{jk\cos\theta} - \frac{2}{\ell}\frac{e^{jkz'\cos\theta}}{k^2\cos^2\theta}\right]_0^{\frac{\ell}{2}}$ $\qquad$ $\left[(1+\frac{2}{\ell}z')\frac{e^{jkz'\cos\theta}}{jk\cos\theta} + \frac{2}{\ell}\frac{e^{jkz'\cos\theta}}{k^2\cos^2\theta}\right]_{-\frac{\ell}{2}}^{0}$

$\left[-\frac{2}{\ell}\frac{e^{jk\frac{\ell}{2}\cos\theta}}{k^2\cos^2\theta} - \frac{1}{jk\cos\theta} + \frac{2}{k^2\cos^2\theta\,\ell}\right]$ $\qquad$ $\left[\frac{1}{jk\cos\theta} + \frac{2}{\ell}\frac{1}{k^2\cos^2\theta} - \frac{2}{\ell}\frac{e^{-jk\frac{\ell}{2}\cos\theta}}{k^2\cos^2\theta}\right]$

$= \frac{\mu_0 I_0}{4\pi r} e^{j(\omega t - kr)}\left[-\frac{4}{\ell}\frac{1}{k^2\cos^2\theta}\left(e^{jk\frac{\ell}{2}\cos\theta} + e^{-jk\frac{\ell}{2}\cos\theta}\right) + \frac{4}{\ell k^2\cos^2\theta}\right]$

$= \frac{\mu_0 I_0}{4\pi r} e^{j(\omega t - kr)} \frac{4}{\ell k^2\cos^2\theta}\left[1 - \left(e^{jk\frac{\ell}{2}\cos\theta} + e^{-jk\frac{\ell}{2}\cos\theta}\right)\right]$

$e^{jx} = \cos x + j\sin x$
$e^{-jx} = \cos x - j\sin x$
$\stackrel{\oplus}{\Rightarrow} 2\cos x = 2\cos(\frac{k\ell}{2}\cos\theta)$
$\Rightarrow 2\cos(\frac{\pi}{2}\cos\theta) \quad (\because \frac{k\ell}{2} = \frac{2\pi}{\lambda}\cdot\frac{\lambda}{4} = \frac{\pi}{2})$

$= \frac{\mu_0 I_0}{4\pi r} e^{j(\omega t - kr)} \frac{4}{\ell k^2\cos^2\theta}\left[1 - \cos(\frac{\pi}{2}\cos\theta)\right]$

$= \frac{\mu_0 I_0}{4\pi r} e^{j(\omega t - kr)} \frac{4\ell}{\frac{\pi^2}{4}\cos^2\theta} \cdot 2\sin^2\left[\frac{\pi}{4}\cos\theta\right] \quad (\because \sin^2\frac{\theta}{2} = \frac{1-\cos\theta}{2})$

$= \frac{\mu_0 I_0}{4\pi r} e^{j(\omega t - kr)}\left[\frac{\sin[\frac{\pi}{4}\cos\theta]}{0.25\pi\cos\theta}\right]^2 \cdot 2\ell \cdot \vec{a_z} \; (wb/m)$ #

$(1+\frac{2}{\ell}z') \quad e^{jkz'\cos\theta}$
$\frac{2}{\ell} \quad + \quad \frac{1}{jk\cos\theta}e^{jkz'\cos\theta}$
$0 \quad - \quad -\left(\frac{1}{k\cos\theta}\right)^2 e^{jkz'\cos\theta}$

$(1-\frac{2}{\ell}z') \quad e^{jkz'\cos\theta}$
$-\frac{2}{\ell} \quad + \quad \frac{1}{jk\cos\theta}e^{jkz'\cos\theta}$
$0 \quad - \quad -\left(\frac{1}{k\cos\theta}\right)^2 e^{jkz'\cos\theta}$

$\vec{E} = \eta_0 \vec{H}$, $\vec{H} = \frac{1}{\mu_0}\nabla \times \vec{A}$ $\quad A_\theta = -A\sin\theta\,\vec{a_\theta}$

$$= \frac{\eta_0 j k I_0}{4\pi r} \cdot e^{j(wt-kr)} \left[\frac{\sin[\frac{\pi}{4}\cos\theta]}{0.25\pi\cos\theta}\right]^2 \cdot 2\ell \cdot \sin\theta\ \vec{a_\theta}$$

$$= \frac{j w \mu_0 I_0}{4\pi r} e^{j(wt-kr)} \left[\frac{\sin[\frac{\pi}{4}\cos\theta]}{0.25\pi\cos\theta}\right]^2 \cdot 2\ell \cdot \sin\theta\ \vec{a_\theta}\ (V/m)_{\#} = -jw\vec{A_\theta}.$$

$$P_{rad} = \int_0^{2\pi}\int_0^{\pi} U(\theta,\phi)\sin\theta\, d\theta\, d\phi \cdot \frac{1}{2}$$

$$= \frac{\eta_0 k^2 I_0^2}{32\pi^2}(2\ell)^2 \cdot 2\pi \int_0^{\pi}\left(\frac{\sin(\frac{\pi}{4}\cos\theta)}{0.25\pi\cos\theta}\right)^2 \sin^3\theta\, d\theta \cdot \frac{1}{2}$$

$$= \frac{377 \cdot \frac{\pi^2}{4} I_0^2 \cdot 4}{32\pi^2} \cdot 2\pi \cdot \frac{1.23}{2}$$

$$= 11.39\ I_0^2\ W_{\#}.$$

$$R_{rad} = 2P_{rad}/I_0^2 = 22.745\ \Omega_{\#}$$