STUDENT'S SOLUTIONS MANUAL

TO ACCOMPANY

ADVANCED ENGINEERING MATHEMATICS 7e

CUSTOM EDITION

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Contents

Chapter 1

First-Order Differential Equations

1.1 Terminology and Separable Equations

1. For $x > 0$, rewrite the equation as

$$2xy' + 2y = e^x.$$

With $y = \varphi(x) = \frac{1}{2}x^{-1}(C - e^x)$, compute

$$y' = \frac{1}{2}\left(-x^{-2}(C - e^x) - x^{-1}e^x\right).$$

Then

$$2xy' + 2y = x\left(-x^{-2}(C - e^x) - x^{-1}e^x\right) + x^{-1}(C - e^x) = e^x.$$

Therefore $\varphi(x)$ is a solution.

3. On any interval not containing $x = 0$ we have

$$x\varphi' = x\left(\frac{1}{2} + \frac{3}{2x^2}\right) = x + \left(\frac{3}{2x} - \frac{x}{2}\right) = x - \left(\frac{x^2 - 3}{2x}\right) = x - \varphi,$$

so φ is a solution.

5. For $x > 1$,

$$2\varphi\varphi' = 2\sqrt{x-1}\frac{1}{2\sqrt{x-1}} = 1,$$

so φ is a solution.

7. This equation is separable because we can write it as

$$\frac{\sin(y)}{\cos(y)}\,dy = \frac{1}{x}\,dx$$

if $\cos(y) \neq 0$ and $x \neq 0$. Integrate both sides of this equation to obtain

$$-\ln|\cos(y)| = \ln|x| + c.$$

Assume for the moment that $x > 0$ and $\cos(y) > 0$. Then

$$\ln\left(\frac{1}{\cos(y)}\right) = \ln(x) + c.$$

Take the exponential of both sides of this equation to obtain

$$\frac{1}{\cos(y)} = e^{\ln(x)+c} = e^{\ln(x)}e^c = kx,$$

in which we have written $k = e^c$, so k is positive constant. This gives us

$$\sec(y) = kx.$$

Now it is routine to check by differentiation that this equation implicitly defines a solution of the differential equation, with k allowed to be positive or negative. Solving for y, we obtain the explicit solution

$$y = \arcsin(kx).$$

There remains the case that $\cos(y) = 0$. This corresponds to

$$y = \frac{(2n+1)\pi}{2},$$

with n any integer. In this case $dy/dx = 0$, so the differential equation is satisfied, and $y = (2n+1)\pi/2$ is a singular solution.

8. **Hint** Use the identities

$$\sin(x - y) = \sin(x)\cos(y) - \cos(x)\sin(y),$$

$$\cos(x + y) = \cos(x)\cos(y) - \sin(x)\sin(y),$$

and

$$\cos(2x) = \cos^2(x) - \sin^2(x)$$

to separate the variables and then integrate to obtain the solution.

9. This differential equation is not separable.

10. **Hint** The differential equation itself assumes that $y \neq 0$ and $x \neq -1$. Write

$$\frac{x}{y}\frac{dy}{dx} = \frac{2y^2+1}{x+1},$$

which separates as

$$\frac{1}{y(2y^2+1)}\,dy = \frac{1}{x(x+1)}\,dx.$$

Use a partial fractions decomposition to write

$$\left(\frac{1}{y} - \frac{2y}{1+2y^2}\right)dy = \left(\frac{1}{x} - \frac{1}{1+x}\right)dx.$$

Now integrate to obtain the general solution.

11. Write
$$3\frac{dy}{dx} = \frac{4x}{y^2}$$
and separate variables:
$$3y^2\,dy = 4x\,dx.$$
Integrate to obtain
$$y^3 = 2x^2 + k,$$
which implicitly defines the general solution. We can also write
$$y = \left(2x^2 + k\right)^{1/3}.$$

13. Write the differential equation as
$$\begin{aligned}\frac{dy}{dx} &= \frac{\sin(x+y)}{\cos(y)} \\ &= \frac{\sin(x)\cos(y) + \cos(x)\sin(y)}{\cos(y)} \\ &= \sin(x) + \cos(x)\frac{\sin(y)}{\cos(y)}.\end{aligned}$$
There is no way to separate the variables in this equation, so the differential equation is not separable.

15. Write the differential equation as
$$x\frac{dy}{dx} = y(y-1).$$
This is separable. If $y \neq 0$ and $y \neq 1$, we can write
$$\frac{1}{x}\,dx = \frac{1}{y(y-1)}\,dy.$$
Use partial fractions to write this as
$$\frac{1}{x}\,dx = \frac{1}{y-1}\,dy - \frac{1}{y}\,dy.$$
Integrate to obtain
$$\ln|x| = \ln|y-1| - \ln|y| + c,$$
or
$$\ln|x| = \ln\left|\frac{y-1}{y}\right| + c.$$
This can be solved for x to obtain the general solution
$$y = \frac{1}{1-kx}.$$
The trivial solution $y(x) = 0$ is a singular solution, as is the constant solution $y(x) = 1$. We assumed that $y \neq 0, 1$ in the algebra of separating the variables.

17. Write $\ln(y^x) = x\ln(y)$ and separate the variables to write

$$\frac{\ln(y)}{y}\,dy = 3x\,dx.$$

Integrate to obtain $(\ln(y))^2 = 3x^2 + c$. Substitute the initial condition to obtain $c = -3$, so the solution is implicitly defined by $(\ln(y))^2 = 3x^2 - 3$.

19. Separate the variables to obtain

$$y\cos(3y)\,dy = 2x\,dx,$$

with solution given implicitly by

$$\frac{1}{3}y\sin(3y) + \frac{1}{9}\cos(3y) = x^2 + c.$$

The initial condition requires that

$$\frac{\pi}{9}\sin(\pi) + \frac{1}{9}\cos(\pi) = \frac{4}{9} + c,$$

so $c = -5/9$. The solution is implicitly defined by

$$3y\sin(3y) + \cos(3y) = 9x^2 - 5.$$

21. If $y \neq -1$ and $x \neq 0$, we obtain the separated equation

$$\frac{y^2}{y+1}\,dy = \frac{1}{x}\,dx.$$

Write this as

$$\left(y - 1 + \frac{1}{1+y}\right)\,dy = \frac{1}{x}\,dx.$$

Integrate to obtain

$$\frac{1}{2}y^2 - y + \ln|1+y| = \ln|x| + c.$$

Now use the initial condition $y(3e^2) = 2$ to obtain

$$2 - 2 + \ln(3) = \ln(3) + 2 + c$$

so $c = -2$ and the solution is implicitly defined by

$$\frac{1}{2}y^2 - y + \ln(1+y) = \ln(x) - 2,$$

in which the absolute values have been removed because the initial condition puts the solution in a part of the $x, y-$ plane where $x > 0$ and $y > -1$.

22. **Hint** By Newton's law of cooling the temperature function $T(t)$ satisfies $T'(t) = k(T - 60)$, with k a constant of proportionality to be determined. Further, from the information of the problem, $T(0) = 90$ and $T(10) = 88$. Solve for $T(t)$, then obtain $T(20$ and also an equation for the time t at which $T(t) = 65$.

23. Compute

$$I'(x) = -\int_0^\infty \frac{2x}{t} e^{-(t^2+(x/t)^2)}\,dt.$$

Let $u = x/t$ to obtain

$$\begin{aligned} I'(x) &= 2\int_\infty^0 e^{-((x/u)^2+u^2)}\,du \\ &= -2\int_0^\infty e^{-(u^2+(x/u)^2)}\,du = -2I(x). \end{aligned}$$

This is the separable equation $I' = -2I$. Write this as

$$\frac{1}{I}\,dI = -2\,dx$$

and integrate to obtain $I(x) = ce^{-2x}$. Now

$$I(0) = \int_0^\infty e^{-t^2}\,dt = \frac{\sqrt{\pi}}{2},$$

a standard result often used in statistics. Then

$$I(x) = \frac{\sqrt{\pi}}{2}e^{-2x}.$$

Put $x = 3$ to obtain

$$\int_0^\infty e^{-t^2-(9/t^2)}\,dt = \frac{\sqrt{\pi}}{2}e^{-6}.$$

24. **Hint** For water h feet deep in the cylindrical hot tub, $V = 25\pi h$, so

$$25\pi\frac{dh}{dt} = -0.6\pi\left(\frac{5}{16}\right)^2\sqrt{64h},$$

with $h(0) = 4$.

The time it will take to drain the tank is

$$T = \int_4^0 \left(\frac{dt}{dh}\right)\,dh$$

seconds. To determine the time it will take to drain the upper half of the tank, integrate from 4 to 2.

25. Model the problem using Torricelli's law and the geometry of the hemispherical tank. Let $h(t)$ be the depth of the liquid at time t, $r(t)$ the radius of the top surface of the draining liquid, and $V(t)$ the volume in the container (See Figure 1.1). Then

$$\frac{dV}{dt} = -kA\sqrt{2gh} \text{ and } \frac{dV}{dt} = \pi r^2\frac{dh}{dt}.$$

Here $r^2 + h^2 = 18^2$, since the radius of the tub is 18. We are given $k = 0.8$ and $A = \pi(1/4)^2 = \pi/16$ is the area of the drain hole. With $g = 32$ feet per second per second, we obtain the initial value problem

$$\pi(324 - h^2)\frac{dh}{dt} = 0.4\pi\sqrt{h}; h(0) = 18.$$

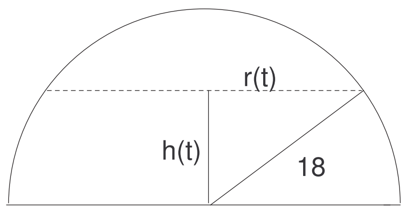

Figure 1.1: Problem 25, Section 1.1.

This is a separable differential equation with the general solution

$$1620\sqrt{h} - h^{5/2} = -t + k.$$

Then $h(0) = 18$ yields $k = 3888\sqrt{2}$, so

$$1620\sqrt{h} - h^{5/2} = 3888\sqrt{2} - t.$$

The hemisphere is emptied at the instant that $h = 0$, hence at $t = 3888\sqrt{2}$ seconds, about 91 minutes, 39 seconds.

26. **Hint** From the geometry of the sphere, $dV/dt = -kA\sqrt{2gh}$ and the problem to solve for $h(t)$ is

$$\pi(324 - (h-18)^2)\frac{dh}{dt} = -0.8\pi\left(\frac{1}{4}\right)^2\sqrt{64h},$$

with $h(0) = 36$. Here $h(t)$ is the height of the upper surface of the fluid above the bottom of the sphere.

27. Let $U(t)$ be the mass present at time t. The initial value, at time designated as time zero, is $U(0) = 10$ kilograms. Then, from the discussion in the text, we have immediately that

$$U(t) = 10e^{kt}.$$

If the half-life is $h = 4.5(10^9)$ years, then

$$h = 4.5(10^9) = -\frac{1}{k}\ln(2),$$

so

$$k = -\frac{\ln(2)}{4.5(10^9)}.$$

Then

$$\begin{aligned} U(t) &= 10e^{-\ln(2)t/4.5(10^9)} \\ &= 10e^{\ln(2^{-t/4.5(10^9)})} \\ &= 10 \cdot 2^{-t/4.5(10^9)} \\ &= 10\left(\frac{1}{2}\right)^{t/4.5(10^9)}. \end{aligned}$$

Since one billion years is 10^9 years, then the mass present in one billion years (from the designated time zero) is

$$U(10^9) = 10(1/2)^{1/4.5} \approx 8.57\text{kg}.$$

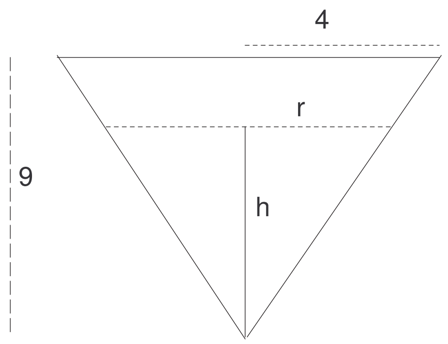

Figure 1.2: Problem 31(a), Section 1.1.

29. Suppose the thermometer was removed from the house at time $t = 0$, and let $t > 0$ denote the time in minutes since then. The house is kept at 70 degrees F. Let A denote the unknown outside ambient temperature, which is assumed constant. The temperature of the thermometer at time t is modeled by

$$T'(t) = k(T - A); T(0) = 70, T(5) = 60 \text{ and } T(15) = 50.4.$$

There are three conditions because we must find k and A, as well as the constant of integration obtained in solving the differential equation.

Separation of variables and the initial condition $T(0) = 70$ yield the expression $T(t) = A + (70 - A)e^{kt}$. The other two conditions now give us

$$T(5) = 60 = A + (70 - A)e^{5k} \text{ and } T(15) = 50.4 = A + (70 - A)e^{15k}.$$

Solve the first equation to obtain

$$e^{5k} = \frac{60 - A}{70 - A}.$$

Substitute this into the second equation to obtain

$$(7 - A)\left(\frac{60 - A}{70 - A}\right)^3 = 50.4 - A.$$

This yields the quadratic equation

$$10.4A^2 - 1156A + 30960 = 0$$

with roots $A = 45$ and 66.16. Clearly we require that $A < 50.4$, so $A = 45$ degrees Fahrenheit.

31. (a) Let $r(t)$ be the radius of the exposed water surface and $h(t)$ the depth of the draining water at time t. Since cross sections of the cone are similar,

$$\pi r^2 \frac{dh}{dt} = -kA\sqrt{2gh},$$

with $h(0) = 9$. From similar triangles (Figure 1.2), $r/h = 4/9$, so $r = (4/9)h$. Substitute $k = 0.6$, $g = 32$ and $A = \pi(1/12)^2$ and simplify the resulting equation to obtain

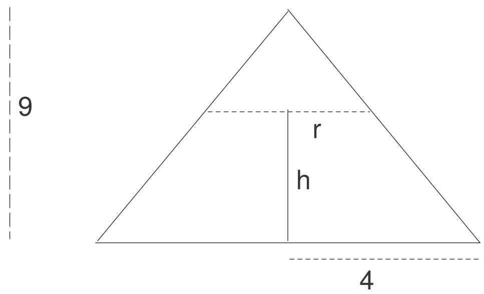

Figure 1.3: Problem 31(b), Section 1.1.

$$h^{3/2}\frac{dh}{dt} = -27/160,$$

with $h(0) = 9$. This separable equation has the general solution given implicitly by

$$h^{5/2} = -\frac{27}{64}t + k.$$

Since $h(0) = 9$, then $k = 243$ and the tank empties out when $h = 0$, so

$$t = 243\left(\frac{64}{27}\right) = 576$$

seconds, about 9 minutes, 36 seconds.

(b) This problem is modeled like part (a), except now the cone is inverted. This changes the similar triangle proportionality (Figure 1.3) to

$$\frac{r}{9-h} = \frac{4}{9}.$$

Then $r = (4/9)(9-h)$. The separable differential equation becomes

$$\frac{(9-h)^2}{\sqrt{h}}\,dh = -\frac{27}{160},$$

with $h(0) = 9$. This initial value problem has the solution

$$162\sqrt{h} - 12h^{3/2} + \frac{2}{5}h^{5/2} = -\frac{27}{160}t + \frac{1296}{5}.$$

The tank runs dry at $h = 0$, which occurs when

$$t = \frac{160}{27}\left(\frac{1296}{5}\right) = 1536$$

seconds, about 25 minutes, 36 seconds.

33. Begin with the logistic equation

$$P'(t) = aP(t) - bP(t)^2$$

in which a and b are positive constants. Then

$$\frac{dP}{dt} = (a - bP)P.$$

This is separable and we can write

$$\frac{1}{(a - bP)P}\, dP = dt.$$

Use a partial fractions decomposition to write

$$\left(\frac{1}{a}\frac{1}{P} + \frac{b}{a}\frac{1}{a - bP}\right) dP = dt.$$

Integrate to obtain

$$\frac{1}{a}\ln(P) - \frac{1}{a}\ln(a - bP) = t + c.$$

Here we assume that $P(t) > 0$ and $a - bP(t) > 0$. Write this equation as

$$\ln\left(\frac{P}{a - bP}\right) = at + k,$$

with $k = ac$ still a constant to be determined. Then

$$\frac{P}{a - bP} = e^{at+k} = e^k e^{at} = Ke^{at},$$

where $K = e^k$ is the constant to be determined. Now $P(0) = p_0$, so

$$K = \frac{p_0}{a - bp_0}.$$

Then

$$\frac{P}{a - bP} = \frac{p_0}{a - bp_0} e^{at}.$$

It is a straightforward algebraic manipulation to solve for P and obtain

$$P(t) = \frac{ap_0}{a - bp_0 + bp_0 e^{at}} e^{at}.$$

Notice that $P(t)$ is a strictly increasing function. Further, by multiplying numerator and denominator by e^{-at}, and using the fact that $a > 0$, we have

$$\begin{aligned}\lim_{t\to\infty} P(t) &= \lim_{t\to\infty} \frac{ap_0}{(a - bp_0)e^{-at} + bp_0}\\ &= \frac{ap_0}{bp_0} = \frac{a}{b}.\end{aligned}$$

34. **Hint** With a and b taking on the given values, and $p_0 = 3,929,214$, the population in 1790, obtain the logistic model for the United States population growth:

$$P(t) = \frac{123,141.5668}{0.03071576577 + 0.0006242342282e^{0.03134t}} e^{0.03134t}.$$

An exponential model can also be constructed as $Q(t) = Ae^{kt}$. Then

$$A = Q(0) = 3,929,214,$$

the initial (1790) population. To find k, use the fact

$$Q(10) = 5308483 = 3929214e^{10k}$$

to solve for k.

1.2 Linear Equations

1. An integrating factor is $e^{\int -2\,dx} = e^{-2x}$. Multiply the differential equation by e^{-2x} to obtain
$$y'e^{-2x} - 2ye^{-2x} = (ye^{-2x})' = -8x^2e^{-2x}.$$
Integrate to obtain
$$ye^{-2x} = \int -8x^2e^{-2x}\,dx = 4x^2e^{-2x} + 4xe^{-2x} + 2e^{-2x} + c.$$
The general solution is
$$y = 4x^2 + 4x + 2 + ce^{2x}.$$

2. **Hint** An integrating factor is
$$e^{\int \sec(x)\,dx} = e^{\ln|\sec(x)+\tan(x)|} = \sec(x) + \tan(x).$$

3. With $p(x) = -3/x$, an integrating factor is
$$e^{\int p(x)\,dx} = e^{-3\ln(x)} = x^{-3}.$$
Multiply the differential equation by x^{-3} to obtain
$$\frac{d}{dx}(yx^{-3}) = \frac{2}{x}.$$
A routine integration gives us $yx^{-3} = 2\ln(x) + c$, or
$$y = cx^3 + 2x^3\ln|x|$$
for $x \neq 0$.

4. **Hint** An integrating factor is $e^{\int dx} = e^x$.

5. $e^{\int 2\,dx} = e^{2x}$ is an integrating factor. Multiply the differential equation by e^{2x} to obtain
$$y'e^{2x} + 2y = (ye^{2x})' = xe^{2x}.$$
Integrate to obtain
$$ye^{2x} = \int xe^{2x}\,dx = \frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + c.$$
The general solution is
$$y = \frac{1}{2}x - \frac{1}{4} + ce^{-2x}.$$

6. **Hint** An integrating factor is
$$e^{\int (5/9x)\,dx} = e^{(5/9)\ln(x)} = e^{\ln(x^{5/9})} = x^{5/9}.$$

7. An integrating factor is
$$e^{\int (2/(x+1))\,dx} = e^{2\ln|x+1|} = e^{\ln((x+1)^2)} = (x+1)^2.$$
Multiply the differential equation by $(x+1)^2$ to obtain
$$(x+1)^2y' + 2(x+1)y = ((x+1)^2y)' = 3(x+1)^2.$$

Integrate to obtain

$$(x+1)^2 y = (x+1)^3 + c.$$

Then

$$y = (x+1) + \frac{c}{(x+1)^2}.$$

Now

$$y(0) = 5 = 1 + c$$

so $c = 4$ and the solution of the initial value problem is

$$y = x + 1 + \frac{4}{(x+1)^2}.$$

9. Notice that, if we multiply the differential equation by $x-2$, we obtain

$$y'(x-2) + y = ((x-2)y)' = 3x(x-2).$$

Integrate to obtain

$$(x-2)y = x^3 - 3x^2 + c.$$

The general solution is

$$y = \frac{1}{x-2}(x^3 - 3x^2 + c).$$

Now

$$y(3) = 27 - 27 + c = 4$$

so the initial value problem has the solution

$$y = \frac{x^3 - 3x^2 + 4}{x-2} = x^2 - x - 2.$$

11. If $A_1(t)$ and $A_2(t)$ are the amounts of salt in tanks one and two, respectively, at time t, we have

$$A_1'(t) = \frac{5}{2} - \frac{5A_1(t)}{100}; A_1(0) = 20$$

and

$$A_2'(t) = \frac{5A_1(t)}{100} - \frac{5A_2(t)}{150}; A_2(0) = 90.$$

Solve the first initial value problem to obtain

$$A_1(t) = 50 - 30e^{-t/20}.$$

Substitute this into the problem for $A_2(t)$ to obtain

$$A_2' + \frac{1}{30}A_2 = \frac{5}{2} - \frac{3}{2}e^{-t/20}; A_2(0) = 90.$$

Solve this to obtain

$$A_2(t) = 75 + 90e^{-t/20} - 75e^{-t/30}.$$

Tank 2 has its minimum when $A_2'(t) = 0$, hence when

$$2.5e^{-t/30} - 4.5e^{-t/20} = 0.$$

Then $e^{t/60} = 9/5$, or $t = 60\ln(9/5)$. Then

$$A_2(t)_{\text{min}} = A_2(60\ln(9/5)) = \frac{5450}{81}$$

pounds.

12. **Hint** If $A(t)$ is the amount of salt in the tank at time $t \geq 0$, then

$$\begin{aligned}\frac{dA}{dt} &= \text{ rate salt is added } - \text{ rate salt is removed}\\ &= 6 - 2\left(\frac{A(t)}{50+t}\right).\end{aligned}$$

Solve this subject to the initial condition $A(0) = 28$.

13. Let (x, y) be a point on the curve. The tangent line at (x, y) must pass through $(0, 2x^2)$, hence must have slope $(y - 2x^2)/x$. But this slope is y', so we have the differential equation

$$y' = \frac{y - 2x^2}{x}.$$

This is the linear differential equation

$$y' - \frac{1}{x}y = -2x,$$

which has the general solution $y = -2x^2 + cx$.

1.3 Exact Equations

In the following we assume that the differential equation has the form $M(x, y) + N(x, y)y' = 0$, or, in differential form, $M\,dx + N\,dy = 0$.

1. $\partial M/\partial y = 1 = \partial N/\partial x$, so the equation is exact for all (x, y) with $x \neq 0$, where the equation is not defined. Integrate $\partial\varphi/\partial x = M$ or $\partial\varphi/\partial y = N$ to obtain the potential function

$$\varphi(x, y) = \ln|x| + xy + y^3.$$

The general solution is defined implicitly by

$$\varphi(x, y) = \ln|x| + xy + y^3 = c$$

for $x \neq 0$.

3. Since

$$\frac{\partial M}{\partial y} = 4y + e^{xy} + xye^{xy} = \frac{\partial N}{\partial x}$$

for all x and y, the equation is exact in the entire plane. One way to find a potential function is to integrate

$$\frac{\partial\varphi}{\partial x} = M(x, y) = 2y^2 + ye^{xy}$$

with respect to x to obtain

$$\varphi(x, y) = 2xy^2 + e^{xy} + \alpha(y).$$

Then we need

$$\frac{\partial\varphi}{\partial y} = 4xy + xe^{xy} + \alpha'(y) = N(x, y) = 4xy + xe^{xy} + 2y.$$

This requires that $\alpha'(y) = 2y$ so we may choose $\alpha(y) = y^2$. A potential function has the form

$$\varphi(x, y) = 2xy^2 + e^{xy} + y^2.$$

The general solution is implicitly defined by

$$\varphi(x,y) = 2xy^2 + e^{xy} + y^2 = c.$$

We could have also started by integrating $\partial N/\partial y = 4xy + xe^{xy} + 2y$ with respect to y.

5. $\partial M/\partial y = 4 + 2x^2$ and $\partial N/\partial x = 4x$, so this equation is not exact.

6. **Hint** For the equation to be exact, we need

$$\frac{\partial M}{\partial y} = \alpha x y^{\alpha-1} = \frac{\partial N}{\partial x} = -2xy^{\alpha-1}.$$

Solve for α and then find the corresponding potential function by integration.

7. For exactness we need

$$\frac{\partial M}{\partial y} = 6xy^2 - 3 = \frac{\partial N}{\partial x} = -3 - 2\alpha x y^2$$

and this requires that $\alpha = -3$. By integration, we find a potential function $\varphi(x,y) = x^2y^3 - 3xy - 3y^2$. The general solution is implicitly defined by

$$x^2y^3 - 3xy - 3y^2 = c.$$

9. Compute

$$\frac{\partial M}{\partial y} = -2x\sin(2y-x) - 2\cos(2y-x) = \frac{\partial N}{\partial x},$$

so the differential equation is exactly. For a potential function, integrate

$$\frac{\partial \varphi}{\partial y} = -2x\cos(2y-x)$$

with respect to y to get

$$\varphi(x,y) = -x\sin(2y-x) + \alpha(x).$$

Then we must have

$$\begin{aligned}\frac{\partial \varphi}{\partial x} &= x\cos(2y-x) - \sin(2y-x)\\ &= -\sin(2y-x) + x\cos(2y-x) + \alpha'(x).\end{aligned}$$

Then $\alpha'(x) = 0$ and we may choose $\alpha(x) = 0$ to obtain

$$\varphi(x,y) = -x\sin(2y-x).$$

The general solution has the form

$$-x\sin(2y-x) = c.$$

For $y(\pi/12) = \pi/8$, we need

$$-\frac{\pi}{12}\sin\left(\frac{\pi}{4} - \frac{\pi}{12}\right) = -\frac{\pi}{12}\sin(\pi/6) = -\frac{\pi}{24} = c.$$

The solution of the initial value problem is implicitly defined by

$$x\sin(2y-x) = \frac{\pi}{24}.$$

11. Since $\partial M/\partial y = 12y^3 = \partial N \partial x$, the differential equation is exact for all x and y. Straightforward integrations yield the potential function

$$\varphi(x, y) = 3xy^4 - x.$$

The general solution is implicitly defined by

$$3xy^4 - x = c.$$

For the initial condition, we need $y = 2$ when $x = 1$, so

$$3(1)(2^4) - 1 = 47 = c.$$

The initial value problem has the unique solution implicitly defined by

$$3xy^4 - x = 47.$$

13. Multiply the differential equation by $\mu(x, y) = x^a y^b$ to obtain

$$x^{a+1}y^{b+1} + x^a y^{b-3/2} + x^{a+2}y^b y' = 0.$$

For this to be exact, we need

$$\begin{aligned}\frac{\partial M}{\partial y} &= (b+1)x^{a+1}y^b + \left(b - \frac{3}{2}\right)x^a y^{b-5/2}\\ &= \frac{\partial N}{\partial x} = (a+2)x^{a+1}y^b.\end{aligned}$$

Divide this by $x^a y^b$ to require that

$$(b+1)x + \left(b - \frac{3}{2}\right)y^{-5/2} = (a+2)x.$$

This will be true for all x and y if we let $b = 3/2$, and then choose a so that $(b+1)x = (a+2)x$, so $b + 1 = a + 2$. Therefore

$$a = \frac{1}{2} \text{ and } b = \frac{3}{2}.$$

Multiply the original differential equation by $\mu(x, y) = x^{1/2}y^{3/2}$ to obtain

$$x^{3/2}y^{5/2} + x^{1/2} + x^{5/2}y^{3/2}y' = 0.$$

Integrate $\partial\varphi/\partial y = x^{5/2}y^{3/2}$ to obtain

$$\varphi(x, y) = \frac{2}{5}x^{5/2}y^{5/2} + \beta(x).$$

Then we need

$$\frac{\partial\varphi}{\partial x} = x^{3/2}y^{5/2} + \beta'(x) = x^{3/2}y^{5/2} + x^{1/2}.$$

Then $\beta(x) = 2x^{3/2}/3$ and a potential function is

$$\varphi(x, y) = \frac{2}{5}x^{5/2}y^{5/2} + \frac{2}{3}x^{3/2}.$$

The general solution of the original differential equation is

$$\varphi(x, y) = \frac{2}{5}x^{5/2}y^{5/2} + \frac{2}{3}x^{3/2} = c.$$

The differential equation multiplied by the integrating factor has the same solutions as the original differential equation because the integrating factor is assumed to be nonzero. Thus we must exclude $x = 0$ and $y = 0$, where $\mu = 0$.

14. Multiply the differential equation by $x^a y^b$ and apply the test for exactness to obtain the following equation for a and b:

$$(2(b+2) - 3(a+1))y = ((9(b+1) - 6(a+2))x.$$

15. $\varphi + c$ is also a potential function if φ is because

$$\frac{\partial(\varphi + c)}{\partial x} = \frac{\partial \varphi}{\partial x}$$

and

$$\frac{\partial(\varphi + c)}{\partial y} = \frac{\partial \varphi}{\partial y}$$

Any function defined implicitly by $\varphi(x, y) = k$ is also defined by $\varphi(x, y) + c = k$, because, if k can assume any real value, so can $k - c$ for any c.

1.4 Integrating Factors

1. A function of y only, $\nu(y)$, is an integrating factor for $M + Ny' = 0$ if only if $\dfrac{\partial}{\partial y}(\nu M) = \dfrac{\partial}{\partial x}(\nu N)$. Since $\nu = \nu(y)$ we get the condition $\nu' M + \nu M_y = \nu N_x$. Solving for $\dfrac{\nu'}{\nu} = \dfrac{1}{M}(N_x - M_y)$, a sufficient condition for such a ν is that $\dfrac{1}{M}(N_x - M_y)$ is a function of y only (since ν'/ν is). If $\dfrac{1}{M}(N_x - M_y) = g(y)$, then $\nu(y) = e^{\int g(y)dy}$ will produce an integrating factor.

3. (a) $M_y = 1$ and $N_x = -1$ so never exact.
(b) Since $\dfrac{1}{N}(M_y - N_x) = -\dfrac{2}{x}, \mu(x) = \dfrac{1}{x^2}$
(c) Since $\dfrac{1}{M}(N_x - M_y) = -\dfrac{2}{y}, \nu(y) = \dfrac{1}{y^2}$
(d) By Problem 2, $M_y - N_x = 2 = a\dfrac{(-x)}{x} - b\dfrac{(y)}{y} = -(a + b)$ for any a, b satisfying $a + b = -2$.

5. (a) $M_y = 0, N_x = 3$; (b) $\nu(y) = e^{3y}$; (c) $xe^{3y} - e^y = c$

7. (a) $M_y = 4x + 12y, N_x = 4x + 6y$
(b) By problem 2, $M_y - N_x = 6y = a(2x + 6y) - b(4x + 6y)$ holds for $a = 2, b = 1$, so $\mu = x^2 y$
(c) $x^4y^2 + 2x^3y^3 = c$.

9. (a) $M_y = 4xy + 2x, N_x = 2xy + 2x$; (b) $\nu(y) = \dfrac{1}{y+1}$; (c) $x^2 y = cy$; (d) $y = -1$

11. (a) $M_y = 1 - 4y^3, N_x = 0$;
(b) The hint produces $\mu(x, y) = e^{-3x}y^{-4}$;
(c) $y^3 - 1 = ky^3e^{3x}$

13. $\mu(x) = \dfrac{1}{x}; \ln|x| + y = c; y = 4 - \ln(x)$;

15. $\mu(x) = x; x^2(y^3 - 2) = c; x^2(y^3 - 2) = -9$

17. $\nu(y) = \dfrac{1}{y}; x^2 + 3\ln|y| = c; y = 4e^{-x^2/3}$

19. $\mu(x) = e^x; e^x \sin(x-y) = c; e^x \sin(x-y) = \dfrac{1}{2}$

21. $\dfrac{\partial}{\partial y}(c\mu M) = c\dfrac{\partial}{\partial y}(\mu M) = c\dfrac{\partial}{\partial x}(\mu N) = \dfrac{\partial}{\partial x}(c\mu N)$

1.5 Homogeneous, Bernoulli and Riccati Equations

1. Write
$$y' = \frac{y}{x+y} = \frac{y/x}{1+y/x}$$
to recognize the differential equation as homogeneous. Put $y = xu$ to obtain
$$u + xu' = \frac{u}{1+u}.$$
Then
$$xu' = \frac{u}{1+u} - u = -\frac{u^2}{1+u}.$$
This is a separable equation and in differential form we have
$$\frac{1+u}{u}\,du = -\frac{1}{x}\,dx,$$
or
$$\left(\frac{1}{u^2} + \frac{1}{u}\right)du = -\frac{1}{x}\,dx.$$
Integrate to obtain
$$-\frac{1}{u} + \ln\left|\frac{y}{x}\right| = -\ln|x| + c.$$
Put $u = y/x$ to obtain
$$-\frac{x}{y} + \ln|y| - \ln|x| = -\ln|x| + c,$$
or
$$y\ln|y| - x = cy.$$
This equation implicitly defines the general solution.

3. This equation is exact, with general solution defined by
$$xy - x^2 - y^2 = c.$$

4. **Hint** The differential equation is Riccati, and by inspection one solution is $S(x) = 4$.

5. The differential equation is Bernoulli, with $\alpha = -3/4$. The general solution is given by
$$5x^{7/4}y^{7/4} + 7x^{-5/4} = c.$$

7. This is a Bernoulli equation with $\alpha = 2$ and we obtain the general solution
$$y = \frac{1}{1 + ce^{x^2/2}}.$$

8. **Hint** This is a Bernoulli equation with $\alpha = -4/3$. Substitute $v = y^{7/3}$, or $y = v^{3/7}$. Solve the resulting equation for v from this obtain the solution for y.

9. The equation is Bernoulli with $\alpha = 2$. We obtain
$$y = 2 + \frac{2}{cx^2 - 1}.$$

11. The equation is Riccati with one solution $S(x) = e^x$. The general solution is
$$y = \frac{2e^x}{ce^{2x} - 1}.$$

13. This is a Riccati equation with solution $S(x) = x$ (by inspection). Put $y = x + 1/z$ and substitute to obtain
$$2 - \frac{z'}{z^2} = \frac{1}{x^2}\left(x + \frac{1}{z}\right)^2 - \frac{1}{x}\left(x + \frac{1}{z}\right) + 1.$$
Simplify this to obtain
$$z' + \frac{1}{x}z = -\frac{1}{x^2}.$$
This linear differential equation can be written $(xz)' = -1/x$ and has the solution
$$z = -\frac{\ln(x)}{x} + \frac{c}{x}.$$
Then
$$y = x + \frac{x}{c - \ln(x)}$$
for $x > 0$.

15. For the first part,
$$F\left(\frac{ax + by + c}{dx + py + r}\right) = F\left(\frac{a + b(y/x) + c/x}{d + p(y/x) + r/x}\right) = f\left(\frac{y}{x}\right)$$
if and only if $c = r = 0$.

Now suppose $x = X + h$ and $y = Y + k$. Then
$$\frac{dY}{dX} = \frac{dY}{dx}\frac{dx}{dX} = \frac{dy}{dx}$$
so
$$\begin{aligned}\frac{dY}{dX} &= F\left(\frac{a(X + h) + b(Y + k) + c}{d(X + h) + p(Y + k) + r}\right)\\ &= F\left(\frac{aX + bY + ah + bk + c}{dX + pY + dh + pk + r}\right)\end{aligned}$$
This equation is homogeneous exactly when
$$ah + bk = -c \text{ and } dh + pk = -r.$$
This two by two system has a solution when the determinant of the coefficients is nonzero: $ap - bd \neq 0$.

17. with $x = X + 2$ and $y = Y - 1$ we obtain
$$(2x + y - 3)^2 = K(y - x + 3).$$

18. **Hint** Here $a = 0, b = 1, c = -3$ and $d = p = 1, r = -1$. Solve

$$k = 3, h + k = 1$$

to obtain $k = 3$ and $h = -2$. Thus let $x = X - 2, y = Y + 3$ to transform the equation to a homogeneous equation in X and Y.

19. Set $x = X + 2, y = Y - 3$ to obtain

$$\frac{dY}{dX} = \frac{3X - Y}{X + Y}.$$

This homogeneous equation has general solution (in terms of x and y)

$$3(x-2)^2 - 2(x-2)(y+3) - (y+3)^2 = K.$$

1.6 Additional Applications

1. (a) Calculate

$$i'(t) = \frac{E}{R}e^{-Rt/L} > 0,$$

implying that the current increases with time.

(b) Note that $(1 - e^{-1}) = 0.63+$, so the inductive time constant is $t_0 = L/R$.

(c) For $i(0) \neq 0$, the time to reach 63 percent of E/R is

$$t_0 = \frac{L}{R}\ln\left(\frac{e(E - Ri(0))}{E}\right),$$

which decreases with $i(0)$.

3. Let θ be the angle the chord makes with the vertical. Then

$$m\frac{dv}{dt} = mg\cos(\theta); v(0) = 0.$$

This gives us $s(t) = \frac{1}{2}gt^2\cos(\theta)$, so the time of descent is

$$t = \left(\frac{2s}{g\cos(\theta)}\right)^{1/2},$$

where s is the length of the chord. By the law of cosines, the length of this chord satisfies

$$s^2 = 2R^2 - 2R^2\cos(\pi - 2\theta) = 2R^2(1 + \cos(2\theta)) = 4R^2\cos^2(\theta).$$

Therefore

$$t = 2\sqrt{\frac{R}{g}},$$

and this is independent of θ.

4. **Hint** Show by analyzing the buoyant force on the box and the mass of the box that the velocity $v(t)$ of the sinking box satisfies

$$12\frac{dv}{dt} = 384 - 375 - \frac{1}{2}v; v(0) = 0.$$

Solve this problem and compute the terminal velocity as

$$\lim_{t\to\infty} v(t).$$

We can also determine $s(t)$ from the velocity. To determine the velocity when the box reaches the bottom, solve $s(t) = 100$ to determine when this occurs and substitute this time into the velocity. Solving $s(t) = 100$ requires a numerical solution using a package such as MAPLE.

5. Until the parachute is opened at $t = 4$ seconds, the velocity $v(t)$ satisfies the initial value problem

$$\left(\frac{192}{32}\right)\frac{dv}{dt} = 192 - 6v; v(0) = 0.$$

This has solution $v(t) = 32(1 - e^{-t})$ for $0 \le t \le 4$. When the parachute opens at $t = 4$, the skydiver has a velocity of $v(4) = 32(1-e^{-4})$ feet per second. Velocity with the open parachute satisfies the initial value problem

$$\left(\frac{192}{32}\right)\frac{dv}{dt} = 192 - 3v^2, v(4) = 32(1 - e^{-4}) \text{ for } t \ge 4.$$

This differential equation is separable and can be integrated using partial fractions:

$$\int \left[\frac{1}{v+8} - \frac{1}{v-8}\right] dv = -\int 8t\, dt.$$

This yields

$$\ln\left(\frac{v+8}{v-8}\right) = -8t + \ln\left(\frac{5-4e^{-4}}{3-4e^{-4}}\right) + 32.$$

Solve for $v(t)$ to obtain

$$v(t) = \frac{8(1 + ke^{-8(t-4)})}{1 - ke^{-8(t-4)}} \text{ for } t \ge 4.$$

We find using the initial condition that

$$k = \frac{3-4e^{-4}}{5-4e^{-4}}.$$

Terminal velocity is $\lim_{t\to\infty} v(t) = 8$ feet per second. The distance fallen is

$$s(t) = \int_0^t v(\xi)\, d\xi = 32(t - 1 + e^{-t})$$

for $0 \le t \le 4$, while

$$s(t) = 32(3 + e^{-4}) + 8(t-4) + 2\ln(1 - ke^{-8(t-4)}) - 2\ln\left(\frac{2}{5-4e^{-4}}\right)$$

for $t \ge 4$.

6. **Hint** Show that the velocity $v(t)$ of the box at time t satisfies the problem

$$\frac{48}{32}\frac{dv}{dt} = -48\left(\frac{24}{25}\right)\left(\frac{1}{3}\right) + 48\left(\frac{7}{25}\right) - \frac{3}{2}v; v(0) = 16.$$

Solve this for the velocity, from which the distance can be computed.

7. If the box loses 32 pounds of material on impact with the bottom, then $m = 11$ slugs. Now

$$11\frac{dv}{dt} = -352 + 375 - \frac{1}{2}v; v(0) = 0$$

in which we have taken up as the positive direction. This gives us

$$v(t) = 46(1 - e^{-t/22})$$

so the distance traveled up from the bottom is

$$s(t) = 46(t + 22e^{-t/22} - 22)$$

feet. Solve $s(t) = 100$ numerically to obtain $t \approx 10.56$ seconds. The surfacing velocity is approximately $v(10.56) \approx 17.5$ feet per second.

9. The capacitor charge is modeled by

$$250(10^3)i + \frac{1}{2(10^{-6})}q = 80; q(0) = 0.$$

Put $i = q'$ to obtain, after some simplification,

$$q' + 2q = 32(10^{-5}),$$

a linear equation with solution $q(t) = 16(10^{-5})(1 - e^{-2t})$. The capacitor voltage is

$$E_C = \frac{1}{C}q = 80(1 - e^{-2t}).$$

The voltage reaches 76 volts when $t = (1/2)\ln(20)$, which is approximately 1.498 seconds after the switch is closed. Calculate the current at this time by

$$\frac{1}{2}\ln(20)i = q'(\ln(20)/2) = 32(10^{-5})e^{-\ln(20)} = 16 \text{ micro amps.}$$

11. Once released, the only force acting on the ballast bag is due to gravity. If $y(t)$ is the distance from the bag to the ground at time t, then $y'' = -g = -32$, with $y(0) = 4$. With two integrations, we obtain

$$y'(t) = 4 - 32t \text{ and } y(t) = 342 + 4t - 16t^2.$$

The maximum height is reached when $y'(t) = 0$, or $t = 1/8$ second. This maximum height is $y(1/8) = 342.25$ feet. The bag remains aloft until $y(t) = 0$, or $-16t^2 + 4t + 342 = 0$. This occurs at $t = 19/4$ seconds, and the bag hits the ground with speed $|y'(19/4)| = 148$ feet per second.

12. At time $t = 0$, assume that the dog is at the origin of an x, y - system and the man is located at $(A, 0)$ on the x - axis. The man moves directly upward into the first quadrant and at time t is at (A, vt). The position of the dog at time $t > 0$ is (x, y) and the dog runs with speed $2v$, always directly toward his master. At time $t > 0$, the man is at (A, vt), the dot is at (x, y), and the tangent to the dog's path joins these two points. Thus

$$\frac{dy}{dx} = \frac{vt - y}{A - x}$$

for $x < A$. To eliminate t from this equation use the fact that during the time the man has moved vt units upward, the dog has run $2vt$ units along his path. Thus

$$2vt = \int_0^x \left[1 + \left(\frac{dy}{d\xi}\right)^2\right]^{1/2} d\xi.$$

Use this integral to eliminate the vt term in the original differential equation to obtain

$$2(A - x)y'(x) = \int_0^x \left[1 + \left(\frac{dy}{d\xi}\right)^2\right]^{1/2} d\xi - 2y.$$

Differentiate this equation to obtain

$$2(A - x)y'' - 2y' = (1 + (y')^2)^{1/2} - 2y',$$

or

$$2(A - x)y'' = (1 + (y')^2)^{1/2},$$

subject to $y(0) = y'(0) = 0$. Let $u = y'$ to obtain the separable equation

$$\frac{1}{\sqrt{1 + u^2}}\, du = \frac{1}{2(A - x)}\, dx.$$

This has the solution

$$\ln(u + \sqrt{1 + u^2}) = -\frac{1}{2}\ln(A - x) + c.$$

Using $y'(0) = u(0) = 0$ gives us

$$u + \sqrt{1 + u^2} = \frac{\sqrt{A}}{\sqrt{A - x}},$$

or, equivalently,

$$y' + \sqrt{(1 + (y')^2)} = \frac{\sqrt{A}}{\sqrt{A - x}}; y(0) = 0.$$

From the equation for y'', we obtain

$$\sqrt{1 + (y')^2} = 2(A - x)y'',$$

so

$$y' + 2(A - x)y'' = \frac{\sqrt{A}}{\sqrt{A - x}}; y(0) = y'(0) = 0$$

for $x < A$. Let $w = y'$ to obtain the linear first order equation

$$w' + \frac{1}{2(A - x)} w = \frac{\sqrt{A}}{2(A - x)^{3/2}}.$$

An integrating factor is $1/\sqrt{A - x}$ and we can write

$$\frac{d}{dx}\left[\frac{w}{\sqrt{A - x}}\right] = \frac{\sqrt{A}}{2(A - x)^2}.$$

The solution, subject to $w(0) = 0$, is

$$w(x) = \frac{A}{\sqrt{2}} \frac{1}{\sqrt{A-x}} - \frac{1}{2\sqrt{A}}\sqrt{A-x} = \frac{dy}{dx}.$$

Integrate one last time to obtain

$$y(x) = -\sqrt{A}\sqrt{A-x} + \frac{1}{3\sqrt{A}}(A-x)^{1/2} + \frac{2}{3}A,$$

in which we have used $y(0) = 0$ to evaluate the constant of integration. The dog catches the man at $x = A$, so they meet at $(A, 2A/3)$. Since this is also (A, vt) when they meet, we conclude that $vt = 2A/3$, so they meet at time

$$t = \frac{2A}{3v}.$$

13. The differential equation of the family is

$$y' = 2kx = \frac{2x(y-1)}{x^2} = \frac{2(y-1)}{x}.$$

Orthogonal trajectories satisfy $y' = x/2(y-1)$ and are the graphs of the family of ellipses

$$(y-1)^2 + \frac{1}{2}x^2 = c.$$

15. The differential equation of the given family is found by solving for k and differentiating to obtain $k = \ln(y)/x$, so

$$\frac{dy}{dx} = \frac{y\ln(y)}{x}.$$

Orthogonal trajectories satisfy

$$\frac{dy}{dx} = -\frac{x}{y\ln(y)}.$$

This is separable with solutions

$$y^2(\ln(y^2) - 1) = c - 2x^2.$$

17. The differential equation of the given family is

$$\frac{dy}{dx} = \frac{4x}{3}.$$

Orthogonal trajectories satisfy

$$\frac{dy}{dx} = -\frac{3}{4x}$$

and are given by

$$y = -\frac{3}{4}\ln|x| + c.$$

19. Let $x(t)$ denote the length of chain hanging down from the table at time t, and note that once the chain starts moving, all 24 feet move with velocity v. The motion is modeled by

$$\rho x = \frac{24\rho}{g}\frac{dv}{dt} = \frac{3\rho}{4}v\frac{dv}{dx},$$

with $v(6) = 0$. Thus $x^2 = \frac{3}{4}v^2 + c$ and $v(6) = 0$ gives $c = 36$, so

$$v^2 = \frac{4}{3}(x^2 - 36).$$

When the end leaves the table, $x = 24$ so $v = 12\sqrt{5} \approx 26.84$ feet per second. The time is

$$\begin{aligned} t_f &= \int_6^{24} \frac{1}{v(x)}\,dx = \int_6^{24} \frac{\sqrt{3}}{2\sqrt{x^2-36}}\,dx \\ &= \frac{\sqrt{3}}{2}\ln(6+\sqrt{35}) \approx 2.15 \end{aligned}$$

seconds.

20. **Hint** The force pulling the chain off the table is due to the four feet of chain hanging between the table and the floor. Let $x(t)$ denote the distance the free end of the chain on the table has moved. The motion is modeled by

$$4\rho = \frac{d}{dt}\left[(22 - x)\frac{\rho}{g}v\right];\ v = 0 \text{ when } x = 0.$$

Solve this for $v(x)$ and recall that the chain leaves the table when $x = 18$.

21. (a) Clearly each bug follows the same curve of pursuit relative to the corner from which it started. Place a polar coordinate system as suggested and determine the pursuit curve for the bug starting at $\theta = 0, r = a/\sqrt{2}$. At any time $t > 0$, the bug will be at $(f(\theta), \theta)$ and its target will be at $(f(\theta), \theta + \pi/2)$, and

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{f'(\theta)\sin(\theta) + f(\theta)\cos(\theta)}{f'(\theta)\cos(\theta) - f(\theta)\sin(\theta)}.$$

On the other hand, the tangent direction must be from $(f(\theta), \theta)$ to $(f(\theta), \theta + \pi/2)$, so

$$\begin{aligned} \frac{dy}{dx} &= \frac{f(\theta)\sin(\theta+\pi/2) - f(\theta)\sin(\theta)}{f(\theta)\cos(\theta+\pi/2) - f(\theta)\cos(\theta)} \\ &= \frac{\cos(\theta) - \sin(\theta)}{-\sin(\theta) - \cos(\theta)} \\ &= \frac{\sin(\theta) - \cos(\theta)}{\sin(\theta) + \cos(\theta)}. \end{aligned}$$

Equate these two expressions for dy/dx and simplify to obtain

$$f'(\theta) + f(\theta) = 0$$

with $f(0) = a/\sqrt{2}$. Then

$$r = f(\theta) = \frac{a}{\sqrt{2}}e^{-\theta}$$

is the polar coordinate equation of the pursuit curve.

(b) The distance traveled by each bug is

$$\begin{aligned} D &= \int_0^\infty \sqrt{(r')^2 + r^2}\,d\theta \\ &= \int_0^\infty \left[\left(\frac{a}{\sqrt{2}}e^{-\theta}\right)^2 + \left(\frac{-a}{\sqrt{2}}e^{-\theta}\right)^2\right]^{1/2} d\theta \\ &= a\int_0^\infty e^{-\theta}\,d\theta = a. \end{aligned}$$

(c) Since $r = f(\theta) = ae^{-\theta}/\sqrt{2} > 0$ for all θ, no bug reaches its quarry. The distance between pursuer and quarry is $ae^{-\theta}$.

1.7 Existence and Uniqueness Questions

1. Both $f(x, y) = x^2 - y^2 + 8x/y$ and

$$\frac{\partial f}{\partial y} = -2y - \frac{8x}{y^2}$$

are continuous on a sufficiently small rectangle centered at $(3, -1)$, for example, on the square of side length 1.

3. Both $f(x, y) = \sin(xy)$ and $\partial f/\partial y = x\cos(xy)$ are continuous (for all (x, y)).

5. By taking $|y'| = y'$, we get $y' = 2y$ and the initial value problem has the solution $y(x) = y_0 e^{2(x-x_0)}$. However, if we take $|y'| = -y'$, then the initial value problem has the solution $y(x) = y_0 e^{-2(x-x_0)}$.

   In this problem we have $|y'| = 2y = f(x, y)$, so we actually have $y' = \pm 2y$, and $f(x, y) = \pm 2y$. This is not even a function, so the terms of Theorem 1.2 do not apply and the theorem offers no conclusion.

7. (a) $f(x, y) = \cos(x)$ and $\partial f/\partial y = 0$ are continuous for all (x, y), so the problem has a unique solution.

   (b) The solution is $y = 1 + \sin(x)$.

   (c)

$$y_0 = 1, y_1 = 1 + \int_\pi^x \cos(t)\, dt = 1 + \sin(x).$$

In this example, $y_n = y_1$ for $n = 2, 3, \cdots$.

   (d) For $n \geq 1$,

$$y = 1 + \sin(x) = 1 + \sum_{k=0}^{\infty} \frac{(-1)^{2k+1} x^{2k+1}}{(2k+1)!}.$$

The nth partial sum T_n of this Taylor series does not agree with the nth Picard iterate $y_n(x)$. However,

$$\lim_{n\to\infty} T_n(x) = \lim_{n\to\infty} y_n(x) = 1 + \sin(x),$$

so both sequences converge to the unique solution.

9. (a) Since both $f(x, y) = 4 + y$ and $\partial f/\partial y = 1$ are continuous everywhere, the initial value problem has a unique solution.

   (b) This linear differential equation is easily solved to yield $y = -4 + 7e^x$ as the unique solution of the initial value problem.

(c)

$$
\begin{aligned}
y_0 &= 3, y_1 = 3 + \int_0^x 7\,dt = 3 + 7x,\\
y_2 &= 3 + \int_0^x (7 + 7t)\,dt = 3 + 7x + 7\frac{x^2}{2},\\
y_3 &= 3 + \int_0^x \left(7 + 7t + 7\frac{t^2}{2}\right)\,dt = 3 + 7x + 7\frac{x^2}{2} + 7\frac{x^3}{3!},\\
y_4 &= 3 + \int_0^x \left(7 + 7t + 7\frac{t^2}{2} + 7\frac{t^3}{3!}\right)\,dt = 3 + 7x + 7\frac{x^2}{2} + 7\frac{x^3}{3!} + 7\frac{x^4}{4!},\\
y_5 &= 3 + \int_0^x y_4(t)\,dt = 3 + 7x + 7\frac{x^2}{2} + 7\frac{x^3}{3!} + 7\frac{x^4}{4!} + 7\frac{x^5}{5!},\\
y_6 &= 3 + \int_0^x y_5(t)\,dt = 3 + 7x + 7\frac{x^2}{2} + 7\frac{x^3}{3!} + 7\frac{x^5}{5!} + 7\frac{x^6}{6!}.
\end{aligned}
$$

(d) We conjecture that

$$y_n(x) = 3 + 7x + 7\frac{x^2}{2} + 7\frac{x^3}{3!} + \cdots + 7\frac{x^n}{n!}.$$

Note that

$$y_n(x) = -4 + 7\sum_{k=0}^{n} \frac{x^k}{k!}$$

and that

$$\lim_{n\to\infty} y_n(x) = -4 + 7\sum_{k=0}^{\infty} \frac{x^k}{k!} = -4 + 7e^x.$$

Thus the Picard iterates converge to the solution.

Chapter 2

Linear Second-Order Equations

2.1 The Linear Second-Order Equation

In Problems 1 - 5, verification that the given functions are solutions of the differential equation is a straightforward differentiation, which we omit.

1. The general solution is $y(x) = c_1 e^x \cos(x) + c_2 e^x \sin(x)$. Then $y(0) = c_1 = 6$. We find that $y'(0) = c_1 + c_2 = 1$, so $c_2 = -5$. The initial value problem has solution
$$y(x) = 6e^x \cos(x) - 5e^x \sin(x).$$

3. The general solution is $y(x) = c_1 e^{-2x} + c_2 e^{-x}$. For the initial conditions, we have
$$y(0) = c_1 + c_2 = -3 \text{ and } y'(0) = -2c_1 - c_2 = -1.$$
Solve these to obtain $c_1 = 4$, $c_2 = -7$. The solution of the initial value problem is
$$y(x) = 4e^{-2x} - 7e^{-x}.$$

5. The general solution is $y(x) = c_1 \sin(6x) + c_2 \cos(6x)$. For the initial conditions, we need $y(0) = c_2 = -5$ and $y'(0) = 6c_1 = 2$. Then $c_1 = 1/3$ and the solution of the initial value problem is
$$y(x) = \frac{1}{3}\sin(6x) - 5\cos(6x).$$

7. The general solution is
$$y(x) = c_1 e^{3x} \cos(2x) + c_2 e^{3x} \sin(2x) - 8e^x.$$

9. The general solution is
$$y(x) = c_1 e^{4x} + c_2 e^{-4x} - \frac{1}{4}x^2 + \frac{1}{2}.$$

11. Suppose $\varphi'(x_0) = 0$. Then φ is the unique solution of the initial value problem
$$y'' + py' + qy = 0; y(x_0) = y'(x_0) = 0$$
on I. But the functions that is identically zero on I is also a solution of this problem. Therefore $\varphi(x) = 0$ for all x in I.

13. It is routine to verify by substitution that x and x^2 are solutions of the given differential equation. The Wronskian is

$$W(x) = \begin{vmatrix} x & x^2 \\ 1 & 2x \end{vmatrix} = -x^2,$$

which vanishes at $x = 0$, but at no other points. However, the theorem only applies to solutions of linear second order differential equations. To write the given differential equation in standard linear form, we must write

$$y'' - \frac{2}{x}y' + \frac{2}{x^2}y = 0,$$

which is not defined at $x = 0$. Thus the theorem does not apply.

15. For conclusion (1), begin with the hint to the problem to write

$$\begin{aligned} y_1'' + py_1' + qy_1 &= 0, \\ y_2'' + py_2' + qy_2 &= 0. \end{aligned}$$

Multiply the first equation by y_2 and the second by $-y_1$ and add the resulting equations to obtain

$$y_1''y_2 - y_2''y_1 + p(y_1'y_2 - y_2'y_1) = 0.$$

Since $W = y_1y_2 - y_2y_1$, then

$$W' = y_1y_2'' - y_1''y_2,$$

so

$$W' + pW = y_1y_2'' - y_1''y_2 + p(y_1y_2' - y_1'y_2) = 0.$$

Therefore the Wronskian satisfies the linear differential equation $W' + pW = 0$. This has integrating factor $e^{\int p(x)\,dx}$ and can be written

$$\left(We^{\int p(x)\,dx}\right)' = 0.$$

Upon integrating we obtain the general solution

$$W = ce^{-\int p(x)\,dx}.$$

If $c = 0$, then this Wronskian is zero for all x in I. If $c \neq 0$, then $W \neq 0$ for x in I because the exponential function does not vanish for any x.

Now turn to conclusion (2). Suppose first that $y_2(x) \neq 0$ on I. By the quotient rule for differentiation it is routine to verify that

$$y_2^2 \frac{d}{dx}\left(\frac{y_1}{y_2}\right) = -W(x).$$

If $W(x)$ vanishes, then the derivative of y_1/y_2 is identically zero on I, so y_1/y_2 is constant, hence y_1 is a constant multiple of y_1, making the two functions linearly dependent. Conversely, if the two functions are linearly independent, then one is a constant multiple of the other, say $y_1 = cy_2$, and then $W(x) = 0$.

If there are points in I at which $y_2(x) = 0$, then we have to use this argument on the open intervals between these points and then make use of the continuity of y_2 on the entire interval. This is a technical argument we will not pursue here.

2.2 Reduction of Order

In problems 1 - 10 we put $y_2(x) = u(x)y_1(x)$, derive the equation satisfied by u, give its solution for $u(x)$, and give the general solution of the second order equation.

1. $u'' \cos(2x) - 4\sin(2x)u' = 0; u(x) = \tan(2x); y = c_1\cos(2x) + c_2\sin(2x)$
3. $u'' = 0; u(x) = x; y = c_1 e^{5x} + c_2 x e^{5x}$
5. $xu'' + u' = 0; u = \ln(x); y = c_1 x^2 + c_2 x^2 \ln(x)$
7. $xu'' + 7u' = 0; u(x) = x^{-6}; y = c_1 x^4 + c_2 x^{-2}$
9. $x^{-1/2}\cos(x)u'' - 2x^{-1/2}\sin(x)u' = 0; u(x) = \tan(x); y = c_1\left(\dfrac{\cos(x)}{\sqrt{x}}\right) + c_2\left(\dfrac{\sin(x)}{\sqrt{x}}\right)$
11. $y = c_1 e^{-ax} + c_2 x e^{-ax}$
13. (a) $yu\dfrac{du}{dy} + 3u^2 = 0$ is separable as $\dfrac{du}{u} = -\dfrac{3dy}{y}$. Integration gives $\ln|u| = -3\ln|y| + c$ or $uy^3 = A$. Thus $y^3 dy = Adx$ and $\dfrac{y^4}{4} = Ax + B$ or $y^4 = c_1 x + c_2$
(b) $(y-1)e^y = c_1 x + c_2$ or $y = c_3$
(c) $y = \dfrac{c_1 e^{c_1 x}}{c_2 - e^{c_1 x}}$ or $y = \dfrac{1}{c_3 - x}$
(d) $y = \ln|\sec(x + c_1)| + c_2$
(e) $y = \ln|c_1 x + c_2|$
15. With $y = uy_1$ we get $y'' + \dfrac{A}{x}y' + \dfrac{B}{x^2}y =$
$\left[u''x^2 + (1-A)xu' - \left(\dfrac{1-A}{2}\right)\left(\dfrac{1+A}{2}\right)u + A\left(xu' + \dfrac{(1-A)}{2}u\right) + Bu\right]x^{-(3+A)/2} =$
$[xu'' + u']x^{(1+A)/2} = 0$ iff $xu'' + u' = 0$. Thus $u = c_1 + c_2\ln(x)$ and $y = c_1 x^{(1-A)/2} + c_2 x^{(1-A)/2}\ln(x)$.

2.3 The Constant Coefficient Case

1. The characteristic equation is $\lambda^2 + 3\lambda + 18 = 0$, with roots $-3/2 \pm 3\sqrt{7}i/2$. The general solution is
$$y = e^{-3x/2}\left[c_1\cos\left(\frac{3\sqrt{7}x}{2}\right) + c_2\sin\left(\frac{3\sqrt{7}x}{2}\right)\right].$$
3. The characteristic equation is $\lambda^2 + 10\lambda + 26 = 0$, with roots $-5 \pm i$. The general solution is
$$y = c_1 e^{-5x}\cos(x) + c_2 e^{-5x}\sin(x).$$
5. The characteristic equation is $\lambda^2 - 14\lambda + 49 = 0$, with repeated root 7. The general solution is
$$y = e^{7x}(c_1 + c_2 x).$$
6. The characteristic equation is $\lambda^2 - 6\lambda + 7 = 0$, with roots $3 \pm \sqrt{2}i$. The general solution is
$$y = e^{3x}[c_1\cos(\sqrt{2}x) + c_2\sin(\sqrt{2}x)].$$
7. The characteristic equation is $\lambda^2 + 6\lambda + 9 = 0$, with repeated root -3. The general solution is
$$y = c_1 e^{-3x} + c_2 x e^{-3x}.$$

9. The characteristic equation is $\lambda^2 - \lambda - 6 = 0$, with roots $-2, 3$. The general solution is

$$y = c_1 e^{-2x} + c_2 e^{3x}.$$

In each of Problems 11 through 20, the solution is obtained by finding the general solution of the differential equation and then solving for the constants to satisfy the initial conditions. We provide the details only for Problem 19, the other problems proceeding similarly.

11.

$$y = \frac{1}{7}[9e^{3(x-2)} + 5e^{-4(x-2)}]$$

13. $y = e^{x-1}(29 - 17x)$

15.

$$y = e^{(x+2)/2}\left[\cos\left(\frac{\sqrt{15}}{2}(x+2)\right) + \frac{5}{\sqrt{15}}\sin\left(\frac{\sqrt{15}}{2}(x+2)\right)\right]$$

17. $y = 0$ for all x

19. The characteristic equation is $\lambda^2 + 3\lambda = 0$, with roots $0, -3$. The general solution of the differential equation is $y = c_1 + c_2 e^{-3x}$. To find a solution satisfying the initial conditions, we need

$$y(0) = c_1 + c_2 = 3 \text{ and } y'(0) = -3c_2 = 6.$$

Then $c_1 = 5$ and $c_2 = -2$, so the solution of the initial value problem is $y = 5 - 2e^{-3x}$.

21. The characteristic equation has roots

$$\lambda_1 = \frac{1}{2}(-a + \sqrt{a^2 - 4b}), \lambda_2 = \frac{1}{2}(-a - \sqrt{a^2 - 4b}).$$

As we have seen, there are three cases.

If $a^2 = 4b$, then

$$y = e^{-ax/2}(c_1 + c_2 x) \to 0 \text{ as } x \to \infty,$$

because $a > 0$.

If $a^2 > 4b$, then $a^2 - 4b < a^2$ and λ_1 and λ_2 are both negative, so

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \to 0 \text{ as } x \to \infty.$$

Finally, if $a^2 < 4b$, then the general solution has the form

$$y(x) = e^{-ax/2}(c_1 \cos(\beta x) + c_2 \sin(\beta x)),$$

where $\beta = \sqrt{4b - a^2}/2$. Because $a > 0$, this solution also has limit zero as $x \to \infty$.

22. We will use the fact that, for any positive integer n,

$$i^{2n} = (i^2)^n = (-1)^n \text{ and } i^{2n+1} = i^{2n} i = (-1)^n i.$$

Now suppose a is real and split the exponential series into two series, one for even values of the summation index, and the other for odd values:

$$\begin{aligned}
e^{ia} &= \sum_{n=0}^{\infty} \frac{1}{n!} i^n a^n \\
&= \sum_{n=0}^{\infty} \frac{1}{(2n)!} i^{2n} a^{2n} + \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} i^{2n+1} a^{2n+1} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{2n!} a^{2n} + \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} i a^{2n+1} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} a^{2n} + i \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} a^{2n+1} \\
&= \cos(a) + i\sin(a).
\end{aligned}$$

23. (a) The characteristic equation is $\lambda^2 - 2\alpha\lambda + \alpha^2 = 0$, with repeated roots $\lambda = \alpha$. The general solution is

$$y(x) = \varphi(x) = (c_1 + c_2 x)e^{\alpha x}.$$

(b) The characteristic equation is $\lambda^2 - 2\alpha\lambda + (\alpha^2 - \epsilon^2) = 0$, with roots $\alpha \pm \epsilon$. The general solution is

$$y_\epsilon(x) = \varphi_\epsilon(x) = e^{\alpha x}(c_1 e^{\epsilon x} + c_2 e^{-\epsilon x}).$$

(c) In general,

$$\lim_{\epsilon \to 0} y_\epsilon(x) = e^{\alpha x}(c_1 + c_2) \neq y(x).$$

2.4 The Nonhomogeneous Equation

For Problems 1 and 3 we will omit some of the details and give an outline of the solution.

1. $y_1 = \cos(3x)$ and $y_2 = \sin(3x)$ are linearly independent solutions of the associated homogeneous equation. Their Wronskian is $W = 3$. With $f(x) = 12\sec(3x)$, carry out the integrations in the equations for u_1 and u_2 to obtain the general solution

$$y(x) = c_1\cos(3x) + c_2\sin(3x) + 4x\sin(3x) + \frac{4}{3}\cos(3x)\ln|\cos(3x)|.$$

3. $y_1 = e^x$ and $y_2 = e^{2x}$, with Wronskian $W = e^{3x}$. With $f(x) = \cos(e^{-x})$, carry out the integrations to obtain u_1 and u_2 to write the general solution

$$y(x) = c_1 e^x + c_2 e^{2x} - e^{2x}\cos(e^{-x})$$

4. $y_1 = e^{3x}$ and $y_2 = e^{2x}$, with Wronskian $W = -e^{5x}$. Use the identity $8\sin^2(4x) = 4\cos(8x) - 4$ to help find u_1 and u_2 and write the general solution

$$y = c_1 e^{3x} + c_2 e^{2x} + \frac{2}{3} + \frac{58}{1241}\cos(8x) + \frac{40}{1241}\sin(8x).$$

5. Two independent solutions of $y'' + y = 0$ are $y_1 = \cos(x)$ and $y_2 = \sin(x)$. The Wronskian is

$$W(x) = \begin{vmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{vmatrix} = 1.$$

To use variation of parameters, seek a particular solution of the differential equation of the form

$$y = u_1 y_1 + u_2 y_2.$$

Let $f(x) = \tan(x)$. We found that we can choose

$$\begin{aligned} u_1(x) &= -\int \frac{y_2(x) f(x)}{W(x)}\, dx = -\int \tan(x)\sin(x)\, dx \\ &= -\int \frac{\sin^2(x)}{\cos(x)}\, dx \\ &= -\int \frac{1-\cos^2(x)}{\cos(x)}\, dx \\ &= \int \cos(x)\, dx - \int \sec(x)\, dx \\ &= \sin(x) - \ln|\sec(x) + \tan(x)| \end{aligned}$$

and

$$\begin{aligned} u_2(x) &= \int \frac{y_1(x) f(x)}{W(x)}\, dx = \int \cos(x)\tan(x)\, dx \\ &= \int \sin(x)\, dx = -\cos(x). \end{aligned}$$

The general solution can be written

$$\begin{aligned} y &= c_1\cos(x) + c_2\sin(x) + \sin(x)\cos(x) \\ &\quad - \cos(x)\ln|\sec(x)+\tan(x)| - \sin(x)\cos(x) \\ &= c_1\cos(x) + c_2\sin(x) - \cos(x)\ln|\sec(x)+\tan(x)| \end{aligned}$$

In Problems 7 - 16 we use the method of undetermined coefficients in writing the general solution. For Problem 15 all the details are included, while for the remaining problems the important details of the solution are outlined.

7. $y_1 = e^x$ and $y_2 = e^{2x}$. With $f(x) = 10\sin(x)$, try $y_p(x) = A\cos(x) + B\sin(x)$ to obtain

$$y = c_1 e^x + c_2 e^{2x} + 3\cos(x) + \sin(x).$$

9. $y_1 = e^{2x}\cos(3x)$ and $y_2 = e^{2x}\sin(3x)$. Since neither e^{2x} nor e^{3x} is a solution of the homogeneous equation, try $y_p(x) = Ae^{2x} + Be^{3x}$ to obtain the general solution

$$y = e^{2x}[c_1\cos(3x) + c_2\sin(3x)] + \frac{1}{3}e^{2x} - \frac{1}{2}e^{3x}.$$

11. $y_1 = e^x\cos(3x)$ and $y_2 = e^x\sin(3x)$. With $f(x)$ a second degree polynomial, try $y_p(x) = Ax^2 + Bx + C$ to obtain

$$y = e^x[c_1\cos(3x) + c_2\sin(3x)] + 2x^2 + x - 1.$$

13. $y_1 = e^{2x}$ and $y_2 = e^{4x}$. With $f(x) = 3e^x$, try $y_p(x) = Ae^x$, noting that e^x is not a solution of the associated homogeneous equation. Obtain the general solution

$$y = c_1 e^{2x} + c_2 e^{4x} + e^x.$$

15. Two independent solutions of the associated homogeneous equation are $y_1 = e^{2x}$ and $y_2 = e^{-x}$. Since $2x^2+5$ is a second degree polynomial, we attempt such a polynomial as a particular solution:
$$y_p(x) = Ax^2 + Bx + C.$$
Substitute this into the (nonhomogeneous) differential equation to obtain
$$2A - (2Ax + B) - 2(Ax^2 + Bx + C) = 2x^2 + 5.$$
Then
$$\begin{aligned} 2A - B - 2C &= 5, \\ -2A - 2B &= 0, \\ -2A &= 2. \end{aligned}$$
Then $A = -1, B = 1$ and $C = -4$. The general solution is
$$y = c_1e^{2x} + c_2e^{-x} - x^2 + x - 4.$$

In Problems 17 through 24, first find the general solution of the differential equation, then solve for the constants to satisfy the initial conditions. Problems 17, 19–20, and 22–24 are well suited to the method of undetermined coefficients, while Problems 18 and 21 can be solved fairly directly by variation of parameters.

17. We find the general solution
$$y(x) = c_1e^{-2x} + c_2e^{-6x} + \frac{1}{5}e^{-x} + \frac{7}{12}.$$
Solve for the constants to obtain the solution
$$y(x) = \frac{3}{8}e^{-2x} - \frac{19}{120}e^{-6x} + \frac{1}{5}e^{-x} + \frac{7}{12}$$

19. The general solution is
$$y(x) = c_1e^{4x} + 2e^{-2x} - 2e^{-x} - e^{2x}.$$
The initial value problem has the solution
$$y = 2e^{4x} + 2e^{-2x} - 2e^{-x} - e^{2x}.$$

21. We find the general solution
$$y(x) = c_1e^{x} + c_2e^{-x} - \sin^2(x) - 2.$$
The initial value problem has the solution
$$y = 4e^{-x} - \sin^2(x) - 2.$$

23. $y_1 = e^{2x}$ and $y_2 = e^{-2x}$. Since e^{2x} is a solution of the homogeneous equation, try $y_p(x) = Axe^{2x} + Bx + C$ to obtain the general solution
$$y = c_1e^{2x} + c_2e^{-2x} - \frac{7}{4}xe^{2x} - \frac{1}{4}x.$$
Now
$$y(0) = c_1 + c_2 = 1 \text{ and } y'(0) = 2c_1 - 2c_2 - \frac{7}{4} = 3.$$
Then $c_1 = 7/4$ and $c_2 = -3/4$. The solution of the initial value problem is
$$y = -\frac{7}{4}e^{2x} - \frac{3}{4}e^{-2x} - \frac{7}{4}xe^{2x} - \frac{1}{4}x.$$

2.5 Spring Motion

1. The solution with initial conditions $y(0) = 5, y'(0) = 0$ is

$$y_1(t) = 5e^{-2t}[\cosh(\sqrt{2}t) + \sqrt{2}\sinh(\sqrt{2}t)].$$

With initial conditions $y(0) = 0, y'(0) = 5$, we obtain

$$y_2(t) = \frac{5}{\sqrt{2}}e^{-2t}\sinh(\sqrt{2}t).$$

Graphs of these solutions are shown in Figure 2.1.

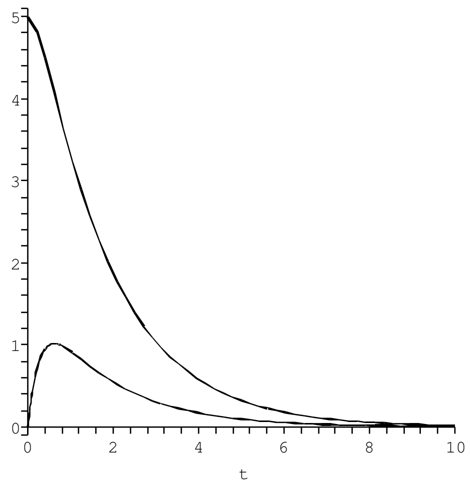

Figure 2.1: Solutions to Problem 1, Section 2.5.

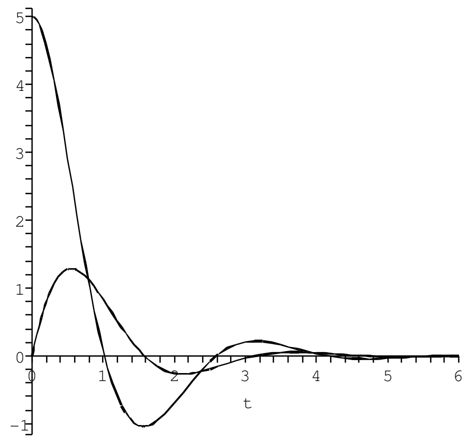

Figure 2.2: Solutions to Problem 3, Section 2.5.

3. With $y(0) = 5$ and $y' = 0$,

$$y_1(t) = \frac{5}{2}e^{-t}[2\cos(2t) + \sin(2t)].$$

   With $y(0) = 0$ and $y'(0) = 5$, $y_2(t) = \frac{5}{2}e^{-t}\sin(2t)$. Graphs are given in Figure 2.2.

4. The solution is $y(t) = Ae^{-2t}[\cosh(\sqrt{2}t) + \sqrt{2}\sinh(\sqrt{2}t)]$.
5. The solution is $y(t) = Ate^{-2t}$, graphed for $A = 1, 3, 6, 10, -4$ and -7 in Figure 2.3.
7. The solution is

$$y(t) = \frac{A}{2}e^{-t}\sin(2t)$$

   and is graphed for $A = 1, 3, 6, 10, -4$ and -7 in Figure 2.4.

9. The solution is

$$y(t) = \frac{A}{\sqrt{2}}e^{-2t}\sinh(\sqrt{(}2)t)$$

   and is graphed for $A = 1, 3, 6, 10, -4$ and -7 in Figure 2.5.

11. The general solution of the overdamped problem

$$y'' + 6y' + 2y = 4\cos(3t)$$

   is

$$y(t) = e^{-3t}[c_1\cosh(\sqrt{7}t) + c_2\sinh(\sqrt{7}t)] - \frac{28}{373}\cos(3t) + \frac{72}{373}\sin(3t).$$

   (a) The initial conditions $y(0) = 6, y'(0) = 0$ give us

$$c_1 = \frac{2266}{373} \text{ and } c_2 = \frac{6582}{373\sqrt{7}}.$$

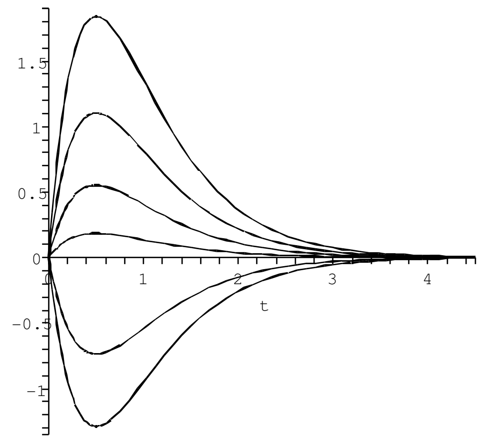

Figure 2.3: Solutions to Problem 5, Section 2.5.

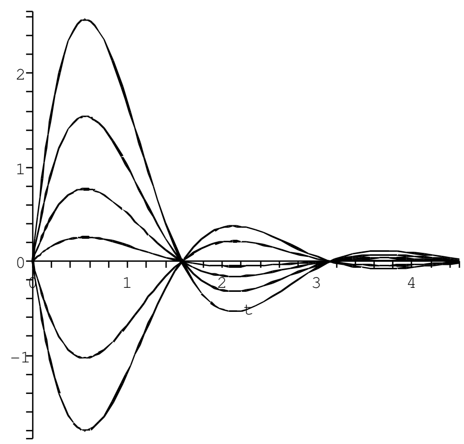

Figure 2.4: Solutions to Problem 7, Section 2.5.

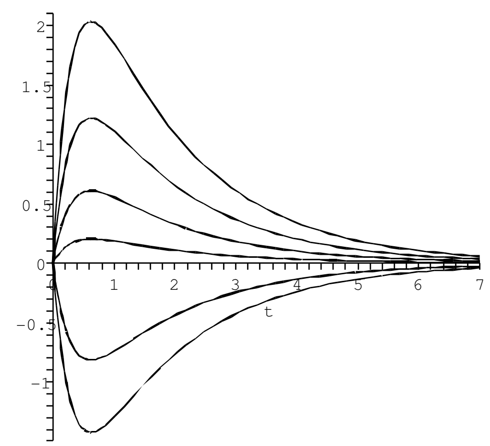

Figure 2.5: Solutions to Problem 9, Section 2.5.

Now the solution is

$$y_a(t) = \frac{1}{373}[e^{-3t}[2266\cosh(\sqrt{7}t) + \frac{6582}{\sqrt{7}}\sinh(\sqrt{7}t)] - 28\cos(3t) + 72\sin(3t)].$$

(b) The initial conditions $y(0) = 0, y'(0) = 6$ give us $c_1 = 28/373$ and $c_2 = 2106/373$ and the unique solution

$$y_b(t) = \frac{1}{373}[e^{-3t}[29\cosh(\sqrt{7}t) + \tfrac{2106}{\sqrt{7}}\sinh(\sqrt{7}t)] - 28\cos(3t) + 72\sin(3t)].$$

These solutions are graphed in Figure 2.6.

13. For overdamped motion the displacement is given by

$$y(t) = e^{-\alpha t}(A + Be^{\beta t}),$$

where α is the smaller of the roots of the characteristic equation and is positive, and β equals the larger root minus the smaller root. The factor $A + Be^{\beta t}$ can be zero at most once and only for some $t > 0$ if $-A/B > 1$. The values of A and B are determined by the initial conditions. In fact, if $y_0 = y(0)$ and $v_0 = y'(0)$, we have

$$A + B = y_0 \text{ and } -\alpha(A + B) + \beta B = v_0.$$

We find from these that

$$-\frac{A}{B} = 1 - \frac{\beta y_0}{v_0 + \alpha y_0}.$$

No condition on only y_0 will ensure that $-A/B \leq 1$. If we also specify that $v_0 > -\alpha y_0$, we ensure that the overdamped bob will never pass through the equilibrium point.

15. For underdamped motion, the solution has the appearance

$$y(t) = e^{-ct/2m}[c_1\cos(\sqrt{4km - c^2}t/2m) + c_2\sin(\sqrt{4km - c^2}t/2m)]$$

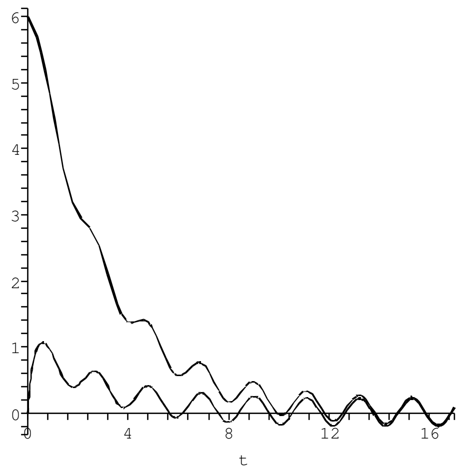

Figure 2.6: Solutions to Problem 11, Section 2.5.

having frequency

$$\omega = \frac{\sqrt{4km - c^2}}{2m}.$$

Thus increasing c decreases the frequency of the the motion, and decreasing c increases the frequency.

17. The general solution of the underdamped problem

$$y''(t) + y' + 3y = 4\cos(3t)$$

is

$$y(t) = e^{-t/2}\left[c_1 \cos\left(\frac{\sqrt{11}t}{2}\right) + c_2 \sin\left(\frac{\sqrt{11}t}{2}\right)\right] - \frac{24}{45}\cos(3t) + \frac{12}{45}\sin(3t).$$

(a) The initial conditions $y(0) = 6, y'(0) = 0$ yield the unique solution

$$y_a(t) = \frac{1}{15}\left[e^{-t/2}\left[98\cos\left(\frac{\sqrt{11}t}{2}\right) + \frac{74}{\sqrt{11}}\sin\left(\frac{\sqrt{11}t}{2}\right)\right] - 8\cos(3t) + 4\sin(3t)\right].$$

(b) The initial conditions $y(0) = 0, y'(0) = 6$ yield the unique solution

$$y_b(t) = \frac{1}{15}\left[e^{-t/2}\left[8\cos\left(\frac{\sqrt{11}t}{2}\right) + \frac{164}{\sqrt{11}}\sin\left(\frac{\sqrt{11}t}{2}\right)\right] - 8\cos(3t) + 4\sin(3t)\right].$$

These solutions are graphed in Figure 2.7.

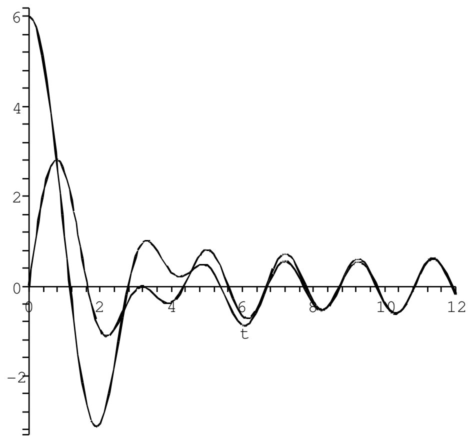

Figure 2.7: Solutions to Problem 17, Section 2.5.

2.6 Euler's Differential Equation

In Problems 7 and 9, details are given with the solution. For the other problems, only the general solution is given. All solutions are for $x > 0$.

1. $y(x) = c_1x^{-2} + c_2x^{-3}$
3. $y(x) = x^{-12}(c_1 + c_2\ln(x))$
5. $y(x) = c_1x^4 + c_2x^{-4}$
7. Solve
$$Y'' + 4Y = 0$$
to obtain
$$Y(t) = c_1\cos(2t) + c_2\sin(2t).$$
Then
$$y(x) = c_1\cos(2\ln(x)) + c_2\sin(2\ln(x)).$$
9. Let $x = e^t$ to obtain
$$Y'' + Y' - 6Y = 0$$
which we can read directly from the original differential equation without further calculation. Then
$$Y(t) = c_1e^{2t} + c_2e^{-3t}.$$
In terms of x,
$$y(x) = c_1e^{2\ln(x)} + c_2e^{-3\ln(x)} = c_1x^2 + c_2x^{-3}.$$
11. $y(x) = x^2(4 - 3\ln(x))$
13. $y(x) = 3x^6 - 2x^4$
15. The general solution of the differential equation is
$$y(x) = c_1x^3 + c_2x^{-7}.$$

We need

$$y(2) = 1 = c_1 2^3 + c_2 2^{-7} \text{ and } y'(2) = 0 = 3c_1 2^2 - 7c_2 2^{-8}.$$

Solve for c_1 and c_2 to obtain the solution of the initial value problem

$$y(x) = \frac{7}{10}\left(\frac{x}{2}\right)^3 + \frac{3}{10}\left(\frac{x}{2}\right)^{-7}$$

17. The transformation $x = e^t$ transforms the Euler equation $x^2y'' + axy' + by = 0$ into

$$Y'' + (a-1)Y' + bY = 0,$$

with characteristic equation

$$\lambda^2 + (a-1)\lambda + b = 0,$$

with roots λ_1 and λ_2. If we substitute $y = x^r$ directly into Euler's equation, we obtain

$$r(r-1)x^r + arx^r + bx^r = 0,$$

or, after dividing by x^r,

$$r^2 + (a-1)r + b = 0.$$

This equation for r is the same as the quadratic equation for λ, so its roots are $r_1 = \lambda_1$ and $r_2 = \lambda_2$. Therefore both the transformation method, and direct substitution of $y = x^r$ into Euler's equation, lead to the same solutions.

Chapter 3

The Laplace Transform

3.1 Definition and Notation

1. From entries (4) and (6) and the linearity of the transform,

$$K(s) = \frac{-10}{(s+4)^3} + \frac{3}{s^2+9}$$

3. From entry (9) of the table,

$$F(s) = \frac{3(s^2-4)}{(s^2+4)^2}$$

5. From entries (2) and (6) and the linearity of the transform,

$$H(s) = \frac{14}{s^2} - \frac{7}{s^2+49}$$

7. From entries (3) and (4),

$$p(t) = e^{-42t} - \tfrac{1}{6}t^3 e^{-3t}.$$

9. From (7) of the table, $q(t) = \cos(8t)$.

11. From the definition,

$$F(s) = \lim_{R\to\infty} \int_0^R e^{-st} f(t)\, dt.$$

For each R, let N be the largest integer so that $(N+1)T \le R$ and use the additivity of the integral to write

$$\int_0^R e^{-st} f(t)\, dt = \sum_{n=0}^{N} \int_{nT}^{(n+1)T} e^{-st} f(t)\, dt + \int_{(N+1)T}^{R} e^{-st} f(t)\, dt.$$

Assuming that $F(s)$ exists, then by choosing R sufficiently large,

$$\int_{(N+1)T}^{R} e^{-st} f(t)\, dt$$

can be made as small as we like. Also, as $R \to \infty$, $N \to \infty$. Therefore

$$\int_0^\infty e^{-st} f(t)\, dt = \sum_{n=0}^{\infty} \int_{nT}^{(n+1)T} e^{-st} f(t)\, dt.$$

13. By the results of Problems 11 and 12,

$$\begin{aligned}\mathcal{L}[f](s) &= \sum_{n=0}^{\infty} \int_{nT}^{(n+1)T} e^{-st} f(t)\,dt \\ &= \sum_{n=0}^{\infty} e^{-snT} \int_{0}^{T} e^{-st} f(t)\,dt \\ &= \left[\sum_{n=0}^{\infty} e^{-snT}\right] \int_{0}^{T} e^{-st} f(t)\,dt,\end{aligned}$$

since the summation is independent of the integral.

15. Since f has period $T = 6$ and

$$\int_0^6 e^{-st} f(t)\,dt = \int_0^3 5e^{-st}\,dt + \int_3^6 e^{-st} \cdot 0\,dt = \frac{5}{s}(1 - e^{-3s}),$$

then

$$\begin{aligned}\mathcal{L}[f](s) &= \frac{5}{s}\frac{1 - e^{-3s}}{1 - e^{-6s}} \\ &= \frac{5}{s}\frac{1 - e^{-3s}}{(1 - e^{-3s})(1 + e^{-3s})} \\ &= \frac{5}{s(1 + e^{-3s})}.\end{aligned}$$

17. f has period $T = 25$, and, from the graph,

$$f(t) = \begin{cases} 0 & \text{for } 0 < t \le 5, \\ 5 & \text{for } 5 < t \le 10, \\ 0 & \text{for } 10 < t \le 25. \end{cases}$$

Now

$$\int_0^{25} e^{-st} f(t)\,dt = \int_5^{10} 5e^{-st}\,dt = \frac{5}{s}e^{-5s}(1 - e^{-5s}).$$

Then

$$\mathcal{L}[f](s) = \frac{5e^{-5s}(1 - e^{-5s})}{s(1 - e^{-25s})}.$$

19. f has period $2\pi/\omega$, and

$$f(t) = \begin{cases} E\sin(\omega t) & \text{for } 0 \le t < \pi/\omega, \\ 0 & \text{for } \pi/\omega \le t < 2\pi/\omega. \end{cases}$$

Compute

$$\begin{aligned}\int_0^{2\pi/\omega} f(t)e^{-st}\,dt &= \int_0^{\pi/\omega} E\sin(\omega t)e^{-st}\,dt \\ &= \frac{E\omega}{s^2 + \omega^2}(1 + e^{-\pi s/\omega}).\end{aligned}$$

Then

$$\begin{aligned}\mathcal{L}[f](s) &= \frac{E\omega}{s^2+\omega^2}\left(\frac{1+e^{-\pi s/\omega}}{1-e^{-2\pi s/\omega}}\right)\\ &= \frac{E\omega}{s^2+\omega^2}\frac{1}{1-e^{-\pi s/\omega}}.\end{aligned}$$

21. We have

$$f(t) = \begin{cases} h & \text{for } 0 < t \le a,\\ 0 & \text{for } a < t \le 2a\end{cases}$$

and $T = 2a$. Now

$$\int_0^{2a} e^{-st} f(t)\,dt = \int_0^a he^{-st}\,dt = \frac{h}{s}(1-e^{-as}).$$

Then

$$\mathcal{L}[f](s) = \frac{h}{s}\frac{1-e^{-as}}{1-e^{-2as}} = \frac{h}{s}\frac{1}{1+e^{-as}}.$$

3.2 Solution of Initial Value Problems

In many of these problems a partial fractions decomposition is used to write $Y(s)$ as a sum of terms whose inverse Laplace transforms can be computed fairly easily (for example, directly from a table). Partial fractions decompositions are reviewed in the text at the end of Chapter 3.

1. Take the transform of the differential equation, insert the initial condition and solve for Y to obtain

$$\begin{aligned}Y(s) &= \frac{1}{s+4}\left(\frac{s}{s^2+1}\right)\\ &= -\frac{4}{17}\left(\frac{1}{s+4}\right) + \frac{1}{17}\frac{4s+1}{s^2+1}.\end{aligned}$$

The solution is

$$y(t) = -\frac{4}{17}e^{-4t} + \frac{4}{17}\cos(t) + \frac{1}{17}\sin(t)$$

3. Transform the differential equation to obtain

$$\begin{aligned}Y(s) &= \frac{1}{(s-2)^2}\left[\frac{s}{s^2+1} + s - 1 + 4\right]\\ &= -\frac{13}{5}\left(\frac{1}{(s-2)^2}\right) + \frac{22}{25}\frac{1}{s-2}\\ &\quad + \frac{3}{25}\frac{s}{s^2+1} - \frac{4}{25}\left(\frac{1}{s^2+1}\right)\end{aligned}$$

The solution is

$$y(t) = -\frac{13}{5}te^{2t} + \frac{22}{25}e^{2t} + \frac{3}{25}\cos(t) - \frac{4}{25}\sin(t)$$

5. Transforming the differential equation, we have

$$\begin{aligned} Y(s) &= \frac{1}{s^2+16}\left[\frac{1}{s}+\frac{1}{s^2}-2s+1\right] \\ &= \frac{1}{16}\left(\frac{1}{s^2}\right)+\frac{1}{16}\left(\frac{1}{s}\right)-\frac{33}{16}\left(\frac{s}{s^2+16}\right)+\frac{15}{64}\left(\frac{4}{s^2+16}\right) \end{aligned}$$

The solution is

$$y(t) = \frac{1}{16}(t+1)-\frac{33}{16}\cos(4t)+\frac{15}{64}\sin(4t)$$

7. Transform the differential equation to obtain (with $y(0)=4$),

$$sY-4-2Y=\frac{1}{s}-\frac{1}{s^2}.$$

Then

$$\begin{aligned} Y(s) &= \frac{1}{s-2}\left(\frac{1}{s}-\frac{1}{s^2}+4\right) \\ &= \frac{1}{2s^2}-\frac{1}{4s}+\frac{17}{4}\left(\frac{1}{s-2}\right) \end{aligned}$$

The solution is

$$y(t)=\frac{1}{2}t-\frac{1}{4}+\frac{17}{4}e^{2t}$$

9. Transform the differential equation to obtain

$$sY(s)-y(0)+4Y(s)=\frac{1}{s}.$$

Set $y(0)=-3$ to obtain

$$sY+3+4Y=\frac{1}{s},$$

or

$$Y(s)=\frac{1}{s+4}\left[\frac{1}{s}-3\right]=\frac{-3s+1}{s(s+4)}.$$

Use a partial fractions decomposition to write this as

$$Y(s)=-\frac{13}{4}\left(\frac{1}{s+4}\right)+\frac{1}{4}\left(\frac{1}{s}\right).$$

The purpose of this decomposition is that we can easily compute the inverse transform of each term on the right, obtaining the solution of the initial value problem:

$$\begin{aligned} y(t) &= -\frac{13}{4}\mathcal{L}^{-1}\left(\frac{1}{s+4}\right)+\frac{1}{4}\mathcal{L}^{-1}\left(\frac{1}{s}\right) \\ &= -\frac{13}{4}e^{-4t}+\frac{1}{4}. \end{aligned}$$

11. Begin with the definition of the Laplace transform and integrate by parts to obtain

$$\begin{aligned} \mathcal{L}[f'(t)](s) &= \int_0^\infty e^{-st}f'(t)\,dt \\ &= e^{-st}f(t)\big]_0^\infty-\int_0^\infty -se^{-st}f(t)\,dt \\ &= -f(0)+s\int_0^\infty e^{-st}f(t)\,dt \\ &= sF(s)-f(0). \end{aligned}$$

3.3 Shifting and the Heaviside Function

In the following, if we shift $f(t)$ by a, replacing t with $t-a$, we may write

$$[f(t)]_{t\to t-a}.$$

In the same spirit, if we want to replace s with $s-a$ in the transform of f, write

$$\mathcal{L}[f(t)]_{s\to s-a}.$$

This notation is sometimes useful in applying a shifting theorem.

1. Replace s with $s+5$ in the transform of t^4+2t^2+t to obtain

$$F(s)=\frac{24}{(s+5)^5}+\frac{4}{(s+5)^3}+\frac{1}{(s+5)^2}.$$

3. Since

$$\mathcal{L}[t\cos(3t)](s)=\frac{s^2-9}{(s^2+9)^2},$$

we obtain the transform of $te^{-t}\cos(3t)$ by replacing s with $s+1$:

$$\mathcal{L}[f](s)=\frac{(s+1)^2-9}{((s+1)^2+9)^2}.$$

5. Replace s with $s+1$ in the transform of $1-t^2+\sin(t)$ to obtain

$$\mathcal{L}[f](s)=\frac{1}{s+1}-\frac{2}{(s+1)^3}+\frac{1}{(s+1)^2+1}$$

7. Write

$$\begin{aligned}f(t)&=t+(1-4t)H(t-3)=t+(1-4((t-3)+3))H(t-3)\\&=t-11H(t-3)-4(t-3)H(t-3).\end{aligned}$$

Then

$$\mathcal{L}[f(t)](s)=\frac{1}{s^2}-\frac{11}{s}e^{-3s}-\frac{4}{s^2}e^{-3s}.$$

9. First write

$$\begin{aligned}f(t)&=(1-H(t-16))(t-2)-H(t-16)\\&=t-2+H(t-16)(2-t-1)=t-2+(1-t)H(t-16).\end{aligned}$$

Then

$$F(s)=\frac{1}{s^2}-\frac{2}{s}+\left(\frac{1}{s}-\frac{1}{s^2}\right)e^{-16s}$$

11. Write

$$\begin{aligned}f(t)&=[1-H(t-7)]+H(t-7)\cos(t)\\&=[1-H(t-7)]+H(t-7)\cos((t-7)+7)\\&=[1-H(t-7)]+\cos(7)H(t-7)\cos(t-7)-\sin(7)H(t-7)\sin(t-7).\end{aligned}$$

Then

$$\mathcal{L}[f(t)](s)=\frac{1}{s}(1-e^{-7s})+\frac{s}{s^2+1}\cos(7)e^{-7s}-\frac{1}{s^2+1}\sin(7)e^{-7s}$$

13. First write
$$f(t) = (1 - H(t - 2\pi))\cos(t) + H(t - 2\pi)(2 - \sin(t))$$
to obtain
$$\mathcal{L}[f](s) = \frac{s}{s^2+1} + \left(\frac{2}{s} - \frac{s}{s^2+1} - \frac{1}{s^2+1}\right)e^{-2\pi s}$$

15.
$$\begin{aligned}\mathcal{L}[(t^3 - 3t + 2)e^{-2t}] &= \mathcal{L}[t^3 - 3t + 2]_{s\to s+2} \\ &= \left[\frac{6}{s^4} - \frac{3}{s^2} + \frac{2}{s}\right]_{s\to s+2} \\ &= \frac{6}{(s+2)^4} - \frac{3}{(s+2)^2} + \frac{2}{s+2}\end{aligned}$$

17. Write
$$\frac{1}{s(s^2+16)} = \frac{1}{16}\frac{1}{s} - \frac{1}{16}\frac{s}{s^2+16}$$
to obtain
$$f(t) = \frac{1}{16}(1 - \cos(4(t-21)))H(t-21).$$

19. Since
$$F(s) = \frac{1}{(s+3)^2 - 2},$$
then
$$f(t) = \frac{1}{\sqrt{2}}\sinh(\sqrt{2}t)e^{-3t}.$$

21. Since $3/(s^2+9)$ is the transform of $\sin(3t)$, then
$$f(t) = \frac{1}{3}\sin(3(t-2))H(t-2).$$

23. Write
$$F(s) = \frac{(s+3) - 1}{(s+3)^2 - 8}$$
to obtain
$$f(t) = e^{-3t}\cosh(2\sqrt{2}t) - \frac{1}{2\sqrt{2}}e^{-3t}\sinh(2\sqrt{2}t).$$

25. Write
$$F(s) = \frac{1}{(s-2)^2 + 1},$$
which we recognize as the transform of $\sin(t)$ with s replaced by $s - 2$. Therefore
$$f(t) = e^{-2t}\sin(t).$$

27. The problem is
$$y^{(3)} - 8y' = 2H(t-6); y(0) = y'(0) = y''(0) = 0.$$
Transform this problem and solve for $Y(s)$ to obtain
$$Y(s) = \left[-\frac{1}{4s} + \frac{1}{12}\frac{1}{s-2} + \frac{1}{6}\frac{s}{s^2+2s+4}\right]e^{-6s}.$$
Invert this to obtain
$$y(t) = \left[-\frac{1}{4} + \frac{1}{12}e^{-2(t-6)} + \frac{1}{6}e^{-(t-6)}\cos(\sqrt{3}(t-6))\right]H(t-6).$$

29. The problem is

$$y^{(3)} - y'' + 4y' - 4y = 1 + H(t-5); y(0) = y'(0) = 0, y''(0) = 1.$$

Transform this and solve for $Y(s)$ to obtain

$$Y(s) = \left[-\frac{1}{4s} + \frac{2}{5}\frac{1}{s-1} - \frac{3}{20}\frac{s}{s^2+4} - \frac{2}{5}\frac{1}{s^2+4}\right](1-e^{-5s}).$$

Invert this to obtain

$$\begin{aligned} y(t) &= -\frac{1}{4} + \frac{2}{5}e^t - \frac{3}{20}\cos(2t) - \frac{1}{5}\sin(2t) \\ &\quad - \left[-\frac{1}{4} + \frac{2}{5}e^{t-5} - \frac{3}{20}\cos(2(t-5)) - \frac{1}{5}\sin(2(t-5))\right]H(t-5). \end{aligned}$$

31. The initial value problem

$$y'' + 4y = 3H(t-4); y(0) = 1, y'(0) = 0$$

transforms to

$$(s^2+4)Y(s) = \frac{3}{s}e^{-4s} + s.$$

Then

$$Y(s) = \frac{3}{4}\left[\frac{1}{s} - \frac{s}{s^2+4}\right]e^{-4s} + \frac{s}{s^2+4}.$$

Inverting this gives the solution

$$y(t) = \cos(2t) + \frac{3}{4}(1 - \cos(2(t-4)))H(t-4).$$

33. Assume that the switch is held in position for B seconds, then switched to position A and left there. The charge q on the capacitor is modeled by the initial value problem

$$250,000q' + 10^6 q = 10H(t-5); q(0) = C, E(0) = 5(10^{-6}).$$

Transform this problem and solve for $Q(s)$ to obtain

$$Q(s) = \frac{5(10^{-6})}{s+4} + 10^{-5}\left[\frac{1}{s} - \frac{1}{s+4}\right]e^{-5s}.$$

Invert this to obtain

$$E_{\text{out}} = \frac{q(t)}{C} = 10^6 q(t) = 5e^{-4t} + 10(1 - e^{-4(t-5)})H(t-5).$$

This output function is graphed in Figure 3.1.

35. The current is modeled by

$$Li' + Ri = k(1 - H(t-5)); i(0) = 0.$$

Transform this problem and solve for $I(s)$ to obtain

$$I(s) = \frac{k}{s(Ls+R)}(1-e^{-5s}) = \frac{k}{R}\left[\frac{1}{s} - \frac{1}{s+R/L}\right](1-e^{-5s}).$$

Invert this to obtain

$$i(t) = \frac{k}{R}(1 - e^{-Rt/L}) - \frac{k}{R}(1 - e^{-R(t-5)/L})H(t-5).$$

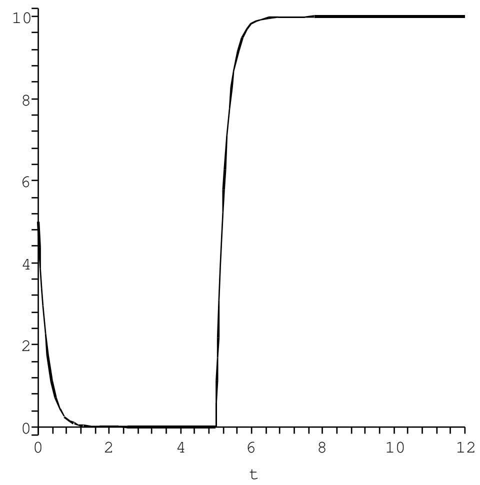

Figure 3.1: Output voltage in Problem 33, Section 3.3.

3.4 Convolution

1.

$$\mathcal{L}^{-1}\left[\frac{1}{(s+2)}\frac{e^{-4s}}{s}\right] = e^{-2t} * H(t-4)$$
$$= \int_4^t e^{-2(t-\tau)} = \frac{1}{2}(1-e^{-2(t-4)})$$

if $t > 4$, and zero if $t \leq 4$. Therefore

$$\mathcal{L}^{-1}\left[\frac{1}{(s+2)}\frac{e^{-4s}}{s}\right] = \frac{1}{2}(1-e^{-2(t-4)})H(t-4).$$

3. First observe that

$$\mathcal{L}^{-1}\left[\frac{1}{s(s^2+a^2)}\right] = \frac{1-\cos(at)}{a^2} \text{ and } \mathcal{L}^{-1}\left[\frac{1}{s^2+a^2}\right] = \frac{\sin(at)}{a}.$$

Then

$$\begin{aligned}
\mathcal{L}^{-1}\left[\frac{1}{s(s^2+a^2)^2}\right] &= \frac{1}{a^3}[1-\cos(at)] * \sin(at) \\
&= \frac{1}{a^3}\int_0^t [1-\cos(a(t-\tau))]\sin(a\tau)\,d\tau \\
&= \frac{1}{a^3}\left[-\frac{\cos(a\tau)}{a} - \frac{\tau\sin(at)}{2} + \frac{\cos(2a\tau-at)}{4a}\right]_0^t \\
&= \frac{1}{a^4}[1-\cos(at)] - \frac{t}{2a^3}\sin(at).
\end{aligned}$$

5. There are two cases. First suppose that $a^2 \neq b^2$. Then

$$\begin{aligned}
\mathcal{L}^{-1}\left[\frac{s}{(s^2+a^2)}\frac{1}{(s^2+b^2)}\right] &= \cos(at) * \frac{\sin(bt)}{b} \\
&= \frac{1}{b}\int_0^t \cos(a(t-\tau))\sin(b\tau)\,d\tau \\
&= \frac{1}{2b}\int_0^t [\sin((b-a)\tau+at)+\sin((b+a)\tau-at)]\,d\tau \\
&= \frac{1}{2b}\left[-\frac{\cos((b-a)\tau+at)}{b-a}-\frac{\cos((b+a)\tau-at)}{b+a}\right]_0^t \\
&= \frac{1}{2b}\left[-\frac{\cos(bt)}{b-a}-\frac{cos(bt)}{b+a}+\frac{\cos(at)}{b-a}+\frac{\cos(at)}{b+a}\right] \\
&= \frac{\cos(at)-\cos(bt)}{(b-a)(b+a)}.
\end{aligned}$$

If $b^2 = a^2$,

$$\begin{aligned}
\mathcal{L}^{-1}\left[\frac{s}{(s^2+a^2)}\frac{1}{(s^2+a^2)}\right] &= \cos(at) * \frac{\sin(at)}{a} \\
&= \frac{1}{a}\int_0^t \cos(a(t-\tau))\sin(a\tau)\,d\tau \\
&= \frac{1}{2a}\int_0^t (\sin(at)+\sin(2a\tau-at))\,d\tau \\
&= \frac{1}{2a}\left[\tau\sin(at)-\frac{\cos(a(2\tau-t))}{2a}\right]_0^t \\
&= \frac{t\sin(at)}{2a}.
\end{aligned}$$

7. Let

$$F(s) = \frac{1}{s^2+4} \text{ and } G(s) = \frac{1}{s^2-4}.$$

Then

$$\mathcal{L}^{-1}[F(s)] = \frac{1}{2}\sin(2t) \text{ and } \mathcal{L}^{-1}[G(s)] = \frac{1}{2}\sinh(2t).$$

By the convolution theorem,

$$\begin{aligned}
&\mathcal{L}^{-1}\left[\left(\frac{1}{s^2+4}\right)\left(\frac{1}{s^2-4}\right)\right] \\
&= \frac{1}{4}\sin(2t) * \sinh(2t) \\
&= \frac{1}{4}\int_0^t \sin(2(t-\tau))\sinh(2\tau)\,d\tau \\
&= \frac{1}{16}\left[\sin(2(t-\tau))\cosh(2\tau)+\cos(2(t-\tau))\sinh(2\tau)\right]_0^t \\
&= \frac{1}{16}[\sinh(2t)-\sin(2t)].
\end{aligned}$$

In Problems 9, 11, and 13, the solution of the initial value problem is given without the details of taking the transform of the differential equation.

9.
$$y(t) = \frac{1}{3}\sin(3t) * f(t) - \cos(3t) + \frac{1}{3}\sin(3t)$$

11.
$$y(t) = \frac{1}{4}e^{2t} * f(t) + \frac{1}{12}e^{-2t} * f(t) - \frac{1}{3}e^{t} * f(t) - \frac{1}{4}e^{2t} - \frac{1}{12}e^{-2t} + \frac{4}{3}e^{t}$$

13. We obtain
$$y(t) = \frac{1}{4}e^{6t} * f(t) - \frac{1}{4}e^{2t} * f(t) + 2e^{6t} - 5e^{2t}.$$

15. Take the transform of the initial value problem to obtain
$$Y(s) = \frac{F(s)}{s^2 - 5s + 6} = \left[\frac{1}{s-3} - \frac{1}{s-2}\right] F(s).$$

By the convolution theorem,
$$y(t) = e^{3t} * f(t) - e^{2t} * f(t).$$

17. The equation is $f(t) = e^{-t} + f(t) * 1$. Transform this and solve for $F(s)$ to obtain
$$F(s) = \frac{s}{(s+1)(s-1)} = \frac{1}{2}\left[\frac{1}{s+1} + \frac{1}{s-1}\right].$$

Now invert to obtain
$$f(t) = \frac{1}{2}e^{-t} + \frac{1}{2}e^{t} = \cosh(t).$$

19. The equation is $f(t) = 3 + f(t) * \cos(2t)$. From this we obtain
$$F(s) = \frac{3(s^2+4)}{s(s^2 - s + 4)} = \frac{3}{s} + \frac{3}{s^2 - s + 4}.$$

Invert this to obtain
$$f(t) = 3 + \frac{2\sqrt{15}}{5}e^{t/2}\sin\left(\frac{\sqrt{15}}{2}t\right).$$

21. The integral equation can be expressed as
$$f(t) = -1 + f(t) * e^{-3t}.$$

Take the transform of this to obtain
$$F(s) = -\frac{1}{s} + \frac{F(s)}{s+3}.$$

Then
$$F(s) = -\frac{s+3}{s(s+2)} = \frac{1}{2(s+2)} - \frac{3}{2s}.$$

Inverting this leads to the solution
$$f(t) = \frac{1}{2}e^{-2t} - \frac{3}{2}.$$

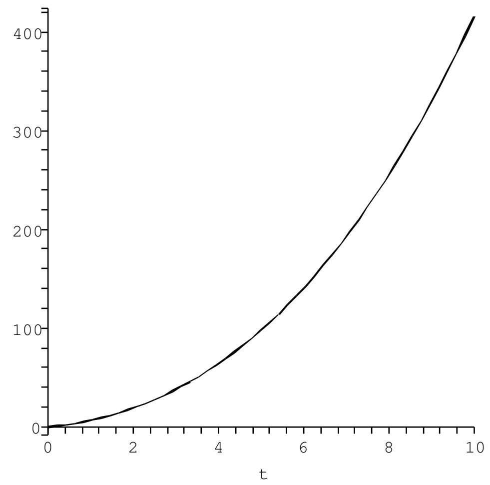

Figure 3.2: Replacement function in Problem 23, Section 3.4.

23. Now $f(t) = A + Bt + Ct^2$ and $m(t) = e^{-kt}$, so

$$F(s) = \frac{A}{s} + \frac{B}{s^2} + \frac{2C}{s^3} \text{ and } M(s) = \frac{1}{s+k}.$$

Then, by a routine algebraic calculation,

$$\begin{aligned} R(s) &= \frac{\frac{A}{s} + \frac{B}{s^2} + \frac{2C}{s^3} - A\left(\frac{1}{s+k}\right)}{\frac{s}{s+k}} \\ &= \frac{Ak+B}{s^2} + \frac{2C+Bk}{s^3} + \frac{2Ck}{s^4}. \end{aligned}$$

Then

$$r(t) = (Ak+B)t + \left(\frac{1}{2}Bk + C\right)t^2 + \frac{1}{3}Ckt^3.$$

Figure 3.2 shows a graph of this replacement function for $A = 1$, $B = 4$, $k = 1/5$ and $C = 2$.

25. We want $r(t)$ if $f(t) = A =$ constant and $m(t) = e^{-kt}$. Begin with

$$R(s) = \frac{F(s) - f(0)M(s)}{sM(s)}.$$

For this problem,

$$F(s) = \frac{A}{s} \text{ and } M(s) = \frac{1}{s+k}.$$

Then

$$R(s) = \frac{\frac{A}{s} - \frac{A}{s+k}}{\frac{s}{s+k}} = \frac{Ak}{s^2}.$$

Therefore $r(t) = Akt$. This function has a straight line graph, shown in Figure 3.3 for $A = 3, k = 1/5$.

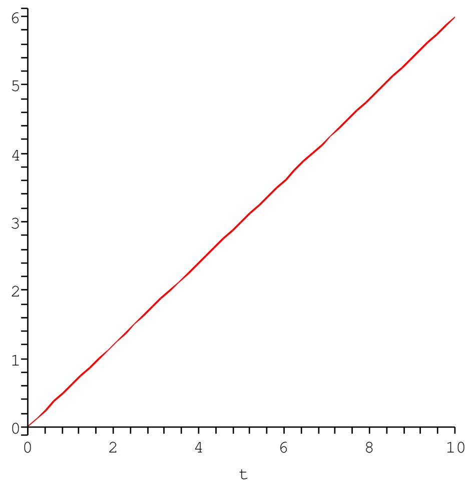

Figure 3.3: Replacement function in Problem 25, Section 3.4.

3.5 Impulses and the Delta Function

For Problems 1 and 3, the details are similar and only the solutions are provided. For Problem 5 the details of the solution are included.

1.
$$y(t) = 6(e^{-2t} - e^{-t} + te^{-t})$$

3.
$$\varphi(t) = (B+9)e^{-2t} - (B+6)e^{-3t}; \varphi(0) = 3, \varphi'(0) = B$$

The Dirac delta function $\delta(t - t_0)$ applied at time t_0 imparts a unit velocity to the unit mass.

5. Transform the initial value problem to obtain
$$(s^2 + 5s + 6)Y(s) = 3e^{-2s} - 4e^{-5s}.$$
Then
$$Y(s) = 3\left[\frac{1}{s+2} - \frac{1}{s+3}\right]e^{-2s} - 4\left[\frac{1}{s+2} - \frac{1}{s+3}\right]e^{-5s}.$$
Invert this to obtain
$$y(t) = 3[e^{-2(t-2)} - e^{-3(t-2)}]H(t-2) - 4[e^{-2(t-5)} - e^{-3(t-5)}]H(t-5).$$

7. Begin by writing, for $\epsilon > 0$,
$$\begin{aligned}\int_0^\infty f(t)\delta_\epsilon(t-a)\,dt &= \int_0^\infty \frac{1}{\epsilon}[H(t-a) - H(t-a-\epsilon)]f(t)\,dt \\ &= \frac{1}{\epsilon}\int_a^{a+\epsilon} f(t)\,dt.\end{aligned}$$

By the mean value theorem for integrals, there is some t_ϵ between a and $a+\epsilon$ such that

$$\int_a^{a+\epsilon} f(t)\,dt = \epsilon f(t_\epsilon).$$

Then

$$\int_0^\infty f(t)\delta_\epsilon(t-a)\,dt = f(t_\epsilon).$$

As $\epsilon \to 0+$, $a+\epsilon \to a$, so $t_\epsilon \to a$ and, by continuity, $f(t_\epsilon) \to f(a)$. Then

$$\begin{aligned}\lim_{\epsilon\to 0+}\int_0^\infty f(t)\delta_\epsilon(t-a)\,dt &= \int_0^\infty f(t)\lim_{\epsilon\to 0+}\delta_\epsilon(t-a)\,dt\\ &= \int_0^\infty f(t)\delta(t-a)\,dt\\ &= \lim_{\epsilon\to 0+} f(t_\epsilon) = f(a).\end{aligned}$$

8. $F(x) = kx$ gives us $k = 2(8/3)(12) = 9$ pounds per foot, and $m = 2/32 = 1/16$ slugs. The motion is modeled by

$$\frac{1}{16}y'' + 9y = \frac{1}{4}\delta(t); y(0) = y'(0) = 0.$$

Transform to obtain

$$Y(s) = \frac{4}{s^2+144},$$

so

$$y(t) = \frac{1}{3}\sin(12t).$$

The initial velocity is $y'(0) = 4$ feet per second. The frequency is $6/\pi$ hertz and the amplitude is $1/3$ feet, or 4 inches.

9. The motion is modeled by the problem

$$my'' + ky = mv_0\delta(t); y(0) = y'(0) = 0.$$

We find that

$$Y(s) = \frac{mv_0}{ms^2+k},$$

so

$$y(t) = v_0\sqrt{\frac{m}{k}}\sin\left(\sqrt{\frac{k}{m}}t\right).$$

3.6 Solution of Systems

1. The transform of the system yields

$$\begin{aligned} sY_1 - 2sY_2 + 3Y_3 &= 0,\\ Y_1 - 4sY_2 + 3Y_3 &= \frac{1}{s^2},\\ Y_1 - 2sY_2 + 3sY_3 &= -\frac{1}{s}.\end{aligned}$$

Then

$$\begin{aligned}
Y_1(s) &= \frac{1+s-s^2}{s^2(s^2-1)} = -\frac{1}{s^2} - \frac{1}{s} + \frac{1}{2}\frac{1}{s-1} + \frac{1}{2}\frac{1}{s+1},\\
Y_2(s) &= -\frac{s+1}{2s^3} = -\frac{1}{2}\frac{1}{s^2} - \frac{1}{2}\frac{1}{s^3},\\
Y_3(s) &= \frac{-2s^2+1}{3s^2(s^2-1)} = -\frac{1}{3}\frac{1}{s^2} - \frac{1}{6}\frac{1}{s-1} + \frac{1}{6}\frac{1}{s+1}.
\end{aligned}$$

Invert these to obtain the solution

$$\begin{aligned}
y_1(t) &= -t - 1 + \frac{1}{2}e^t + \frac{1}{2}e^{-t},\\
y_2(t) &= -\frac{1}{2}t - \frac{1}{4}t^2,\\
y_3(t) &= -\frac{1}{3}t - \frac{1}{6}e^t + \frac{1}{6}e^{-t}.
\end{aligned}$$

3. The transform of the system yields

$$(s+1)X + (s-1)Y = 0 \text{ and } (s+1)X + 2sY = \frac{1}{s}.$$

Then

$$\begin{aligned}
X(s) &= \frac{1-s}{s(s+1)^2} = \frac{1}{s} - \frac{1}{s+1} - \frac{2}{(s+1)^2},\\
Y(s) &= \frac{1}{s(s+1)} = \frac{1}{s} - \frac{1}{s+1}.
\end{aligned}$$

The solution is

$$x(t) = 1 - e^{-t} - 2te^{-t} \text{ and } y(t) = 1 - e^{-t}.$$

5. From the system, obtain

$$(s+2)X - sY = 0 \text{ and } (s+1)X + Y = \frac{2}{s^3}.$$

Then

$$\begin{aligned}
X(s) &= \frac{2}{s^2(s^2+2s+2)} = \frac{1}{s^2} - \frac{1}{s} + \frac{s+1}{(s+1)^2+1},\\
Y(s) &= \frac{2(s+2)}{s^3(s^2+2s+2)} = \frac{2}{s^3} - \frac{1}{s^2} + \frac{1}{(s+1)^2+1}.
\end{aligned}$$

Invert these to obtain

$$x(t) = t - 1 + e^{-t}\cos(t) \text{ and } y(t) = t^2 - t + e^{-t}\sin(t).$$

7. Take the Laplace transform:

$$3sX - Y = \frac{2}{s^2} \text{ and } sX + (s-1)Y = 0.$$

Then

$$X(s) = \frac{2(s-1)}{s^3(3s-2)} = \frac{1}{s^3} + \frac{1}{2}\frac{1}{s^2} + \frac{3}{4}\frac{1}{s} - \frac{9}{4}\frac{1}{3s-2},$$
$$Y(s) = \frac{-2}{s^2(3s-2)} = \frac{1}{s^2} + \frac{3}{2}\frac{1}{s} - \frac{9}{2}\frac{1}{3s-2}.$$

The solution is

$$x(t) = \frac{1}{2}t^2 + \frac{1}{2}t + \frac{3}{4} - \frac{3}{4}e^{2t/3},$$
$$y(t) = t + \frac{3}{2} - \frac{3}{2}e^{2t/3}.$$

9. After taking the transform of the system, we have

$$sX + (2s-1)Y = \frac{1}{s} \text{ and } sX + Y = 0.$$

Then

$$X(s) = \frac{-1}{s^2(4s-3)} = \frac{1}{3}\frac{1}{s^2} + \frac{4}{9}\frac{1}{s} - \frac{16}{9}\frac{1}{4s-3},$$
$$Y(s) = \frac{2}{s(4s-3)} = -\frac{2}{3}\frac{1}{s} + \frac{8}{3}\frac{1}{4s-3}.$$

The solution is

$$x(t) = \frac{1}{3}t + \frac{4}{9} - \frac{4}{9}e^{3t/4} \text{ and } y(t) = -\frac{2}{3} + \frac{2}{3}e^{3t/4}.$$

11. Take the Laplace transform of the system:

$$sX - 2sY = \frac{1}{s},$$
$$sX - X + Y = 0.$$

Solve these for X and Y:

$$X(s) = \frac{1}{s^2(2s-1)} = -\frac{1}{s^2} - \frac{2}{s} + \frac{4}{2s-1},$$
$$Y(s) = \frac{1-s}{s^2(2s-1)} = -\frac{1}{s^2} - \frac{1}{s} + \frac{2}{2s-1}.$$

Apply the inverse transform to obtain the solution

$$x(t) = -t - 2 + 2e^{t/2},$$
$$y(t) = -t - 1 + e^{t/2}.$$

13. The loop currents satisfy

$$5i_1' + 5i_1 - 5i_2' = 1 - H(t-4)\sin(2(t-4)),$$
$$-5i_1' + 5i_2' + 5i_2 = 0.$$

Simplify these equations and solve take the Laplace transform to obtain

$$\begin{aligned}
I_1(s) &= \frac{s+1}{5(2s+1)}\left[\frac{1}{s} - \frac{2e^{-4s}}{s^2+4}\right] \\
&= \frac{1}{5}\left[\frac{1}{s} - \frac{1}{2s+1}\right] - \frac{2}{85}\left[\frac{2}{2s+1} - \frac{s}{s^2+4} + \frac{9}{s^2+4}\right] e^{-4s}, \\
I_2(s) &= \frac{1}{5(2s+1)}\left[1 - \frac{2se^{-4s}}{s^2+4}\right] \\
&= \frac{1}{5(2s+1)} + \frac{2}{85}\left[\frac{2}{2s+1} - \frac{s}{s^2+4} - \frac{8}{s^2+4}\right] e^{-4s}.
\end{aligned}$$

Apply the inverse transform to obtain the currents:

$$\begin{aligned}
i_1(t) &= \frac{1}{5}\left[1 - \frac{1}{2}e^{-t/2}\right] \\
&\quad - \frac{2}{85}\left[e^{-(t-4)/2} - \cos(2(t-4)) + \frac{9}{2}\sin(2(t-4))\right] H(t-4), \\
i_2(t) &= \frac{1}{10}e^{-t/2} \\
&\quad + \frac{2}{85}\left[e^{-(t-4)/2} - \cos(2(t-4)) - 4\sin(2(t-4))\right] H(t-4).
\end{aligned}$$

14. **Hint** Let x_1 and x_2 be the downward displacements of the masses m_1 and m_2, respectively. By Newton's second law of motion and from the equilibrium of the system, the equations of motion are

$$\begin{aligned}
m_1 x_1'' &= -k_1 x_1 + k_2(x_2 - x_1) + f_1(t), \\
m_2 x_2'' &= -k_3 x_2 + k_2(x_1 - x_2) + f_2(t).
\end{aligned}$$

The initial positions and velocities are zero.

15. As in the solution to Problem 14, write the equations of motion

$$\begin{aligned}
x_1'' + 8x_1 - 2x_2 &= 1 - H(t-2), \\
x_2'' - 2x_1 + 5x_2 &= 0.
\end{aligned}$$

Initial positions and velocities are zero. Transforming these yields

$$\begin{aligned}
(s^2+8)X_1 - 2X_2 &= \frac{1}{s}(1 - e^{-2s}), \\
-2X_1 + (s^2+5)X_2 &= 0.
\end{aligned}$$

Then

$$\begin{aligned}
X_1(s) &= \frac{s^2+5}{s(s^4+13s^2+36)}(1 - e^{-2s}), \\
X_2(s) &= \frac{2}{s(s^4+13s^2+36)}(1 - e^{-2s}).
\end{aligned}$$

Invert these to obtain the solution

$$\begin{aligned}
x_1(t) &= \frac{5}{36} - \frac{1}{20}\cos(2t) - \frac{4}{45}\cos(3t) \\
&\quad - \left[\frac{5}{36} - \frac{1}{20}\cos(2(t-2)) - \frac{4}{45}\cos(3(t-2))\right] H(t-2), \\
x_2(t) &= \frac{1}{18} - \frac{1}{10}\cos(2t) + \frac{2}{45}\cos(3t) \\
&\quad - \left[\frac{1}{18} - \frac{1}{10}\cos(2(t-2)) + \frac{2}{45}\cos(3(t-2))\right] H(t-2).
\end{aligned}$$

17. The equations of motion are

$$\begin{aligned}
my_1'' &= k(y_2 - y_1), \\
m_2 y_2'' &= k(y_1 - y_2),
\end{aligned}$$

with initial conditions

$$y_1(0) = y_1'(0) = y_2'(0) = 0, y_2(0) = d.$$

Apply the transform to the system and solve for $Y_1(s)$ and $Y_2(s)$ to obtain

$$\begin{aligned}
Y_1(s) &= \frac{kd}{m_1 s\left(s^2 + \frac{m_1+m_2}{(m_1 m_2)}k\right)}, \\
Y_2(s) &= \frac{d(m_1 s^2 + k)}{m_1 s\left(s^2 + \frac{m_1+m_2}{(m_1 m_2)}k\right)}.
\end{aligned}$$

The quadratic factor in the denominator shows that the motion has frequency

$$\omega = \sqrt{\left(\frac{m_1 + m_2}{m_1 m_2}\right) k},$$

and therefore period

$$2\pi\sqrt{\frac{m_1 m_2}{(m_1 + m_2)k}}.$$

19. As in the solution of Problem 18, except with $E(t) = 5\delta(t-1)$, the transformed equations yield

$$\begin{aligned}
I_1(s) &= \frac{5(30s + 20)e^{-s}}{600s^2 + 700s + 100} \\
&= \left[\frac{1}{10(s+1)} + \frac{9}{10(6s+1)}\right] e^{-s}, \\
I_2(s) &= \frac{50e^{-s}}{600s^2 + 700s + 100} \\
&= \left[-\frac{1}{10(s+1)} + \frac{3}{5(6s+1)}\right] e^{-s}.
\end{aligned}$$

The currents are

$$\begin{aligned}
i_1(t) &= \left[\frac{1}{10}e^{-(t-1)} + \frac{3}{20}e^{-(t-1)/6}\right] H(t-1), \\
i_2(t) &= \left[-\frac{1}{10}e^{-(t-1)} + \frac{1}{10}e^{-(t-1)/6}\right] H(t-1).
\end{aligned}$$

21. Using the notation of the solution of Problem 20, we can write the system

$$x_1' = -\frac{6}{200}x_1 + \frac{3}{100}x_2,$$
$$x_2' = \frac{4}{200}x_1 - \frac{4}{100}x_2 + 5\delta(t-3),$$

with initial conditions

$$x_1(0) = 10, x_2(0) = 5.$$

Simplify these equations and apply the Laplace transform to obtain

$$(100s+3)X_1 - 3X_2 = 1000,$$
$$-2X_1 + (100s+4)X_2 = 500 + 500e^{-3s}.$$

Then

$$\begin{aligned} X_1(s) &= \frac{100000s + 5500 + 1500e^{-3s}}{10000s^2 + 700s + 6} \\ &= \frac{50}{50s+3} + \frac{900}{100s+1} + \left[\frac{300}{100s+1} - \frac{150}{50s+3}\right]e^{-3s}, \\ X_2(s) &= \frac{50000s + 3500 + (50000s + 1500)e^{-3s}}{10000s^2 + 700s + 6} \\ &= -\frac{50}{50s+3} + \frac{600}{100s+1} + \left[\frac{150}{50s+3} + \frac{200}{100s+1}\right]e^{-3s}. \end{aligned}$$

Invert these equations to obtain the solution

$$x_1(t) = e^{-3t/50} + 9e^{-t/100} + 3(e^{-(t-3)/100} - e^{-3(t-3)/50})H(t-3),$$
$$x_2(t) = -e^{-3t/50} + 6e^{-t/100} + (3e^{-3(t-3)/50} + 2e^{-(t-3)/100})H(t-3).$$

3.7 Polynomial Coefficients

In Problems 1, 3, 7, and 9, the details of the solution are like those of Problem 5, and only the solution is given.

1. $y(t) = 3t^2/2$
2. Transform the differential equation to obtain

$$s^2Y - sy(0) - y'(0) + \frac{d}{ds}(s^2Y(s) - sy(0) - y'(0)) - \frac{d}{ds}(sY(s) - y(0)) - Y = 0.$$

Since $y(0) = 3$ and $y'(0) = -1$, this is

$$(s^2 - s)Y' + (s^2 + 2s - 2)Y = 3s + 2,$$

a first order linear differential equation for $Y(s)$. An integrating factor is $\mu = se^s$. Multiplying by this factor gives us

$$\frac{d}{ds}(e^s(s^3 - s^2)Y) = 3s^2e^s + 2se^s.$$

Integrate this equation to obtain

$$\begin{aligned} Y(s) &= \frac{3s^2-4s+4}{s^2(s-1)} + K\frac{e^{-s}}{s^2(s-1)} \\ &= \frac{3}{s-1} - \frac{4}{s^2} + K\left(\frac{1}{s-1} - \frac{1}{s} - \frac{1}{s^2}\right)e^{-s}. \end{aligned}$$

Invert this equation to obtain the solution

$$y(t) = 3e^t - 4t + K(e^t - 1 - t)H(t-1).$$

K is arbitrary and can be given any real value. This illustrates a *bifurcation* in the solution. At $t = 1$, the solution splits off and travels along different curves, depending on the choice of K. Notice that the existence/uniqueness theorem for solutions of this differential equation does not apply at $t = 1$, which is a singular point.

3. $y(t) = 4$

5. Before transforming the equation, make the change of variable $u = 1/t$. Let $z(u) = y(t(u)) = y(1/u)$. Then

$$\frac{dy}{dt} = \frac{dz}{du}\frac{du}{dt} = -\frac{1}{t^2}\frac{dz}{du},$$

and $t^2(dy/dt) - 2y = 2$ transforms to

$$-\frac{dz}{du} - 2z = 2.$$

Apply the transform to this differential equation to obtain

$$-sZ + z(0) - 2Z = \frac{2}{s}.$$

Then

$$Z(s) = -\frac{2}{s(s+2)} + \frac{z(0)}{s+2} = \frac{1+z(0)}{s+2} - \frac{1}{s}.$$

Invert this equation to obtain

$$z(u) = ce^{-2u} - 1$$

or

$$y(t) = -1 + ce^{-2/t}.$$

This problem can also be solved as a first order linear differential equation, after dividing it by t^2.

7. $y(t) = ct^2e^{-t}$

9. $y(t) = 7t^2$

11. When we wrote factorials and inverted terms of the form $1/s^{2n+k}$ in the binomial expansion used in the derivation, we assumed that n is a nonnegative integer.

Chapter 4

Series Solutions

4.1 Power Series Solutions

1. Write

$$y'' - xy' + y = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=1}^{\infty} na_n x^n + \sum_{n=0}^{\infty} a_n x^n$$
$$= (2a_2 + a_0) + \sum_{n=3}^{\infty}(n(n-1)a_n - (n-3)a_{n-2})x^{n-2} = 3.$$

Then a_0 and a_1 are arbitrary, $a_2 = -(a_0 - 3)/2$, and, as the recurrence relation,

$$a_n = \frac{(n-3)}{n(n-1)}a_{n-2} \text{ for } n = 3, 4, \cdots .$$

This yields the general solution

$$y(x) = 3 + a_1 x + (a_0 - 3)\left(1 + \sum_{n=1}^{\infty} \frac{(-1)(1)(3)\cdot(2n-3)}{(2n)!}x^{2n}\right).$$

Here $a_1 = y'(0)$ and $a_0 = y(0)$.

3. We have

$$y'' + (1-x)y' + 2y = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}$$
$$+ \sum_{n=1}^{\infty} na_n x^{n-1} - \sum_{n=1}^{\infty} na_n x^n + \sum_{n=0}^{\infty} 2a_n x^n$$
$$= (2a_2 + a_1 + 2a_0) + \sum_{n=3}^{\infty}(n(n-1)a_n + (n-1)a_{n-1} - (n-4)a_{n-2})x^{n-2}$$
$$= 1 - x^2,$$

Then a_0 and a_1 are arbitrary, $2a_2 + a_1 + 2a_0 = 1$, $6a_3 + 2a_2 + a_1 = 0$, $12a_4 + 3a_3 = -1$, and

$$a_n = \frac{-(n-1)a_{n-1} + (n-4)a_{n-2}}{n(n-1)}$$

for $n = 5, 6, \cdots$. The general solution is

$$y(x) = a_0\left[1 - x^2 + \frac{1}{3}x^3 - \frac{1}{12}x^4 + \frac{1}{30}x^5 - \cdots\right]$$
$$+ a_1\left(x - \frac{1}{2}x^2\right) + \frac{1}{2}x^2 - \frac{1}{6}x^3 - \frac{1}{24}x^4 - \frac{1}{360}x^6 + \frac{1}{2520}x^7 + \cdots,$$

where $a_0 = y(0)$ and $a_1 = y'(0)$. The last series is a particular solution of the nonhomogeneous equation.

5. Write

$$y'' - x^2y' + 2y = \sum_{n=2}^{\infty} n(n-1)a_nx^{n-2}$$
$$- \sum_{n=1}^{\infty} na_nx^{n+1} + \sum_{n=0}^{\infty} 2a_nx^n = (2a_2 + 2a_0) + (6a_3 + 2a_1)x$$
$$+ \sum_{n=4}^{\infty}(n(n-1)a_n - (n-3)a_{n-3} + 2a_{n-2})x^{n-2} = x.$$

Then a_0 and a_1 are arbitrary, $a_2 = -a_0$, $6a_3 + 2a_1 = 1$, and, for the recurrence relation,

$$a_n = \frac{(n-3)a_{n-3} - 2a_{n-2}}{n(n-1)} \text{ for } n = 4, 5, \cdots.$$

The general solution has the form

$$y(x) = a_0\left[1 - x^2 + \frac{1}{6}x^4 - \frac{1}{10}x^5 - \frac{1}{90}x^6 + \cdots\right]$$
$$+ a_1\left[x - \frac{1}{3}x^3 + \frac{1}{12}x^4 + \frac{1}{30}x^5 - \frac{7}{180}x^6 + \cdots\right]$$
$$+ \frac{1}{6}x^3 - \frac{1}{60}x^5 + \frac{1}{60}x^6 + \frac{1}{1260}x^7 - \frac{1}{480}x^8 + \cdots.$$

Here $a_0 = y(0)$ and $a_1 = y'(0)$. The third series in the solution represents a particular solution obtained from the recurrence by putting $a_0 = a_1 = 0$.

7. Put $y(x) = \sum_{n=0}^{\infty} a_nx^n$ into the differential equation to obtain

$$y' - xy = \sum_{n=1}^{\infty} na_nx^{n-1} - \sum_{n=0}^{\infty} a_nx^{n+1}$$
$$= a_1 + (2a_2 - a_0)x + \sum_{n=3}^{\infty}(na_n - a_{n-2})x^{n-1}$$
$$= 1 - x.$$

Then a_0 is arbitrary, $a_1 = 1$, $2a_2 - a_0 = -1$ and

$$a_n = \frac{a_{n-2}}{n} \text{ for } n = 3, 4, \cdots.$$

This is the recurrence relation. If we set $a_0 = c_0 + 1$, we obtain the coefficients

$$a_2 = \frac{c_0}{2}, a_4 = \frac{c_0}{2 \cdot 4}, a_6 = \frac{c_0}{2 \cdot 4 \cdot 6},$$

and so on, and $a_1 = 1$, $a_3 = 1/3$, $a_5 = 1/(3 \cdot 5)$, $a_7 = 1/(3 \cdot 5 \cdot 7)$, and so on. In general, we obtain

$$y(x) = 1 + \sum_{n=0}^{\infty} \frac{1}{1 \cdot 3 \cdot 5 \cdots (2n+1)} x^{2n+1} + c_0 \left(1 + \sum_{n=1}^{\infty} \frac{1}{2 \cdot 4 \cdot 6 \cdots 2n} x^{2n} \right).$$

9. Write

$$\begin{aligned} y' + (1-x^2)y &= \sum_{n=1}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n - \sum_{n=0}^{\infty} a_n x^{n+2} \\ &= (a_1 + a_0) + (2a_2 + a_1)x + \sum_{n=3}^{\infty} (n a_n + a_{n-1} - a_{n-3}) x^{n-1} \\ &= x. \end{aligned}$$

The recurrence relation is

$$n a_n + a_{n-1} - a_{n-3} = 0 \text{ for } n \geq 3$$

and we also have a_0 arbitrary, $a_1 + a_0 = 0$, and $2a_2 + a_1 = 1$. This yields the solution

$$\begin{aligned} y = a_0 &\left(1 - x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 - \frac{7}{4!}x^4 + \frac{19}{5!}x^5 + \cdots \right) \\ &+ \frac{1}{2!}x^2 - \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{11}{5!}x^5 - \frac{31}{6!}x^6 + \cdots. \end{aligned}$$

4.2 Frobenius Solutions

1. The indicial equation is $r^2 - 3r + 2 = 0$ with roots $r_1 = 2$ and $r_2 = 1$. There are solutions

$$y_1(x) = \sum_{n=0}^{\infty} c_n x^{n+2} \text{ and } y_2(x) = k y_1 \ln(x) + \sum_{n=0}^{\infty} c_n^* x^{n+1}.$$

Substitute these in turn into the differential equation to obtain

$$y_1(x) = x^2 + \frac{1}{3!}x^4 + \frac{1}{5!}x^6 + \frac{1}{7!}x^8 + \cdots = x \sinh(x)$$

and

$$y_2(x) = x - x^2 + \frac{1}{2!}x^3 - \frac{1}{3!}x^4 + \frac{1}{4!}x^5 - \cdots = xe^{-x}.$$

3. The indicial equation is $4r^2 - 2r = 0$, with roots $r_1 = 1/2$ and $r_2 = 0$. There are solutions of the form

$$y_1(x) = \sum_{n=0}^{\infty} c_n x^{n+1/2}$$

and

$$y_2(x) = \sum_{n=0}^{\infty} c_n^* x^n.$$

Substitute these into the differential equation in turn to obtain

$$\begin{aligned} y_1(x) &= x^{1/2}\left[1+\sum_{n=1}^{\infty}\frac{(-1)^n}{2^n n!(3\cdot 5\cdot 7\cdots(2n+1))}x^n\right] \\ &= x^{1/2}\left[1-\frac{1}{6}x+\frac{1}{120}x^2-\frac{1}{5040}x^3+\frac{1}{362880}x^4+\cdots\right] \end{aligned}$$

and

$$\begin{aligned} y_2(x) &= 1+\sum_{n=1}^{\infty}\frac{(-1)^n}{2^n n!(1\cdot 3\cdot 5\cdots(2n-1))}x^n \\ &= 1-\frac{1}{2}x+\frac{1}{24}x^2-\frac{1}{720}x^3+\frac{1}{40320}x^4-\cdots . \end{aligned}$$

5. The indicial equation is $2r^2 = 0$ with roots $r_1 = r_2 = 0$. There are solutions of the form

$$y_1(x) = \sum_{n=0}^{\infty} c_n x^n$$

and

$$y_2(x) = y_1(x)\ln(x) + \sum_{n=1}^{\infty} c_n^* x^n.$$

Upon substituting these in turn into the differential equation, we obtain the simple solutions

$$y_1(x) = 1 - x \text{ and } y_2(x) = (1-x)\ln\left(\frac{x}{x-2}\right) - 2.$$

7. The indicial equation is $r^2 - 4r = 0$, with roots $r_1 = 4$ and $r_2 = 0$. There are solutions of the form

$$y_1(x) = \sum_{n=0}^{\infty} c_n x^{n+4} \text{ and } y_2(x) = ky_1(x)\ln(x) + \sum_{n=0}^{\infty} c_n^* x^n.$$

With $r = 4$ we obtain the recurrence relation

$$c_n = \frac{n+1}{n}c_{n-1}$$

and the first solution

$$\begin{aligned} y_1(x) &= x^4(1+2x+3x^2+4x^3+\cdots) \\ &= x^4\frac{d}{dx}(1+x+x^2+x^3+x^4+\cdots) \\ &= x^4\frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{x^4}{(1-x)^2}. \end{aligned}$$

A second solution is

$$y_2(x) = \frac{3-4x}{(1-x)^2}.$$

9. Substitute $y(x) = \sum_{n=0}^{\infty} c_n x^{n+r}$ into the differential equation to obtain

$$\begin{aligned}
xy'' + (1-x)y' + y &= \sum_{n=0}^{\infty}(n+r)(n+r-1)c_n x^{n+r-1} \\
&+ \sum_{n=0}^{\infty}(n+r)c_n x^{n+r-1} - \sum_{n=0}^{\infty}(n+r)c_n x^{n+r} + \sum_{n=0}^{\infty} c_n x^{n+r} \\
&= r^2 c_0 x^{r-1} + \sum_{n=1}^{\infty}((n+r)^2 c_n - (n+r-2)c_{n-1})x^{n+r-1} \\
&= 0.
\end{aligned}$$

Since c_0 is assumed to be nonzero, then r must satisfy the indicial equation $r^2 = 0$, with equal roots $r_1 = r_2 = 0$. One solution has the form

$$y_1(x) = \sum_{n=0}^{\infty} c_n x^n$$

while a second solution has the form

$$y_2(x) = y_1(x)\ln(x) + \sum_{n=1}^{\infty} c_n^* x^n.$$

For the first solution, choose the coefficients to satisfy $c_0 = 1$ and

$$c_n = \frac{n-2}{n^2} c_{n-1} \text{ for } n = 1, 2, \cdots.$$

This yields the solution $y_1(x) = 1 - x$. Therefore

$$y_2(x) = (1-x)\ln(x) + \sum_{n=1}^{\infty} c_n^* x^n.$$

Substitute this into the differential equation to obtain

$$\begin{aligned}
&x\left[-\frac{2}{x} - \frac{1-x}{x^2}\right] + (1-x)\left[-\ln(x) + \frac{1-x}{x}\right] \\
&+ (1-x)\ln(x) + \sum_{n=2}^{\infty} n(n-1)c_n^* x^{n-1} + (1-x)\sum_{n=1}^{\infty} nc_n^* x^{n-1} + \sum_{n=1}^{\infty} c_n^* x^n \\
&= (-3 + c_1^*) + (1 + 4c_2^*)x + \sum_{n=3}^{\infty}(n^2 c_n^* - (n-2)c_{n-1}^*)x^{n-1} = 0.
\end{aligned}$$

The coefficients c_n^* are determined by $c_1^* = 3$, $c_2^* = -1/4$, and

$$c_n^* = \frac{n-2}{n^2} \text{ for } n \geq 3.$$

A second solution is

$$y_2(x) = (1-x)\ln(x) + 3x - \sum_{n=2}^{\infty} \frac{1}{n(n-1)n!} x^n.$$

Chapter 5

Approximation of Solutions

5.1 Direction Fields

1. The direction field is in Figure 5.1.

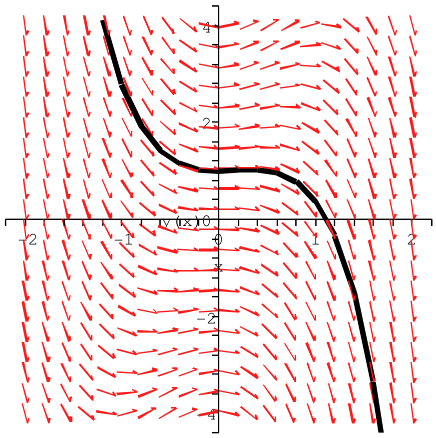

Figure 5.1: Problem 1, Section 5.1.

3. The direction field is in Figure 5.2.
5. The direction field is shown in Figure 5.3.

5.2 Euler's Method

In each of Problems 1, 3, and 5, approximate solutions were computed by Euler's method with $h = 0.05$ and $n = 10$. In these problems the exact solution can be written, allowing comparisons between the approximate and exact solution values.

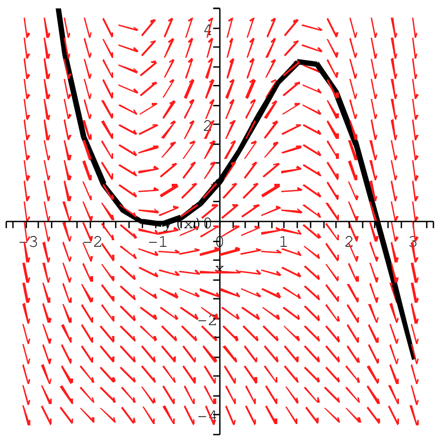

Figure 5.2: Problem 3, Section 5.1.

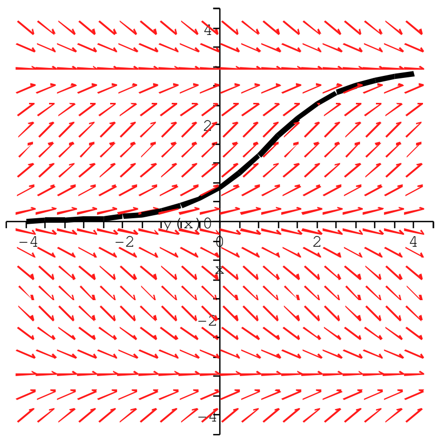

Figure 5.3: Problem 5, Section 5.1.

x_k	Euler approx. y_k	Exact $y(x_k)$
0.0	5	5
0.05	5	5.018785200
0.10	5.0375	5.075565325
0.15	5.1132605	5.171629965
0.20	5.228106406	5.309182735
0.25	5.384949598	5.491425700
0.30	5.586885208	5.722683920
0.35	5.838295042	6.008576785
0.40	6.141805532	6.356245750
0.45	6.513493864	6.774651405
0.50	6.953154700	7.274957075

Table 5.1: Problem 1, Section 5.2.

x_k	Euler approx. y_k	Exact $y(x_k)$
1	-2	-2
1.05	-2.127015115	-2.129163317
1.10	-2.258244423	-2.262726022
1.15	-2.393836450	-2.400852694
1.20	-2.534057644	-2.543722054
1.25	-2.678878414	-2.691527844
1.30	-2.828588453	-2.844479698
1.35	-2.983392817	-3.002804084
1.40	-3.143512792	-3.166745253
1.45	-3.309186789	-3.336566226
1.50	-3.480671266	-3.512549830

Table 5.2: Problem 3, Section 5.2.

1. The exact solution is
$$y(x) = 5e^{3x^2/2}.$$
See Table 5.1 for computed values.

3. The exact solution is
$$y(x) = \left[\frac{\sin(1) - \cos(1)}{2} - 2\right] e^{x-1} + \frac{1}{2}(\cos(x) - \sin(x)).$$
See Table 5.2 for computed values.

5. The exact solution is
$$y = e^{1-\cos(x)}.$$
See Table 5.3 for the approximate values.

5.3 Taylor and Modified Euler Methods

In each of Problems 1, 3, and 5, approximate solution values are computed using the Runge-Kutta method with $h = 0.2$ and $n = 10$.

x_k	Euler approx. y_k	Exact $y(x_k)$
0.0	1	1
0.05	1	1.00125021
0.1	1.002498958	1.005008335
0.15	1.007503130	1.011292203
0.20	1.015031072	1.020133420
0.25	1.025113849	1.031575844
0.30	1.037794811	1.045675942
0.35	1.053129278	1.062502832
0.40	1.071185064	1.082138316
0.45	1.092042020	1.104676904
0.50	1.115792052	1.130225803

Table 5.3: Problem 5, Section 5.2.

x_k	Runge-Kutta approximation
0.0	1
0.2	1.26465161
0.4	1.45389723
0.6	1.58483705
0.8	1.67216598
1.0	1.72743772
1.2	1.75944359
1.4	1.77479969
1.6	1.77846513
1.8	1.77414403
2.0	1.76458702

Table 5.4: Problem 1, Section 5.3.

1. Table 5.4 gives approximate values for Problem 1.
3. Table 5.5 lists approximate values for Problem 3, along with computed exact values, which can be obtained in this problem. The exact solution for this problem is
$$y(x) = (x+4)e^{-x}.$$
5. Table 5.6 lists the approximate values for Problem 5.

x_k	Runge-Kutta approx.	Exact
0.0	4	4
0.2	3.34867474	3.43866916
0.4	2.94941776	2.9494082
0.6	2.52454578	2.52453353
0.8	2.15679297	2.15677903
1.0	1.83941205	1.83939721
1.2	1.56622506	1.5662099
1.4	1.33163683	1.33162361
1.6	1.13063507	1.1306205
1.8	0.958747437	0.958733552
2.0	0.812024757	0.812011699

Table 5.5: Problem 3, Section 5.3.

x_k	Runge-Kutta approximation
0.0	2
0.2	2.162573
0.4	2.27782433
0.6	2.34197299
0.8	2.35937518
1.0	2.33748836
1.2	2.28390814
1.4	2.20518645
1.6	2.10658823
1.8	1.99221666
2.0	1.86523474

Table 5.6: Problem 5, Section 5.3.

Chapter 6

Vectors and Vector Spaces

6.1 Vectors in the Plane and 3-Space

1. $\mathbf{F}+\mathbf{G}=3\mathbf{i}-\mathbf{k}$, $\mathbf{F}-\mathbf{G}=\mathbf{i}-10\mathbf{j}+\mathbf{k}$, $2\mathbf{F}=4\mathbf{i}-10\mathbf{j}$, $3\mathbf{G}=3\mathbf{i}+15\mathbf{j}-3\mathbf{k}$, $\parallel \mathbf{F} \parallel=\sqrt{29}$
3. $\mathbf{F}+\mathbf{G}=3\mathbf{i}-\mathbf{j}+3\mathbf{k}$, $\mathbf{F}-\mathbf{G}=-\mathbf{i}-3\mathbf{j}-\mathbf{k}$, $2\mathbf{F}=2\mathbf{i}+2\mathbf{j}+2\mathbf{k}$, $3\mathbf{G}=6\mathbf{i}-6\mathbf{j}+6\mathbf{k}$, $\parallel \mathbf{F} \parallel=\sqrt{3}$
5. $\mathbf{F}+\mathbf{G}=(2+\sqrt{2})\mathbf{i}+3\mathbf{j}$, $\mathbf{F}-\mathbf{G}=(2-\sqrt{2})\mathbf{i}-9\mathbf{j}+10\mathbf{k}$, $2\mathbf{F}=4\mathbf{i}-6\mathbf{j}+10\mathbf{k}$, $3\mathbf{G}=3\sqrt{2}\mathbf{i}+18\mathbf{j}-15\mathbf{k}$, $\parallel \mathbf{F} \parallel=\sqrt{38}$

In Problems 6 through 9, the strategy is to first write a vector from the first point to the second. Divide this vector by its length to obtain a unit vector in the direction from the first to the second point. Finally, multiply this unit vector by a positive scalar α to obtain a vector of length α in the direction from the first point to the second.

7.
$$\frac{4}{9}(-4\mathbf{i}+7\mathbf{j}+4\mathbf{k})$$

In problems involving finding parametric equations of a line through given points, the strategy outlined in the text will work. However, it is worth noting that any line in $3-$ space can be described by infinitely many different parametric equations. For example, if we have equations of a line in terms of a parameter t, and replace t by, we obtain slightly different looking parametric equations of the same line, since $2t$ takes on all real values as t does.

9. A vector from the first to the second point is
$$\mathbf{F}=-5\mathbf{i}-4\mathbf{j}+2\mathbf{k}.$$
This specifies the correct direction. For a unit vector in this direction, divide by $\parallel \mathbf{F} \parallel$, or $\sqrt{45}$, to obtain the vector
$$\frac{1}{\sqrt{45}}(\mathbf{F}=-5\mathbf{i}-4\mathbf{j}+2\mathbf{k}).$$
A vector of length 9 in this direction is
$$\frac{9}{\sqrt{45}}(-5\mathbf{i}-4\mathbf{j}+2\mathbf{k})$$

11. $x = 0, y = 1 - t, z = 3 - 2t$ for $-\infty < t < \infty$.
13. $x = 2 - 3t, y = -3 + 9t, z = 6 - 2t$ for $-\infty < t < \infty$.
15. The vector
$$\mathbf{F} = -6\mathbf{i} + \mathbf{j}$$
is represented by the arrow from $(3, 0, 0)$ to $(-3, 1, 0)$. If (x, y, z) is on the line through these two points, then
$$(x - 3)\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$
must be parallel to $\mathbf{F}$, so for some t,
$$x - 3 = -6t, y = t, z = 0.$$
The line has parametric equations
$$x = 3 - 6t, y = t, z = 0 \text{ for } -\infty < t < \infty.$$
This line is in the $x, y-$ plane, since each of the given points has zero third coordinate.

6.2 The Dot Product

For Problems 1, 3, and 5, $\mathbf{F}$ is the first given vector, $\mathbf{G}$ the second, and θ is the angle between these vectors.

1. $\mathbf{F} \cdot \mathbf{G} = -23$, $\cos(\theta) = -23/\sqrt{29}\sqrt{41}$, not orthogonal
3. $\mathbf{F} \cdot \mathbf{G} = -18$, $\cos(\theta) = -9/10$, not orthogonal
5. $\mathbf{F} \cdot \mathbf{G} = 2$ and
$$\cos(\theta) = \frac{\mathbf{F} \cdot \mathbf{G}}{\parallel \mathbf{F} \parallel \parallel \mathbf{G} \parallel} = \frac{2}{\sqrt{14}}.$$
The vectors are not orthogonal.

For Problems 7, 9, and 11, if the given point is (x_0, y_0, z_0) and the normal vector is $\mathbf{N} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$, then the equation of the plane is
$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0,$$
because (x, y, z) is on the plane if and only if the vector $(x - x_0)\mathbf{i} + (y - y_0)\mathbf{j} + (z - z_0)\mathbf{k}$ is orthogonal to $\mathbf{N}$, hence their dot product is zero. It is common practice to accumulate the constant term $ax_0 + by_0 + cz_0$ on the other side of the equation to write the plane in the form
$$ax + by + cz = k.$$

7. $4x - 3y + 2z = 25$
9. $7x + 6y - 5z = -26$
11. If (x, y, z) is in the plane, then $(x + 1)\mathbf{i} + (y - 1)\mathbf{j} + (z - 2)\mathbf{k}$ is orthogonal to $3\mathbf{i} - \mathbf{j} + 4\mathbf{k}$, so
$$3(x + 1) - (y - 1) + 4(z - 2) = 0.$$
This is one equation of the plane. We can write this equation as
$$3x - y + 4z = 4.$$

For Problems 13, 14, and 15, the projection of $\mathbf{v}$ onto $\mathbf{u}$ is calculated as

$$\frac{\mathbf{u} \cdot \mathbf{v}}{\| \mathbf{u} \|^2}\mathbf{u}.$$

13.

$$\frac{1}{62}\mathbf{u}$$

15.

$$\text{proj}_{\mathbf{u}}\mathbf{v} = \frac{-9}{14}\mathbf{u}$$

6.3 The Cross Product

1. $\mathbf{F} \times \mathbf{G} = -8\mathbf{i} - 12\mathbf{j} - 5\mathbf{k}$

3.

$$\mathbf{F} \times \mathbf{G} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3 & 6 & 1 \\ -1 & -2 & 1 \end{vmatrix} = 8\mathbf{i} + 2\mathbf{j} + 12\mathbf{k}.$$

$$\mathbf{G} \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & -2 & 1 \\ -3 & 6 & 1 \end{vmatrix} = -8\mathbf{i} - 2\mathbf{j} - 12\mathbf{k} = -\mathbf{F} \times \mathbf{G}.$$

In Problems 5, 7, and 9, the three given points are used to find two vectors in the plane that is wanted. Their cross product produces a normal vector to this plane, and then, knowing a point on the plane and a normal vector, we can find an equation of the plane, as in Section 6.1. This procedure results in $\mathbf{N} = \mathbf{O}$ exactly when the three points are collinear and do not define a unique plane, hence no nonzero normal vector. The details of this procedure are included only for Problem 7.

5. The points are not collinear and the plane containing them has equation $29x + 37y - 12z = 30$.

7. Form vectors $\mathbf{F} = 4\mathbf{i} - \mathbf{j} - 6\mathbf{k}$ and $\mathbf{G} = \mathbf{i} - \mathbf{k}$. Take the cross product to form a normal vector:

$$\mathbf{N} = \mathbf{F} \times \mathbf{G} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & -1 & -6 \\ 1 & 0 & -1 \end{vmatrix} = \mathbf{i} - 2\mathbf{j} + \mathbf{k}.$$

Of course, other normals could be used. The fact that $\mathbf{N} \neq \mathbf{O}$ means that the points are not collinear. The plane containing these points has equation

$$x + 1 - 2(y - 1) + z - 6 = 0$$

or

$$x - 2y + z = 3.$$

9. The points are not collinear and the plane containing them has equation $2x - 11y + z = 0$.

For Problem 11, recall that the vector $a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ is normal to the plane $ax + by + cz = d$. Any nonzero scalar multiple of this normal vector is also a normal vector.

11. $\mathbf{N} = \mathbf{i} - \mathbf{j} + 2\mathbf{k}$

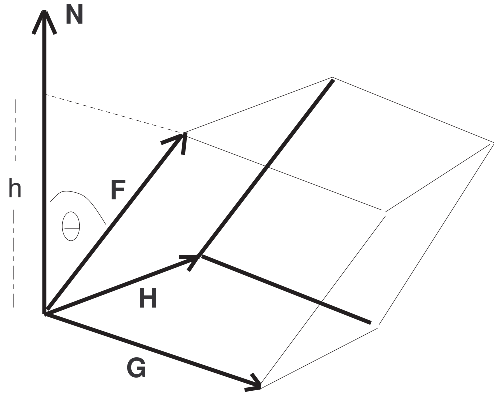

Figure 6.1: Parallelopiped in Problem 14, Section 6.3.

13. The area of a parallelogram in which two incident sides have an angle of θ between them is the product of the lengths of the sides times the cosine of θ. If the sides are along the vectors $\mathbf{F}$ and $\mathbf{G}$, drawn from a common point, then this area is

$$\| \mathbf{F} \| \| \mathbf{G} \| \cos(\theta),$$

and this is exactly $\| \mathbf{F} \times \mathbf{G} \|$.

14. The vector $\mathbf{N} = \mathbf{G} \times \mathbf{H}$ is normal to the base of the parallelopiped having sides along the vectors $\mathbf{F}$ and $\mathbf{G}$ (Figure 6.1). We know (Problem 13) that the area of this base is $\| \mathbf{N} \|$. Now

$$(\mathbf{G} \times \mathbf{H}) \cdot \mathbf{F} = \mathbf{N} \cdot \mathbf{F} = \| \mathbf{N} \| \| \mathbf{F} \| \cos(\theta)$$

is in magnitude the volume of the box having incident edges $\mathbf{F}$, $\mathbf{G}$, $\mathbf{H}$ as incident sides, because

$$\| \mathbf{F} \| \cos(\theta) = \pm \text{ altitude of the box.}$$

This altitude is denoted h in Figure 6.1.

6.4 The Vector Space R^n

In Problems 1–10, the standard approach is to determine if there is a linear combination of the given vectors, with at least one nonzero coefficient, that equals the zero vector. In such a case the vectors are linearly dependent. If the only linear combination of the vectors that equals the zero vector has all zero coefficients, then the given vectors are linearly dependent.

Equivalently, we can show that a set of vectors is linearly dependent if any one of the vectors is a linear combination of the others.

1. The vectors are linearly independent.
3. The vectors are linearly dependent because

$$2 < 1, 2, -3, 1 > + < 4, 0, 0, 2 > - < 6, 4, -6, 4 > = < 0, 0, 0, 0 > .$$

5. The vectors are linearly independent.
7. If $\alpha(3\mathbf{i}+2\mathbf{j})+\beta(\mathbf{i}-\mathbf{j})=\mathbf{0}$, then $3\alpha+\beta=0$ and $2\alpha-\beta=0$. Then $\alpha=\beta=0$, so the given vectors are linearly independent.
9. The vectors are linearly dependent, since

$$2<1,-2> -2 <4,1> + <6,6> = <0,0> .$$

In each of Problems 11, 13, and 15, it is routine to check that S is not empty and that a linear combination of vectors in S is again in S. Thus S is a subspace of R^n for the appropriate n. We then produce a basis for the subspace.

11. Every vector in S is a scalar multiple of $<0,1,0,2,0,3,0>$, so S has dimension 1.
13. By choosing $x=1, y=0$, then $x=0, y=1$ we obtain linearly independent vectors $<1,0,0,-1>$ and $<0,1,-1,0>$ that span S. These vectors form a basis for S.
15. The vectors $<1,0,0,0>$, $<0,0,1,0>$ and $<0,0,0,1>$ form a basis for S, which has dimension 3.
17. Write

$$<-3,-2,5,1,-4> = a<1,1,1,1,0> + b<-1,1,0,0,0>$$
$$+c<1,1,-1,-1,0> + d<0,0,2,-2,0> + e<0,0,0,0,2> .$$

Then

$$\begin{aligned} a-b+c&=-3,\\ a+b+c&=-2,\\ a-c+2d&=5,\\ a-c-2d&=1,\\ 2e&=4. \end{aligned}$$

Solve these linear equations to obtain $a=1/4$, $b=1/2$, $c=-11/4$, $d=1$, $e=2$. The coordinates of $\mathbf{X}$ with respect to the given vectors in S are

$$(1/4, 1/2, -11/4, 1, 2).$$

19. Since $\mathbf{V}_1,\cdots,\mathbf{V}_k$ span S, there are numbers $c_1,\cdots,c_k$ such that

$$\mathbf{U}=c_1\mathbf{V}_1+\cdots+c_k\mathbf{V}_k.$$

But then $\mathbf{U},\mathbf{V}_1,\cdots,\mathbf{V}_k$ are linearly dependent, because one of these vectors is a linear combination of the others.

21. Let $\mathbf{V}_1,\cdots,\mathbf{V}_m$ be a spanning set for R^n. If these vectors are linearly independent, then they form a basis. Thus consider the case that the vectors are linearly dependent. In this case one of the vectors is a linear combination of the others, say (by renumbering if needed)

$$\mathbf{V}_m=c_1\mathbf{V}_1+\cdots+c_{m-1}\mathbf{V}_{m-1}.$$

Then $\mathbf{V}_1,\cdots,\mathbf{V}_{m-1}$ span R^n. If these vectors are linearly independent, they form a basis. If not, one of the vectors is a linear combination of the others, say

$$\mathbf{V}_{m-1}=k_1\mathbf{V}_1+\cdots+k_{m-1}\mathbf{V}_{m-2}.$$

But then $\mathbf{V}_1, \cdots, \mathbf{V}_{m-2}$ span R^n. Now repeat this argument. If $\mathbf{V}_1, \cdots, \mathbf{V}_{m-2}$ are linearly independent, they form a basis. If not, one of these vectors is a linear combination of the others, and we can drop this vector from the list of $m-2$ vectors to form a spanning set with $m-3$ vectors. Eventually we reach a spanning set from which no vector can be left out and still span the same subspace. At this point we have removed $m-n$ vectors, and the remaining n vectors form a basis for R^n.

23. First,

$$\begin{aligned}(\mathbf{X}-\mathbf{Y})\cdot(\mathbf{X}+\mathbf{Y}) &= \mathbf{X}\cdot\mathbf{X}+\mathbf{X}\cdot\mathbf{Y}-\mathbf{Y}\cdot\mathbf{X}-\mathbf{Y}\cdot\mathbf{Y}\\ &=\parallel \mathbf{X}\parallel^2-\parallel \mathbf{Y}\parallel^2=0,\end{aligned}$$

so $\mathbf{X}-\mathbf{Y}$ is orthogonal to $\mathbf{X}+\mathbf{Y}$.

24. **Hint** Let

$$\mathbf{Y}=\mathbf{X}-\sum_{j=1}^{k}(\mathbf{X}\cdot\mathbf{V}_j)\mathbf{V}_j.$$

Show that

$$\begin{aligned}0 &\le \parallel \mathbf{Y}\parallel^2=\mathbf{Y}\cdot\mathbf{Y}\\ &=\left(\mathbf{X}-\sum_{j=1}^{k}(\mathbf{X}\cdot\mathbf{V}_j)\mathbf{V}_j\right)\cdot\left(\mathbf{X}-\sum_{j=1}^{k}(\mathbf{X}\cdot\mathbf{V}_j)\mathbf{V}_j\right)\\ &=\mathbf{X}\cdot\mathbf{X}-2\mathbf{X}\cdot\sum_{j=1}^{k}(\mathbf{X}\cdot\mathbf{V}_j)\mathbf{V}_j\\ &+\sum_{j=1}^{k}\sum_{r=1}^{k}(\mathbf{X}\cdot\mathbf{V}_j)(\mathbf{X}\cdot\mathbf{V}_r)\mathbf{V}_j\cdot\mathbf{V}_r.\end{aligned}$$

Now apply the fact that the vectors $\mathbf{V}_1, \cdots \mathbf{V}_k$ are orthonormal.

25. If $\mathbf{V}_1, \cdots, \mathbf{V}_n$ is an orthonormal basis for R^n, and $\mathbf{X}$ is in R^n, then

$$\mathbf{X}=\sum_{j=1}^{n}(\mathbf{X}\cdot\mathbf{V}_j)\mathbf{V}_j.$$

Now reason as in the solution to Problem 24, using the fact

$$\mathbf{V}_j\cdot\mathbf{V}_k=\begin{cases}0 & \text{if } j\neq k,\\ 1 & \text{if } j=k.\end{cases}$$

We have

$$\begin{aligned}\parallel \mathbf{X}\parallel^2 &= \mathbf{X}\cdot\mathbf{X}\\ &=\left(\sum_{j=1}^{n}(\mathbf{X}\cdot\mathbf{V}_j)\mathbf{V}_j\right)\cdot\left(\sum_{j=1}^{n}(\mathbf{X}\cdot\mathbf{V}_j)\mathbf{V}_j\right)\\ &=\sum_{j=1}^{n}\sum_{r=1}^{n}(\mathbf{X}\cdot\mathbf{V}_j)(\mathbf{X}\cdot\mathbf{V}_r)\mathbf{V}_j\cdot\mathbf{V}_r\\ &=\sum_{j=1}^{n}(\mathbf{X}\cdot\mathbf{V}_j)^2.\end{aligned}$$

6.5 Orthogonalization

The arithmetic involved in carrying out the Gram-Schmidt process can be tedious and computations are most easily carried out using a software package such as MAPLE.

In each problem, the given vectors are denoted $\mathbf{X}_1, \cdots, \mathbf{X}_k$ in the given order.

1. $\mathbf{V}_1 = \mathbf{X}_1$,

$$\begin{aligned}\mathbf{V}_2 &= \mathbf{X}_2 + \frac{3}{2}\mathbf{X}_1 \\ &= \frac{1}{2} < 0, 0, -3, 3, 0, 0 > .\end{aligned}$$

3. $\mathbf{V}_1 = \mathbf{X}_1$,

$$\begin{aligned}\mathbf{V}_2 &= \mathbf{X}_2 - \frac{5}{7}\mathbf{X}_1 \\ &= \frac{1}{9} < 0, 0, -1, -19, 40 >,\end{aligned}$$

$$\begin{aligned}\mathbf{V}_3 &= \mathbf{X}_3 + \frac{2}{9}\mathbf{V}_1 + \frac{17}{9}\mathbf{V}_2 \\ &= \frac{1}{218} < 0, 218, -341, 279, 62 >,\end{aligned}$$

$$\begin{aligned}\mathbf{V}_4 &= \frac{6}{9}\mathbf{X}_1 + \frac{13}{3}\mathbf{V}_2 - \frac{435}{1179}\mathbf{V}_3 \\ &= \frac{1}{393} < 0, 248, 88, -24, -32 > .\end{aligned}$$

5. Let $\mathbf{V}_1 = \mathbf{X}_1$, then

$$\mathbf{V}_2 = \mathbf{X}_1 - \frac{-7}{6}\mathbf{X}_1 =< 0, 4/3, 13/6, 29/6 > .$$

Finally,

$$\begin{aligned}\mathbf{V}_3 &= \mathbf{X}_3 - \frac{3}{6}\mathbf{V}_1 - \frac{43/2}{179/6}\mathbf{V}_2 \\ &= \mathbf{X}_3 - \frac{1}{2}\mathbf{V}_1 - \frac{129}{179}\mathbf{V}_2 \\ &= \frac{1}{179} < 0, 7, -11, 3 > .\end{aligned}$$

7. Let $\mathbf{V}_1 = \mathbf{X}_1$ and then let

$$\begin{aligned}\mathbf{V}_2 &= \mathbf{X}_2 - \frac{\mathbf{X}_2 \cdot \mathbf{X}_1}{\mathbf{X}_1 \cdot \mathbf{X}_1}\mathbf{X}_1 \\ &= \mathbf{X}_2 + \frac{18}{17}\mathbf{X}_1 \\ &=< 52/17, -13/17, 0 > .\end{aligned}$$

6.6 Orthogonal Complements and Projections

1.
$$\mathbf{u}_S = \frac{7}{2}\mathbf{V}_1 + \mathbf{V}_2 - 3\mathbf{v}_3 =< 9/2, -1/2, 0, 5/2, -13/2 >,$$
$$\mathbf{u}^{\perp} =< -1/2, -1/2, 3, -1/2, -1/2 > .$$

2.
$$\mathbf{u}_S = -3\mathbf{V}_1 + \frac{31}{39}\mathbf{V}_2 =< -86/39, 148/39, 62/13, 31/39 >,$$
$$\mathbf{u}^{\perp} =< 203/309, 203/309, -10/13, -226/39 > .$$

5. Let $\mathbf{V}_1 =< 1, -1, 0, 0 >$ and $\mathbf{V}_2 =< 1, 1, 0, 0 >$. These form an orthogonal basis for S. Let
$$\begin{aligned}\mathbf{u}_S &= \frac{\mathbf{u}\cdot\mathbf{V}_1}{\mathbf{V}_1\cdot\mathbf{V}_1}\mathbf{V}_1 + \frac{\mathbf{u}\cdot\mathbf{V}_2}{\mathbf{V}_2\cdot\mathbf{V}_2}\mathbf{V}_2\\ &= -4\mathbf{V}_1 + 2\mathbf{V}_2 =< -2, 6, 0, 0 >\end{aligned}$$
and
$$\mathbf{u}^{\perp} = \mathbf{u} - \mathbf{u}_S =< 0, 0, 1, 7 > .$$
Then $\mathbf{u}_S$ is in S and $\mathbf{u}^{\perp}$ is in $S^{\perp}$, and $\mathbf{u} = \mathbf{u}_S + \mathbf{u}^{\perp}$.

7. The idea is to use an orthogonal basis for S to produce $\mathbf{u}_S$, which is the vector we want. Let
$$\mathbf{V}_1 =< 2, 1, -1, 0, 0 >, \mathbf{V}_2 =< -1, 2, 0, 1, 0 > \text{ and } \mathbf{V}_3 =< 0, 1, 1, -2, 0 > .$$
These form an orthogonal basis for S. With $\mathbf{u} =< 4, 3, -3, 4, 7 >$, compute
$$\begin{aligned}\mathbf{u}_S &= \frac{7}{3}\mathbf{V}_1 + \mathbf{V}_2 - \frac{4}{3}\mathbf{V}_3\\ &=< 11/3, 3, -11/3, 11/3, 0 > .\end{aligned}$$

9. Let $\mathbf{v}_1, \cdots, \mathbf{v}_k$ be an orthogonal basis for S, and $\mathbf{u}_1, \cdots, \mathbf{u}_r$ an orthogonal basis for $S^{\perp}$. If $\mathbf{u}$ is any vector in R^n, then $\mathbf{u}$ has a unique representation as a sum of a vector in S and a vector in $S^{\perp}$, $\mathbf{u} = \mathbf{u}_S + \mathbf{u}^{\perp}$. Therefore every vector in R^n is a linear combination of the vectors
$$\mathbf{v}_1, \cdots, \mathbf{v}_k, \mathbf{u}_1, \cdots, \mathbf{u}_r.$$
Further, each $\mathbf{u}_i$ is orthogonal to each $\mathbf{v}_j$, because every vector in $S^{\perp}$ is orthogonal to each vector in S. Now $\mathbf{u}_S$ is a linear combination of $\mathbf{v}_1, \cdots, \mathbf{v}_k$ and $\mathbf{u}^{\perp}$ is a linear combination of $\mathbf{u}_1, \cdots, \mathbf{u}_r$, so $\mathbf{v}_1, \cdots, \mathbf{u}_r$ span R^n. Further,
$$\mathbf{v}_1, \cdots, \mathbf{v}_k, \mathbf{u}_1, \cdots, \mathbf{u}_r$$
are orthogonal, hence linearly independent. The vectors
$$\mathbf{v}_1, \cdots, \mathbf{v}_k, \mathbf{u}_1, \cdots, \mathbf{u}_r$$
form a basis for R^n. But then $k + r = n$. We conclude that
$$\text{dimension}(S) + \text{ dimension}(S^{\perp}) = \text{ dimension}R^n.$$

6.7 The Function Space $C[a, b]$

For Problems 1 and 3 we use the Gram-Schmidt orthogonalization process, except the setting is now a function space. The only difference this makes in applying the Gram-Schmidt process is that the vectors are now functions and the dot products are defined by integrals of the form

$$f \cdot g = \int_a^b p(x) f(x) g(x)\, dx,$$

in which the weight function $p(x)$ must be specified.

Problems 5 and 7 involve finding a function "closest" to a given set of functions in the same sense that a vector $\mathbf{u}_S$ is closest to a subspace spanned by a given set of vectors. Again, the only difference is that now the vectors are functions and the dot products are integrals. Thus in these problems we must determine an orthogonal projection f_S, given $f(x)$ and a spanning set for the subspace S of $C[a, b]$.

1. Let $X_1(x) = 1$, $X_2(x) = x$ and $X_3(x) = x^2$. Choose $V_1(x) = 1$,

$$V_2(x) = x - \frac{x \cdot 1}{1 \cdot 1}(1) = x - \frac{2}{3}$$

and

$$\begin{aligned} V_3(x) &= x^2 - \frac{x^2 \cdot x}{x \cdot x} x - \frac{x^2 \cdot 1}{1 \cdot 1}(1) \\ &= x^2 - \frac{6}{5}\left(x - \frac{2}{3}\right) - \frac{1}{2}. \end{aligned}$$

3. Denote $X_1(x) = e^x$ and $X_2(x) = e^{-x}$. These span a subset of $C[0, 1]$ consisting of all functions of the form $ae^x + be^{-x}$. However, these functions are not orthogonal, since

$$X_1 \cdot X_2 = \int_0^1 X_1(x) X_2(x)\, dx = \int_0^1 dx = 1 \neq 0.$$

For an orthogonal basis, first choose

$$V_1(x) = X_1(x) = e^x.$$

Next choose

$$\begin{aligned} V_2(x) &= X_2(x) - \frac{X_2 \cdot X_1}{X_1 \cdot X_1} X_1 \\ &= e^{-x} - \frac{\int_0^1 1\, dx}{\int_0^1 e^{2x}\, dx} e^x \\ &= e^{-x} - \frac{2}{e^2 - 1} e^x. \end{aligned}$$

It is routine to check that indeed V_1 and V_2 are orthogonal, since

$$V_1 \cdot V_2 = \int_0^1 V_1(x) V_2(x)\, dx = 0.$$

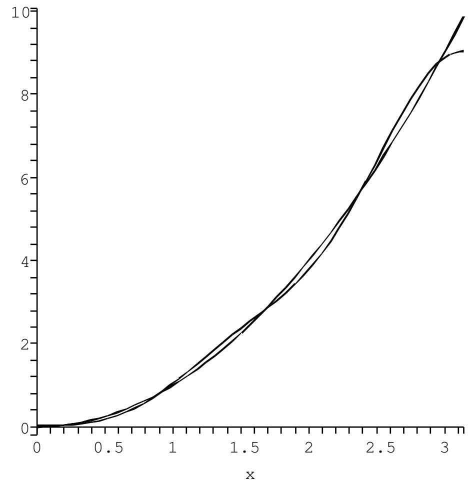

Figure 6.2: $f(x)$ and $f_S(x)$ in Problem 5, Section 6.7.

5. Here we are computing the orthogonal projection of $f(x) = x^2$ onto the subspace of $C[0,\pi]$ spanned by $1, \cos(x)$, $\cos(2x)$, $\cos(3x)$ and $\cos(4x)$. It is routine to verify that the given functions form an orthogonal basis for S with respect to the given dot product. This orthogonal projection is

$$f_S(x) = \sum_{k=0}^{n} c_k X_k(x),$$

where $X_k(x) = \cos(kx)$ for $k = 0, 1, 2, 3, 4$, where

$$c_k = \frac{\int_0^\pi x^2 X_k(x)\,dx}{\int_0^\pi X_k^2(x)\,dx}.$$

Routine integrations yield

$$c_0 = \frac{\pi^2}{3} \text{ and } c_k = \frac{4(-1)^k}{k^2}$$

for $k = 1, 2, 3, 4$. Then

$$f_S(s) = \frac{\pi^2}{3} - 4\cos(x) + \cos(2x) - \frac{1}{2}\cos(3x) + \frac{1}{4}\cos(4x).$$

Figure 6.2 compares a graph of $f(x)$ and $f_S(x)$. It happens that these graphs are fairly close, but in applications $f(x)$ is probably not approximated closely enough by $f_S(x)$ for reliable calculations. The point, however, is that $f_S(x)$ is the function in $C[0,\pi]$ nearest to the subspace S spanned by the five given functions, in the sense of distance in this function space. If we wanted a better numerical approximation (graphs closer together), we could change S and include more functions $\cos(kx)$. This is the idea of a Fourier cosine expansion, treated in Chapter Fourteen.

7. We want the function f_S in S that is closest (in the distance defined on this function space) to $f(x) = x(2-x)$, where S is the subspace spanned by the orthogonal functions 1, $\cos(k\pi x/2$

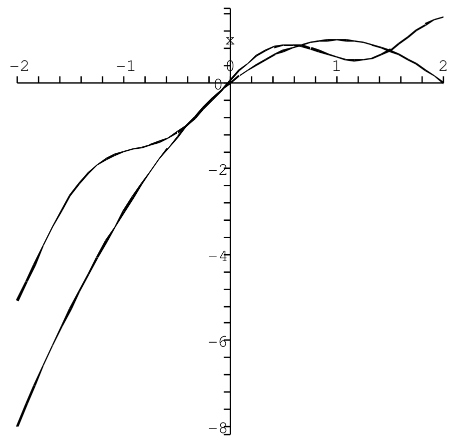

Figure 6.3: f and f_S in Problem 7, Section 6.7.

and $\sin(k\pi x/2$ for $k = 1, 2, 3$. This orthogonal projection has the form

$$f_S(x) = c_0 + \sum_{k=1}^{3}(c_k \cos(k\pi x/2) + d_k \sin(k\pi x/2)).$$

Routine integrations yield $c_0 = -4/3$ and, for $k = 1, 2, 3$,

$$c_k = \frac{16(-1)^{k+1}}{\pi^2 k^2} \text{ and } d_k = \frac{8(-1)^{k+1}}{\pi k}.$$

Figure 6.3 shows graphs of f and f_S.

Chapter 7

Matrices and Linear Systems

7.1 Matrices

1.
$$4\mathbf{A}+8\mathbf{B}=\begin{pmatrix}-36 & 0 & 68 & 196 & 20\\ 128 & -40 & -36 & -8 & 72\end{pmatrix}$$

3.
$$2\mathbf{A}-3\mathbf{B}=\begin{pmatrix}14 & -2 & 6\\ 10 & -5 & -6\\ -26 & -43 & -8\end{pmatrix}$$

5.
$$\mathbf{A}^2+2\mathbf{AB}=\begin{pmatrix}2+2x-x^2 & -12x+(1-x)(x+e^x+2\cos(x))\\ 4+2x+2e^x+2xe^x & -22-2x+e^{2x}+2e^x\cos(x)\end{pmatrix}$$

7.
$$\mathbf{AB}=(115);\mathbf{BA}=\begin{pmatrix}3 & -18 & -6 & -42 & 66\\ -2 & 12 & 4 & 28 & -44\\ -6 & 36 & 12 & 84 & -132\\ 0 & 0 & 0 & 0 & 0\\ 4 & -24 & -8 & -56 & 88\end{pmatrix}$$

9.
$$\mathbf{AB}=\begin{pmatrix}-10 & -34 & -16 & -30 & -14\\ 10 & -2 & -11 & -8 & -45\\ -5 & 1 & 15 & 61 & -63\end{pmatrix};\mathbf{BA}\text{ is not defined.}$$

11. $\mathbf{BA}$ is not defined,
$$\mathbf{AB}=\begin{pmatrix}39 & -84 & 21\\ -23 & 38 & 3\end{pmatrix}$$

13.
$$\mathbf{AB}\text{ is not defined};\mathbf{BA}=\begin{pmatrix}410 & 36 & -56 & 227\\ 17 & 253 & 40 & -1\end{pmatrix}$$

15. $\mathbf{AB}$ is not defined and
$$\mathbf{BA}=\begin{pmatrix}-16 & -13 & -5\end{pmatrix}$$

18. $\mathbf{AB}$ is 1×3, $\mathbf{BA}$ is not defined.

19. $\mathbf{AB}$ is not defined, $\mathbf{BA}$ is 7×6.

21. $\mathbf{AB}$ is 14×14, $\mathbf{BA}$ is 21×21.

22. (a) **Hint** The i, i element of $\mathbf{A}^2$ is the number of $v_i - v_i$ walks of length 2 in the graph, and each such walk has the form $v_i - v_j - v_i$, for some $j \neq i$.

    (b) The i, i element of $\mathbf{A}^3$ is the number of walks $v_i - v_i$ walks of length 3 in G. Any such walk has the form $v_i - v_j - v_k - v_i$, for some $j \neq k$, and neither j nor k equal to i.

23. For the given graph G the adjacency matrix is

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \end{pmatrix}.$$

Compute

$$\mathbf{A}^3 = \begin{pmatrix} 2 & 7 & 7 & 4 & 4 \\ 7 & 8 & 9 & 9 & 9 \\ 7 & 9 & 8 & 9 & 9 \\ 4 & 9 & 9 & 6 & 7 \\ 4 & 9 & 9 & 7 & 6 \end{pmatrix} \text{ and } \mathbf{A}^4 = \begin{pmatrix} 14 & 17 & 17 & 18 & 18 \\ 17 & 34 & 33 & 26 & 26 \\ 17 & 33 & 34 & 26 & 26 \\ 18 & 26 & 26 & 25 & 24 \\ 18 & 26 & 26 & 24 & 25 \end{pmatrix}.$$

The number of $v_1 - v_4$ walks of length 3 is $(A^3)_{14} = 4$ and the number of $v_1 - v_4$ walks of length 4 is $(A^4)_{14} = 18$. The number of $v_2 - v_3$ walks of length 3 is 9, and the number of $v_2 - v_4$ walks of length 4 is 26.

25. The adjacency matrix is

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \end{pmatrix}.$$

Then

$$\mathbf{A}^2 = \begin{pmatrix} 4 & 2 & 3 & 3 & 2 \\ 2 & 3 & 2 & 2 & 3 \\ 3 & 2 & 4 & 3 & 2 \\ 3 & 2 & 3 & 4 & 2 \\ 2 & 3 & 2 & 2 & 3 \end{pmatrix}, \mathbf{A}^3 = \begin{pmatrix} 10 & 10 & 11 & 11 & 10 \\ 10 & 6 & 10 & 10 & 6 \\ 11 & 10 & 10 & 11 & 10 \\ 11 & 10 & 11 & 10 & 10 \\ 10 & 6 & 10 & 10 & 6 \end{pmatrix},$$

and

$$\mathbf{A}^4 = \begin{pmatrix} 42 & 32 & 41 & 41 & 32 \\ 32 & 30 & 32 & 32 & 30 \\ 41 & 32 & 42 & 41 & 32 \\ 41 & 32 & 41 & 42 & 32 \\ 32 & 30 & 32 & 32 & 30 \end{pmatrix}.$$

The number of $v_4 - v_5$ walks of length 2 is 2, the number of $v_2 - v_3$ walks of length 3 is 10, the number of $v_1 - v_2$ walks of length 4 is 32, and the number of $v_4 - v_5$ walks of length 4 is 32.

27. Let $\mathcal{M}$ be the set of all real $n \times m$ matrices.

    First, each $n \times m$ matrix has nm elements in its n rows and m columns. If we string out the rows of an $n \times m$ real matrix $\mathbf{A}$ into one long row (row 2 following row 1, then row 3, and so on), we form an $nm-$ vector. In this way, we form a one-to-one correspondence matrices in $\mathcal{M}$ and vectors in R^{nm}.

Notice that we add two matrices by adding corresponding components, so the nm vector formed from $\mathbf{A}+\mathbf{B}$ is the sum of the nm vectors formed from $\mathbf{A}$ and $\mathbf{B}$. Further, if we multiply $\mathbf{A}$ by a real number c, the rows of $c\mathbf{A}$, when strung out in this way, form the components of c times the nm vector formed from the rows of $\mathbf{A}$. Thus we can identify the set of all real $n \times m$ matrices with R^{nm}, with this identification preserving the operations of addition of matrices (vectors) and multiplication by scalars. The dimension of this vector space of matrices is therefore the same as the dimension of R^{nm}, namely nm.

As an example of this correspondence, the 2 real matrix

$$\begin{pmatrix} 3 & 2 & -4 \\ 6 & 1 & 8 \end{pmatrix}$$

corresponds to the $6-$ vector $< 3, 2, -4, 6, 1, 8 >$.

We can also see this dimension by explicitly constructing a basis for $\mathcal{M}$. Let $\mathbf{K}_{ij}$ be the matrix having a 1 in the i, j entry, and zeros everywhere else. These nm matrices correspond to the nm unit vectors in R^{nm} having one component 1 and all other components zero. The matrices $\mathbf{K}_{ij}$ form a basis for $\mathcal{M}$.

7.2 Elementary Row Operations

In each of Problems 1 - 8, if a single row operations is applied to $\mathbf{A}$, then the resulting matrix is $\mathbf{\Omega A}$, where $\mathbf{\Omega}$ is the elementary matrix formed by performing the operation on $\mathbf{I}_n$. If a sequence of k elementary row operations is performed, then $\mathbf{\Omega} = \mathbf{E}_k \cdots \mathbf{E}_1$, where $\mathbf{E}_1$ is the elementary matrix performing the first operation, and so on.

1.
$$\mathbf{\Omega} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 14 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 4 \\ 14 & 1 & 0 \end{pmatrix}$$

and

$$\mathbf{\Omega A} = \begin{pmatrix} -1 & 0 & 3 & 0 \\ -36 & 28 & -20 & 28 \\ -13 & 3 & 44 & 9 \end{pmatrix}$$

It is sometimes useful to use the delta notation, defined by

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

For example, $\mathbf{I}_n$ is the $n \times n$ matrix whose $i, j-$ element is δ_{ij}.

3. $\mathbf{A}$ is 3×4. To multiply row two of $\mathbf{A}$ by $\sqrt{3}$, multiply $\mathbf{a}$ on the left by the 3×3 matrix $\mathbf{\Omega}$ formed from $\mathbf{I}_3$ by multiplying row two of this matrix by $\sqrt{3}$. Thus form

$$\mathbf{\Omega} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sqrt{3} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

As a check, observe that

$$\mathbf{\Omega A} = \begin{pmatrix} -2 & 1 & 4 & 2 \\ 0 & \sqrt{3} & 16\sqrt{3} & 3\sqrt{3} \\ 1 & -2 & 4 & 8 \end{pmatrix}$$

5.

$$\mathbf{\Omega} = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & \sqrt{13} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 5 & 0 \\ 1 & 0 & \sqrt{13} \\ 0 & 0 & 1 \end{pmatrix}$$

and

$$\mathbf{\Omega A} = \begin{pmatrix} 40 & 5 & -15 \\ -2+2\sqrt{13} & 14+9\sqrt{13} & 6+5\sqrt{13} \\ 2 & 9 & 5 \end{pmatrix}$$

6. Because $\mathbf{A}$ is 4×2, perform this row operation by adding 6 times row two to row three of $\mathbf{I}_4$ to obtaim

$$\mathbf{\Omega} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 6 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Then

$$\mathbf{\Omega A} = \begin{pmatrix} 3 & -6 \\ 1 & 1 \\ 14 & 4 \\ 0 & 5 \end{pmatrix},$$

and this is the matrix obtained by performing the given row operation on $\mathbf{A}$.

7.

$$\mathbf{\Omega} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 15 \end{pmatrix} \begin{pmatrix} 1 & \sqrt{3} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 15 \\ 1 & \sqrt{3} \end{pmatrix}$$

and

$$\mathbf{\Omega A} = \begin{pmatrix} 30 & 120 \\ -3+2\sqrt{3} & 15+8\sqrt{3} \end{pmatrix}$$

9. Let $\mathbf{A}$ be $n \times m$. Now $\mathbf{B}$ and $\mathbf{E}$ are obtained, respectively, from $\mathbf{A}$ and $\mathbf{I}_n$ by adding α times row s to row t. Then, for $i \neq t$, $b_{ij} = a_{ij}$ and $e_{ij} = \delta_{ij}$, while for $i = t$, $b_{tj} = a_{tj} + \alpha a_{sj}$ and $e_{tj} = \delta_{tj} + \alpha\delta_{sj}$.

Now consider the $i, j-$ element of $\mathbf{EA}$. For $i \neq t$,

$$(\mathbf{EA})_{ij} = \sum_{k=1}^{n} e_{ik}a_{kj} = \sum_{k=1}^{n} \delta_{ik}a_{kj} = a_{ij}$$

while, for $i = t$,

$$\begin{aligned}(\mathbf{EA})_{tj} &= \sum_{k=1}^{n} e_{tk}a_{kj} = \sum_{k=1}^{n} (\delta_{tk} + \alpha\delta_{sj})a_{kj} \\ &= a_{tj} + \alpha a_{sj} = b_{sj}.\end{aligned}$$

Therefore $\mathbf{EA} = \mathbf{B}$.

11. Let $\mathbf{A} = [a_{ij}]$ be $n \times m$. Since $\mathbf{B}$ and $\mathbf{E}$ are obtained, respectively, by interchanging rows s and t of $\mathbf{A}$ and $\mathbf{I}_n$ then, for $i \neq s$ and $i \neq t$, $b_{ij} = a_{ij}$ and $e_{ij} = \delta_{ij}$. For $i = s$, $b_{sj} = a_{tj}$ and $e_{sj} = \delta_{tj}$. And for $i = t$, $b_{ij} = a_{sj}$ and $e_{ij} = \delta_{sj}$.

    Now consider the $i, j-$ element of $\mathbf{EA}$. For $i \neq s$ and $i \neq t$,

$$(\mathbf{EA})_{ij} = \sum_{k=1}^{n} e_{ik} a_{kj} = a_{ij} = b_{ij}.$$

    For $i = s$,

$$(\mathbf{EA})_{sj} = \sum_{k=1}^{n} e_{sk} a_{kj} = \sum_{k=1}^{n} \delta_{tk} a_{kj} = a_{tj} = b_{sj}.$$

    And for $i = t$,

$$(\mathbf{EA})_{tj} = \sum_{k=1}^{n} e_{ik} a_{kj} = \sum_{k=1}^{n} \delta_{sk} a_{kj} = a_{sj} = b_{tj}$$

    for $j = 1, 2, \cdots, m$. Therefore $\mathbf{EA} = \mathbf{B}$.

7.3 Reduced Row Echelon Form

For Problems 9, 11, and 12, a sequence of row operations that reduces the matrix is given, along with $\mathbf{\Omega}$ that reduces $\mathbf{A}$ by multiplication on the left. $\mathbf{\Omega}$ is formed by applying the reducing sequence in order, beginning with $\mathbf{I}_n$. For Problems 1-8 and 10, only $\mathbf{\Omega}$ and the reduced matrix $\mathbf{A}_R$ are given.

It should be kept in mind that many different sequences of operations can be used to reduced a matrix. However, the final reduced matrix $\mathbf{A}_R$ will be the same regardless of the sequence used.

1.
$$\mathbf{\Omega} = \frac{1}{270}\begin{pmatrix} -8 & -2 & 38 \\ 37 & 43 & -7 \\ 19 & -29 & 11 \end{pmatrix}, \mathbf{A}_R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \mathbf{I}_3$$

3.
$$\mathbf{\Omega} = \begin{pmatrix} 0 & 1 \\ 1/2 & 1/2 \end{pmatrix}, \mathbf{A}_R = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 3/2 & 1/2 \end{pmatrix}$$

5.
$$\mathbf{\Omega} = \begin{pmatrix} 0 & 1/2 & -1 \\ 0 & 0 & 1 \\ -1/7 & 2/7 & -3/7 \end{pmatrix}, \mathbf{A}_R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \mathbf{I}_3$$

7.
$$\mathbf{\Omega} = \begin{pmatrix} 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & -6 & 17 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \mathbf{A}_R = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

9. We can reduce $\mathbf{A}$ by the following sequence of operations, starting with $\mathbf{I}_4$: interchange rows one and two, then (on the resulting matrix), multiply row one by -1, then add row two to row one. Thus form

$$\mathbf{I}_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$\to \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} = \mathbf{\Omega}.$$

Then

$$\mathbf{A}_R = \begin{pmatrix} -1 & -4 & -1 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

11. $\mathbf{A}$ is reduced simply by adding row two to row one. Thus

$$\mathbf{\Omega} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \mathbf{A}_R = \begin{pmatrix} 1 & 0 & 5 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

7.4 Row and Column Spaces

1. $\mathbf{A}_R = \mathbf{I}_3$, so $\mathbf{A}$ has rank 3. All of the rows form a basis for the row space and all of the columns form a basis for the column space.

3. We find that

$$\mathbf{A}_R = \begin{pmatrix} 1 & 0 & -3/5 \\ 0 & 1 & 3/5 \end{pmatrix}$$

so $\mathbf{A}$ has rank 2.

The rows of $\mathbf{A}$ are

$$\mathbf{R}_1 = (-4, 1, 3) \text{ and } \mathbf{R}_2 = (2, 2, 0).$$

These are linearly independent as vectors in R^3 and form a basis for the row space of $\mathbf{A}$.

The columns of $\mathbf{A}$ are

$$\mathbf{C}_1 = \begin{pmatrix} -4 \\ 2 \end{pmatrix}, \mathbf{C}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \mathbf{C}_3 = \begin{pmatrix} 3 \\ 0 \end{pmatrix}.$$

$\mathbf{C}_1$ and $\mathbf{C}_2$ are linearly independent as vectors in R^2, while

$$\mathbf{C}_3 = -\frac{3}{5}\mathbf{C}_1 + \frac{3}{5}\mathbf{C}_2.$$

Therefore $\mathbf{C}_1$ and $\mathbf{C}_2$ form a basis for the column space, which also has dimension 2.

Note that we can actually read the row and column space dimensions from the reduced matrix, since the rank of $\mathbf{A}$ is the number of nonzero rows of $\mathbf{A}_R$, and this rank is equal to both the row and column ranks.

In addition, as an example, we looked at the row and column vectors explicitly in this solution, but this is not necessary if all we want is the rank of the matrix. For this, either the row rank or the column rank is sufficient, since these numbers must be equal.

5. $$\mathbf{A}_R = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix},$$

so $\mathbf{A}$ has rank 2. The first two rows and the two columns of $\mathbf{A}$ are bases for the row and column spaces, respectively.

7.

$$\mathbf{A}_R = \begin{pmatrix} 1 & 0 & -1/4 & 1/2 \\ 0 & 1 & -5/4 & 1/2 \end{pmatrix},$$

so $\mathbf{A}$ has dimension 2. The two rows of $\mathbf{A}$ form a basis for the row space in R^4 and the first two columns form a basis for the column space in R^2.

9.

$$\mathbf{A}_R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$

so $\mathbf{A}$ has rank 3. The first, second and fourth rows are linearly independent and form a basis for the row space in R^3. All three columns are linearly independent and form a basis for the column space in R^4.

11.

$$\mathbf{A}_R = \begin{pmatrix} 1 & 0 & -11 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{pmatrix},$$

so $\mathbf{A}$ has rank 2. The first two rows are linearly independent and form a basis for the row space in R^3, and the first two columns form a basis for the column space in R^3.

13. We find that $\mathbf{A}_R = \mathbf{I}_3$, so $\mathbf{A}$ has rank 3. The row space has all the rows for a basis and the column space has all the columns.

15. Use the fact that, for any matrix, the rank, row rank and column rank are the same. Since the rows of $\mathbf{A}$ are the columns of $\mathbf{A}^t$, then

$$\begin{aligned} \text{rank of } \mathbf{A} &= \text{row rank of } \mathbf{A} \\ \text{column rank of } \mathbf{A}^t &= \text{rank of } \mathbf{A}^t. \end{aligned}$$

7.5 Homogeneous Systems

For these problems, use the facts that (1) $\mathbf{AX} = \mathbf{O}$ has the same solutions as $\mathbf{A}_R\mathbf{X} = \mathbf{O}$, and (2) the solution of the reduced system can be read by inspection from the reduced coefficient matrix $\mathbf{A}_R$.

1. Notice that the equations have unknowns x_1, x_2, x_4, x_5, but no x_3. Thus we have a system of three equations in four unknowns, but the unknowns are called x_1, x_2, x_4, x_5. The coefficient matrix is

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & -3 & 1 \\ 2 & -1 & 1 & 0 \\ 2 & -3 & 0 & 4 \end{pmatrix}$$

with reduced form

$$\mathbf{A}_R = \begin{pmatrix} 1 & 0 & 0 & -5/14 \\ 0 & 1 & 0 & -11/17 \\ 0 & 0 & 1 & -6/7 \end{pmatrix}.$$

The first three unknowns, x_1, x_2, x_4, depend on the fourth, x_5, which can be given any value α. The general solution is read from $\mathbf{A}_R$:

$$\mathbf{X} = \alpha \begin{pmatrix} 5/14 \\ 11/7 \\ 6/7 \\ 1 \end{pmatrix}.$$

The solution space is clearly one-dimensional. We can also see this dimension from $m - \text{rank}(\mathbf{A}) = 4 - 3 = 1$.

3. The coefficient matrix is

$$\mathbf{A} = \begin{pmatrix} 1 & -2 & 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & -1 & 1 & -2 & 3 \\ 1 & 0 & 0 & 0 & -1 & 2 & 0 \\ 2 & 0 & 0 & -3 & 1 & 0 & 0 \end{pmatrix}$$

and

$$\mathbf{A}_R = \begin{pmatrix} 1 & 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 1 & 0 & 0 & -1 & 3/2 & -1/2 \\ 0 & 0 & 1 & 0 & 0 & -2/3 & 3 \\ 0 & 0 & 0 & 1 & -1 & 4/3 & 0 \end{pmatrix}.$$

With $x_5 = \alpha$, $x_6 = \beta$ and $x_7 = \gamma$, the general solution is

$$\mathbf{X} = \alpha \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} -2 \\ -3/2 \\ 2/3 \\ -4/3 \\ 0 \\ 1 \\ 0 \end{pmatrix} + \gamma \begin{pmatrix} 0 \\ 1/2 \\ -3 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

The solution space has dimension 3, consistent with $m - \text{rank}(\mathbf{A}) = 7 - 4 = 3$.

5. The coefficient matrix

$$\mathbf{A} = \begin{pmatrix} -10 & -1 & 4 & -1 & 1 & -1 \\ 0 & 1 & -1 & 3 & 0 & 0 \\ 2 & -1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \end{pmatrix}$$

has reduced matrix

$$\mathbf{A}_R = \begin{pmatrix} 1 & 0 & 0 & 0 & 5/6 & 5/9 \\ 0 & 1 & 0 & 0 & 2/3 & 10/9 \\ 0 & 0 & 1 & 0 & 8/3 & 13/9 \\ 0 & 0 & 0 & 1 & 2/3 & 1/9 \end{pmatrix}.$$

With $x_5 = \alpha$ and $x_6 = \beta$ the general solution is

$$\mathbf{X} = \alpha \begin{pmatrix} -5/6 \\ -2/3 \\ -8/3 \\ -2/3 \\ 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} -5/9 \\ -10/9 \\ -13/9 \\ -1/9 \\ 0 \\ 1 \end{pmatrix}.$$

The solution space has dimension 2, which is also $m - \text{rank}(\mathbf{A}) = 6 - 4 = 2$.

7. The coefficient matrix

$$\mathbf{A} = \begin{pmatrix} -2 & 1 & 2 \\ 1 & -1 & 0 \\ 1 & 1 & 0 \end{pmatrix}$$

has reduced matrix

$$\mathbf{A}_R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The unique solution of the system is $\mathbf{X} = \mathbf{O}$, the trivial solution. Since rank$(\mathbf{A}) = 3$, the solution space has dimension $3 - 3 = 0$.

9. The coefficient matrix

$$\mathbf{A} = \begin{pmatrix} 1 & -1 & 3 & -1 & 4 \\ 2 & -2 & 1 & 1 & 0 \\ 1 & 0 & -2 & 0 & 1 \\ 0 & 0 & 1 & 1 & -1 \end{pmatrix}$$

has the reduced matrix

$$\mathbf{A}_R = \begin{pmatrix} 1 & 0 & 0 & 0 & 9/4 \\ 0 & 1 & 0 & 0 & 7/4 \\ 0 & 0 & 1 & 0 & 5/8 \\ 0 & 0 & 0 & 1 & -13/8 \end{pmatrix}.$$

With $x_5 = \alpha$ the general solution is

$$\mathbf{X} = \alpha \begin{pmatrix} -9/4 \\ -7/4 \\ -5/8 \\ 13/8 \\ 1 \end{pmatrix}.$$

The solution space has dimension 1, which is indeed equal to $m - \text{rank}(\mathbf{A}) = 5 - 4$.

11. The coefficient matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & -1 & 1 \\ 0 & 1 & -1 & 1 \end{pmatrix}$$

has reduced form

$$\mathbf{A}_R = \begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & -1 & 1 \end{pmatrix}.$$

Since rank $(\mathbf{A}) = 2$, the general solution will have $m - \text{rank}(\mathbf{A}) = 4 - 2 = 2$ arbitrary constants. This is the dimension of the solution space. From the reduced system, we read that

$$x_1 = -x_3 + x_4,$$
$$x_2 = x_3 - x_4.$$

This system is solved by giving x_3 and x_4 any values (hence the solution space has dimension 2), and choosing x_1 and x_2 according to the last equations. Thus,

$$\mathbf{X} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -x_3 + x_4 \\ x_3 - x_4 \\ x_3 \\ x_4 \end{pmatrix} = x_3 \begin{pmatrix} -1 \\ 1 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \end{pmatrix}.$$

It looks nicer to write $x_3 = \alpha$ and $x_4 = \beta$ (both arbitrary numbers) and write the general solution as

$$\mathbf{X} == \alpha \begin{pmatrix} -1 \\ 1 \\ 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \end{pmatrix}.$$

13. (a) Let $\mathbf{R}_1, \cdots, \mathbf{R}_n$ be the rows of $\mathbf{A}$. These vectors span R, the row space of the matrix. Now, $\mathbf{X}$ is in the solution space if and only if $\mathbf{X} = \mathbf{O}$, and this is true exactly when $\mathbf{R}_j \cdot \mathbf{X} = 0$ for $j = 1, \cdots, n$, which in turn is true if and only if $\mathbf{X}$ is orthogonal to each row of $\mathbf{A}$. But this is equivalent to $\mathbf{X}$ being orthogonal to every linear combination of the rows of $\mathbf{A}$, hence to every vector in the row space of $\mathbf{A}$. Therefore the solution space of $\mathbf{A}$ is the orthogonal complement of the row space, or

$$R^{\perp} = S(\mathbf{A}).$$

Since the columns of $\mathbf{A}^t$ are the rows of $\mathbf{A}$, the conclusion that $C^{\perp} = S(\mathbf{A}^t)$ follow immediately from the reasoning of part (a).

14. **Hint** The proposition follows from the following observation. Suppose $\mathbf{A}$ is $n \times m$. Let the columns of $\mathbf{A}$ be $\mathbf{C}_1, \cdots, \mathbf{C}_m$, written as column matrices. If

$$\mathbf{X} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{pmatrix}$$

then $\mathbf{AX} = \mathbf{O}$ is equivalent to

$$a_1\mathbf{C}_1 + a_2\mathbf{C}_2 + \cdots + a_m\mathbf{C}_m = \mathbf{O},$$

the $n \times 1$ zero matrix.

15. Yes. All that is required is that $m - \text{rank}(\mathbf{A}) > 0$, so that the solution space has something in it. As a specific example, consider the system $\mathbf{AX} = \mathbf{O}$, with

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & -1 \\ 3 & 0 & 9 \end{pmatrix}.$$

This is a homogeneous system with three equations in three unknowns. We find that

$$\mathbf{A}_R = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix},$$

so $\mathbf{A}$ has rank 2. The solution space has dimension $3 - 2 = 1$, hence has nonzero vectors in it. The general solution is

$$\mathbf{X} = \alpha \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}.$$

7.6 Nonhomogeneous Systems

1. The augmented matrix is

$$\begin{pmatrix} 2 & -3 & 0 & 1 & 0 & -1 & \vdots & 0 \\ 3 & 0 & -2 & 0 & 1 & 0 & \vdots & 1 \\ 0 & 1 & 0 & -1 & 0 & 6 & \vdots & 3 \end{pmatrix}.$$

This reduces to

$$\begin{pmatrix} 1 & 0 & 0 & -1 & 0 & 17/2 & \vdots & 9/2 \\ 0 & 1 & 0 & -1 & 0 & 6 & \vdots & 3 \\ 0 & 0 & 1 & -3/2 & -1/2 & 51/4 & \vdots & 25/4 \end{pmatrix}.$$

Then $\text{rank}(\mathbf{A}) = \text{rank}([\mathbf{A}\vdots\mathbf{B}])$, the system has solutions. From the reduced augmented matrix we read the general solution

$$\mathbf{X} = \begin{pmatrix} 9/2 \\ 3 \\ 25/4 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \alpha \begin{pmatrix} 1 \\ 1 \\ 3/2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ 0 \\ 1/2 \\ 0 \\ 1 \\ 0 \end{pmatrix} + \gamma \begin{pmatrix} -17/2 \\ -6 \\ -51/4 \\ 0 \\ 0 \\ 1 \end{pmatrix},$$

with α, β and γ arbitrary.

3. The augmented matrix is

$$\begin{pmatrix} 8 & -4 & 0 & 0 & 10 & \vdots & 1 \\ 0 & 1 & 0 & 1 & -1 & \vdots & 2 \\ 0 & 0 & 1 & -3 & 2 & \vdots & 0 \end{pmatrix}.$$

(The x_1 column has been omitted since x_1 does not appear in the equations). The reduced form of this matrix is

$$\begin{pmatrix} 1 & 0 & 0 & 1/2 & 3/4 & \vdots & 9/8 \\ 0 & 1 & 0 & 1 & -1 & \vdots & 2 \\ 0 & 0 & 1 & -3 & 2 & \vdots & 0 \end{pmatrix}.$$

Since $\text{rank}(\mathbf{A}) = \text{rank}([\mathbf{A}\vdots\mathbf{B}])$, this system has solutions, which we read as

$$\mathbf{X} = \begin{pmatrix} 9/8 \\ 2 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \alpha \begin{pmatrix} -1/2 \\ -1 \\ 3 \\ 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} -3/4 \\ 1 \\ -2 \\ 0 \\ 1 \end{pmatrix},$$

in which α and β are arbitrary.

5. The augmented matrix is

$$\begin{pmatrix} 4 & -1 & 4 & \vdots & 1 \\ 1 & 1 & -5 & \vdots & 0 \\ -2 & 1 & 7 & \vdots & 4 \end{pmatrix},$$

with reduced form

$$\begin{pmatrix} 1 & 0 & 0 & \vdots & 16/57 \\ 0 & 1 & 0 & \vdots & 99/57 \\ 0 & 0 & 1 & \vdots & 23/57 \end{pmatrix}.$$

Since

$$\text{rank}(\mathbf{A}) = \text{rank}([\mathbf{A} \vdots \mathbf{B}]) = \text{number of unknowns} = 3,$$

the system has the unique solution

$$\mathbf{X} = \begin{pmatrix} 16/57 \\ 99/57 \\ 23/57 \end{pmatrix}.$$

7. The augmented matrix

$$\begin{pmatrix} 0 & 0 & 14 & 0 & -3 & 0 & 1 & \vdots & 2 \\ 1 & 1 & 1 & -1 & 0 & 1 & 0 & \vdots & -4 \end{pmatrix}.$$

This has reduced form

$$\begin{pmatrix} 1 & 1 & 0 & -1 & 3/14 & 1 & -1/14 & \vdots & -29/7 \\ 0 & 0 & 1 & 0 & -3/14 & 0 & 1/14 & \vdots & 1/7 \end{pmatrix}.$$

Note that $\text{rank}(\mathbf{A}) = \text{rank}(\mathbf{A} \vdots \mathbf{B}])$, so there are solutions. We read from the augmented matrix that the general solution has the form

$$\mathbf{X} = \begin{pmatrix} -29/7 \\ 0 \\ 1/7 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \alpha \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \gamma \begin{pmatrix} -3/14 \\ 0 \\ 3/14 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \delta \begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + \epsilon \begin{pmatrix} 1/14 \\ 0 \\ -1/14 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix},$$

with α, β, γ, δ and ϵ arbitrary.

9. The augmented matrix is

$$\begin{pmatrix} 7 & -3 & 4 & 0 & \vdots & -7 \\ 2 & 1 & -1 & 4 & \vdots & 6 \\ 0 & 1 & 0 & -3 & \vdots & -5 \end{pmatrix}$$

with reduced form

$$\begin{pmatrix} 1 & 0 & 0 & 19/15 & \vdots & 22/15 \\ 0 & 1 & 0 & -3 & \vdots & -5 \\ 0 & 0 & 1 & -67/13 & \vdots & -121/15 \end{pmatrix}.$$

Now

$$\text{rank }(\mathbf{A}) = 3 = \text{ rank}([\mathbf{A} \vdots \mathbf{B}]),$$

so the system has solutions. We read from the reduced system that

$$\mathbf{X} = \begin{pmatrix} 22/15 \\ -5 \\ -121/15 \\ 0 \end{pmatrix} + \alpha \begin{pmatrix} -19/15 \\ 3 \\ 67/15 \\ 1 \end{pmatrix},$$

in which α is arbitrary.

11. The augmented matrix is

$$\begin{pmatrix} 0 & 3 & 0 & -4 & 0 & 0 & \vdots & 10 \\ 1 & -3 & 0 & 0 & 4 & -1 & \vdots & 8 \\ 0 & 1 & 1 & -6 & 0 & 1 & \vdots & -9 \\ 1 & -1 & 0 & 0 & 0 & 1 & \vdots & 0 \end{pmatrix}.$$

The reduced form of this is

$$\begin{pmatrix} 1 & 0 & 0 & 0 & -2 & 2 & \vdots & -4 \\ 0 & 1 & 0 & 0 & -2 & 1 & \vdots & -4 \\ 0 & 0 & 1 & 0 & -7 & 9/2 & \vdots & -38 \\ 0 & 0 & 0 & 1 & -3/2 & 3/4 & \vdots & -11/2 \end{pmatrix}.$$

Since $\text{rank}(\mathbf{A}) = \text{rank}([\mathbf{A} \vdots \mathbf{B}])$, the system has solutions, which we read from the reduced augmented matrix. The general solution is

$$\mathbf{X} = \begin{pmatrix} -4 \\ -4 \\ -38 \\ -11/2 \\ 0 \\ 0 \end{pmatrix} + \alpha \begin{pmatrix} 2 \\ 2 \\ 7 \\ 3/2 \\ 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} -2 \\ -1 \\ -9/2 \\ -3/4 \\ 0 \\ 1 \end{pmatrix},$$

with α and β arbitrary.

13. The augmented matrix is

$$\begin{pmatrix} 3 & -2 & 1 & \vdots & 6 \\ 1 & 10 & -1 & \vdots & 2 \\ -3 & -2 & 1 & \vdots & 0 \end{pmatrix}$$

with reduced matrix

$$\begin{pmatrix} 1 & 0 & 0 & \vdots & 1 \\ 0 & 1 & 0 & \vdots & 1/2 \\ 0 & 0 & 1 & \vdots & 4 \end{pmatrix}.$$

Since $\text{rank}(\mathbf{A}) = \text{rank}([\mathbf{A}\vdots\mathbf{B}]) = 3$, and this is the number of unknowns, the system has the unique solution

$$\mathbf{X} = \begin{pmatrix} 1 \\ 1/2 \\ 4 \end{pmatrix}.$$

15. Write

$$\mathbf{X} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{pmatrix}$$

and let $\mathbf{C}_1, \cdots, \mathbf{C}_n$ be the columns of $\mathbf{A}$. Now $\mathbf{AX} = \mathbf{B}$ if and only if

$$a_1\mathbf{C}_1 + a_2\mathbf{C}_2 + \cdots + a_m\mathbf{C}_m = \mathbf{B}.$$

This means that the system has a solution $\mathbf{X}$ if and only if $\mathbf{X}$ is a linear combination of the columns of $\mathbf{A}$, hence is in the column space of $\mathbf{A}$.

7.7 Matrix Inverses

The most efficient way of computing a matrix inverse is by a software routine, such as MAPLE. Here we provide the details of the reduction method for Problem 9 and then just give the inverse matrix for the remaining problems.

1.
$$\mathbf{A}^{-1} = \frac{1}{12}\begin{pmatrix} -2 & 2 \\ 1 & 5 \end{pmatrix}$$

3.
$$\mathbf{A}^{-1} = \frac{1}{31}\begin{pmatrix} -6 & 11 & 2 \\ 3 & 10 & -1 \\ 1 & -7 & 10 \end{pmatrix}$$

5.
$$\mathbf{A}^{-1} = -\frac{1}{12}\begin{pmatrix} 6 & -6 & 0 \\ -3 & -9 & 2 \\ 3 & -3 & -2 \end{pmatrix}$$

7.
$$\mathbf{A}^{-1} = \frac{1}{12}\begin{pmatrix} 3 & -2 \\ -3 & 6 \end{pmatrix}$$

9. Reduce

$$\begin{pmatrix} -1 & 2 & \vdots & 1 & 0 \\ 2 & 1 & \vdots & 0 & 1 \end{pmatrix} \rightarrow \text{add two times row one to row two} \rightarrow \begin{pmatrix} -1 & 2 & \vdots & 1 & 0 \\ 0 & 5 & \vdots & 2 & 1 \end{pmatrix}$$

$$\to \text{ multiply row one by } -1 \to \begin{pmatrix} 1 & -2 & \vdots & -1 & 0 \\ 0 & 5 & \vdots & 2 & 1 \end{pmatrix}$$

$$\to \text{ multiply row two by } 1/5 \to \begin{pmatrix} 1 & -2 & \vdots & -1 & 0 \\ 0 & 1 & \vdots & 2/5 & 1/5 \end{pmatrix}$$

$$\to \text{ add 2 times row two to row one } \to \begin{pmatrix} 1 & 0 & \vdots & -1/5 & 2/5 \\ 0 & 1 & \vdots & 2/5 & 1/5 \end{pmatrix}.$$

Because $\mathbf{I}_2$ has appeared on the left, the right two columns form the inverse matrix:

$$\mathbf{A}^{-1} = \frac{1}{5}\begin{pmatrix} -1 & 2 \\ 2 & 1 \end{pmatrix}.$$

11.
$$\mathbf{X} = \mathbf{A}^{-1}\mathbf{B} = -\frac{1}{25}\begin{pmatrix} 5 & -15 & -15 \\ -10 & 15 & 10 \\ -5 & 10 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ -7 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} -21 \\ 14 \\ 0 \end{pmatrix}$$

13.
$$\mathbf{X} = \mathbf{A}^{-1}\mathbf{B} = \frac{1}{11}\begin{pmatrix} -1 & -1 & 8 & 4 \\ -9 & 2 & -5 & 14 \\ 2 & 2 & -5 & 3 \\ 3 & 3 & -2 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 0 \\ -5 \end{pmatrix} = \frac{1}{11}\begin{pmatrix} -23 \\ -75 \\ -9 \\ 14 \end{pmatrix}$$

15.
$$\mathbf{X} = \mathbf{A}^{-1}\mathbf{B}$$
$$= -\frac{1}{28}\begin{pmatrix} -11 & -12 & -9 \\ -3 & -16 & -5 \\ -8 & -24 & -4 \end{pmatrix}\begin{pmatrix} -4 \\ 5 \\ 8 \end{pmatrix} = \frac{1}{7}\begin{pmatrix} 22 \\ 27 \\ 30 \end{pmatrix}$$

7.8 Least Squares Vectors and Data Fitting

1. Compute
$$\mathbf{A}^t\mathbf{A} = \begin{pmatrix} 37 & 12 \\ 12 & 4 \end{pmatrix} \text{ and } (\mathbf{A}^t\mathbf{A})^{-1} = \begin{pmatrix} 1 & -3 \\ -3 & 37/4 \end{pmatrix}.$$
Next,
$$\mathbf{A}^t\mathbf{B} = \begin{pmatrix} -26 \\ -8 \end{pmatrix}.$$
Then
$$\mathbf{X}^* = (\mathbf{A}^t\mathbf{A})^{-1}(\mathbf{A}^t\mathbf{B}) = \begin{pmatrix} -2 \\ 4 \end{pmatrix}.$$

3. As in Problem 2, we find that $\mathbf{A}^t\mathbf{A}$ is singular. Further, we also find that $\mathbf{B}_S = \mathbf{B}$, so solve
$$\mathbf{A}\mathbf{X}^* = \mathbf{B}$$
to obtain
$$\mathbf{X}^* = \alpha\begin{pmatrix} 7 \\ 6 \\ 7 \\ 1 \end{pmatrix} + \begin{pmatrix} -15 \\ -31/3 \\ -44/3 \\ 0 \end{pmatrix},$$
with α an arbitrary constant.

5. We have

$$\mathbf{A} = \begin{pmatrix} 1 & 1 \\ -2 & 3 \end{pmatrix} \text{ and } \mathbf{B} = \begin{pmatrix} 4 \\ -1 \end{pmatrix}.$$

Compute

$$\mathbf{A}^t\mathbf{A} = \begin{pmatrix} 5 & -5 \\ -5 & 10 \end{pmatrix} \text{ and } (\mathbf{A}^t\mathbf{A})^{-1} = \begin{pmatrix} 2/5 & 1/5 \\ 1/5 & 1/5 \end{pmatrix}.$$

Finally,

$$\mathbf{A}^t\mathbf{B} = \begin{pmatrix} 6 \\ 1 \end{pmatrix}.$$

The solution is

$$\mathbf{X}^* = (\mathbf{A}^t\mathbf{A})^{-1}(\mathbf{A}^t\mathbf{B}) = \begin{pmatrix} 13/5 \\ 7/5 \end{pmatrix}.$$

7. We have

$$\mathbf{A} = \begin{pmatrix} 1 & -3 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 4 \\ 1 & 7 \end{pmatrix} \text{ and } \mathbf{B} = \begin{pmatrix} -23 \\ -8.2 \\ -4.6 \\ -0.5 \\ 7.3 \\ 19.2 \end{pmatrix}.$$

Then

$$\mathbf{A}^t\mathbf{A} = \begin{pmatrix} 6 & 11 \\ 11 & 79 \end{pmatrix} \text{ and } (\mathbf{A}^t\mathbf{A})^{-1} = \begin{pmatrix} 79/353 & -11/353 \\ -11/353 & 6/353 \end{pmatrix}.$$

Next, compute

$$\mathbf{A}^t\mathbf{B} = \begin{pmatrix} -9.79999 \\ 227 \end{pmatrix}.$$

Then

$$\mathbf{X}^* = \begin{pmatrix} -9.266855 \\ 4.167394 \end{pmatrix}.$$

The equation of the line is $y = a + bx$, with $a = 4.167394$ and $b = -9.266855$.

9. We have

$$\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 1 & 3 \\ 1 & 5 \\ 1 & 7 \\ 1 & 9 \end{pmatrix} \text{ and } \mathbf{B} = \begin{pmatrix} 3.8 \\ 11.7 \\ 20.6 \\ 26.5 \\ 35.2 \end{pmatrix}.$$

Compute

$$\mathbf{A}^t\mathbf{A} = \begin{pmatrix} 5 & 25 \\ 25 & 165 \end{pmatrix} \text{ and } (\mathbf{A}^t\mathbf{A})^{-1} = \begin{pmatrix} 33/40 & -1/8 \\ -1/8 & 1/40 \end{pmatrix}.$$

Further,

$$\mathbf{A}^t\mathbf{B} = \begin{pmatrix} 97.80000 \\ 644.20000 \end{pmatrix}.$$

Then

$$\mathbf{X}^* = (\mathbf{A}^t\mathbf{A})^{-1}(\mathbf{A}^t\mathbf{B}) = \begin{pmatrix} 0.1599 \\ 3.8799 \end{pmatrix}.$$

The line has the equation $y = a + bx$, with $a = 3.8799$ and $b = 0.1599$.

7.9 LU Factorization

For Problems 1 and 5 the same algorithm is used and we give only the matrices **L** and **U**.

1.
$$\mathbf{U} = \begin{pmatrix} 1 & 4 & 2 & -1 & 4 \\ 0 & -5 & 2 & 0 & 0 \\ 0 & 0 & 88/5 & 4 & 6 \\ 0 & 0 & 0 & 195/22 & -691/44 \end{pmatrix}, \mathbf{L} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ -2 & -14/5 & 1 & 0 \\ 4 & 14/5 & -63/88 & 1 \end{pmatrix}$$

3. Given **A**, first produce **U**. Proceed

$$\mathbf{A} = \begin{pmatrix} \mathbf{2} & 4 & -6 \\ \mathbf{8} & 2 & 1 \\ \mathbf{-4} & 4 & 10 \end{pmatrix} \rightarrow \text{ add } -4 \text{ row one to row two, } 2 \text{ row one to row three}$$

$$\rightarrow \begin{pmatrix} 2 & 4 & -6 \\ 0 & \mathbf{-14} & 25 \\ 0 & \mathbf{12} & -2 \end{pmatrix}$$

$$\rightarrow \text{ add } 6/7 \text{ row two to row three } \rightarrow \begin{pmatrix} 2 & 4 & -6 \\ 0 & -14 & 25 \\ 0 & 0 & \mathbf{136/7} \end{pmatrix}.$$

This is **U**:

$$\mathbf{U} = \begin{pmatrix} 2 & 4 & -6 \\ 0 & -14 & 25 \\ 0 & 0 & 136/7 \end{pmatrix}.$$

Now use the boldface entries in the formation of **U** to obtain **L**. Start with

$$\mathbf{D} = \begin{pmatrix} 2 & 0 & 0 \\ 8 & -14 & 0 \\ -4 & 12 & 136/7 \end{pmatrix}.$$

Here we have listed the boldface elements from the formation of **U**, with zeros above, to form a lower triangular matrix. This is not yet **L**. In **D**, divide each column by the reciprocal of the diagonal element of that column to obtain

$$\mathbf{L} = \begin{pmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ -2 & -6/7 & 1 \end{pmatrix}.$$

It is routine to check that $\mathbf{LU} = \mathbf{A}$.

4. **Hint** This problem is the only problem in which the number of rows exceeds the number of columns. If the algorithm is carried out starting with **A**, a difficulty occurs. However, we can still write the $LU-$ decomposition of **A** by working with $\mathbf{A}^t$, which is 3×4. The strategy is to find upper and lower triangular matrices **U** and **L** so that

$$\mathbf{A}^t = \mathbf{LU}.$$

Then

$$\mathbf{A} = \mathbf{U}^t\mathbf{L}^t.$$

But the transpose of an upper triangular matrix is lower triangular, and the transpose of a lower triangular matrix is upper triangular, so this is the decomposition we want for **A**.

5.

$$\mathbf{U} = \begin{pmatrix} -2 & 1 & 12 \\ 0 & -5 & 13 \\ 0 & 0 & 119/5 \end{pmatrix}, \mathbf{L} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & -3/5 & 1 \end{pmatrix}$$

Problems 7 - 12 are small in the sense that the matrices are low-dimensional and the entries are integers. In such cases it would be just as efficient to solve the system $\mathbf{AX} = \mathbf{B}$ directly. The $LU-$ factorization method only reveals computational efficiencies when the systems are large. However, these problems are intended to promote familiarity with the method.

7. Obtain

$$\mathbf{U} = \begin{pmatrix} 6 & 1 & -1 & 3 \\ 0 & 4/3 & 5/3 & 3 \\ 0 & 0 & 13/4 & 13/4 \\ 0 & 0 & 0 & 5 \end{pmatrix}, \mathbf{L} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2/3 & 1 & 0 & 0 \\ -2/3 & 5/4 & 1 & 0 \\ 1/3 & -1 & 4/13 & 1 \end{pmatrix}.$$

Solve $\mathbf{LY} = \mathbf{B}$ to obtain

$$\mathbf{Y} = \begin{pmatrix} 4 \\ 28/3 \\ -7 \\ 93/13 \end{pmatrix}$$

and then solve $\mathbf{UX} = \mathbf{Y}$:

$$\mathbf{X} = \begin{pmatrix} -263/130 \\ 537/65 \\ -233/65 \\ 93/65 \end{pmatrix}.$$

9. We want to solve $\mathbf{AX} = \mathbf{B}$, where

$$\mathbf{A} = \begin{pmatrix} 4 & 4 & 2 \\ 1 & -1 & 3 \\ 1 & 4 & 2 \end{pmatrix} \text{ and } \mathbf{B} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

We find that $\mathbf{A} = \mathbf{LU}$, where

$$\mathbf{U} = \begin{pmatrix} 4 & 4 & 2 \\ 0 & -2 & 5/2 \\ 0 & 0 & 21/4 \end{pmatrix} \text{ and } \mathbf{L} = \begin{pmatrix} 1 & 0 & 0 \\ 1/4 & 1 & 0 \\ 1/4 & -3/2 & 1 \end{pmatrix}.$$

Next solve the system $\mathbf{LY} = \mathbf{B}$ to obtain

$$\mathbf{Y} = \begin{pmatrix} 1 \\ -1/4 \\ 3/8 \end{pmatrix}.$$

Finally, solve $\mathbf{UX} = \mathbf{Y}$ to obtain

$$\mathbf{X} = \begin{pmatrix} 0 \\ 3/14 \\ 1/14 \end{pmatrix}.$$

11. We find that

$$\mathbf{U} = \begin{pmatrix} -1 & 1 & 1 & 6 \\ 0 & 3 & 2 & 16 \\ 0 & 0 & 17/3 & 52/3 \end{pmatrix}, \mathbf{L} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & -1/3 & 1 \end{pmatrix}.$$

Solve $\mathbf{LY} = \mathbf{B}$ to obtain

$$\mathbf{Y} = \begin{pmatrix} 2 \\ 5 \\ 29/3 \end{pmatrix}$$

and then solve $\mathbf{UX} = \mathbf{Y}$ for

$$\mathbf{X} = \alpha \begin{pmatrix} 1 \\ 28/3 \\ 26/3 \\ -17/6 \end{pmatrix} + \begin{pmatrix} 0 \\ -5/3 \\ -1/3 \\ 2/3 \end{pmatrix}.$$

7.10 Linear Transformations

1. T is not linear because of the constant fourth and fifth components of $T(x, y, u, v, w)$. Note also that the zero vector does not map to the zero vector by T.
3. T is not linear because of the $\sin(xy)$ term.
5. T is linear and

$$T(1,0,0) =< 3,1,0 >, T(0,1,0) =< 0,-1,0 >, T(0,0,1) =< 0,0,2 >$$

so

$$\mathbf{A}_T = \begin{pmatrix} 3 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

Because $\mathbf{A}_T$ has rank 3, T is one-to-one and onto and the dimension of the null space is $3-3=0$ (contains only the zero vector).

7. T is nonlinear because of the $2xy$ term.
9. T is linear and

$$\mathbf{A}_T = \begin{pmatrix} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}.$$

T is not one-to-one, but is onto. The null space has dimension $5-3=2$.

Chapter 8

Determinants

8.1 Definition of the Determinant

1. If p is a permutation of $1, 2, \cdots, n$, then it is impossible for each $p(j) \geq j$ or for each $p(j) \leq j$ for $j = 1, 2, \cdots, n$, unless each $p(j) = j$ and p is the identity permutation. Thus, the only (possibly) nonzero term in the sum defining the determinant is

$$\begin{aligned}|\mathbf{A}| &= \sigma(p)\mathbf{A}_{1p(1)}\mathbf{A}_{2p(2)}\cdots\mathbf{A}_{np(n)}\\ &= \mathbf{A}_{11}\mathbf{A}_{22}\cdots\mathbf{A}_{nn}.\end{aligned}$$

2. **Hint** From the definition of determinant,

$$|\mathbf{I}_n| = \sum_p \sigma(p)(\mathbf{I}_n)_{1p(1)}(\mathbf{I}_n)_{2p(2)}\cdots(\mathbf{I}_n)_{np(n)}.$$

Observe that the only way a term of this sum can be nonzero for a particular permutation p is for each $(\mathbf{I}_n)_{ip(i)} \neq 0$. Now think about how this can happen.

4. **Hint** A square $\mathbf{A}$ is nonsingular if and only if $|\mathbf{A}| = 0$. Now use the result of Problem 1.

5. Suppose $\mathbf{A} = -\mathbf{A}^t$. Use property (1) of determinants, and the conclusion of Problem 7. Since $\mathbf{A}^t = -\mathbf{A}$, then

$$|\mathbf{A}| = |\mathbf{A}^t| = |-\mathbf{A}| = (-1)^n|\mathbf{A}|.$$

If n is odd, then

$$|\mathbf{A}| = -|\mathbf{A}|,$$

and this implies that $|\mathbf{A}| = 0$.

6. In the 2×2 case,

$$\mathbf{B} = \begin{pmatrix} a_{11} & (1/\alpha)a_{12} \\ \alpha a_{21} & a_{22} \end{pmatrix}$$

so

$$\begin{aligned}|\mathbf{B}| &= a_{11}a_{22} - \left(\frac{1}{\alpha}\right)(\alpha)(a_{12} - a_{21})\\ &= a_{11}a_{22} - a_{12}a_{21} = |\mathbf{A}|.\end{aligned}$$

In the 3×3 case,

$$\mathbf{B} = \begin{pmatrix} a_{11} & (1/\alpha)a_{12} & (1/\alpha^2)a_{13} \\ \alpha a_{21} & a_{22} & (1/\alpha)a_{23} \\ \alpha^2 a_{31} & (1/\alpha)a_{32} & a_{33} \end{pmatrix}$$

and by expanding this determinant we find that $|\mathbf{B}| = |\mathbf{A}|$.

What we observe in these small cases is that each factor of α is matched with a factor of $1/\alpha$ in the terms of the sum defining the determinant, so these cancel. This leads us to conjecture that $|\mathbf{B}| = |\mathbf{A}|$ in the $n \times n$ case.

7. Each factor $a_{jp(j)}$ in a typical term of the sum defining $|\mathbf{A}|$ is replaced by $\alpha a_{jp(j)}$ in the corresponding term of $|\mathbf{B}|$. Since there are n factors in each such term, then each term in the sum defining $|\mathbf{B}|$ is α^n times the corresponding term in $|\mathbf{B}|$. Therefore $|\mathbf{B}| = \alpha^n|\mathbf{A}|$.

8.2 Evaluation of Determinants I

The most efficient way to evaluate a determinant is by using a software package. In many kinds of general computations, however, it is useful to understand row and column operations and cofactor expansions and how these are used to manipulate determinants, and this is the purpose of these problems.

There are many sequences of row and/or column operations that can be used to evaluate a given determinant. Regardless of the sequence used, the value of the determinant depends only on the original matrix.

1. 72

3. Add 2 times column three to column one and then add column three to column two to obtain

$$\begin{vmatrix} 17 & -2 & 5 \\ 1 & 12 & 0 \\ 14 & 7 & -7 \end{vmatrix} = \begin{vmatrix} 27 & 3 & 5 \\ 1 & 12 & 0 \\ 0 & 0 & -7 \end{vmatrix} = (-1)^{3+3}(-7)\begin{vmatrix} 27 & 3 \\ 1 & 12 \end{vmatrix} = -2,247$$

The determinants in Problems 1-2 and 9-10 are treated similarly, and we list only the value of the determinant for Problems 1 and 9.

5. Add column two to column one, then 3 times column two to column three:

$$\begin{vmatrix} -4 & 5 & 6 \\ -2 & 3 & 5 \\ 2 & -2 & 6 \end{vmatrix} = \begin{vmatrix} 1 & 5 & 21 \\ 1 & 3 & 14 \\ 0 & -2 & 0 \end{vmatrix} = (-1)^{3+2}(-2)\begin{vmatrix} 1 & 21 \\ 1 & 14 \end{vmatrix} = -14$$

7. Add 2 times row two to row one and -7 times row two to row three to write

$$\begin{vmatrix} -2 & 4 & 1 \\ 1 & 6 & 3 \\ 7 & 0 & 4 \end{vmatrix} = \begin{vmatrix} 0 & 16 & 7 \\ 1 & 6 & 3 \\ 0 & -42 & -17 \end{vmatrix} = (-1)^{2+1}(1)\begin{vmatrix} 16 & 7 \\ -42 & -17 \end{vmatrix} = -22$$

9. -122

8.3 Evaluation of Determinants II

For these problems, use a combination of row and column operations to obtain a row or column with some zeros, then expand by that row or column. Depending on the size of the resulting determinants, it may be useful to apply the cofactor method to each of these in turn.

For Problem 7 and 10, the cofactor expansion is written out in detail. For Problems 1, 3, 5, and 9, only the value of the determinant is given.

1. $1,693$
3. -773
5. -152
7. Expand the determinant by the third column:

$$\begin{vmatrix} -4 & 2 & -8 \\ 1 & 1 & 0 \\ 1 & -3 & 0 \end{vmatrix} = (-1)^{1+3}(-8)\begin{vmatrix} 1 & 1 \\ 1 & -3 \end{vmatrix} = (-8)(-4) = 32$$

9. 3
11. Define a function

$$L(x,y) = \begin{vmatrix} 1 & x & y \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix} = (y_2 - y_3)x + (x_3 - x_2)y + x_2y_3 - x_3y_2.$$

Thus $L(x,y)$ has the form $L(x,y) = ax + by + c$, with a, b and c constants. The graph of the equation $L(x,y) = 0$ is a straight line in the plane. Since $L(x_2,y_2) = L(x_3,y_3) = 0$, both points (x_2,y_2) and $x_3,y_3)$ are on this line.

Finally, $L(x_1,y_1) = 0$ if and only if (x_1,y_1) is also on this line, and this occurs if and only if

$$\begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix} = 0.$$

13.

$$\begin{vmatrix} 1 & \alpha & \alpha^2 \\ 1 & \beta & \beta^2 \\ 1 & \gamma & \gamma^2 \end{vmatrix} = \begin{vmatrix} 1 & \alpha & \alpha^2 \\ 0 & \beta-\alpha & \beta^2-\alpha^2 \\ 0 & \gamma-\alpha & \gamma^2-\alpha^2 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & \alpha & \alpha^2 \\ 0 & \beta-\alpha & (\beta^-\alpha)(\beta+\alpha) \\ 0 & \gamma-\alpha & (\gamma-\alpha)(\gamma+\alpha) \end{vmatrix} = (\beta-\alpha)(\gamma-\alpha)\begin{vmatrix} 1 & \alpha & \alpha^2 \\ 0 & 1 & \beta+\alpha \\ 0 & 1 & \gamma+\alpha \end{vmatrix}$$

$$= (\beta-\alpha)(\gamma-\alpha)\begin{vmatrix} 1 & \beta+\alpha \\ 1 & \gamma+\alpha \end{vmatrix} = (\beta-\alpha)(\gamma-\alpha)(\gamma-\beta).$$

8.4 A Determinant Formula for $\mathbf{A}^{-1}$

1.

$$\mathbf{A}^{-1} = \frac{1}{32}\begin{pmatrix} 5 & 3 & 1 \\ -8 & -24 & 24 \\ -2 & -14 & 6 \end{pmatrix}$$

3.

$$\mathbf{A}^{-1} = \frac{1}{29}\begin{pmatrix} -1 & 25 & -21 \\ -8 & -3 & 6 \\ -1 & -4 & 8 \end{pmatrix}$$

5.

$$\mathbf{A}^{-1} = \frac{1}{378}\begin{pmatrix} 210 & -42 & 42 & 0 \\ 899 & -124 & 223 & -135 \\ 275 & -64 & 109 & -27 \\ -601 & 122 & -131 & 81 \end{pmatrix}$$

7.

$$\mathbf{A}^{-1} = \frac{1}{13}\begin{pmatrix} 6 & 1 \\ -1 & 2 \end{pmatrix}$$

9.

$$\mathbf{A}^{-1} = \frac{1}{5}\begin{pmatrix} -4 & 1 \\ 1 & 1 \end{pmatrix}$$

8.5 Cramer's Rule

1. $|\mathbf{A}| = -6$ and the solution is

$$x_1 = \frac{5}{6}, x_2 = -\frac{10}{3}, x_3 = -\frac{5}{6}$$

3. $|\mathbf{A}| = 4$ and the solution is

$$x_1 = -\frac{172}{2} = -86, x_2 = -\frac{109}{2}, x_3 = -\frac{43}{2}, x_4 = \frac{37}{2}$$

5. $|\mathbf{A}| = 93 \neq 0$ and the solution is

$$x_1 = \frac{33}{93}, x_2 = -\frac{409}{33}, x_3 = -\frac{1}{93}, x_4 = \frac{116}{93}.$$

7. $|\mathbf{A}| = 132$ and the solution is

$$x_1 = \frac{1}{132}\begin{vmatrix} 0 & -4 & 3 \\ -5 & 5 & -1 \\ -4 & 6 & 1 \end{vmatrix} = -\frac{66}{132} = -\frac{1}{2},$$

$$x_2 = \frac{1}{132}\begin{vmatrix} 8 & 0 & 3 \\ 1 & -5 & -1 \\ -2 & -4 & 1 \end{vmatrix} = -\frac{114}{132} = -\frac{19}{22},$$

$$x_3 = \frac{1}{132}\begin{vmatrix} 8 & -4 & 0 \\ 1 & 5 & -5 \\ -2 & 6 & -4 \end{vmatrix} = \frac{24}{132} = \frac{2}{11}$$

9. Since $|\mathbf{A}| = 47 \neq 0$, Cramer's rule applies. The solution is

$$x_1 = \frac{1}{47}\begin{vmatrix} 5 & -4 \\ -4 & 1 \end{vmatrix} = -\frac{11}{47},\ x_2 = \frac{1}{47}\begin{vmatrix} 15 & 5 \\ 8 & -4 \end{vmatrix} = -\frac{100}{47}$$

8.6 The Matrix Tree Theorem

1.
$$\mathbf{T} = \begin{pmatrix} 3 & -1 & 0 & 0 & -1 & -1 \\ -1 & 3 & -1 & 0 & -1 & 0 \\ 0 & -1 & 4 & -1 & -1 & -1 \\ 0 & 0 & -1 & 2 & -1 & 0 \\ -1 & 0 & -1 & 0 & 0 & 2 \end{pmatrix}$$

and each cofactor is equal to 61.

3. The tree matrix for this graph is

$$\mathbf{T} = \begin{pmatrix} 2 & 0 & -1 & 0 & -1 \\ 0 & 2 & -1 & -1 & 0 \\ -1 & -1 & 4 & -1 & -1 \\ 0 & -1 & -1 & 3 & -1 \\ -1 & 0 & -1 & -1 & 3 \end{pmatrix}.$$

Evaluate any 4×4 cofactor of T to obtain 21 as the number of spanning trees in G.

5.
$$\mathbf{T} = \begin{pmatrix} 4 & -1 & 0 & -1 & -1 & -1 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 3 & -1 & -1 & 0 \\ -1 & 0 & -1 & 4 & -1 & -1 \\ -1 & 0 & -1 & -1 & 3 & 0 \\ -1 & 0 & 0 & -1 & 0 & 2 \end{pmatrix}$$

and each cofactor equals 61.

6. **Hint** The tree matrix for the complete graph K_n is

$$\mathbf{T} = \begin{pmatrix} n-1 & -1 & -1 & \cdots & -1 \\ -1 & n-1 & -1 & \cdots & -1 \\ -1 & -1 & n-1 & \cdots & -1 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ -1 & -1 & -1 & \cdots & n-1 \end{pmatrix}.$$

Compute $(-1)^{1+1}M_{11}$, which is the $n-1 \times n-1$ determinant formed by deleting row one and column one of $\mathbf{T}$. In M_{11}, add the last $n-2$ rows to row one to obtain a new $n-1 \times n-1$ determinant equal to M_{11}, then subtract column one of this determinant from each other column, resulting in a lower triangular matrix whose determinant equals the number of spanning trees in K_n. Evaluate the determinant of this lower triangular matrix.

Chapter 9

Eigenvalues, Diagonalization, and Special Matrices

9.1 Eigenvalues and Eigenvectors

It is possible in some of these problems for a Gershgorin circle to have zero radius. In such a case, think of the circle as degenerate, consisting of only its center (the eigenvalue).

1.

$$p_{\mathbf{A}}(\lambda) = (\lambda - 1)(\lambda - 2)(\lambda^2 + \lambda - 13),$$

$$\lambda_1 = 1, \mathbf{V}_1 = \begin{pmatrix} -2 \\ -11 \\ 0 \\ 1 \end{pmatrix}, \lambda_2 = 2, \mathbf{V}_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix},$$

$$\lambda_3 = \frac{-1 + \sqrt{53}}{2}, \mathbf{V}_3 = \begin{pmatrix} \sqrt{53} - 7 \\ 0 \\ 0 \\ 2 \end{pmatrix}, \lambda_4 = \frac{-1 - \sqrt{53}}{2}, \mathbf{V}_4 = \begin{pmatrix} -\sqrt{53} - 7 \\ 0 \\ 0 \\ 2 \end{pmatrix}.$$

The Gershgorin circles have radius 2, center $(-4, 0)$ and radius 1 and center $(3, 0)$.

3.

$$p_{\mathbf{A}}(\lambda) = (\lambda + 14)(\lambda - 2)^2,$$

$$\lambda_1 = -14, \mathbf{V}_1 = \begin{pmatrix} -16 \\ 0 \\ 1 \end{pmatrix}$$

$$\lambda_2 = \lambda_3 = 2, \mathbf{V}_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

with only one independent eigenvector associated with the multiple eigenvalue λ_2. The Gershgorin circles have radius 1, center $(-14, 0)$ and radius 3, center $(2, 0)$.

5.

$$p_{\mathbf{A}}(\lambda) = \lambda^3 - 5\lambda^2 + 6\lambda,$$

$$\lambda_1 = 0, \mathbf{V}_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \lambda_2 = 2, \mathbf{V}_2 = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \lambda_3 = 3, \mathbf{V}_3 = \begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix}.$$

The Gershgorin circle has radius 3, center the origin.

7.

$$p_{\mathbf{A}}(\lambda) = \lambda^2(\lambda + 3),$$

$$\lambda_1 = -3, \mathbf{V}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \lambda_2 = \lambda_3 = 0, \mathbf{V}_2 = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}.$$

There is only one independent eigenvector associated with eigenvalue 0. The Gershgorin circle has radius 2, center $(-3, 0)$.

9. $p_{\mathbf{A}}(\lambda) = \lambda^2 - 3\lambda + 14$,

$$\lambda_1 = (3 + \sqrt{14}i)/2, \mathbf{V}_1 = \begin{pmatrix} -1 + \sqrt{47}i \\ 4 \end{pmatrix}$$

$$\lambda_2 = (3 - \sqrt{14}i)/2, \mathbf{V}_2 = \begin{pmatrix} -1 - \sqrt{47}i \\ 4 \end{pmatrix}.$$

The Gershgorin circles have radius 6, center $(1, 0)$ and radius 2, center $(2, 0)$.

11.

$$p_{\mathbf{A}}(\lambda) = \lambda(\lambda^2 - 8\lambda + 7),$$

$$\lambda_1 = 0, \mathbf{V}_1 \begin{pmatrix} 14 \\ 7 \\ 10 \end{pmatrix}, \lambda_2 = 1, \mathbf{V}_2 = \begin{pmatrix} 6 \\ 0 \\ 5 \end{pmatrix}, \lambda_3 = 7, \mathbf{V}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

The Gershgorin circles have radius 2, center $(1, 0)$ and radius 5, center $(7, 0)$.

13.

$$p_{\mathbf{A}}(\lambda) = |\lambda\mathbf{I} - \mathbf{A}| = \lambda^2 - 2\lambda - 5$$

with roots (eigenvalues of $\mathbf{A}$) $\lambda_1 = 1 + \sqrt{6}$ and $\lambda_2 = 1 - \sqrt{6}$. Corresponding eigenvectors are

$$\mathbf{V}_1 = \begin{pmatrix} \sqrt{6} \\ 2 \end{pmatrix}, \mathbf{V}_2 = \begin{pmatrix} -\sqrt{6} \\ 2 \end{pmatrix}.$$

The Gershgorin circles are of radius 3 about $(1, 0)$ and radius 2 about $(1, 0)$. These enclose the eigenvalues.

15.

$$p_{\mathbf{A}}(\lambda) = \lambda^2 + 3\lambda - 10,$$

$$\lambda_1 = -5, \mathbf{V}_1 = \begin{pmatrix} 7 \\ -1 \end{pmatrix}, \lambda_2 = 2, \mathbf{V}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The Gershgorin circle has radius 1 and center $(2, 0)$.

17.

$$p_{\mathbf{A}}(\lambda) = \lambda^2 - 10\lambda - 23,$$

so eigenvalues are $\lambda_1 = 5 + \sqrt{2}$ and $\lambda_2 = 5 - \sqrt{2}$. Corresponding eigenvectors are

$$\mathbf{V}_1 = \begin{pmatrix} 1 + \sqrt{2} \\ 1 \end{pmatrix} \text{ and } \mathbf{V}_2 = \begin{pmatrix} 1 - \sqrt{2} \\ 1 \end{pmatrix}.$$

These eigenvectors are orthogonal.

19.
$$p_{\mathbf{A}}(\lambda) = (\lambda - 3)(\lambda^2 + 2\lambda - 1),$$
so eigenvalues are $\lambda_1 = 3$, $\lambda_2 = 1 + \sqrt{2}$ and $\lambda_3 = -1 - \sqrt{2}$. Corresponding eigenvectors are
$$\mathbf{V}_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \mathbf{V}_2 = \begin{pmatrix} 1+\sqrt{2} \\ 1 \\ 0 \end{pmatrix}, \text{ and } \mathbf{V}_3 = \begin{pmatrix} 1-\sqrt{2} \\ 1 \\ 0 \end{pmatrix}.$$
These eigenvectors are mutually orthogonal.

21.
$$p_{\mathbf{A}}(\lambda) = \lambda^2 - 5\lambda,$$
$$\lambda_1 = 0, \mathbf{V}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \lambda_2 = 5, \mathbf{V}_2 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}.$$
By taking the dot product of the eigenvectors, we see that they are orthogonal.

23. We know that $\mathbf{AE} = \lambda\mathbf{E}$. Then
$$\mathbf{A}^2\mathbf{E} = \mathbf{A}(\mathbf{AE}) = \mathbf{A}(\lambda\mathbf{E}) = \lambda(\mathbf{AE}) = \lambda^2\mathbf{E}.$$
This means that λ^2 is an eigenvalue of $\mathbf{E}^2$, with eigenvector $\mathbf{E}$. The general result follows now from an induction on k to show that $\mathbf{A}^k = \lambda^k\mathbf{E}$.

24. **Hint** The characteristic polynomial of $\mathbf{A}$ is $p_{\mathbf{A}}(\lambda) = |\lambda\mathbf{I} - \mathbf{A}| = 0$, and the constant term is obtained by setting $\lambda = 0$.

9.2 Diagonalization

1.
$$p_{\mathbf{A}}(\lambda) = \lambda(\lambda - 5)(\lambda + 2),$$
and the eigenvalues of $\mathbf{A}$ are $\lambda_1 = 0$, $\lambda_2 = 5$ and $\lambda_3 = -2$. Corresponding eigenvectors are
$$\mathbf{V}_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \mathbf{V}_2 = \begin{pmatrix} 5 \\ 1 \\ 0 \end{pmatrix} \text{ and } \mathbf{V}_3 = \begin{pmatrix} 0 \\ -3 \\ 2 \end{pmatrix}.$$
Form
$$\mathbf{P} = \begin{pmatrix} 0 & 5 & 0 \\ 1 & 1 & -3 \\ 0 & 0 & 2 \end{pmatrix}$$
Then
$$\mathbf{P}^{-1}\mathbf{AP} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

3.
$$p_{\mathbf{A}}(\lambda) = (\lambda + 2)^2(\lambda - 1),$$
so eigenvalues and corresponding eigenvectors are
$$\lambda_1 = 1, \mathbf{V}_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \lambda_2 = \lambda_3 = -2, \mathbf{V}_2 = \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix}.$$
$\mathbf{A}$ does not have three linearly independent eigenvectors (the repeated eigenvalue has only one independent eigenvector), and so is not diagonalizable.

5.
$$p_{\mathbf{A}}(\lambda) = (\lambda - 1)(\lambda - 4)(\lambda^2 + 5\lambda + 5),$$
so eigenvalues and eigenvectors are $\lambda_1 = 1$, $\lambda_2 = 4$, $\lambda_3 = (-5 + \sqrt{5})/2$ and $\lambda_4 = (-5 - \sqrt{5})/2$. Corresponding eigenvectors are
$$\mathbf{V}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \mathbf{V}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix},$$
$$\mathbf{V}_3 = \begin{pmatrix} 0 \\ (2 - 3\sqrt{5})/41 \\ (-1 + \sqrt{5})/2 \\ 1 \end{pmatrix} \text{ and } \mathbf{V}_4 = \begin{pmatrix} 0 \\ (2 + 3\sqrt{5})/41 \\ (-1 - \sqrt{5})/2 \\ 1 \end{pmatrix}.$$
Let
$$\mathbf{P} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & (2 - 3\sqrt{5})/41 & (2 + 3\sqrt{5})/41 \\ 0 & 0 & (-1 + \sqrt{5})/2 & (-1 - \sqrt{5})/2 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$
Then
$$\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & (-5 + \sqrt{5})/2 & 0 \\ 0 & 0 & 0 & (-5 - \sqrt{5})/2 \end{pmatrix}.$$

7.
$$p_{\mathbf{A}}(\lambda) = \lambda^2 - 3\lambda + 4$$
is the characteristic polynomial, with roots $\lambda_1 = (3 + \sqrt{7}i)/2$ and $\lambda_2 = (3 - \sqrt{7}i)/2$. Corresponding eigenvectors are
$$\mathbf{V}_1 = \begin{pmatrix} -3 + \sqrt{7}i \\ 8 \end{pmatrix} \text{ and } \mathbf{V}_2 = \begin{pmatrix} -3 - \sqrt{7}i \\ 8 \end{pmatrix}.$$
The matrix
$$\mathbf{P} = \begin{pmatrix} -3 + \sqrt{7}i & -3 - \sqrt{7}i \\ 8 & 8 \end{pmatrix}$$
diagonalizes $\mathbf{A}$ and
$$\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \begin{pmatrix} (3 + \sqrt{7}i)/2 & 0 \\ 0 & (3 - \sqrt{7}i)/2 \end{pmatrix}.$$
If we wrote the eigenvectors in the other order in forming $\mathbf{P}$, then the columns of $\mathbf{P}^{-1}\mathbf{A}\mathbf{P}$ would be reversed.

9.
$$p_{\mathbf{A}}(\lambda) = \lambda^2 - 2\lambda + 1,$$
so the eigenvalues are $\lambda_1 = \lambda_2 = 1$. Every eigenvector is a scalar multiple of
$$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$
so $\mathbf{A}$ is not diagonalizable.

11. Since $\mathbf{P}$ diagonalizes $\mathbf{A}$, $\mathbf{P}^{-1}\mathbf{AP} = \mathbf{D}$, a diagonal matrix having the eigenvalues of $\mathbf{A}$ along its main diagonal. Then $\mathbf{A} = \mathbf{PDP}^{-1}$, and

$$\begin{aligned}\mathbf{A}^k &= (\mathbf{PDP}^{-1})^k \\ &= (\mathbf{PDP}^{-1})(\mathbf{PDP}^{-1})\cdots(\mathbf{PDP}^{-1}) \\ &= \mathbf{PD}^k\mathbf{P}^{-1},\end{aligned}$$

with the interior pairings of $\mathbf{P}^{-1}\mathbf{P}$ canceling.

13. Eigenvalues of $\mathbf{A}$ are $\lambda_1 = \sqrt{2}$ and $\lambda_2 = -\sqrt{2}$, with corresponding eigenvectors

$$\mathbf{V}_1 = \begin{pmatrix}\sqrt{2}\\1\end{pmatrix}, \mathbf{V}_2 = \begin{pmatrix}-\sqrt{2}\\1\end{pmatrix}.$$

Let

$$\mathbf{P} = \begin{pmatrix}\sqrt{2} & -\sqrt{2}\\ 1 & 1\end{pmatrix}.$$

We find that

$$\mathbf{P}^{-1} = \begin{pmatrix}\sqrt{2}/4 & 1/2\\ -\sqrt{2}/4 & 1/2\end{pmatrix}.$$

Then

$$\begin{aligned}\mathbf{A}^{43} &= \begin{pmatrix}\sqrt{2} & -\sqrt{2}\\ 1 & 1\end{pmatrix}\begin{pmatrix}(\sqrt{2})^{43} & 0\\ 0 & (-\sqrt{2})^{43}\end{pmatrix}\begin{pmatrix}\sqrt{2}/4 & 1/2\\ -\sqrt{2}/4 & 1/2\end{pmatrix} \\ &= \begin{pmatrix}0 & 2^{22}\\ 2^{21} & 0\end{pmatrix}.\end{aligned}$$

15.

$$p_{\mathbf{A}}(\lambda) = \lambda^2 + 6\lambda + 5,$$

so the eigenvalues are -1 and -5. Form $\mathbf{P}$ using corresponding eigenvectors as columns:

$$\mathbf{P} = \begin{pmatrix}4 & 0\\ 1 & 1\end{pmatrix}.$$

Then

$$\begin{aligned}\mathbf{A}^{18} &= \mathbf{PAP}^{-1} \\ &= \begin{pmatrix}4 & 0\\ 1 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & 5^{18}\end{pmatrix}\begin{pmatrix}1/4 & 0\\ -1/4 & 1\end{pmatrix} \\ &= \begin{pmatrix}1 & 0\\ (1-5^{18})/4 & 5^{18}\end{pmatrix}.\end{aligned}$$

16. **Hint** Since $\mathbf{A}^2$ is diagonalizable, $\mathbf{A}^2$ has n linearly independent eigenvectors $\mathbf{X}_1, \cdots, \mathbf{X}_n$, with associated eigenvalues $\lambda_1, \cdots, \lambda_n$, respectively. Show that, for $j = 1, 2, \cdots, n$,

$$p_{\mathbf{A}^2}(\lambda) = p_{\mathbf{A}}(\sqrt{\lambda_j})p_{\mathbf{A}}(-\sqrt{\lambda_j}) = O.$$

for $j = 1, 2, \cdots, n$. Use this to show that $\mathbf{A}$ has n linearly independent eigenvectors.

9.3 Some Special Types of Matrices

In Problems 1 - 12, begin by finding an orthogonal set of eigenvectors. Since any nonzero constant times an eigenvector is also an eigenvector, multiply each eigenvector by the reciprocal of its magnitude to obtain an orthonormal set of eigenvectors. The matrix $\mathbf{Q}$ is an orthogonal matrix that diagonalizes the given matrix.

For Problems 7 - 12, orthogonal eigenvectors were requested in Problems 17 - 22 of Section 9.1, so for these problems all that is needed is to normalize these eigenvectors.

1.
$$p_{\mathbf{A}}(\lambda) = \lambda(\lambda^2 - \lambda - 4)$$
and the eigenvalues are 0, $(1+\sqrt{17})/2$, $(1-\sqrt{17})/2$. Corresponding eigenvectors are
$$\begin{pmatrix}1\\0\\0\end{pmatrix}, \mathbf{V}_2 \begin{pmatrix}0\\-1-\sqrt{17}\\4\end{pmatrix}, \mathbf{V}_3 = \begin{pmatrix}0\\-1-\sqrt{17}\\4\end{pmatrix}.$$
Then
$$\mathbf{Q} = \begin{pmatrix}1 & 0 & 0\\ 0 & (-1-\sqrt{17})/\sqrt{34+2\sqrt{17}} & (-1+\sqrt{17})/\sqrt{34+2\sqrt{17}}\\ 0 & 4/\sqrt{34+2\sqrt{17}} & 4/\sqrt{34+2\sqrt{17}}\end{pmatrix}.$$

3.
$$p_{\mathbf{A}}(\lambda) = \lambda^2(\lambda^2 - 2\lambda - 3)$$
so the eigenvalues are $0, 0, -1$ and 3. Corresponding eigenvectors are
$$\mathbf{V}_1 = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \mathbf{V}_2 = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}, \mathbf{V}_3 = \begin{pmatrix}0\\1\\1\\0\end{pmatrix}, \mathbf{V}_4 = \begin{pmatrix}0\\-1\\1\\0\end{pmatrix}.$$
Then
$$\mathbf{Q} = \begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 0 & 1/\sqrt{2} & -1/\sqrt{2}\\ 0 & 0 & 1/\sqrt{2} & 1/\sqrt{2}\\ 0 & 1 & 0 & 0\end{pmatrix}.$$

5.
$$p_{\mathbf{A}}(\lambda) = \lambda(\lambda^2 - 5\lambda - 4)$$
and $\mathbf{A}$ has eigenvalues 0, $(5+\sqrt{41})/2$ and $5-\sqrt{41})/2$, with corresponding eigenvectors
$$\mathbf{V}_1 = \begin{pmatrix}0\\1\\0\end{pmatrix}, \mathbf{V}_2 = \begin{pmatrix}5+\sqrt{41}\\0\\4\end{pmatrix}, \mathbf{V}_3 = \begin{pmatrix}5-\sqrt{41}\\0\\4\end{pmatrix}.$$
Normalize these to find an orthogonal matrix that diagonalizes $\mathbf{A}$:
$$\mathbf{Q} = \begin{pmatrix}0 & (5+\sqrt{41})/\sqrt{82+10\sqrt{41}} & (5-\sqrt{41})/\sqrt{82-10\sqrt{41}}\\ 1 & 0 & 0\\ 0 & 4/\sqrt{82+10\sqrt{41}} & 4/\sqrt{82-10\sqrt{41}}\end{pmatrix}.$$

7. In Problem 21 of Section 9.1 we found the orthogonal eigenvectors

$$\mathbf{V}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \mathbf{V}_2 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}.$$

Divide each by its length $\sqrt{5}$ and use the resulting orthonormal vectors as columns of $\mathbf{Q}$:

$$\mathbf{Q} = \begin{pmatrix} 1\sqrt{5} & -2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{pmatrix}.$$

$\mathbf{Q}$ is an orthogonal matrix that diagonalizes $\mathbf{A}$.

9. We know the eigenvectors

$$\mathbf{V}_1 = \begin{pmatrix} 1+\sqrt{2} \\ 1 \end{pmatrix} \text{ and } \mathbf{V}_2 = \begin{pmatrix} 1-\sqrt{2} \\ 1 \end{pmatrix}.$$

Divide each by its length to form

$$\mathbf{Q} = \begin{pmatrix} \frac{1+\sqrt{2}}{\sqrt{4+2\sqrt{2}}} & \frac{1-\sqrt{2}}{\sqrt{4-2\sqrt{2}}} \\ \frac{1}{\sqrt{4+2\sqrt{2}}} & \frac{1}{\sqrt{4-2\sqrt{2}}} \end{pmatrix}$$

.

11. Eigenvectors are

$$\mathbf{V}_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \mathbf{V}_2 = \begin{pmatrix} 1+\sqrt{2} \\ 1 \\ 0 \end{pmatrix}, \text{ and } \mathbf{V}_3 = \begin{pmatrix} 1-\sqrt{2} \\ 1 \\ 0 \end{pmatrix}.$$

Normalize these to form columns of $\mathbf{Q}$:

$$\mathbf{Q} = \begin{pmatrix} 0 & \frac{1+\sqrt{2}}{\sqrt{4+2\sqrt{2}}} & \frac{1-\sqrt{2}}{\sqrt{4-2\sqrt{2}}} \\ 0 & \frac{1}{\sqrt{4+2\sqrt{2}}} & \frac{1}{\sqrt{4-2\sqrt{2}}} \\ 1 & 0 & 0 \end{pmatrix}.$$

13. This matrix S is skew-hermitian, since

$$\mathbf{S}^t = -\overline{\mathbf{S}}.$$

$$p_{\mathbf{s}}(\lambda) = \lambda(\lambda^2 + 3)$$

so the eigenvalues are 0, $\sqrt{3}i$ and $-\sqrt{3}i$, with corresponding eigenvectors

$$\begin{pmatrix} 2 \\ 0 \\ 1+i, \end{pmatrix} \begin{pmatrix} 1 \\ \sqrt{3}i \\ -1-i \end{pmatrix}, \text{ and } \begin{pmatrix} 1 \\ -\sqrt{3}i \\ -1-i \end{pmatrix}.$$

Let $\mathbf{P}$ have these eigenvectors as columns (in the given order). Then

$$\mathbf{P}^{-1}\mathbf{S}\mathbf{P} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \sqrt{3}i & 0 \\ 0 & 0 & -\sqrt{3}i \end{pmatrix}.$$

15. The matrix is not hermitian, skew-hermitian or unitary. Compute

$$p_{\mathbf{A}}(\lambda) = (\lambda - 2)^2.$$

The eigenvalue is 2 with multiplicity 2 and only one independent eigenvector,

$$\begin{pmatrix} i \\ 1 \end{pmatrix}.$$

Therefore $\mathbf{A}$ is not diagonalizable.

17. $\mathbf{H}$ is hermitian with eigenvalues $0, 4 + 3\sqrt{2}$ and $4 - 3\sqrt{2}$. Corresponding eigenvectors are

$$\begin{pmatrix} 0 \\ i \\ 1 \end{pmatrix}, \begin{pmatrix} 4 + 3\sqrt{2} \\ -1 \\ -i \end{pmatrix} \text{ and } \begin{pmatrix} 4 - 3\sqrt{2} \\ -1 \\ -i \end{pmatrix}.$$

The matrix having these eigenvectors as columns diagonalizes $\mathbf{H}$.

19. The matrix is hermitian, since $\mathbf{H}^t = \overline{\mathbf{H}}$. The eigenvalues are approximately $\lambda_1 = 4.051374$, $\lambda_2 = 0.482696$, $\lambda_3 = -1.53407$. These are distinct, so the matrix is diagonalizable.

21. The matrix $\mathbf{S}$ is skew-hermitian with approximate eigenvalues $-2.164248i$, $0.772866i$ and $2.39182i$. Since these are distinct, $\mathbf{S}$ is diagonalizable.

23. The matrix is

$$\begin{pmatrix} 5 & 2 \\ 2 & 2 \end{pmatrix}$$

with eigenvalues $1, 6$. The standard form is

$$y_1^2 + 6y_2^2.$$

25. The matrix of this form is

$$\mathbf{A} = \begin{pmatrix} 4 & -6 \\ -6 & 1 \end{pmatrix}.$$

The eigenvalues are $(5 + \sqrt{153})/2$ and $(5 - \sqrt{153})/2$. The standard form is

$$\left(\frac{5 + \sqrt{153}}{2}\right) y_1^2 + \left(\frac{5 - \sqrt{153}}{2}\right) y_2^2.$$

27. The matrix is

$$\begin{pmatrix} 4 & -2 \\ -2 & 1 \end{pmatrix}$$

with eigenvalues $(3 + \sqrt{17})/2$ and $(3 - \sqrt{17})/2$. The standard form is

$$\left(\frac{3 + \sqrt{17}}{2}\right) y_1^2 + \left(\frac{3 - \sqrt{17}}{2}\right) y_2^2.$$

29. If S is skew-hermitian, then $\mathbf{S}^t = -\overline{\mathbf{S}}$, so $s_{jj} = -\overline{s_{jj}}$ for $j = 1, \cdots, n$. Now write $s_{jj} = a_{jj} + ib_{jj}$. Then $a_{jj} = -a_{jj}$, so each $a_{jj} = 0$. This makes the diagonal elements zero (if $b = 0$) or pure imaginary (if $b \neq 0$).

In computations involving complex matrices, we assume the easily verified facts that the conjugate of a product of complex matrices is the product of the conjugates of these matrices, and the conjugate of a transpose is the transpose of the conjugate.

31. If $\mathbf{A}$ is hermitian, then $\mathbf{A}^t = \overline{\mathbf{A}}$, so

$$\overline{(\mathbf{A}\mathbf{A}^t)} = \overline{\mathbf{A}}(\overline{\mathbf{A}})^t = \overline{\mathbf{A}}(\mathbf{A}^t)^t = \overline{\mathbf{A}}\mathbf{A}.$$

32. **Hint** If $\mathbf{H}$ is hermitian, then $\mathbf{H}^t = \overline{\mathbf{H}}$. What does this say about h_{jj}?

Chapter 10

Systems of Linear Differential Equations

10.1 Linear Systems

1. The coefficient matrix is

$$\mathbf{A} = \begin{pmatrix} 3 & 8 \\ 1 & -1 \end{pmatrix}.$$

A fundamental matrix is

$$\mathbf{\Omega}(t) = \begin{pmatrix} 4e^{(1+2\sqrt{3})t} & 4e^{(1-2\sqrt{3})t} \\ (-1+\sqrt{3})e^{(1+2\sqrt{3})t} & (-1-\sqrt{3})e^{(1-2\sqrt{3})t} \end{pmatrix}.$$

Notice that $|\mathbf{\Omega}(0)| = -8\sqrt{3} \neq 0$. The general solution is $\mathbf{X}(t) = \mathbf{\Omega}(t)\mathbf{C}$. For the initial value problem, choose

$$\mathbf{C} = \mathbf{\Omega}^{-1}(0)\mathbf{X}(0) = -\frac{1}{8\sqrt{3}} \begin{pmatrix} -1-\sqrt{3} & -4 \\ 1-\sqrt{3} & 4 \end{pmatrix} \begin{pmatrix} 2 \\ 2 \end{pmatrix}$$

$$= \frac{1}{12} \begin{pmatrix} 3+5\sqrt{3} \\ 3-5\sqrt{3} \end{pmatrix}.$$

The unique solution of the initial value problem is (after some manipulation),

$$\mathbf{X}(t) = \mathbf{\Omega}(t)\mathbf{C} = \begin{pmatrix} 2e^t \cosh(2\sqrt{3}t) + (10/\sqrt{3})e^t \sinh(2\sqrt{3}t) \\ 2e^t \cosh(2\sqrt{3}t) - (1/\sqrt{3})e^t \sinh(2\sqrt{3}t) \end{pmatrix}.$$

3. The coefficient matrix is

$$\mathbf{A} = \begin{pmatrix} 5 & -4 & 4 \\ 12 & -11 & 12 \\ 4 & -4 & 5 \end{pmatrix}$$

A fundamental matrix is

$$\mathbf{\Omega}(t) = \begin{pmatrix} e^t & 0 & e^{-3t} \\ 0 & e^t & 3e^{-3t} \\ -e^t & e^t & e^{-3t} \end{pmatrix}.$$

Then $|\mathbf{\Omega}(0)| = -1 \neq 0$. The general solution is $\mathbf{X}(t) = \mathbf{\Omega}(t)\mathbf{C}$. For the initial value problem, choose

$$\mathbf{C} = \mathbf{\Omega}^{-1}(0)\mathbf{X}(0) = \begin{pmatrix} 2 & -1 & 1 \\ 3 & -2 & 3 \\ -1 & 1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ -3 \\ 5 \end{pmatrix} = \begin{pmatrix} 10 \\ 24 \\ -9 \end{pmatrix}.$$

This gives us the unique solution

$$\mathbf{X}(t) = \mathbf{\Omega}(t) = \begin{pmatrix} 10e^t - 9e^{-3t} \\ 24e^t - 27e^{-3t} \\ 14e^t - 9e^{-3t} \end{pmatrix}.$$

5. The system is $\mathbf{X}' = \mathbf{AX}$, where

$$\mathbf{A} = \begin{pmatrix} 5 & 3 \\ 1 & 3 \end{pmatrix}.$$

Two linearly independent solutions are

$$\mathbf{\Phi}_1(t) = \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{2t} \text{ and } \mathbf{\Phi}_2(t) = \begin{pmatrix} 3 \\ 1 \end{pmatrix} e^{6t}.$$

Using $t = 0$, form the determinant having columns $\mathbf{\Phi}_1(0)$ and $\mathbf{\Phi}_1(0)$:

$$\begin{vmatrix} -1 & 3 \\ 1 & 1 \end{vmatrix} = -4 \neq 0,$$

Therefore these solutions are linearly independent. We can form the fundamental matrix using these solutions as columns:

$$\mathbf{\Omega}(t) = \begin{pmatrix} -e^{2t} & 3e^{6t} \\ e^{2t} & e^{6t} \end{pmatrix}.$$

In terms of the fundamental matrix, the general solution of the system is $\mathbf{X}(t) = \mathbf{\Omega}(t)\mathbf{C}$, where

$$\mathbf{C} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}.$$

To satisfy the initial condition $x_1(0) = 0, x_2(0) = 4$, solve for $\mathbf{C}$ in $\mathbf{X}(0) = \mathbf{\Omega}(0)\mathbf{C}$, which is

$$\begin{pmatrix} 0 \\ 4 \end{pmatrix} = \begin{pmatrix} -1 & 3 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}.$$

Then

$$\begin{pmatrix} 0 \\ 4 \end{pmatrix} = \mathbf{\Omega}^{-1}(0) \begin{pmatrix} 0 \\ 4 \end{pmatrix}$$

$$= -\frac{1}{4} \begin{pmatrix} 1 & -3 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 4 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}.$$

The solution of the initial value problem is

$$\mathbf{X}(t) = \mathbf{\Omega}(t) \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} -3e^{2t} + 3e^{6t} \\ 3e^{2t} + e^{6t} \end{pmatrix}.$$

10.2 Solution of $\mathbf{X}' = \mathbf{AX}$ for Constant $\mathbf{A}$

In each of Problems 1 and 9, we give just the unique solution of the initial value problem.

1.

$$\mathbf{X}(t) = \begin{pmatrix} 0 & e^{2t} & 3e^{3t} \\ 1 & e^{2t} & e^{3t} \\ 1 & 0 & e^{3t} \end{pmatrix} \begin{pmatrix} 0 & 1 & 3 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 5 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 4e^{2t} - 3e^{3t} \\ 2 + 4e^{2t} - e^{3t} \\ 2 - e^{3t} \end{pmatrix}$$

For Problems 3 and 5, we give the solution in the form $\mathbf{X}(t) = \mathbf{\Omega}(t)\mathbf{C}$, with $\mathbf{\Omega}(t)$ a fundamental matrix. Note that we can read the eigenvalues and corresponding eigenvectors of the coefficient matrix from the fundamental matrix.

3.

$$\mathbf{X}(t) = \mathbf{\Omega}(t)\mathbf{C} = \begin{pmatrix} 1 & 2e^{3t} & -e^{-4t} \\ 6 & 3e^{3t} & 2e^{-4t} \\ -13 & -2e^{3t} & e^{-4t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$$

$$= \begin{pmatrix} c_1 + 2c_2e^{3t} - c_3e^{-4t} \\ 6c_1 + 3c_2e^{3t} + 2c_3e^{-4t} \\ -13c_1 - 2c_2e^{3t} + c_3e^{-4t} \end{pmatrix}$$

5.

$$\mathbf{X}(t) = \mathbf{\Omega}(t)\mathbf{C} = \begin{pmatrix} 1 & e^{2t} \\ -1 & e^{2t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} c_1 + c_2e^{2t} \\ -c_1 + c_2e^{2t} \end{pmatrix}$$

7. The coefficient matrix is

$$\mathbf{A} = \begin{pmatrix} 3 & 0 \\ 5 & -4 \end{pmatrix}.$$

The characteristic polynomial of $\mathbf{A}$ is $p_{\mathbf{A}}(\lambda) = (\lambda - 3)(\lambda + 4)$. Eigenvalues and corresponding eigenvectors are

$$3, \begin{pmatrix} 7 \\ 5 \end{pmatrix}, -4, \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

A fundamental matrix is

$$\mathbf{\Omega}(t) = \begin{pmatrix} 7e^{3t} & 0 \\ 5e^{3t} & e^{-4t} \end{pmatrix}.$$

The general solution is

$$\mathbf{X}(t) = \mathbf{\Omega}(t)\mathbf{C} = \begin{pmatrix} 7c_1e^{3t} \\ 5c_1e^{3t} + c_2e^{-4t} \end{pmatrix}.$$

9.

$$\mathbf{X}(t) = \begin{pmatrix} 2e^{4t} & e^{-3t} \\ -3e^{4t} & 2e^{-3t} \end{pmatrix} \begin{pmatrix} 2 & 1 \\ -3 & 2 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ -19 \end{pmatrix} = \begin{pmatrix} 6e^{4t} - 5e^{-3t} \\ -9e^{4t} - 10e^{-3t} \end{pmatrix}$$

11. Eigenvalues of $\mathbf{A}$ are roots of $\lambda^2-2\lambda+2$, and are $\lambda_1 = 1+i$ and $\lambda_2 = 1-i$, with corresponding eigenvectors

$$\begin{pmatrix} 5 \\ 2-i \end{pmatrix} \text{ and } \begin{pmatrix} 5 \\ 2+i \end{pmatrix}.$$

A real fundamental matrix is

$$\mathbf{\Omega}(t) = \begin{pmatrix} 5e^t\cos(t) & 5e^t\sin(t) \\ e^t(2\cos(t)+\sin(t)) & e^t(2\sin(t)-\cos(t)) \end{pmatrix}.$$

13. Eigenvalues are $\mathbf{A}$ are roots of $(\lambda+2)(\lambda^2+2\lambda+5)$ and are $\lambda_1 = -2$, $\lambda_2 = -1+2i$ and $\lambda_3 = -1-2i$, with corresponding eigenvectors

$$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1+2i \\ 3 \end{pmatrix} \text{ and } \begin{pmatrix} 1 \\ 1-2i \\ 3 \end{pmatrix}.$$

A real fundamental matrix is

$$\mathbf{\Omega}(t) = \begin{pmatrix} 0 & e^{-t}\cos(2t) & e^{-t}\sin(2t) \\ 0 & e^{-t}(\cos(2t)-2\sin(2t)) & e^{-t}(\sin(2t)+2\cos(2t)) \\ e^{-2t} & 3e^{-t}\cos(2t) & 3e^{-t}\sin(2t) \end{pmatrix}.$$

15. Eigenvalues of $\mathbf{A}$ are roots of $\lambda^2-4\lambda+8$, and are $\lambda_1 = 2+2i$ and $\lambda_2 = 2-2i$, with corresponding eigenvectors

$$\begin{pmatrix} 2 \\ -i \end{pmatrix} \text{ and } \begin{pmatrix} 2 \\ i \end{pmatrix}.$$

A real fundamental matrix is

$$\mathbf{\Omega}(t) = \begin{pmatrix} 2e^{2t}\cos(2t) & 2e^{2t}\sin(2t) \\ e^{2t}\sin(2t)) & -e^{2t}\cos(2t) \end{pmatrix}.$$

In Problems 17, 19, and 21, we omit some of the details given in the solution of Problem 20.

17. The coefficient matrix has eigenvalue 2 with eigenvector

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

and eigenvalue 5 of multiplicity 2, with eigenvector

$$\begin{pmatrix} -3 \\ -3 \\ 1 \end{pmatrix}.$$

A fundamental matrix is

$$\mathbf{\Omega}(t) = \begin{pmatrix} e^{2t} & 3e^{5t} & 27te^{5t} \\ 0 & 3e^{5t} & (3+27t)e^{5t} \\ 0 & -e^{5t} & (2-9t)e^{5t} \end{pmatrix}.$$

19. The coefficient matrix has eigenvalue 0 with eigenvector

$$\begin{pmatrix}2\\0\\1\\0\end{pmatrix}$$

and eigenvalue 3 with eigenvector

$$\begin{pmatrix}3\\2\\2\\0\end{pmatrix}$$

and eigenvalue 1 of multiplicity 2 and two linearly independent eigenvectors

$$\begin{pmatrix}1\\0\\0\\0\end{pmatrix} \text{ and } \begin{pmatrix}0\\-2\\-2\\1\end{pmatrix}.$$

A fundamental matrix is

$$\mathbf{\Omega}(t) = \begin{pmatrix}2 & 3e^{3t} & e^t & 0\\ 0 & 2e^{3t} & 0 & -2e^t\\ 1 & 2e^{3t} & 0 & -2e^t\\ 0 & 0 & 0 & e^t\end{pmatrix}.$$

20. The coefficient matrix is

$$\mathbf{A} = \begin{pmatrix}2 & 0\\5 & 2\end{pmatrix}.$$

The eigenvalue is 2 with multiplicity 2 and one independent eigenvector, which we take to be

$$\mathbf{E}_1 = \begin{pmatrix}0\\1\end{pmatrix}.$$

One solution is

$$\mathbf{\Phi}_1(t) = e^{2t}\begin{pmatrix}0\\1\end{pmatrix}.$$

Try a second solution

$$\mathbf{\Phi}_2(t) = \mathbf{E}_1 te^{2t} + \mathbf{E}_2 e^{2t}.$$

Substitute this into the differential equation $\mathbf{X}' = \mathbf{AX}$ to obtain

$$\mathbf{E}_1 e^{2t} + 2t\mathbf{E}_1 e^{2t} + 2\mathbf{E}_2 e^{2t} = \mathbf{AE}_1 te^{2t} + \mathbf{AE}_2 e^{2t}.$$

Divide by e^{2t}. Further, $\mathbf{AE}_1 = 2\mathbf{E}_1$, so two terms in the last equation cancel. This leaves

$$\mathbf{E}_1 + 2\mathbf{E}_2 = \mathbf{AE}_2.$$

The unknown here is

$$\mathbf{E}_2 = \begin{pmatrix}e_1\\e_2\end{pmatrix}.$$

This system of equations reduces to

$$\begin{aligned}2e_1 &= 2e_1\\ 1 + 2e_2 &= 5e_1 + 2e_2.\end{aligned}$$

Then $e_1 = 1/5$ and e_2 can be any number. Choose $e_2 = 1$. Then

$$\mathbf{E}_2 = \begin{pmatrix} 1/5 \\ 1 \end{pmatrix}.$$

The second solution is

$$\mathbf{\Phi}_2(t) = \mathbf{E}_1 te^{2t} + \mathbf{E}_2 e^{2t} = \begin{pmatrix} (1/5)e^{2t} \\ te^{2t} + e^{2t} \end{pmatrix}.$$

A fundamental matrix has these two solutions as columns:

$$\mathbf{\Omega}(t) = \begin{pmatrix} 0 & (1/5)e^{2t} \\ e^{2t} & te^{2t} + e^{2t} \end{pmatrix}.$$

Different fundamental matrices can be obtained by making different choices of arbitrary constants in the derivation of this solution.

21. The coefficient matrix has eigenvalue 3 of multiplicity 2, with eigenvector

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

A fundamental matrix is

$$\mathbf{\Omega}(t) = \begin{pmatrix} e^{3t} & 2te^{3t} \\ 0 & e^{3t} \end{pmatrix}.$$

10.3 Solution of $\mathbf{X}' = \mathbf{AX} + \mathbf{G}$

For a linear system of differential equations, a fundamental matrix is not unique, and different fundamental matrices may be derived using different methods. Of course, the general solution can be written using any such fundamental matrix.

In Problems 1 and 5, where initial values are given, the method is to find the general solution of the system and then solve for the constants to satisfy the initial values. For these problems only the solution is given.

1.

$$\mathbf{X}(t) = \begin{pmatrix} (6 + 12t + (1/2)t^2)e^{-2t} \\ (2 + 12t + (1/2)t^2)e^{-2t} \\ (3 + 38t + 66t^2 + (13/6)t^3)e^{-2t} \end{pmatrix}$$

3. $\mathbf{A}$ has eigenvalue 1 with multiplicity 2 and single associated independent eigenvector

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix},$$

and eigenvalue 3 with multiplicity 2 and two associated linearly independent eigenvectors

$$\begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} \text{ and } \begin{pmatrix} 0 \\ -9 \\ 2 \\ 0 \end{pmatrix}.$$

A fundamental matrix is

$$\boldsymbol{\Omega}(t) = \begin{pmatrix} 0 & e^t & 0 & 0 \\ 0 & -2e^t & e^{3t} & -9e^{3t} \\ 0 & 0 & 0 & 2e^{3t} \\ e^t & -5te^t & e^{3t} & 0 \end{pmatrix}$$

The general solution is

$$\mathbf{X}(t) = \begin{pmatrix} c_2e^t \\ -2c_2e^t + (c_3 - 9c_4)e^{3t} + e^t \\ 2c_4e^{3t} \\ (c_1 - 5c_2t)e^t + c_3e^{3t} + (1+3t)e^t \end{pmatrix}.$$

5.

$$\mathbf{X}(t) = \begin{pmatrix} (-1-14t)e^t \\ (3-14t)e^t \end{pmatrix}$$

7. The coefficient matrix $\mathbf{A}$ has eigenvalue 3 of multiplicity 2, with one independent eigenvector

$$\begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Using the methods of Section 10.2, we find a fundamental matrix for the homogeneous system $\mathbf{X}' = \mathbf{AX}$:

$$\boldsymbol{\Omega}(t) = e^{3t}\begin{pmatrix} 1+2t & 2t \\ -2t & 1-2t \end{pmatrix}.$$

Compute

$$\boldsymbol{\Omega}^{-1}(t) = e^{-3t}\begin{pmatrix} 1-2t & -2t \\ 2t & 1+2t \end{pmatrix}.$$

Now compute a particular solution of the given nonhomogeneous system as

$$\mathbf{u}(t) = \int \boldsymbol{\Omega}^{-1}(t)\mathbf{G}(t)\,dt = \int e^{-3t}\begin{pmatrix} 1-2t & -2t \\ 2t & 1+2t \end{pmatrix}\begin{pmatrix} -3e^t \\ e^{3t} \end{pmatrix}\,dt$$

$$= \int \begin{pmatrix} 6te^{-2t} - 3e^{-2t} - 2t \\ -6te^{-2t} + 1 + 2t \end{pmatrix}\,dt = \begin{pmatrix} -3te^{-2t} - t^2 \\ (3/2)(1+2t)e^{-2t} + t + t^2 \end{pmatrix}.$$

The general solution is

$$\begin{aligned} \mathbf{X}(t) &= \boldsymbol{\Omega}(t)\mathbf{C} + \boldsymbol{\Omega}(t)\mathbf{u}(t) \\ &= e^{3t}\begin{pmatrix} 1+2t & 2t \\ -2t & 1-2t \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \\ &+e^{3t}\begin{pmatrix} 1+2t & 2t \\ -2t & 1-2t \end{pmatrix}\begin{pmatrix} -3te^{-2t} - t^2 \\ (3/2)(1+2t)e^{-2t} + t + t^2 \end{pmatrix} \\ &= \begin{pmatrix} e^{3t}(c_1(1+2t) + 2c_2t) + t^2e^{3t} \\ e^{3t}(-2c_1t + c_2(1-2t)) + (t-t^2)e^{3t} + 3e^t/2 \end{pmatrix}. \end{aligned}$$

9. $\mathbf{A}$ has repeated eigenvalue 6 and eigenvector

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

A fundamental matrix is

$$\mathbf{\Omega}(t) = e^{6t}\begin{pmatrix} 1 & 1+t \\ 1 & t \end{pmatrix}$$

and the general solution is

$$\mathbf{X}(t) = \begin{pmatrix} e^{6t}\left(c_1 + c_2(1+t) + 2t + t^2 - t^3\right) \\ e^{6t}\left(c_1 + c_2 t + 4t^2 - t^3\right) \end{pmatrix}.$$

For Problems 11, 13, 15, 17, and 19, the solution is expressed in the form $\mathbf{X}(t) = \mathbf{PZ}(t)$, where $\mathbf{P}$ is a matrix having eigenvectors of $\mathbf{A}$ as columns, $\mathbf{A}(t)$ is the solution of the uncoupled system $\mathbf{Z}' = \mathbf{DZ} + \mathbf{P}^{-1}\mathbf{G}$, and $\mathbf{D}$ is a diagonal matrix having the eigenvalues of $\mathbf{A}$ on its main diagonal.

11. The coefficient matrix is

$$\mathbf{A} = \begin{pmatrix} 6 & 5 \\ 1 & 2 \end{pmatrix}$$

with eigenvalues $1, 7$. Form $\mathbf{P}$ from corresponding eigenvectors:

$$\mathbf{P} = \begin{pmatrix} 1 & 5 \\ -1 & 1 \end{pmatrix}.$$

Then

$$\mathbf{P}^{-1} = \frac{1}{6}\begin{pmatrix} 1 & -5 \\ 1 & 1 \end{pmatrix}$$

and the system for $\mathbf{Z}$ is

$$\mathbf{Z}' = \begin{pmatrix} 1 & 0 \\ 0 & 7 \end{pmatrix}\mathbf{Z} + \frac{1}{6}\begin{pmatrix} 1 & -5 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} -4\cos(3t) \\ 8 \end{pmatrix}.$$

Solving for $\mathbf{Z}$, we obtain

$$\mathbf{Z}(t) = \begin{pmatrix} c_1 e^t + (1/15)\cos(3t) - (3/15)\sin(3t) + (20/3 \\ c_2 e^{7t} + (7/87)\cos(3t) - (2/58)\sin(3t) - 4/21 \end{pmatrix}.$$

Then

$$\mathbf{X}(t) = \mathbf{PZ}(t) =$$

$$\begin{pmatrix} c_1 e^t + 5c_2 e^{7t} + (68/145)\cos(3t) - (54/145)\sin(3t) + 40/7 \\ -c_1 e^t + c_2 e^{7t} + (2/145)\cos(3t) + (24/145)\sin(3t) - 48/7 \end{pmatrix}.$$

13. The coefficient matrix has eigenvalues 1, 2, 2. A diagonalizing matrix is

$$\mathbf{P} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

Obtain

$$\mathbf{P}^{-1} = \begin{pmatrix} -1 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{pmatrix}.$$

With $\mathbf{X} = \mathbf{PZ}$, the uncoupled system is

$$\mathbf{Z}' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}\mathbf{Z} + \begin{pmatrix} -1 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ t \\ 2e^t \end{pmatrix}.$$

The initial conditions are

$$\mathbf{Z}(0) = \mathbf{P}^{-1}\mathbf{x}(0) = \begin{pmatrix} -1 \\ 3 \\ -1 \end{pmatrix}.$$

Solve for $\mathbf{Z}$ to obtain

$$\mathbf{X} = \mathbf{PZ} = \begin{pmatrix} (-1/4)e^{2t} + (2+2t)e^t - (3/4) - (1/2)t \\ e^{2t} + (2+2t)e^t - 1 - t \\ -(5/4)e^{2t} + 2te^t - (3/4) - (1/2)t \end{pmatrix}.$$

15. The coefficient matrix

$$\mathbf{A} = \begin{pmatrix} 3 & 3 \\ 1 & 5 \end{pmatrix}$$

has eigenvalues 2 and 6. Form a matrix of eigenvectors

$$\mathbf{P} = \begin{pmatrix} 3 & 1 \\ -1 & 1 \end{pmatrix}.$$

Then

$$\mathbf{P}^{-1} = \frac{1}{4}\begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix}$$

and the uncoupled system for $\mathbf{Z}$ is

$$\mathbf{Z}' = \begin{pmatrix} 2 & 0 \\ 0 & 6 \end{pmatrix}\mathbf{Z} + \frac{1}{4}\begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 8 \\ 4e^{3t} \end{pmatrix}.$$

Solve for $\mathbf{Z}$ to obtain

$$\mathbf{Z}(t) = \begin{pmatrix} c_1e^{2t} - 1 - e^{3t} \\ c_2e^{6t} - 1/3 - e^{3t} \end{pmatrix}.$$

Then

$$\mathbf{X}(t) = \mathbf{PZ}(t) = \begin{pmatrix} 3c_1e^{2t} + c_2e^{6t} - 4e^{3t} - 10/3 \\ -c_1e^{2t} + c_2e^{6t} + 2/3 \end{pmatrix}.$$

17. The coefficient matrix is

$$\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

with eigenvalues 0 and 2. Use independent corresponding eigenvectors to form

$$\mathbf{P} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}.$$

Then

$$\mathbf{P}^{-1} = \begin{pmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{pmatrix}.$$

The uncoupled system is

$$\mathbf{Z}' = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}\mathbf{Z} + \begin{pmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{pmatrix}\begin{pmatrix} 6e^{2t} \\ 2e^{2t} \end{pmatrix},$$

with initial condition

$$\mathbf{Z}(0) = \mathbf{P}^{-1}\mathbf{X}(0) = \begin{pmatrix} 3 \\ 3 \end{pmatrix}.$$

Solve for $\mathbf{Z}$ to obtain

$$\mathbf{Z}(t) = \begin{pmatrix} 2 + e^{2t} \\ 3e^{2t} + 4te^{2t} \end{pmatrix}.$$

Then

$$\mathbf{X}(t) = \mathbf{PZ}(t) = \begin{pmatrix} 2 + 4(1+t)e^{2t} \\ -2 + 2(1+2t)e^{2t} \end{pmatrix}.$$

19. With coefficient matrix

$$\mathbf{A} = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix}$$

the eigenvalues are $\pm i$ and, from the eigenvectors, we let

$$\mathbf{P} = \begin{pmatrix} 5 & 5 \\ 2-i & 2+i \end{pmatrix}.$$

With $\mathbf{X} = \mathbf{PZ}$ the uncoupled system is

$$\mathbf{Z}' = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \mathbf{Z} + \begin{pmatrix} (1-2i)/10 & i/2 \\ (1+2i)/10 & -i/2 \end{pmatrix} \begin{pmatrix} 5\sin(t) \\ 0 \end{pmatrix},$$

with initial condition

$$\mathbf{Z}(0) = \mathbf{P}^{-1}\mathbf{X}(0) = \begin{pmatrix} 1 + i/2 \\ 1 - i/2 \end{pmatrix}.$$

Solve for z_1 and z_2 and then obtain

$$\mathbf{X} = \mathbf{PZ} = \begin{pmatrix} 10\cos(t) - (5/2)t\sin(t) - 5t\cos(t) \\ 5\cos(t) + (5/2)\sin(t) - (5/2)t\cos(t) \end{pmatrix}.$$

10.4 Exponential Matrix Solutions

Problems 1, 3, and 5 were done using MAPLE.

1. $e^{\mathbf{A}t}$ is the 2×2 matrix having elements $\alpha_{ij}(t)$, where

$$\alpha_{11}(t) = e^{13t/2}\left(\cos(\sqrt{23}t/2) - \frac{3}{\sqrt{23}}\sin(\sqrt{23}t/2)\right),$$
$$\alpha_{12}(t) = -\frac{4}{\sqrt{23}}e^{13t/2}\sin(\sqrt{23}t/2),$$
$$\alpha_{21}(t) = \frac{8}{\sqrt{23}}e^{13t/2}\sin(\sqrt{23}t/2),$$
$$\alpha_{22}(t) = e^{13t/2}\left(\cos(\sqrt{23}t/2) + \frac{3}{\sqrt{23}}\sin(\sqrt{23}t/2)\right).$$

3. $e^{\mathbf{A}t}$ has elements $a_{ij}(t)$, where

$$\begin{aligned}
a_{11}(t) &= \frac{2}{5}e^{2t} + \frac{2}{5}\cos(t) - \frac{1}{5}\sin(t),\\
a_{12}(t) &= -\frac{1}{5}e^{2t} + \frac{2}{5}\sin(t) + \frac{1}{5}\cos(t),\\
a_{13}(t) &= \frac{1}{5}e^{2t} + \frac{3}{5}\sin(t) - \frac{1}{5}\cos(t),\\
a_{21}(t) &= -\frac{3}{5}e^{2t} + \frac{3}{5}\cos(t) - \frac{4}{5}\sin(t),\\
a_{22}(t) &= \frac{1}{5}e^{2t} + \frac{4}{5}\cos(t) + \frac{3}{5}\sin(t),\\
a_{23}(t) &= -\frac{1}{5}e^{2t} + \frac{7}{5}\sin(t) + \frac{1}{5}\cos(t),\\
a_{31}(t) &= \frac{3}{5}e^{2t} - \frac{3}{5}\cos(t) - \frac{1}{5}\sin(t),\\
a_{32}(t) &= -\frac{1}{5}e^{2t} + \frac{1}{5}\cos(t) - \frac{3}{5}\sin(t),\\
a_{33}(t) &= \frac{1}{5}e^{2t} + \frac{4}{5}\cos(t) - \frac{2}{5}\sin(t).
\end{aligned}$$

5.

$$e^{\mathbf{A}t} = \begin{pmatrix} \cos(2t) - (1/2)\sin(2t) & (1/2)\sin(2t) \\ -(5/2)\sin(2t) & \cos(2t) + (1/2)\sin(2t) \end{pmatrix}$$

7. Notice that

$$\begin{aligned}
\mathbf{B}^n &= (\mathbf{P}^{-1}\mathbf{A}\mathbf{P})^n\\
&= (\mathbf{P}^{-1}\mathbf{A}\mathbf{P})(\mathbf{P}^{-1}\mathbf{A}\mathbf{P})\cdots(\mathbf{P}^{-1}\mathbf{A}\mathbf{P})\\
&= \mathbf{P}^{-1}\mathbf{A}^n\mathbf{P}.
\end{aligned}$$

Then

$$\begin{aligned}
e^{\mathbf{B}t} &= \sum_{n=0}^{\infty} \frac{1}{n!}(\mathbf{P}^{-1}\mathbf{A}\mathbf{P})^n t\\
&= \sum_{n=0}^{\infty} \frac{1}{n!}\mathbf{P}^{-1}\mathbf{A}^n\mathbf{P}t\\
&= \mathbf{P}^{-1}\left((\sum_{n=0}^{\infty} \frac{1}{n!}\mathbf{A}^n t \right)\mathbf{P}\\
&= \mathbf{P}^{-1}e^{\mathbf{A}t}\mathbf{P}.
\end{aligned}$$

9. First deal with the matrix $\mathbf{A}$ of Problem 5. The eigenvalues, with corresponding eigenvectors, are

$$2i, \begin{pmatrix} 1-2i \\ 5 \end{pmatrix}, -2i, \begin{pmatrix} 1+2i \\ 5 \end{pmatrix}.$$

The matrix

$$\mathbf{P} = \begin{pmatrix} 1-2i & 1+2i \\ 5 & 5 \end{pmatrix}$$

diagonalizes $\mathbf{A}$, so

$$\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \mathbf{D} = \begin{pmatrix} 2i & 0 \\ 0 & -2i \end{pmatrix}.$$

Now,

$$e^{\mathbf{D}t} = \begin{pmatrix} e^{2it} & 0 \\ 0 & e^{-2it} \end{pmatrix}.$$

Further, we find that

$$\mathbf{P}^{-1} = \begin{pmatrix} (1/4)i & 1/10 - i/20 \\ -(1/4)i & 1/10 + i/20 \end{pmatrix}.$$

Then

$$e^{\mathbf{A}t} = \mathbf{P}e^{\mathbf{D}t}\mathbf{P}^{-1}$$

$$= \begin{pmatrix} 1-2i & 1+2i \\ 5 & 5 \end{pmatrix} \begin{pmatrix} e^{2it} & 0 \\ 0 & e^{-2it} \end{pmatrix} \begin{pmatrix} (1/4)i & 1/10 - i/20 \\ -(1/4)i & 1/10 + i/20 \end{pmatrix}$$

$$= \begin{pmatrix} (1/2 + i/4)e^{2it} + (1/2 - i/4)e^{-2it} & (i/4)(e^{-2it} - e^{2it}) \\ (5i/4)(e^{2it} - e^{-2it}) & (1/2 - i/4)e^{2it} + (1/2 + i/4)e^{-2it} \end{pmatrix}.$$

This appears to be different from the solution obtained using MAPLE. However, recall that

$$e^{2it} = \cos(2t) + i\sin(2t) \text{ and } \sin(2t) = \cos(2t) - i\sin(2t).$$

If these are substituted into the exponential matrix we have just found, we obtain the exponential matrix produced by MAPLE.

Now turn to the matrix of Problem 4. Eigenvalues and eigenvectors are

$$-3, \begin{pmatrix} -1 \\ 1 \end{pmatrix}, 0, \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

Let

$$\mathbf{P} = \begin{pmatrix} -1 & 1 \\ 1 & 2 \end{pmatrix}.$$

Then

$$\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \mathbf{D} = \begin{pmatrix} -3 & 0 \\ 0 & 0 \end{pmatrix}.$$

Now

$$e^{\mathbf{D}t} = \begin{pmatrix} e^{-3t} & 0 \\ 0 & 1 \end{pmatrix}.$$

Then

$$\mathbf{P}e^{\mathbf{D}t}\mathbf{P}^{-1} = \frac{1}{3}\begin{pmatrix} 1 + 2e^{-3t} & 1 - e^{-3t} \\ 2 - 2e^{-3t} & 2 + e^{-3t} \end{pmatrix}.$$

Because only real quantities were involved in the computation, we obtain the same result as that returned by MAPLE.

10.5 Applications and Illustrations of Techniques

1. Let $x_j(t)$ be the number of pounds of salt in tank j at time t. Then

$$\begin{aligned} x_1' &= -\frac{4}{50}x_1 + \frac{1}{50}x_2 + 1, \\ x_2' &= \frac{1}{50}x_1 - \frac{4}{50}x_2 + 2, \\ x_1(0) &= 40, x_2(0) = 0. \end{aligned}$$

   This initial value problem has the unique solution

$$\begin{aligned} x_1(t) &= 20 + 25e^{-t/10} - 5e^{-3t/50}, \\ x_2(t) &= 30 - 25e^{-t/10} - 5e^{-3t/50}. \end{aligned}$$

   The brine in tank 1 has minimum concentration when $t = 25\ln(25/3)$ minutes. At this time there is $20-6\sqrt{3}/125$ pounds of salt in tank 1 (about 19.9 pounds). The initial amount of salt in the tank is 40 pounds and this quantity decreases to this value and then rises toward the terminal amount of 20 pounds of salt (the limit as $t \to \infty$ of $x_1(t)$).

2. **Hint** Denote the amount (in pounds) of salt in tank j at time t by $x_j(t)$. The system modeling this problem is

$$\begin{aligned} x_1' &= -16\left(\frac{x_1}{200}\right) + 12\left(\frac{x_2}{300}\right) + \frac{1}{4}(4), \\ x_2' &= 12\left(\frac{x_1}{200}\right) - 18\left(\frac{x_2}{300}\right). \end{aligned}$$

3. The capacitor charge is maximum when the capacitor voltage is maximum. This voltage is

$$V_C = \frac{q_2 - q_3}{10^{-1}} = 10(q_2 - q_3) = 5i_3.$$

   Therefore

$$V_C = 180(e^{-2t} - e^{-20t/9}).$$

   Then

$$\frac{dV_C}{dt} = 10(i_2 - i_3) = 40(1 - e^{-20t/9} - 9e^{-2t}) = 0.$$

   This occurs if

$$t = \frac{9}{2}\ln\left(\frac{10}{9}\right) \approx 0.474$$

   seconds. The capacitor voltage at this time is

$$V_C\left(\frac{9}{2}\ln\left(\frac{10}{9}\right)\right) = 20\left(\frac{9}{10}\right)^{10} \approx 6.97$$

   volts.

5. Designate down as positive, $y_1(t)$ the position of the upper weight relative to the equilibrium position of this weight, and $y_2(t)$ the position of the lower weight relative to the equilibrium position of the lower weight. Then

$$\begin{aligned} y_1'' &= -22y_1 + 6y_2, \\ y_2'' &= 6y_1 - 6y_2, \end{aligned}$$

with initial conditions

$$y_1(0) = y_2(0) = 1, y_1'(0) = y_2'(0) = 0.$$

Let $x_1 = y_1$, $x_2 = y_2$, $x_3 = y_1'$, and $x_4 = y_2'$. This converts the system of two second-order differential equations to a system of four first-order equations:

$$\begin{aligned} x_1' &= x_3, \\ x_2' &= x_4, \\ x_3' &= -22x_1 + 6x_2, \\ x_4' &= 6x_1 - 6x_2, \end{aligned}$$

with initial conditions

$$x_1(0) = x_2(0) = 1, x_3(0) = x_4(0) = 0.$$

The matrix of this system is

$$\mathbf{A} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -22 & 6 & 0 & 0 \\ 6 & -6 & 0 & 0 \end{pmatrix}$$

with eigenvalues $\pm 2i$ and $\pm 2\sqrt{6}i$. One eigenvector associated with $2i$ is

$$\begin{pmatrix} 1 \\ 3 \\ 2i \\ 6i \end{pmatrix}$$

and an eigenvector associated with $2\sqrt{6}i$ is

$$\begin{pmatrix} 3 \\ -1 \\ 6\sqrt{6}i \\ -2\sqrt{6}i \end{pmatrix}.$$

Use these to write the general solution of the system of differential equations in terms of real functions:

$$\mathbf{X}(t) = \begin{pmatrix} c_1\cos(2t) + c_2\sin(2t) + 3c_3\cos(2\sqrt{6}t) + 3c_4\sin(2\sqrt{6}t) \\ 3c_1\cos(2t) + 3c_2\sin(2t) - c_3\cos(2\sqrt{6}t) - c_4\sin(2\sqrt{6}t) \\ 2c_2\cos(2t) - 2c_1\sin(2t) + 6\sqrt{6}c_4\cos(2\sqrt{6}t) - 6\sqrt{6}c_3\sin(2\sqrt{6}t) \\ 6c_2\cos(2t) - 6c_1\sin(2t) - 2\sqrt{6}c_4\cos(2\sqrt{6}t) + 2\sqrt{6}c_3\sin(2\sqrt{6}t) \end{pmatrix}.$$

Substitute the initial conditions and recall that $y_1 = x_1$ and $y_2 = x_2$ to obtain

$$\begin{aligned} y_1(t) &= \frac{2}{5}\cos(2t) + \frac{3}{5}\cos(2\sqrt{6}t), \\ y_2(t) &= \frac{6}{5}\cos(2t) - \frac{1}{5}\cos(2\sqrt{6}t). \end{aligned}$$

7. From Kirchhoff's laws,

$$\begin{aligned} 50i_1 + 100(i_1' - i_2') &= 5, \\ 50i_1 + 1000q_2 &= 5, \\ 10(i_1' - i_2') &= 1000q_2, \\ i_1(0+) = i_2(0+) &= \frac{1}{10}. \end{aligned}$$

Using the first equation and the derivative of the second, we have the system

$$\begin{pmatrix} 2 & -2 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} i_1 \\ i_2 \end{pmatrix}' = \begin{pmatrix} -10 & 0 \\ 0 & -20 \end{pmatrix} \begin{pmatrix} i_1 \\ i_2 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

with

$$i_1(0+) = i_2(0+) = \frac{1}{10}.$$

Multiply the system on the left by

$$\begin{pmatrix} 2 & -2 \\ 1 & 0 \end{pmatrix}^{-1}$$

which is the matrix

$$\begin{pmatrix} 0 & 1 \\ -1/2 & 1 \end{pmatrix}.$$

We obtain

$$\begin{pmatrix} i_1 \\ i_2 \end{pmatrix}' = \begin{pmatrix} 0 & -20 \\ 5 & -20 \end{pmatrix} - \begin{pmatrix} 0 \\ 1/2 \end{pmatrix},$$

with the given initial conditions. The coefficient matrix of this system has repeated eigenvalue -10, and only one independent eigenvector

$$\begin{pmatrix} 2 \\ 1 \end{pmatrix},$$

or any nonzero constant multiple of this eigenvector. One solution of the associated homogeneous system is

$$\begin{pmatrix} i_1 \\ i_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{-10t}.$$

To find a second, linearly independent solution, we can apply the method of Section 10.2, obtaining

$$\begin{pmatrix} 1 + 10t \\ 5t \end{pmatrix} e^{-10t}.$$

A fundamental matrix for the homogeneous system is

$$\mathbf{\Omega}(t) = \begin{pmatrix} 2e^{-10t} & (1 + 10t)e^{-10t} \\ e^{-10t} & 5te^{-10t} \end{pmatrix}.$$

In order to use variation of parameters, we need

$$\mathbf{\Omega}^{-1}(t) = \begin{pmatrix} -5te^{10t} & (1 + 10t)e^{10t} \\ e^{10t} & -2e^{10t} \end{pmatrix}.$$

We need to integrate

$$\mathbf{\Omega}^{-1}\mathbf{G}(t) = \mathbf{\Omega}^{-1} \begin{pmatrix} 0 \\ -1/2 \end{pmatrix} = \begin{pmatrix} -(1/2)(1 + 10t)e^{10t} \\ e^{10t} \end{pmatrix}.$$

This integration gives us

$$\mathbf{u}(t) = \int \mathbf{\Omega}^{-1}\mathbf{G}(t)\, dt = \begin{pmatrix} -(1/2)te^{10t} \\ (1/10)e^{10t} \end{pmatrix}.$$

Then

$$\boldsymbol{\Omega}(t)\mathbf{u}(t) = \begin{pmatrix} 1/10 \\ 0 \end{pmatrix}$$

is a particular solution. The general solution of the nonhomogeneous system is

$$\begin{aligned} i_1(t) &= 2c_1 e^{-10t} + c_2(1+10t)e^{-10t} + \frac{1}{10}, \\ i_2(t) &= c_1 e^{-10t} + c_2 t e^{-10t}. \end{aligned}$$

Upon inserting the initial conditions, we obtain the solution for the current functions:

$$\begin{aligned} i_1(t) &= \frac{1}{10} - 2te^{-10t}, \\ i_2(t) &= \left(\frac{1}{10} - t\right) e^{-10t} \end{aligned}$$

amperes.

9. Consider the direction to the right as positive and let y_1 be the displacement of the left weight from the equilibrium position and y_2 the displacement of the right weight from its equilibrium position. The spring/mass system is modeled by the initial value problem

$$\begin{aligned} 2y_1'' &= -8y_1 + 5(y_2 - y_1), \\ 2y_2'' &= -5(y_2 - y_1) - 8y_2, \\ y_1(0) = 1, y_2(0) &= -1, y_1'(0) = y_2'(0) = 0. \end{aligned}$$

As we have done before, let

$$x_1 = y_1, x_2 = y_2, x_3 = y_1', x_4 = y_2'.$$

This gives us the first-order system

$$\begin{aligned} x_1' &= x_3, \\ x_2' &= x_4, \\ x_3' &= -\frac{13}{2}x_1 + \frac{5}{2}x_2, \\ x_4' &= \frac{5}{2}x_1 - \frac{13}{2}x_2. \\ x_1(0) = 1, x_2(0) &= -1, x_3(0) = x_4(0) = 0. \end{aligned}$$

The matrix of this system has eigenvalues $\pm 2i$ and $\pm 3i$. Eigenvectors corresponding to $2i$ and one for $3i$ are, respectively,

$$\begin{pmatrix} 1 \\ 1 \\ 2i \\ 2i \end{pmatrix} \text{ and } \begin{pmatrix} 1 \\ -1 \\ 3i \\ -3i \end{pmatrix}.$$

Using these, write the general solution of the system:

$$\mathbf{X}(\mathbf{t}) = \begin{pmatrix} c_1\cos(2t) + c_2\sin(2t) + c_3\cos(3t) + c_4\sin(4t) \\ c_1\cos(2t) + c_2\sin(2t) - c_3\cos(3t) - c_4\sin(3t) \\ 2c_2\cos(2t) - 2c_1\sin(2t) + 3c_4\cos(3t) - 3c_3\sin(3t) \\ 6c_2\cos(2t) - 6c_1\sin(2t) - 3c_4\cos(3t) + 3c_3\sin(3t) \end{pmatrix}.$$

Upon using the initial conditions and setting $y_1 = x_1$ and $y_2 = x_2$ we obtain the solution for the displacement functions:

$$\begin{aligned} y_1(t) &= \cos(3t), \\ y_2(t) &= -\cos(3t). \end{aligned}$$

11. The spring/mass system is modeled by the initial value problem

$$\begin{aligned} 5y_1'' &= -(65-\alpha)y_1 + \alpha(y_2 - y_1) - 30y_1', \\ 13y_2'' &= -\alpha(y_2 - y_1) - (65-\alpha)y_2 + 39\sin(t), \\ y_1(0) = y_2(0) &= y_1'(0) = y_2'(0) = 0. \end{aligned}$$

Let

$$x_1 = y_1, x_2 = y_2, x_3 = y_1', x_4 = y_2'.$$

This produces the first order system

$$\begin{aligned} x_1' &= x_3, \\ x_2' &= x_4, \\ x_3' &= -13x_1 + 2\sqrt{26}x_2 - 6x_3, \\ x_4' &= \frac{10\sqrt{26}}{13}x_1 - 5x_2 + 3\sin(t), \\ x_1(0) = x_2(0) &= x_3(0) = x_4(0) = 0. \end{aligned}$$

The coefficient matrix of this system has characteristic polynomial

$$p_{\mathbf{A}}(\lambda) = \lambda^4 + 6\lambda^3 + 18\lambda^2 + 30\lambda + 25$$

with roots (eigenvalues of $\mathbf{A}$) $-1 \pm 2i$ and $-2 \pm i$. An eigenvector associated with $-1 + 2i$ is

$$\begin{pmatrix} \sqrt{26} - 2\sqrt{26}i \\ 10 \\ 3\sqrt{26} + 4\sqrt{26}i \\ -10 + 20i \end{pmatrix}$$

and an eigenvector associated with $-2 + i$ is

$$\begin{pmatrix} 2\sqrt{26} - \sqrt{26}i \\ 5 \\ -3\sqrt{26} + 4\sqrt{26} \\ -10 + 5i \end{pmatrix}.$$

Use these to write the general solution in terms of real-valued functions of the associated homogeneous system:

$$
\begin{aligned}
x_1(t) &= \sqrt{26}\left[e^{-t}(c_1 - 2c_2)\cos(2t) + (2c_1 + c_2)\sin(2t)\right] \\
&+ e^{-2t}\left[(2c_3 - c_4)\cos(t) + (c_3 + 2c_4)\sin(t)\right], \\
x_2(t) &= 5e^{-t}\left[2c_1\cos(2t) + 2c_2\sin(2t)\right] \\
&+ e^{-2t}\left[c_3\cos(t) + c_4\sin(t)\right], \\
x_3(t) &= \sqrt{26}e^{-t}\left[(3c_1 + 4c_2)\cos(2t)\right] + (4c_1 + 3c_2)\sin(2t), \\
&+ e^{-2t}\left[(-3c_3 + 4c_4)\cos(t) + (-4c_3 - 3c_4)\sin(t)\right], \\
x_4(t) &= 5e^{-t}\left[(-2c_1 + 4c_2)\cos(2t) + (-4c_1 - 2c_2)\sin(2t)\right] \\
&+ e^{-2t}\left[(-2c_3 + c_4)\cos(t) + (-c_3 - 2c_4)\sin(t)\right].
\end{aligned}
$$

We also find the following solution of the nonhomogeneous system:

$$
\begin{aligned}
x_1(t) &= -\frac{9\sqrt{26}}{40}\cos(t) + \frac{3\sqrt{26}}{40}\sin(t), \\
x_2(t) &= -\frac{9}{8}\cos(t) + \frac{9}{8}\sin(t), \\
x_3(t) &= \frac{3\sqrt{26}}{40}\cos(t) + \frac{9\sqrt{26}}{40}\sin(t), \\
x_4(t) &= \frac{9}{8}\cos(t) + \frac{9}{8}\sin(t).
\end{aligned}
$$

Add this particular solution to the general solution of the associated homogeneous equation and then insert the initial conditions to solve for the constants, obtaining

$$c_1 = \frac{6}{100}, c_2 = \frac{3}{100}, c_3 = \frac{21}{200}, c_4 = -\frac{3}{200}.$$

Upon inserting these constants and putting $y_1 = x_1$ and $y_2 = x_2$ we obtain the displacement functions for the weights:

$$
\begin{aligned}
y_1(t) &= \frac{3\sqrt{26}}{40}\left[2e^{-t}\sin(2t)\right. \\
&\left.+e^{-2t}(3\cos(t) + \sin(t)) - 3\cos(t) + \sin(t)\right] \\
y_2(t) &= \frac{3}{40}\left[e^{-t}(8\cos(2t) + 4\sin(2t))\right. \\
&\left.+e^{-2t}(7\cos(t) - \sin(t)) - 15\cos(t) + 15\sin(t)\right].
\end{aligned}
$$

10.6 Phase Portraits

1. The eigenvalues are $-2 \pm \sqrt{3}i$, so the origin is a spiral point. The general solution is

$$\mathbf{X} = \begin{pmatrix} c_1e^{-2t}\cos(\sqrt{3}t) - c_2e^{-2t}\sin(\sqrt{3}t) \\ c_1e^{-2t}\sin(\sqrt{3}t) + 3c_2e^{-2t}\cos(\sqrt{3}t) \end{pmatrix}$$

Figure 10.1 is a phase portrait for this system.

3. The eigenvalues are $4 \pm 5i$ and the origin is a spiral point. The general solution is

$$\mathbf{X} = \begin{pmatrix} (3c_1 - 5c_2)e^{4t}\sin(5t) + (5c_1 + 3c_2)e^{4t}\cos(5t) \\ 2c_1e^{4t}\sin(5t) + 2c_2e^{4t}\cos(5t) \end{pmatrix}$$

Figure 10.2 is a phase portrait.

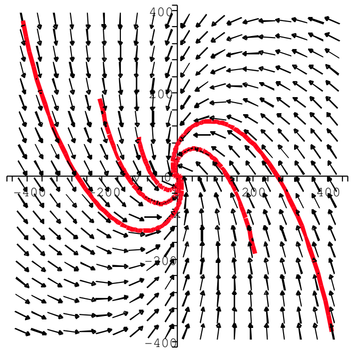

Figure 10.1: Phase portrait for Problem 1, Section 10.6.

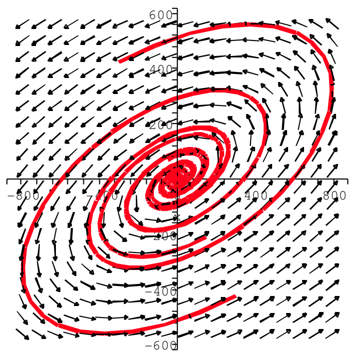

Figure 10.2: Phase portrait for Problem 3, Section 10.6.

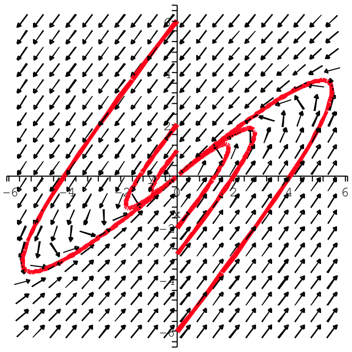

Figure 10.3: Phase portrait for Problem 7, Section 10.6.

5. The eigenvalues are $3, 3$ and the origin is an improper node. The general solution is

$$\mathbf{X} = \begin{pmatrix} c_1 e^{3t} + c_2 t e^{3t} \\ (c_1 + c_2) e^{3t} + c_2 t e^{3t} \end{pmatrix}$$

7. Eigenvalues of $\mathbf{A}$ are $-2, -2$ and the origin is an improper node. The general solution is

$$\mathbf{X}(t) = \begin{pmatrix} c_1 e^{-2t} + 5(c_1 - c_2) t e^{-2t} \\ c_2 e^{-2t} + 5(c_1 - c_2) t e^{-2t} \end{pmatrix}$$

   A phase portrait is given in Figure 10.3.

9. Eigenvalues are $\pm 2i$; the origin is a center. The general solution is

$$\mathbf{X} = \begin{pmatrix} (c_1 - 2c_2)\sin(2t) + (2c_1 + c_2)\cos(2t) \\ c_1 \sin(2t) + c_2 \cos(2t) \end{pmatrix}$$

   Figure 10.4 is a phase portrait.

11. (a) Figure 10.5.
    (c) Figure 10.6.

12. (a) Figure 10.7.
    (c) Figure 10.8.

13. Let H be the constant of proportionality for the outside agent that at any time removes members of both species at a rate proportional to their population at that time. Coupling this term with a predator/prey model, we have the system.

$$x_1' = ax_1 - bx_1x_2 - Hx_1,$$
$$x_2' = -kx_2 + cx_1x_2 - Hx_2.$$

14. (a) Figure 10.9.
    (c) Figure 10.10.

In generating phase portraits, it is sometimes necessary to experiment with various parameters, initial values, and values of the variable. Different choices can cause the program to terminate. For

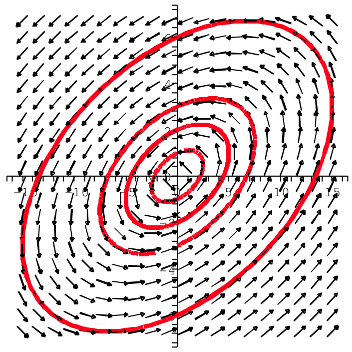

Figure 10.4: Phase portrait for Problem 9, Section 10.6.

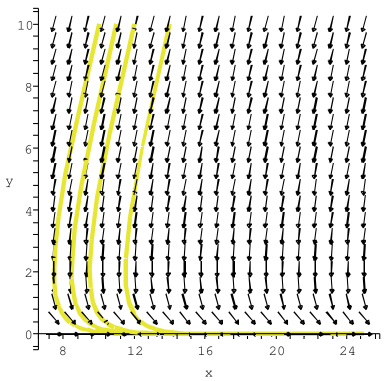

Figure 10.5: Phase portrait for Problem 11(a), Section 10.6.

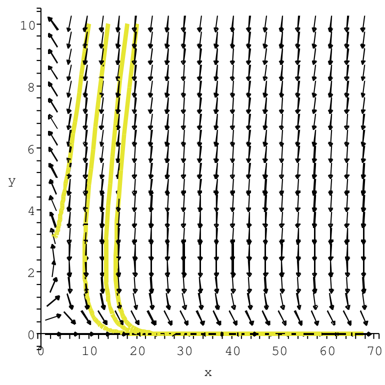

Figure 10.6: Phase portrait for Problem 11(c), Section 10.6.

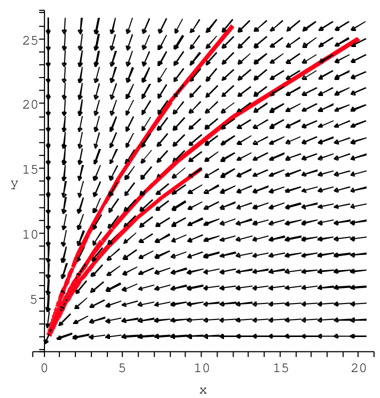

Figure 10.7: Phase portrait for Problem 12(a), Section 10.6.

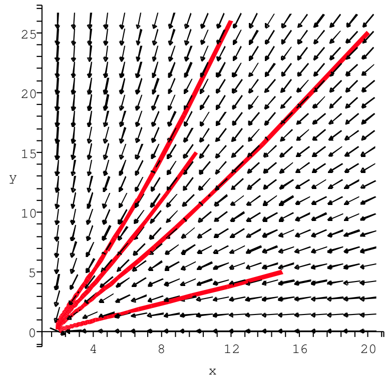

Figure 10.8: Phase portrait for Problem 12(c), Section 10.6.

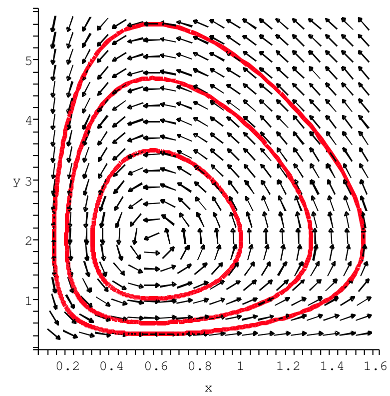

Figure 10.9: Phase portrait for Problem 14(a), Section 10.6.

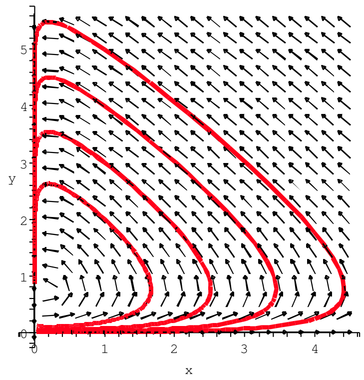

Figure 10.10: Phase portrait for Problem 14(c), Section 10.6.

example, if we specify a value of t for which values of $x(t)$ and/or $y(t)$ are undefined, then no phase portrait will be generated. In such a case, try a different range of values for t. It may also be that the initial values do not correspond to points at which trajectories are interesting or informative. In such a case experiment with different initial values.

For some types of problems, phase portraits are quantitatively similar even for different choices of various constants occurring in the differential equations. We can see this with predator/prey models, whose phase portraits have certain similarities (closed, periodic orbits in the first quadrant). However, differences caused by different choices of coefficients become apparent if the scales on the axes are noted. Often, if phase portraits appearing to be similar for two systems were drawn on the same set of axes, we would see differences in scale in the orbits.

Chapter 11

Vector Differential Calculus

11.1 Vector Functions of One Variable

For Problems 3, 5, and 8, we provide the details of the differentiation carried out both ways. For Problems 1 - 2, 4, and 6 - 7, just the derivative is given.

1.
$$(\mathbf{F}(t) \times \mathbf{G}(t))' = te^t(2+t)(\mathbf{j}-\mathbf{k})$$

3. First, applying the "product rule" for a scalar function times a vector function,

$$\begin{aligned}(f(t)\mathbf{F}'(t)) &= f'(t)\mathbf{F}(t) + f(t)\mathbf{F}'(t)\\ &= (-12\sin(3t))\mathbf{F}(t) + 4\cos(3t)[6t\mathbf{j}+2\mathbf{k}]\\ &= -12\sin(3t)\mathbf{i} + [24t\cos(3t) - 36t^2\sin(3t)]\mathbf{j} + [8\cos(3t) - 24t\sin(3t)]\mathbf{k}.\end{aligned}$$

Now first carry out the product

$$f(t)\mathbf{F}(t) = 4\cos(3t)\mathbf{i} + 12t^2\cos(3t)\mathbf{j} + 8t\cos(3t)\mathbf{k},$$

so

$$\begin{aligned}(f(t)\mathbf{F}'(t)) = -12\sin(3t)\mathbf{i} &+ (24t\cos(3t) - 36t^2\sin(3t))\mathbf{j}\\ &+ (8\cos(3t) - 24t\sin(3t))\mathbf{k}.\end{aligned}$$

5. Applying the product rule for cross products, we have

$$\begin{aligned}(\mathbf{F}(t) \times \mathbf{G}(t))' &= \mathbf{F}'(t) \times \mathbf{G}(t) + \mathbf{F}(t) \times \mathbf{G}'(t)\\ &= \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 1 & 0 & 0\\ 1 & -\cos(t) & t\end{vmatrix} + \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ t & 1 & 4\\ 0 & \sin(t) & 1\end{vmatrix}\\ &= -t\mathbf{j} - \cos(t)\mathbf{k} + ((1-4\sin(t))\mathbf{i} - t\mathbf{j} + t\sin(t)\mathbf{k})\\ &= (1-4\sin(t))\mathbf{i} - 2t\mathbf{j} - (\cos(t) - t\sin(t))\mathbf{k}.\end{aligned}$$

To carry out the cross product and then differentiate, first compute

$$\begin{aligned}\mathbf{F}(t) \times \mathbf{G}(t) &= \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ t & 1 & 4\\ 1 & -\cos(t) & t\end{vmatrix}\\ &= (t+4\cos(t))\mathbf{i} + (4-t^2)\mathbf{j} - (t\cos(t)+1)\mathbf{k}.\end{aligned}$$

Then

$$(\mathbf{F}(t) \times \mathbf{G}(t))' = (1 - 4\sin(t))\mathbf{i} \\ - 2t\mathbf{j} - (\cos(t) - t\sin(t))\mathbf{k}.$$

7.

$$(f(t)\mathbf{F}(t))' = (1 - 8t^3)\mathbf{i} + (6t^2\cosh(t) - (1 - 2t^3)\sinh(t))\mathbf{j} \\ + (e^t - 6t^2e^t - 2t^3e^t)\mathbf{k}$$

9.

$$\mathbf{F}(t) = t^2(2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k})$$

is a position vector, for $1 \leq t \leq 3$, and

$$\mathbf{F}'(t) = 2t(2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k})$$

is a tangent vector. A distance function along the curve is given by

$$s(t) = \int_1^t \| \mathbf{F}'(\tau) \| \, d\tau = 2\sqrt{29}\int_1^t \tau\, d\tau = \sqrt{29}(t^2 - 1).$$

Then $t = t(s) = \sqrt{1 + s/\sqrt{29}}$ for $0 \leq s \leq 8\sqrt{29}$. Let

$$\mathbf{F}(s) = \mathbf{G}(s) = \left(\frac{s}{\sqrt{29}} + 1\right)(2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}).$$

Then

$$\mathbf{G}'(s) = \frac{1}{\sqrt{29}}(2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k})$$

and this is a unit tangent vector because $\| \mathbf{G}'(s) \| = 1$.

11.

$$\mathbf{F}(t) = \sin(t)\mathbf{i} + \cos(t)\mathbf{j} + 45t\mathbf{k}, 0 \leq t \leq 2\pi$$

is a position vector for the curve C. Then

$$\mathbf{F}'(t) = \cos(t)\mathbf{i} - \sin(t)\mathbf{j} + 45\mathbf{k}$$

is a tangent vector. The distance function along the curve is

$$s(t) = \int_0^t \| \mathbf{F}'(\tau) \| \, d\tau = \int_0^t \sqrt{2026}\, d\tau = \sqrt{2026}t.$$

Then $t = s/\sqrt{2026}$, so in terms of the distance function, a position vector is

$$\mathbf{G}(s) = \mathbf{F}(t(s)) = \sin\left(\frac{s}{\sqrt{2026}}\right)\mathbf{i} + \cos\left(\frac{s}{\sqrt{2026}}\right)\mathbf{j} + \frac{45s}{\sqrt{2026}}\mathbf{k}.$$

This gives the tangent vector

$$\mathbf{G}'(s) = \frac{1}{\sqrt{2026}}\left[\cos\left(\frac{s}{\sqrt{2026}}\right)\mathbf{i} - \sin\left(\frac{s}{\sqrt{2026}}\right)\mathbf{j} + 45\mathbf{k}\right].$$

This is a unit tangent vector, since $\| \mathbf{G}'(s) \| = 1$.

11.2 Velocity and Curvature

In Problems 1 - 10, we can compute

$$\mathbf{v}(t) = \mathbf{F}'(t), \mathbf{a}(t) = \mathbf{F}''(t), v(t) = \| \mathbf{v}(t) \|$$

by straightforward differentiations and computation of a magnitude. In terms of t, we can compute the unit tangent vector as

$$\mathbf{T}(t) = \frac{1}{v(t)}\mathbf{v}(t) = \frac{1}{\| \mathbf{F}'(t) \|}\mathbf{F}'(t).$$

The tangential and normal components of the acceleration can be obtained as

$$a_T = \frac{dv}{dt} \text{ and } a_N = \sqrt{\| \mathbf{a} \|^2 - a_T^2}.$$

The unit normal is then

$$\mathbf{N}(t) = \frac{1}{a_N}(\mathbf{a}(t) - a_T\mathbf{T}(t)).$$

In this way we do not have to compute s and attempt to write vectors in terms of s, which is often quite awkward or even impossible in terms of elementary functions. We could also compute

$$\mathbf{N}(t) = \frac{d\mathbf{T}/dt}{\| d\mathbf{T}/dt \|},$$

which is a fairly straightforward calculation for finding a unit normal vector.

Finally, the curvature is conveniently computed in terms of t as

$$\kappa = \frac{a_N}{v^2}.$$

We can also compute curvature by

$$\kappa(t) = \frac{\| \mathbf{T}'(t) \|}{\| \mathbf{F}'(t) \|}$$

or, from the formula requested in Problem 11,

$$\kappa(t) = \frac{\| \mathbf{F}'(t) \times \mathbf{F}''(t) \|}{\| \mathbf{F}'(t) \|^3}.$$

Full details are provided for Problem 9, while Problems 1, 3, 5, and 7 are done similarly and only final results are given.

1.

$$\begin{gathered}
\mathbf{v}(t) = 2t(\alpha\mathbf{i} + \beta\mathbf{j} + \gamma\mathbf{k}), \\
\mathbf{a}(t) = 2(\alpha\mathbf{i} + \beta\mathbf{j} + \gamma\mathbf{k}), \\
v(t) = 2|t|\sqrt{\alpha^2 + \beta^2 + \gamma^2}, \\
\mathbf{T}(t) = \frac{1}{\sqrt{\alpha^2 + \beta^2 + \gamma^2}}(\alpha\mathbf{i} + \beta\mathbf{j} + \gamma\mathbf{k}), \\
a_N = 0, \kappa = 0,
\end{gathered}$$

and

$$a_T = 2(\text{sgn}(t))\sqrt{\alpha^2 + \beta^2 + \gamma^2},$$

where

$$\text{sgn}(t) = \begin{cases} 1 & \text{if } t > 0, \\ -1 & \text{if } t < 0. \end{cases}$$

3.
$$\mathbf{v}(t) = 2\cosh(t)\mathbf{j} - 2\sinh(t)\mathbf{k}, v(t) = 2\sqrt{\cosh(2t)},$$
$$\mathbf{a}(t) = 2\sinh(t)\mathbf{j} - 2\cosh(t)\mathbf{k},$$
$$\mathbf{T}(t) = \frac{1}{\sqrt{\cosh(2t)}}(\cosh(t)\mathbf{j} - \sinh(t)\mathbf{k})$$
$$a_T = \frac{2\sinh(2t)}{\cosh(2t)}, a_N = \frac{2}{\sqrt{\cosh(2t)}}$$
$$\kappa = \frac{1}{2(\cosh(2t))^{3/2}}$$

Here we have used the hyperbolic identity
$$\cosh(2t) = \cosh^2(t) + \sinh^2(t).$$

5.
$$\mathbf{v}(t) = 2\mathbf{i} - 2\mathbf{j} + \mathbf{k}, v = 3,$$
$$\mathbf{T} = \frac{1}{3}(2\mathbf{i} - 2\mathbf{j} + \mathbf{k})$$
$$a_T = a_N = \kappa = 0$$

7.
$$\mathbf{v}(t) = -3e^{-t}(\mathbf{i} + \mathbf{j} - 2\mathbf{k}), \mathbf{a}(t) = 3e^{-t}(\mathbf{i} + \mathbf{j} - 2\mathbf{k}),$$
$$v(t) = 3\sqrt{6}e^{-t}, \mathbf{T}(t) = \frac{1}{\sqrt{6}}(-\mathbf{i} - \mathbf{j} + 2\mathbf{k}),$$
$$a_T = -3\sqrt{6}e^{-t}, a_N = 0, \kappa = 0$$

9. The velocity is
$$\mathbf{v}(t) = \mathbf{F}'(t) = 3\mathbf{i} + 2t\mathbf{k},$$
the speed is $\| \mathbf{v}(t) \| = \sqrt{9 + 4t^2}$, acceleration is
$$\mathbf{a}(t) = \mathbf{F}''(t) = 2\mathbf{k},$$
and a unit tangent is
$$\mathbf{T}(t) = \frac{1}{\| \mathbf{F}'(t) \|}\mathbf{F}'(t) = \frac{1}{\sqrt{9 + 4t^2}}(3\mathbf{i} + 2t\mathbf{k}).$$
The curvature is
$$\kappa(t) = \frac{\| \mathbf{T}'(t) \|}{\| \mathbf{F}'(t) \|} = \frac{6}{(9 + 4t^2)^{3/2}},$$
in which we have omitted routine differentiations. The normal and tangential components of the acceleration are given by
$$a_T = \frac{dv}{dt} = \frac{4t}{\sqrt{9 + 4t^2}}$$
and
$$a_N = \sqrt{\| \mathbf{a} \|^2 - a_T^2} = \frac{6}{\sqrt{9 + 4t^2}}.$$

11. First write

$$\mathbf{T}(t) = \frac{1}{\| \mathbf{F}'(t) \|}\mathbf{F}'(t) = \frac{1}{v(t)}\mathbf{F}'(t).$$

Thus

$$v\mathbf{T} = \mathbf{F}'.$$

Now $\mathbf{F}''(t)$ is the acceleration $\mathbf{a}(t)$, and $\mathbf{T} \times \mathbf{T} = \mathbf{O}$, so

$$\begin{aligned} v\mathbf{T} \times \mathbf{F}'' &= v\mathbf{T}(a_T\mathbf{T} + a_N\mathbf{N}) \\ &= va_T\mathbf{T} \times \mathbf{T} + va_N\mathbf{T} \times \mathbf{N} \\ &= va_N\mathbf{T} \times \mathbf{N} = v(v^2\kappa)\mathbf{T} \times \mathbf{N}. \end{aligned}$$

Now $\mathbf{T}$ and $\mathbf{N}$ are orthogonal unit vectors, so $\| \mathbf{T} \times \mathbf{N} \| = 1$ and we have

$$\| \mathbf{F}' \times \mathbf{F}'' \| = v^3\kappa.$$

Finally, $v = \| \mathbf{F}' \|$, so

$$\kappa = \frac{\| \mathbf{F(t)}' \times \mathbf{F(t)}'' \|}{\| \mathbf{F(t)}' \|^3}.$$

12. **Hint** We can translate any circle in R^3 to a circle about the origin in the $x, y-$ plane without affecting the curvature. Thus suppose C has position vector $\mathbf{F}(t) = r\cos(t)\mathbf{i} + r\sin(t)\mathbf{j}$. Compute the curvature of C.

13. A position vector for a straight line has the form

$$\mathbf{F}(t) = (a + bt)\mathbf{i} + (c + dt)\mathbf{j} + (p + ht)\mathbf{k}.$$

The tangent vector $\mathbf{F}'(t)$ is the constant vector $b\mathbf{i} + d\mathbf{j} + h\mathbf{k}$, so $\mathbf{T}(t)$ is a constant vector. Then $\mathbf{T}'(t) = \mathbf{O}$, so $\kappa = 0$.

Conversely, suppose a smooth curve has curvature zero. Then

$$\kappa = \| \mathbf{T}'(s) \| = \| \mathbf{F}''(s) \| = 0.$$

If

$$\mathbf{F}(s) = f(s)\mathbf{i} + g(s)\mathbf{j} + h(s)\mathbf{k},$$

then

$$f''(s) = g''(s) = h''(s) = 0$$

which means that $f(s) = a + bs$, $g(s) = c + ds$ and $h(s) = p + qs$ for some constants a, b, c, d, p, h. Then $\mathbf{F}$ is the position vector for a straight line.

11.3 Vector Fields and Streamlines

1. Circular streamlines about the origin in the $x, y-$ plane can be written as $x^2 + y^2 = r^2$, so $x\,dx + y\,dy = 0$, or

$$\frac{dx}{y} = -\frac{dy}{x}, dz = 0.$$

A vector field having these streamlines is

$$\mathbf{F(x, y)} = \frac{1}{x}\mathbf{i} - \frac{1}{y}\mathbf{j}.$$

3. Streamlines satisfy $dx = 0$ and

$$\frac{dy}{2e^z} = -\frac{dz}{\cos(y)}.$$

Integration of the separable differential equation

$$\cos(y)\,dy = -2e^z\,dz$$

gives $x = c_1$ and $\sin(y) = c_2 - 2e^z$. To pass through $(3, \pi/4, 0)$, we need $c_1 = 3$ and $c_2 = 2 + \sqrt{2}/2$. With y as parameter, this curve has parametric equations

$$x = 3, y = y, z = \ln\left[\frac{\sqrt{2}}{4} + 1 - \frac{1}{2}\sin(y)\right].$$

5. Streamlines satisfy

$$x\,dx = \frac{dy}{e^x} = \frac{dz}{-1}.$$

Integration $xe^x\,dx = dy$ to obtain $y = xe^x - e^x + c_1$. Integrate $x\,dx = -dz$ to obtain $x^2 = -2z + c_2$. Using x as parameter, streamlines are given by

$$y = xe^x - e^x + c_1, z = \frac{1}{2}(c_2 - x^2).$$

For the streamline passing through $(2, 0, 4)$, we need

$$e^2 + c_1 = 0 \text{ and } 4 = \frac{1}{2}(c_2 - 4).$$

Then $c_1 = -e^2$ and $c_2 = 12$. This yields the streamline

$$x = x, y = xe^x - e^x - e^2, z = \frac{1}{2}(12 - x^2).$$

7. The streamlines satisfy

$$dx = -\frac{dy}{y^2} = \frac{dz}{z}.$$

Integrate $dx = -(1/y^2)\,dy$ to obtain $x = 1/y + c_1$. Next integrate $dx = (1/z)\,dz$ to obtain $x = \ln|z| + c_2$. In terms of x, we can write parametric equations of the streamlines:

$$x = x, y = \frac{1}{x - c_1}, z = e^{x - c_2}.$$

For the streamline through $(2, 1, 1)$, we need $x = 2$ and

$$1 = \frac{1}{2 - c_1}, 1 = e^{2 - c_2}.$$

Solve these to obtain $c_1 = 1$ and $c_2 = 2$. The streamline through $(2, 1, 1)$ has parametric equations

$$x = x, y = \frac{1}{x - 1}, z = e^{x-2}.$$

11.4 The Gradient Field

1.

$$\nabla\varphi(x,y,z) = 2y\sinh(2xy)\mathbf{i} + 2x\sinh(2xy)\mathbf{j} - \cosh(z)\mathbf{k},$$
$$\nabla\varphi(0,1,1) = -\cosh(1)\mathbf{k},$$

$$D_{\mathbf{u}}\varphi(0,1,1)_{\text{max}} = \parallel \nabla\varphi(0,1,1) \parallel = \cosh(1),$$
$$D_{\mathbf{u}}\varphi(0,1,1)_{\text{min}} = - \parallel \nabla\varphi(0,1,1) \parallel = -\cosh(1)$$

3.

$$\nabla\varphi(x,y,z) = \frac{\partial}{\partial x}(xyz)\mathbf{i} + \frac{\partial}{\partial y}(xyz)\mathbf{j} + \frac{\partial}{\partial z}(xyz)\mathbf{k}$$
$$= yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$$

and

$$\nabla\varphi(1,1,1) = \mathbf{i} + \mathbf{j} + \mathbf{k}.$$

The maximum value of $D_{\mathbf{u}}\varphi(1,1,1)$ is $\parallel \nabla\varphi(1,1,1) \parallel = \sqrt{3}$. The minimum value is $-\sqrt{3}$.

5.

$$\nabla\varphi(x,y,z) = (2y + e^z)\mathbf{i} + 2x\mathbf{j} + xe^z\mathbf{k}$$
$$\nabla\varphi(-2,1,6) = (2 + e^6)\mathbf{i} - 4\mathbf{j} - 2e^6\mathbf{k}$$

The maximum value of $D_{\mathbf{u}}\varphi(-2,1,6)$ is

$$\parallel \nabla\varphi(-2,1,6) \parallel = \sqrt{20 + 4e^6 + 5e^{12}},$$

and the minimum value is the negative of this maximum value.

7.

$$D_{\mathbf{u}}\varphi(x,y,z) = \nabla\varphi(x,y,z) \cdot \mathbf{u}$$
$$= (2xyz^3\mathbf{i} + x^2y^3\mathbf{j} + 3x^2yz^2\mathbf{k}) \cdot \frac{1}{\sqrt{5}}(2\mathbf{j} + \mathbf{k})$$
$$= \frac{1}{\sqrt{5}}(2x^2z^3 + 3x^2yz^2)$$

9.

$$D_{\mathbf{u}}\varphi(x,y,z) = \nabla\varphi(x,y,z) \cdot \mathbf{u}$$
$$= ((8y^2 - z)\mathbf{i} + 16xy\mathbf{j} - x\mathbf{k}) \cdot \frac{1}{\sqrt{3}}(\mathbf{i} + \mathbf{j} + \mathbf{k})$$
$$= \frac{1}{\sqrt{3}}(8y^2 - z + 16xy - x)$$

11. Since $\nabla\varphi(x,y,z) = \mathbf{i} + \mathbf{k}$ for all (x,y,z), the normal to the level surface $\varphi(x,y,z) = K$ is the constant vector $\mathbf{N} = \mathbf{i} + \mathbf{k}$, so the surface must be the plane $x + z = K$. The streamlines of the vector field $\nabla\varphi(x,y,z) = \mathbf{i} + \mathbf{k}$ are solutions of

$$dx = dz, dy = 0.$$

Integrate to obtain

$$x = z + c_1, y = c_2.$$

Using t as parameter,

$$x = t + c_1, y = c_2, z = t \text{ for } -\infty < t < \infty.$$

These streamlines are lines in $3-$ space which are orthogonal to the surface $x + z = K$.

13. The normal vector is

$$\mathbf{N} = \nabla(2x - \cos(x, y, z))|_{(1,\pi,1)} = 2\mathbf{i}.$$

The tangent plane has the equation $x = 1$ and the normal line has parametric equations

$$x = 1 + 2t, y = \pi, z = 1 \text{ for } -\infty < t < \infty.$$

15. The normal vector is

$$\mathbf{N} = \nabla(x^2 - y^2 - z^2)|_{(1,1,0)} = 2\mathbf{i} - 2\mathbf{j}$$

and the tangent plane at $(1, 1, 0)$ has equation $2x - 2y = 0$, or $y = x$. The normal line at this point has parametric equations

$$x = 1 + 2t, y = 1 - 2t, z = 0 \text{ for } -\infty < t < \infty.$$

17. Let $\varphi(x, y, z) = x^2+y^2+z^2$, so the level surface is the locus of points satisfying $\varphi(x, y, z) = 4$. A normal vector at $(1, 1, \sqrt{2})$ is

$$\mathbf{N} = \nabla\varphi(1, 1, \sqrt{2}) = 2\mathbf{i} + 2\mathbf{j} + 2\sqrt{2}\mathbf{k}.$$

The tangent plane to the surface at $(1, 1, \sqrt{2})$ has equation

$$2(x - 1) + 2(y - 1) + 2\sqrt{2}(z - \sqrt{2}) = 0,$$

or

$$x + y + \sqrt{2}z = 4.$$

The normal line to the surface at this point has parametric equations

$$x = y = 1 + 2t, z = \sqrt{2}(1 + 2t) \text{ for } -\infty < t < \infty.$$

11.5 Divergence and Curl

1.

$$\nabla \cdot \mathbf{F} = 2y + xe^y + 2$$
$$\nabla \times \mathbf{F} = (e^y - 2x)\mathbf{k}$$
$$\nabla \cdot (\nabla \times \mathbf{F}) = \frac{\partial}{\partial z}(e^y - 2x) = 0$$

3.

$$\nabla \cdot \mathbf{F} = \cosh(x) + xz\sinh(xyz) - 1$$
$$\nabla \times \mathbf{F} = (-1 - xy\sinh(xyz))\mathbf{i} - \mathbf{j} + yz\sinh(xyz)\mathbf{k}$$

$$\begin{aligned}\nabla \cdot \nabla \times \mathbf{F} &= \frac{\partial}{\partial x}(-1 - xy\sinh(xyz)) + \frac{\partial}{\partial z}(yz\sinh(xyz)) \\ &= (-y + y)\sinh(xyz) + \cosh(xyz)(-xy^2z + xy^2z) = 0\end{aligned}$$

5.
$$\nabla \cdot \mathbf{F} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(2z) = 4$$
$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x & y & 2z \end{vmatrix} = 0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k} = \mathbf{O}$$
$$\nabla \cdot (\nabla \times \mathbf{F}) = 0$$

7.
$$\begin{aligned} \nabla\varphi = {} & (\cos(x+y+z) - x\sin(x+y+z))\mathbf{i} \\ & - x\sin(x+y+z)\mathbf{j} - x\sin(x+y+z)\mathbf{k} \end{aligned}$$

$$\begin{aligned} & \nabla \times (\nabla\varphi) = (-x\cos(x+y+z) + x\cos(x+y+z))\mathbf{i} \\ & + (-\sin(x+y+z) - x\cos(x+y+z) + \sin(x+y+z) + x\cos(x+y+z))\mathbf{j} \\ & + (-\sin(x+y+z) - x\cos(x+y+z) + \sin(x+y+z) + x\cos(x+y+z))\mathbf{k} \\ & = \mathbf{O} \end{aligned}$$

8.
$$\nabla\varphi = e^{x+y+z}(\mathbf{i} + \mathbf{j} + \mathbf{k})$$
$$\nabla \times (\nabla\varphi) = (e^{x+y+z} - e^{x+y+z})(\mathbf{i} + \mathbf{j} + \mathbf{k}) = \mathbf{O}$$

9.
$$\nabla\varphi = \mathbf{i} - \mathbf{j} + 4z\mathbf{k}$$
$$\nabla \times (\nabla\varphi) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 1 & -1 & 4z \end{vmatrix} = \mathbf{O}$$

11.
$$\nabla\varphi = -6x^2yz^2\mathbf{i} - 2x^3z^2\mathbf{j} - 4x^3yz\mathbf{k}$$

and

$$\begin{aligned} & \nabla \times (\nabla\varphi) = (-4x^3z + 4x^3z)\mathbf{i} \\ & + (-12x^2yz + 12x^2yz)\mathbf{j} + (-6x^2z^2 + 6x^2z^2)\mathbf{k} = \mathbf{O} \end{aligned}$$

13. Let $\mathbf{F} = f\mathbf{i} + g\mathbf{j} + h\mathbf{k}$. Then

$$\begin{aligned} \nabla \cdot (\varphi\mathbf{F}) &= \nabla \cdot (\varphi f\mathbf{i} + \varphi g\mathbf{j} + \varphi h\mathbf{k}) \\ &= \frac{\partial}{\partial x}(\varphi f) + \frac{\partial}{\partial y}(\varphi g) + \frac{\partial}{\partial z}(\varphi h) \\ &= \left(\frac{\partial\varphi}{\partial x}f + \frac{\partial\varphi}{\partial y}g + \frac{\partial\varphi}{\partial z}h\right) \\ &+ \varphi\left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}\right) \\ &= \nabla\varphi \cdot \mathbf{F} + \varphi\nabla \cdot \mathbf{F}. \end{aligned}$$

Next,

$$
\begin{aligned}
\nabla \times (\varphi \mathbf{F}) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \varphi f & \varphi g & \varphi h \end{vmatrix} \\
&= \left[\frac{\partial}{\partial y}(\varphi h) - \frac{\partial}{\partial z}(\varphi g)\right]\mathbf{i} \\
&+ \left[\frac{\partial}{\partial z}(\varphi f) - \frac{\partial}{\partial x}(\varphi h)\right]\mathbf{j} \\
&+ \left[\frac{\partial}{\partial x}(\varphi g) - \frac{\partial}{\partial y}(\varphi f)\right]\mathbf{k} \\
&= \left[\frac{\partial \varphi}{\partial y}h - \frac{\partial \varphi}{\partial z}g\right]\mathbf{i} + \left[\frac{\partial \varphi}{\partial z}f - \frac{\partial \varphi}{\partial x}h\right]\mathbf{j} \\
&+ \left[\frac{\partial \varphi}{\partial x}g - \frac{\partial \varphi}{\partial y}f\right]\mathbf{k} \\
&+ \varphi\left[\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}\right]\mathbf{i} + \left[\frac{\partial f}{\partial z} - \frac{\partial h}{\partial x}\right]\mathbf{j} + \left[\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right]\mathbf{k} \\
&= \nabla \varphi \times \mathbf{F} + \varphi(\nabla \times \mathbf{F}).
\end{aligned}
$$

Chapter 12

Vector Integral Calculus

12.1 Line Integrals

1.

$$\begin{aligned}\int_C -xyz\,dz &= \int_4^9 -z\sqrt{z}\,dz \\ &= -\frac{2}{5}z^{5/2}\Big]_4^9 = -\frac{422}{5}\end{aligned}$$

3. On C, $x = t, y = t, z = t^3$, so

$$\begin{aligned}\int_C x\,dx - dy + z\,dz &= \int_0^1 (t(1) - (1) + t^3(3t^2))\,dt \\ &= \int_0^1 (t - 1 + 3t^5)\,dt = 0.\end{aligned}$$

5.

$$\begin{aligned}\int_C (x+y)\,ds &= \int_0^2 (t+t)\sqrt{1+1+4t^2}\,dt \\ &= \int_0^2 2t\sqrt{2+4t^2}\,dt = \frac{1}{6}\,(2+4t^2)^{3/2}\Big]_0^2 = \frac{26\sqrt{2}}{3}\end{aligned}$$

7.

$$\begin{aligned}\int_C \mathbf{F}\cdot\,d\mathbf{R} &= \int_0^3 (\cos(t)\mathbf{i} + t^2\mathbf{j} + t\mathbf{k})\cdot(\mathbf{i} - 2t\mathbf{j} + 0\mathbf{k})\,dt \\ &= \int_0^3 (\cos(t) - 2t^3)\,dt = \sin(3) - \frac{81}{2}.\end{aligned}$$

9. Parametrize C as $x = 2\cos(t), y = 2\sin(t), z = 0$ for $0 \le t \le 2\pi$. Then

$$\begin{aligned}\int_C \mathbf{F}\cdot d\mathbf{R} &= \int_0^{2\pi} (2\cos(t)\mathbf{i} + 2\sin(t)\mathbf{j})\cdot(-2\sin(t)\mathbf{i} + 2\cos(t)\mathbf{j})\,dt \\ &= \int_0^{2\pi} (-4\cos(t)\sin(t) + 4\sin(t)\cos(t))\,dt = 0.\end{aligned}$$

11. Take $\mathbf{F}(x) = f(x)\mathbf{i}$ and $\mathbf{R}(t) = t\mathbf{i}$, for $a \le t \le b$. The graph of the curve defined by this position vector is $[a, b]$, and

$$\int_C \mathbf{F} \cdot d\mathbf{R} = \int_a^b f(x)\, dx.$$

13. Parametrize the line segment as $x = y = z = 1 + 3t$ for $0 \le t \le 1$. The work done is

$$\begin{aligned}\int_C \mathbf{F} \cdot d\mathbf{R} &= \int_0^1 ((1+3t)^2 - 2(1+3t)^2 + (1+3t))(3)\, dt \\ &= \left[\frac{(1+3t)^2}{2} - \frac{(1+3t)^3}{3}\right]_0^1 = -\frac{27}{2}.\end{aligned}$$

12.2 Green's Theorem

1. By Green's theorem,

$$\begin{aligned}\oint_C -\frac{\partial u}{\partial y}\, dx + \frac{\partial u}{\partial x}\, dy &= \iint_D \left[\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) - \frac{\partial}{\partial y}\left(-\frac{\partial u}{\partial y}\right)\right] dA \\ &= \iint_D \left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right] dA.\end{aligned}$$

3. The work done by $\mathbf{F}$ is

$$\begin{aligned}\text{work } &= \oint_C xy\, dx + x\, dy = \iint_D \left[\frac{\partial}{\partial x}(x) - \frac{\partial}{\partial y}(xy)\right] dA \\ &= \int_0^1 \int_0^{6x} (1-x)\, dy\, dx + \int_1^4 \int_0^{8-2x} (1-x)\, dy\, dx \\ &= \int_0^1 6x(1-x)\, dx + \int_1^4 (8-2x)(1-x)\, dx = -8.\end{aligned}$$

5.

$$\begin{aligned}\oint_C \mathbf{F} \cdot d\mathbf{R} &= \iint_D \left[\frac{\partial}{\partial x}(e^x \sin(y)) - \frac{\partial}{\partial y}(e^x \cos(y))\right] dA \\ &= \iint_D (-e^x \sin(y) + e^x \sin(y))\, dA = 0.\end{aligned}$$

7.

$$\begin{aligned}\oint_C \mathbf{F} \cdot d\mathbf{R} &= \iint_D \left[\frac{\partial}{\partial x}(xy^2 - e^{\cos(y)}) - \frac{\partial}{\partial y}(xy)\right] dA \\ &= \iint_D (y^2 - x)\, dA = \int_0^3 \int_0^{5-5x/3} (y^2 - x)\, dy\, dx \\ &= \int_0^3 \frac{1}{3}\left(5 - \frac{5x}{3}\right)^3 dx - \int_0^3 x\left(5 - \frac{5x}{3}\right) dx \\ &= \frac{95}{4}.\end{aligned}$$

9.

$$\oint_C \mathbf{F} \cdot d\mathbf{R} = \iint_D \frac{\partial}{\partial x}(8xy^2) = \iint_D 8y^2 \, dA.$$

Change to polar coordinates $x = r\cos(\theta), y = r\sin(\theta)$, with $0 \le \theta \le 2\pi, 0 \le r \le 4$ to obtain

$$\begin{aligned}\iint_D 8y^2 \, dA &= \int_0^{2\pi} \int_0^4 8r^2 \sin^2(\theta) r \, dr \, d\theta \\ &= \int_0^{2\pi} \sin^2(\theta) \, d\theta \int_0^4 8r^3 \, dr = 512\pi.\end{aligned}$$

11.

$$\begin{aligned}&\oint_C \mathbf{F} \cdot d\mathbf{R} \\ &= \iint_D \left[\frac{\partial}{\partial x}(-2xy) - \frac{\partial}{\partial y}(x^2)\right] dA \\ &= \iint_D (-2y) \, dA = \int_1^6 \int_{(y+4)/5}^{(22-2y)/5} -2y \, dx \, dy \\ &= \int_1^6 (3y - 18)\frac{2y}{5} \, dy = -40.\end{aligned}$$

12. Assume that C is a join of two curves in two ways. First, C has an upper piece $y = p(x)$ and a lower piece $y = q(x)$ for $a \le x \le b$, so D consists of all (x, y) with

$$a \le x \le b, q(x) \le y \le p(x).$$

Second, C also has a right piece $y = \beta(x)$ and a left piece $y = \alpha(x)$ for $c \le y \le d$, so, looking left to right instead of bottom to top, D can also be described as consisting of all (x, y) with

$$c \le y \le d, \alpha(y) \le x \le \beta(y).$$

Now use both of these descriptions in turn as follows. Using the second description of C (look at C from left to right),

$$\oint_C g(x, y) \, dy = \int_c^d g(\beta(y), y) \, dy + \int_d^c g(\alpha(y), y) \, dy.$$

Note that, on the right part of C, y varies from c to d for a counterclockwise orientation, while, to retain this orientation, y varies from d to c on the left part of the boundary curve. Further,

$$\begin{aligned}\iint_D \frac{\partial g}{\partial x} \, dA &= \int_c^d \int_{\alpha(y)}^{\beta(y)} \frac{\partial g}{\partial y} \, dy \\ &= \int_c^d (g(\beta(y), y) - g(\alpha(y), y)) \, dy.\end{aligned}$$

Therefore

$$\oint_C g(x, y) \, dy = \iint_D \frac{\partial g}{\partial x} \, dA.$$

This is ”half” of the conclusion of Green’s theorem. For the rest, use the first description of C. Now, looking from bottom to top, we have (keeping in mind the counterclockwise orientation on C),

$$\begin{aligned}\oint_C f(x,y)\,dx &= \int_b^a f(x,p(x))\,dx + \int_a^b f(x,q(x))\,dx \\ &= -\int_a^b (f(x,p(x)) - f(x,q(x)))\,dx\end{aligned}$$

and

$$\iint_D \frac{\partial f}{\partial y}\,dA = \int_a^b \int_{q(x)}^{p(x)} f(x,p(x)) - f(x,q(x))\,dA.$$

Then

$$\oint_C f(x,y)\,dx = -\iint_D \frac{\partial f}{\partial y}\,dA.$$

Upon adding these two equations, we obtain

$$\oint_C f(x,y)\,dx + g(x,y)\,dx = \iint_D \left[\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right] dA.$$

13.

$$\begin{aligned}\text{work } &= \oint_C (-\cosh(4x^4) + xy)\,dx + (e^{-y} + x)\,dy \\ &\iint_D \left[\frac{\partial}{\partial x}(e^{-y} + x) - \frac{\partial}{\partial y}(-\cosh(4x^4) + xy)\right] dA \\ &= \iint_D (1-x)\,dA = \int_1^3 \int_1^7 (1-x)\,dy\,dx \\ &= \int_1^3 6(1-x)\,dx = -12.\end{aligned}$$

12.3 An Extension of Green’s Theorem

1. If C does not enclose the origin, then by Green’s theorem,

$$\begin{aligned}&\oint_C \mathbf{F}\cdot d\mathbf{R} = \oint_K \mathbf{F}\cdot d\mathbf{R} \\ &= \int_0^{2\pi} \left[\frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2} - 2y\right) - \frac{\partial}{\partial y}\left(\frac{-y}{x^2+y^2} + x^2\right)\right] dA = 0\end{aligned}$$

because the partial derivatives in this double integral are equal. If C encloses the origin, choose a smaller circle K, of radius r, enclosed by C and enclosing the origin. Then

$$\begin{aligned}\oint_C \mathbf{F}\cdot d\mathbf{R} &= \oint_K \mathbf{F}\cdot d\mathbf{R}\\ &= \int_0^{2\pi}\left[\left(\frac{-r\sin(\theta)}{r^2}+r^2\cos^2(\theta)\right)(-r\sin(\theta))\right]d\theta\\ &+\int_0^{2\pi}\left[\left(\frac{r\cos(\theta)}{r^2}-2r\sin(\theta)\right)(r\cos(\theta))\right]d\theta\\ &= \int_0^{2\pi}(1-r^3\cos^2(\theta)\sin(\theta)-2r^2\sin(\theta)\cos(\theta))\,d\theta\\ &= \theta+\frac{r^3}{3}\cos^3(\theta)-r^2\sin^2(\theta)\Big]_0^{2\pi}=2\pi.\end{aligned}$$

3. If C does not enclose the origin, then by Green's theorem,

$$\oint_C \mathbf{F}\cdot d\mathbf{R} = \iint_D\left[\frac{\partial}{\partial x}\left(\frac{y}{\sqrt{x^2+y^2}}-3y^2\right)-\frac{\partial}{\partial y}\left(\frac{x}{\sqrt{x^2+y^2}}+2x\right)\right]dA=0,$$

since the partial derivatives in the integral are equal and therefore cancel. If C does enclose the origin, use the extension of Green's theorem. If K is a circle of radius r enclosing the origin and enclosed by C, we obtain

$$\begin{aligned}\oint_C \mathbf{F}\cdot d\mathbf{R} &= \oint_K \mathbf{F}\cdot d\mathbf{R}\\ &= \int_0^{2\pi}\left[\left(\frac{r\cos(\theta)}{r}+2r\cos(\theta)\right)(-r\sin(\theta))\right]d\theta\\ &+\int_0^{2\pi}\left[\left(\frac{r\sin(\theta)}{r}-3r^2\sin^2(\theta)\right)r\cos(\theta)\right]d\theta\\ &= -r^2\int_0^{2\pi}(2\cos(\theta)\sin(\theta)+3\sin^2(\theta)\cos(\theta))\,d\theta\\ &= -r^2(\sin^2(\theta)+\sin^3(\theta)]_0^{2\pi}=0.\end{aligned}$$

5. If C does not enclose the origin, then by Green's theorem we have

$$\oint_C \mathbf{F}\cdot d\mathbf{R} = \iint_D\left[\frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right)-\frac{\partial}{\partial y}\left(\frac{x}{x^2+y^2}\right)\right]dA=0,$$

because the integrand is identically zero. If C encloses the origin, use the extended form of Green's theorem, where K is a circle of radius r lying entirely within C and enclosing the origin. Then

$$\begin{aligned}\oint_C \mathbf{F}\cdot d\mathbf{R} &= \oint_K \mathbf{F}\cdot d\mathbf{R}\\ &= \int_0^{2\pi}\left[\frac{r\cos(\theta)}{r^2}(-r\sin(\theta))+\frac{r\sin(\theta)}{r^2}(r\cos(\theta))\right]d\theta=0.\end{aligned}$$

12.4 Independence of Path and Potential Theory

1. We find that
$$\nabla \times \mathbf{F} = (-z^2 - xy)\mathbf{i} + yz\mathbf{k} \neq \mathbf{O}$$
so $\mathbf{F}$ is not conservative and there is no potential function.

3. Since
$$\frac{\partial}{\partial y}\left(\frac{2x}{x^2+y^2}\right) = -\frac{4xy}{(x^2+y^2)^2} = \frac{\partial}{\partial x}\left(\frac{2y}{x^2+y^2}\right),$$
we know that $\mathbf{F}$ is conservative on any region not containing the origin. A potential function $\varphi(x, y)$ must satisfy
$$\frac{\partial \varphi}{\partial x} = \frac{2x}{x^2+y^2} \text{ and } \frac{\partial \varphi}{\partial y} = \frac{2y}{x^2+y^2}.$$
Integrate one of these. If we integrate the second, we obtain
$$\varphi(x, y) = \ln(x^2 + y^2) + c(x).$$
Then we need
$$\frac{\partial \varphi}{\partial x} = \frac{2x}{x^2+y^2} + c'(x) = \frac{2x}{x^2+y^2}.$$
Then $c'(x) = 0$ and we may choose $c(x) = 0$, yielding the potential function
$$\varphi(x, y) = \ln(x^2 + y^2).$$

5. By inspection in this simple case, $\varphi(x, y, z) = x - 2y + z$ is a potential function for $\mathbf{F}$.

7. Since
$$\frac{\partial}{\partial y}(16x) = 0 = \frac{\partial}{\partial x}(2 - y^2),$$
then $\mathbf{F}$ is conservative. Integrate $\partial\varphi/\partial x = 16x$ with respect to x to obtain
$$\varphi(x, y) = 8x^2 + k(y).$$
Then
$$\frac{\partial \varphi}{\partial y} = 2 - y^2 = k'(y)$$
so we may choose $k(y) = 2y - y^3/3$ to obtain the potential function
$$\varphi(x, y) = 8x^2 + 2y - \frac{1}{3}y^3.$$

9. Since
$$\frac{\partial}{\partial y}(y^3) = 3y^2 = \frac{\partial}{\partial x}(3x^2y - 4),$$
then $\mathbf{F}$ is conservative (in the entire plane, where the components of $\mathbf{F}$ are defined). To find a potential function φ, begin with $\partial\varphi/\partial x = y^3$ and integrate with respect to x to obtain
$$\varphi(x, y) = xy^3 + k(y)$$
in which $k(y)$ is the "constant" of the integration with respect to x. Next
$$\frac{\partial \varphi}{\partial y} = 3x^2y + k'(y) = 3x^2y - 4$$

so $k'(y) = -4$ and we can choose $k(y) = -4y$. A potential function is

$$\varphi(x, y) = xy^3 - 4y.$$

Of course $xy^3 - 4y + c$ is also a potential function for any constant c.

In Problems 11, 13, 15, 17, and 19, the details of deriving the potential function are omitted.

11. In any region not containing the points of the $y-$ axis,

$$\varphi(x, y) = x^2 y - \ln|y|$$

is a potential function for $\mathbf{F}$. If C does not cross the $x-$ axis, then

$$\int_C \mathbf{F} \cdot d\mathbf{R} = \varphi(2,2) - \varphi(1,3) = 8 - \ln(2) - 3 + \ln(3) = 5 + \ln(3/2).$$

13. $\varphi(x, y, z) = 2x^3 e^{yz}$ is a potential function for $\mathbf{F}$, so

$$\int_C \mathbf{F} \cdot d\mathbf{R} = \varphi(1, 2, -1) - \varphi(0,0,0) = 2e^{-2}.$$

15. $\varphi(x, y, z) = x - 3y^3 z$ is a potential function for $\mathbf{F}$, so

$$\int_C \mathbf{F} \cdot d\mathbf{R} = \varphi(0,3,5) - \varphi(1,1,1) = -403.$$

17. $\mathbf{F}$ has potential function $\varphi(x, y) = x^3 y^2 - 6xy^3$, so

$$\int_C \mathbf{F} \cdot d\mathbf{R} = \varphi(1,1) - \varphi(0,0) = -5.$$

19. By integrating, we find the potential function

$$\varphi(x, y) = x^3(y^2 - 4y).$$

Then

$$\int_C \mathbf{F} \cdot d\mathbf{R} = \varphi(2,3) - \varphi(-1,1) = -24 - 3 = -27.$$

21. Let C be a smooth path of motion given by $\mathbf{R}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$ and let L be the kinetic energy plus the potential energy. Then

$$L(t) = \frac{m}{2} \parallel \mathbf{R}'(t) \parallel^2 - \varphi(x(t), y(t), z(t)) = \frac{m}{2}\mathbf{R}'(t) \cdot \mathbf{R}'(t) - \varphi(x(t), y(t), z(t)).$$

Then

$$\begin{aligned} \frac{dL}{dt} &= \frac{m}{2}(2\mathbf{R}''(t) \cdot \mathbf{R}'(t)) - \frac{\partial \varphi}{\partial x}x'(t) - \frac{\partial \varphi}{\partial y}y'(t) - \frac{\partial \varphi}{\partial z}z'(t) \\ &= (m\mathbf{R}''(t) \cdot \mathbf{R}'(t)) - \nabla\varphi \cdot \mathbf{R}'(t) \\ &= (m\mathbf{R}''(t) - \nabla\varphi) \cdot \mathbf{R}'(t). \end{aligned}$$

But $\nabla\varphi$ is the force acting on the particle, so by Newton's second law, $m\mathbf{R}'' = \nabla\varphi$, and therefore $dL/dt = 0$. Therefore $L(t)$ is a constant of the motion.

22. We want to show that, in Theorem 12.5, a potential function exists if

$$\frac{\partial g}{\partial x} = \frac{\partial f}{\partial y}.$$

We will use this condition to explicitly construct a potential function. First observe that, if K is any closed path in D, then

$$\oint_K \mathbf{F} \cdot d\mathbf{R} = \iint_D \left[\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right] dA = 0.$$

This means that $\int_C \mathbf{F} \cdot d\mathbf{R}$ is independent of path in D. If we fix a point $P : (a, b)$ in D, we can define a function

$$\varphi(x, y) = \int_P^{(x,y)} \mathbf{F} \cdot d\mathbf{R}.$$

This is a function because its value depends only on (x, y) and not on the path in D from P to (x, y). We claim that $\nabla\varphi = \mathbf{F}$.

To show this, we will first show that

$$\frac{\partial \varphi}{\partial x} = f(x, y).$$

Choose Δx small enough that $(x + \Delta x, y)$ is in D. Now

$$\begin{aligned}
&\varphi(x + \Delta x, y) - \varphi(x, y) \\
&= \int_P^{(x+\Delta x,y)} \mathbf{F} \cdot d\mathbf{R} - \int_P^{(x,y)} \mathbf{F} \cdot d\mathbf{R} \\
&= \int_P^{(x,y)} \mathbf{F} \cdot d\mathbf{R} + \int_{(x,y)}^{(x+\Delta x,y)} \mathbf{F} \cdot d\mathbf{R} - \int_P^{(x,y)} \mathbf{F} \cdot d\mathbf{R} \\
&= \int_{(x,y)}^{(x+\Delta x,y)} \mathbf{F} \cdot d\mathbf{R} \\
&= \int_{(x,y)}^{(x+\Delta x,y)} f(\xi, \eta)\, d\xi + g(\xi, \eta)\, d\eta.
\end{aligned}$$

This is a line integral over over a horizontal line segment from (x, y) to $(x + \Delta x, y)$, with y fixed on this segment. Parametrize this segment by

$$\xi = x + t\Delta x, \eta = y \text{ for } 0 \le t \le 1.$$

On this segment,

$$d\xi = (\Delta x)\, dt \text{ and } d\eta = 0.$$

Then

$$\varphi(x + \Delta x, y) - \varphi(x, y) = \Delta x \int_0^1 f(x + t\Delta x, y)\, dt.$$

Then

$$\frac{\varphi(x + \Delta x, y) - \varphi(x, y)}{\Delta x} = \int_0^1 f(x + t\Delta x, y)\, dt.$$

By the mean value theorem for integrals, there is some t_0 in $(0, 1)$ such that

$$\int_0^1 f(x + t\Delta x, y)\, dt = f(x + t_0\Delta x, y).$$

Therefore

$$\frac{\varphi(x+\Delta x, y)-\varphi(x, y)}{\Delta x}=f(x+t_0 \Delta x, y).$$

As $\Delta x \to 0$, $x+t_0\Delta x \to x$ and, by continuity, $f(x+t_0\Delta x, y) \to f(x, y)$. Therefore,

$$\begin{aligned} \frac{\partial \varphi}{\partial x} &= \lim_{\Delta x \to 0} \frac{\varphi(x+\Delta x, y)-\varphi(x, y)}{\Delta x} \\ &= \lim_{\Delta x \to 0} f(x+t_0\Delta x, y)=f(x, y). \end{aligned}$$

A similar argument, using a vertical path from (x, y) to $(x, y+\Delta y)$ shows that

$$\frac{\partial \varphi}{\partial y}=g(x, y).$$

12.5 Surface Integrals

1. On Σ, $d\sigma=\sqrt{1+4x^2}\,dA$, so

$$\begin{aligned} \iint_\Sigma y\,d\sigma &= \iint_D y\sqrt{1+4x^2}\,dA \\ &= \int_0^2 \int_0^3 y\sqrt{1+4x^2}\,dy\,dx=\frac{9}{2}\int_0^2 \sqrt{1+4x^2}\,dx=\frac{9}{8}(\ln(4+\sqrt{17})+4\sqrt{17}). \end{aligned}$$

3. On the surface, $z^2=x^2+y^2$, so

$$2z\frac{\partial z}{\partial x}=2x \text{ and } 2z\frac{\partial z}{\partial y}z=2y.$$

Then

$$\frac{\partial z}{\partial x}=\frac{x}{z} \text{ and } \frac{\partial z}{\partial y}=\frac{y}{z}.$$

Then

$$d\sigma=\sqrt{1+\frac{x^2}{z^2}+\frac{y^2}{z^2}}\,dA=\sqrt{2}\,dA.$$

Then

$$\begin{aligned} \iint_\Sigma z\,d\sigma &= \int_D \sqrt{2}\sqrt{x^2+y^2}\,dA \\ &= \sqrt{2}\int_0^{\pi/2}\int_2^4 r^2\,dr\,d\theta=\frac{28\pi}{3}\sqrt{2}. \end{aligned}$$

5. On this surface, $d\sigma=\sqrt{3}\,dA$ and $z=x-y$ so

$$\begin{aligned} \iint_\Sigma z\,d\sigma &= \iint_D \sqrt{3}(x-y)\,dA \\ &= \sqrt{3}\int_0^1\int_0^5 (x-y)\,dy\,dx=-10\sqrt{3}. \end{aligned}$$

7. On Σ,
$$d\sigma = \sqrt{1^2 + (2x)^2 + (2y)^2}\, dA = \sqrt{1 + 4(x^2 + y^2)}\, dA.$$
D is the annular region $2 \le x^2 + y^2 \le 7$. Then
$$\iint_\Sigma d\sigma = \iint_D \sqrt{1 + 4(x^2 + y^2)}\, dA.$$
Use polar coordinates. Now D is given by $\sqrt{2} \le r \le \sqrt{7}$, $0 \le \theta \le 2\pi$ and
$$\begin{aligned}\iint_\Sigma d\sigma &= \int_0^{2\pi} \int_{\sqrt{2}}^{\sqrt{7}} r\sqrt{1 + 4r^2}\, dr\, d\theta \\ &= 2\pi \left[\frac{1}{12}(1 + 4r^2)^{3/2}\right]_{\sqrt{2}}^{\sqrt{7}} = \frac{\pi}{6}((29)^{3/2} - 27).\end{aligned}$$

9. On the surface, $z = 10 - x - 4y$, so
$$d\sigma = \sqrt{1 + (\partial z/\partial x)^2 + (\partial z/\partial y)^2}\, dA = 3\sqrt{2}\, dA,$$
and
$$\begin{aligned}\iint_\Sigma x\, d\sigma &= \iint_D 3\sqrt{2}x\, dA \\ &= 3\sqrt{2} \int_0^{5/2} \int_0^{10-4y} x\, dx\, dy = \frac{3\sqrt{2}}{2} \int_0^{5/2} (10 - 4y)^2\, dy \\ &= -\frac{\sqrt{2}}{8}(10 - 4y)^3\Big]_0^{5/2} = 125\sqrt{2}.\end{aligned}$$

12.6 Applications of Surface Integrals

1. By symmetry of Σ and the density function, $\overline{x} = \overline{y} = 0$. Further, $d\sigma = \sqrt{1 + 4x^2 + 4y^2}\, dA$. For the mass, compute
$$\begin{aligned}m &= \iint_\Sigma \sqrt{1 + 4x^2 + 4y^2}\, d\sigma = \iint_D (1 + 4x^2 + 4y^2)\, dA \\ &= \int_0^{2\pi} \int_0^{\sqrt{6}} (1 + 4r^2 + 4r^2) r\, dr\, d\theta = 78\pi.\end{aligned}$$
Finally,
$$\begin{aligned}\overline{z} &= \frac{1}{m}\iint_\Sigma z\delta(x, y, z)\, d\sigma = \frac{1}{m}\iint_D (6 - x^2 - y^2)(1 + 4x^2 + 4y^2)\, dA \\ &= \frac{1}{m}\int_0^{2\pi} \int_0^{\sqrt{6}} (6 - r^2)(1 + 4r^2) r\, dr\, d\theta = \frac{162\pi}{m} = \frac{27}{13}.\end{aligned}$$

3. The triangular shell is part of the plane having equation $6x + 2y + 3z = 6$ (this is the plane through the given points). The projection of Σ onto the $x, y-$ plane is the set D of points (x, y) such that $0 \le y \le 3 - 3x$, $0 \le x \le 1$. On the surface,
$$z = 2 - \frac{2}{3}y - 2x$$

so $d\sigma = \sqrt{1 + \frac{4}{9} + 4}\, dA = \frac{7}{3}\, dA$. The mass is

$$\begin{aligned} m &= \iint_\Sigma (xz+1)\, d\sigma = \frac{7}{3}\iint_D x\left[\left(2 - \frac{2}{3}y - 2x\right) + 1\right] dA \\ &= \frac{7}{3}\int_0^1 \int_0^{3-3x} x\left[\left(2 - \frac{2}{3}y - 2x\right) + 1\right] dy\, dx. \end{aligned}$$

This integral is routine and we obtain $m = 49/12$. In similar fashion, evaluate

$$\overline{x} = \iint_\Sigma x(xz+1)\, d\sigma = \frac{12}{35},$$

$$\overline{y} = \iint_\Sigma y(xz+1)\, d\sigma = \frac{33}{35},$$

and

$$\overline{z} = \iint_\Sigma z(xz+1)\, d\sigma = \frac{24}{35}.$$

Observe that, because Σ is part of a plane, the center of mass is a point of Σ. This is not true of surfaces in general. For example, the center of mass of a homogeneous sphere is its center.

5. By symmetry of the shell, and the fact that the density is constant, we have $\overline{x} = \overline{y} = 0$. On this surface, $d\sigma = \sqrt{1 + (x/z)^2 + (y/z)^2}\, dA = \sqrt{2}\, dA$. Then

$$\text{mass } = m = \iint_\Sigma K\, d\sigma = K\sqrt{2}\int_0^{2\pi}\int_0^3 r\, dr\, d\theta = 9\pi K\sqrt{2}.$$

Then

$$\begin{aligned} \overline{z} &= \frac{1}{m}\iint_\Sigma z\, d\sigma \\ &= \frac{\sqrt{2}K}{m}\int_0^{2\pi}\int_0^3 r^2\, dr\, d\theta = \frac{18K\pi\sqrt{2}}{m} = 2. \end{aligned}$$

The center of mass is $(0, 0, 2)$.

7. A unit normal to the plane $x + 2y + z = 8$ is

$$\mathbf{n} = \frac{1}{\sqrt{6}}(\mathbf{i} + 2\mathbf{j} + \mathbf{k}).$$

Then

$$\mathbf{F} \cdot \mathbf{n} = \frac{1}{\sqrt{6}}(x + 2y - z).$$

On Σ, $z = 8 - x - 2y$, so

$$\mathbf{F} \cdot \mathbf{n} = \frac{1}{\sqrt{6}}(2x + 4y - 8).$$

Further, $d\sigma = \sqrt{1+4+1}\, dA = \sqrt{6}\, dA$. Therefore, the flux of $\mathbf{F}$ across Σ is

$$\iint_\Sigma \mathbf{F} \cdot \mathbf{n}\, d\sigma = \iint_D (2x + 4y - 8)\, dA = \int_0^4 \int_0^{8-2y} (2x + 4y - 8)\, dx\, dy = \frac{128}{3}.$$

12.7 Lifting Green's Theorem to R^3

1. By Green's theorem,

$$\begin{aligned}\oint_C -\varphi\frac{\partial\psi}{\partial y}\,dx+\varphi\frac{\partial\psi}{\partial x}\,dy &= \iint_D\left[\frac{\partial}{\partial x}\left(\varphi\frac{\partial\psi}{\partial x}\right)-\frac{\partial}{\partial y}\left(-\varphi\frac{\partial\psi}{\partial y}\right)\right]dA\\ &= \iint_D\left[\frac{\partial\varphi}{\partial x}\frac{\partial\psi}{\partial x}+\frac{\partial\varphi}{\partial y}\frac{\partial\psi}{\partial y}+\frac{\partial\varphi}{\partial z}\frac{\partial\psi}{\partial z}\right]dA\\ &+\iint_D\varphi\left[\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}\right]dA\\ &= \iint_D\nabla\varphi\cdot\nabla\psi\,dA+\iint_D\varphi\nabla^2\psi\,dA.\end{aligned}$$

Upon rearranging terms at both ends of this equation, we obtain the requested identity.

3. Under the given conditions,

$$\mathbf{N}=\frac{dy}{ds}\mathbf{i}-\frac{dx}{ds}\mathbf{j}\text{ and }\varphi_{\mathbf{N}}=\nabla\varphi\cdot\mathbf{N}.$$

Then

$$\begin{aligned}\oint_C\varphi_{\mathbf{N}}(x,y)\,ds &= \oint_C\left[\frac{\partial\varphi}{\partial x}\frac{dy}{ds}-\frac{\partial\varphi}{\partial y}\frac{dx}{ds}\right]ds\\ &= \oint_C-\frac{\partial\varphi}{\partial y}\,dx+\frac{\partial\varphi}{\partial x}\,dy.\end{aligned}$$

Apply Green's theorem to this line integral to obtain

$$\begin{aligned}\oint_C\varphi_{\mathbf{N}}(x,y)\,ds &= \iint_D\left[\frac{\partial}{\partial x}\left(\frac{\partial\varphi}{\partial x}\right)-\frac{\partial}{\partial y}\left(-\frac{\partial\varphi}{\partial y}\right)\right]dA\\ &= \iint_D\nabla^2\varphi\,dA.\end{aligned}$$

12.8 The Divergence Theorem of Gauss

1. Since $\nabla \cdot \mathbf{F} = 0$, $\int_M \nabla \cdot \mathbf{F}\, dV = 0$.
3. $\nabla \cdot \mathbf{F} = 1$, so compute

$$\int_M \nabla \cdot \mathbf{F}\, dV = \text{volume of } M = \frac{4}{3}\pi(4^3) = \frac{256\pi}{3}.$$

5. Compute $\nabla \cdot \mathbf{F} = 2(x + y + z)$, so, using cylindrical coordinates, we have

$$\int_M \nabla \cdot \mathbf{F}\, dV = 2\int_0^{2\pi}\int_0^{\sqrt{2}}\int_r^{\sqrt{2}} (r\cos(\theta) + r\sin(\theta) + z) r\, dz\, dr\, d\theta.$$

We will do these integrations one at a time. First,

$$\int_r^{\sqrt{2}} (r^2(\cos(\theta) + \sin(\theta)) + rz)\, dz = r^2(\cos(\theta) + \sin(\theta))(\sqrt{2} - r) + \frac{1}{2}r(2 - r^2).$$

Next,

$$\int_0^{\sqrt{2}} \left[r^2(\cos(\theta) + \sin(\theta))(\sqrt{2} - r) + \frac{1}{2}r(2 - r^2)\right] dr = \frac{1}{3}(\cos(\theta) + \sin(\theta)) + \frac{1}{2}.$$

Finally,

$$\int_0^{2\pi} \left(\frac{1}{3}(\cos(\theta) + \sin(\theta)) + \frac{1}{2}\right) d\theta = \pi.$$

Therefore

$$\int_M \nabla \cdot \mathbf{F}\, dV = 2\pi.$$

7. since $\nabla \cdot \mathbf{F} = 4$, compute

$$\int_M \nabla \cdot \mathbf{F}\, dV = \int_M 4\, dV = 4(\text{volume of } M) = \frac{8\pi}{3}.$$

9. With the given conditions on $\mathbf{F}$, Σ and M, we have

$$\iint_\Sigma (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, d\sigma = \int_M (\nabla \cdot \nabla \times \mathbf{F})\, dV.$$

But $\nabla \cdot \nabla \times \mathbf{F} = 0$, so

$$\iint_\Sigma (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, d\sigma = 0.$$

12.9 Stokes's Theorem

1. Compute $\nabla \times \mathbf{F} = \mathbf{i} + \mathbf{j} + \mathbf{k}$. A unit normal to Σ is

$$\mathbf{n} = \frac{1}{\sqrt{x^2 + y^2 + z^2}}(x\mathbf{i} + y\mathbf{j} + z\mathbf{k}).$$

This is

$$\mathbf{n} = \frac{1}{\sqrt{2}\sqrt{x^2 + y^2}}(x\mathbf{i} + y\mathbf{j} - 2\mathbf{k}).$$

Further

$$d\sigma = \sqrt{1 + \frac{x^2}{/}x^2 + y^2 + \frac{y^2}{x^2+y^2}}\,dA = \sqrt{2}\,dA.$$

Then

$$\iint_\Sigma (\nabla \times \mathbf{F}) \cdot \mathbf{n}\,d\sigma = \iint_D \frac{x+y-z}{\sqrt{x^2+y^2}}\,dA = \iint_D \left(\frac{x+y}{\sqrt{x^2+y^2}} - 1\right)dA,$$

in which we used the fact that, on Σ, $z = \sqrt{x^2+y^2}$. This integral is easily evaluated using polar coordinates, obtaining -16π.

For the line integral, parametrize C by $x = 4\sin(t)$, $y = 4\cos(t)$, $z = 4$ for $0 \le t \le 2\pi$. This orientation is consistent with the choice of the unit normal $\mathbf{n}$ on Σ. This gives us

$$\oint_C \mathbf{F} \cdot d\mathbf{R} = \int_0^{2\pi} (-16\cos(t)\sin(t) - 16\sin^2(t))\,dt = -16\pi.$$

3. Notice that the boundary curve C is piecewise smooth and must be parametrized in three smooth curves. This is not difficult but is tedious. We therefore try to compute $\int\int_\Sigma (\nabla \times \mathbf{F}) \cdot \mathbf{n}\,d\sigma$. First,

$$\nabla \times \mathbf{F} = (x-y)\mathbf{i} - y\mathbf{j} - x\mathbf{k}.$$

And

$$\mathbf{n} = \frac{1}{\sqrt{21}}(2\mathbf{i} + 4\mathbf{j} + \mathbf{k}).$$

Finally, $d\sigma = \sqrt{21}\,dA$. Then

$$\iint_\Sigma (\nabla \times \mathbf{F}) \cdot \mathbf{n} d\sigma = \iint_D (x - 6y)\,dA = \int_0^2 \int_0^{4-2y} (x-6y)\,dx\,dy = -\frac{32}{3}.$$

5. The boundary curve C can be parametrized by $x = 2\cos(t)$, $y = 2\sin(t)$, $z = 0$ for $0 \le t \le 2\pi$. Further, on C,

$$\mathbf{R} = 2\cos(t)\mathbf{i} + 2\sin(t)\mathbf{j} + 0\mathbf{k}$$

so

$$\mathbf{F} \cdot d\mathbf{R} = (-16\cos^2(t)\sin^2(t) - 16\cos^2(t)\sin^2(t))\,dt = -32\cos^2(t)\sin^2(t)\,dt.$$

Then

$$\oint_C \mathbf{F} \cdot d\mathbf{R} = \int_0^{2\pi} -32\cos^2(t)\sin^2(t)\,dt = -8\pi.$$

If we use the surface integral, then evaluate $\int\int_\Sigma (\nabla \times \mathbf{F}) \cdot \mathbf{n}\,d\sigma$. Compute

$$\nabla \times \mathbf{F} = -(x^2+y^2)\mathbf{k}.$$

Further, a normal to Σ is

$$\nabla(x^2+y^2+z^2) = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}.$$

A unit normal vector is

$$\mathbf{n} = \frac{1}{2}(x\mathbf{i} + y\mathbf{j} + z\mathbf{k}).$$

Finally, $d\sigma = \sqrt{1 + (x/z)^2 + (y/z)^2}\,dA = (2/z)\,dA$. Then

$$\iint_\Sigma (\nabla \times \mathbf{F}) \cdot \mathbf{n}\,d\sigma = -\iint_D (x^2+y^2)\,dA = -\int_0^{2\pi}\int_0^2 r^3\,dr\,d\theta = -8\pi.$$

7. Compute

$$\nabla \times \mathbf{F} = -\mathbf{i} - \mathbf{j} - \mathbf{k}.$$

A normal to the surface is

$$\mathbf{N} = \mathbf{i} + 4\mathbf{j} + \mathbf{k}$$

and the unit normal is

$$\mathbf{n} = \frac{1}{\sqrt{18}}(\mathbf{i} + 4\mathbf{j} + \mathbf{k}).$$

Here C is the boundary of the part of the plane $x + 4y + z = 12$ in the first octant, consisting of three straight line segments: the line from $(0, 0, 12$ to $(12, 0, 0)$, then from $(12, 0, 0)$ to $(0, 3, 0)$, then from $(0, 3, 0)$ to $(0, 0, 12)$. We may think of this portion of the plane in the first octant as having equation $z = 12-x-4y$, with (x, y) varying over the triangle D bounded by the segment $[0, 12]$ on the $x-$ axis, the segment $[0, 3]$ on the $y-$ axis, and the line $x + 4y = 12$. D has area $(1/2)(12)(3) = 18$.

By Stokes's theorem, the circulation is

$$\oint_C \mathbf{F} \cdot \mathbf{T}\, ds = \iint_D (\nabla \times \mathbf{F}) \cdot \mathbf{n} d\sigma.$$

Now

$$(\nabla \times \mathbf{F}) \cdot \mathbf{n} = -\frac{6}{\sqrt{18}}$$

and

$$d\sigma = \| \mathbf{N} \| \; dx\, dy = \sqrt{18}$$

so

$$\begin{aligned} \iint_D (\nabla \times \mathbf{F}) \cdot \mathbf{n} d\sigma &= \iint_D \frac{-6}{\sqrt{18}} \sqrt{18}\, dx\, dy \\ &= -6(\text{ area of } D) = -6(18) = -108. \end{aligned}$$

12.10 Curvilinear Coordinates

In these problems, the scale factors may be denoted h_1, h_2, h_3 or, h_u, h_v, h_w if the orthogonal coordinates are denotes u, v, w. The unit vectors along the axes in the new coordinate system (the orthogonal curvilinear coordinates version of **i**, **j**, **k**), are

$$\mathbf{u}_\alpha = \frac{1}{h_\alpha}\left(\frac{\partial x}{\partial q_\alpha}\mathbf{i} + \frac{\partial y}{\partial q_\alpha}\mathbf{j} + \frac{\partial z}{\partial z_\alpha}\mathbf{k}\right).$$

if the orthogonal coordinates are denotes q_1, q_2, q_3. Sometimes mildly clumsy notation is tolerated in the context of curvilinear coordinates. For example, if $q_1 = u$, we might write

$$\mathbf{u}_{q_1} = \mathbf{u}_u,$$

in which we have to use the boldface notation to distinguish the vector **u** from the coordinate u.

1. In cylindrical coordinates, we often see the notations

$$u_1 = u_r = r, u_2 = u_\theta = \theta, u_3 = u_z = z.$$

From Example 12.31, we know that, for cylindrical coordinates, the scale factors are

$$h_1 = h_r = 1, h_2 = h_\theta = r, h_3 = h_z = 1.$$

Given $g(r, \theta, z)$, we can compute the gradient and Laplacian in cylindrical coordinates as

$$\nabla g = \frac{\partial g}{\partial r}\mathbf{u}_r + \frac{1}{r}\frac{\partial \mathbf{g}}{\partial \theta}\mathbf{u}_\theta + \frac{\partial g}{\partial z}\mathbf{u}_z$$

and

$$\begin{aligned}\nabla^2 g &= \frac{1}{r}\left(\frac{\partial}{\partial r}\left(r\frac{\partial g}{\partial r}\right) + \frac{\partial}{\partial \theta}\left(\frac{1}{r}\frac{\partial g}{\partial \theta}\right) + \frac{\partial}{\partial z}\left(r\frac{\partial g}{\partial z}\right)\right) \\ &= \frac{1}{r}\frac{\partial g}{\partial r} + \frac{\partial^2 g}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 g}{\partial \theta^2} + \frac{\partial^2 g}{\partial z^2}.\end{aligned}$$

Given a vector field $\mathbf{F}(r, \theta, z)$ in cylindrical coordinates, write

$$\mathbf{F} = f_1\mathbf{u}_1 + f_2\mathbf{u}_2 + f_3\mathbf{u}_3.$$

The divergence is given by

$$\begin{aligned}\nabla \cdot \mathbf{F} &= \frac{1}{r}\left(\frac{\partial}{\partial r}(f_1 r) + \frac{\partial}{\partial \theta}(f_2) + \frac{\partial}{\partial z}(r f_1)\right) \\ &= \frac{1}{r}\left(f_1 + r\frac{\partial f_1}{\partial r} + \frac{\partial f_2}{\partial \theta} + r\frac{\partial f_3}{\partial r}\right).\end{aligned}$$

Finally, the curl is given by

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{u}_r & \mathbf{u}_\theta & \mathbf{u}_z \\ \partial/\partial r & \partial/\partial \theta & \partial/\partial z \\ f_1 & r f_2 & f_3 \end{vmatrix}.$$

3. Bipolar coordinates are defined by

$$x = \frac{a\sinh(v)}{\cosh(v) - \cos(u)}, y = \frac{a\sin(u)}{\cosh(v) - \cos(u)}, z = z.$$

It is routine to compute the scale factors

$$h_u = \frac{a}{\cosh(v) - \cos(u)} = h_v, h_z = 1.$$

If $g(u, v, z)$ is a scalar function, then

$$\nabla g = \frac{1}{a}(\cosh(v) - \cos(u))\frac{\partial g}{\partial u}\mathbf{u}_u + \frac{1}{a}(\cosh(v) - \cos(u))\frac{\partial g}{\partial v}\mathbf{u}_v + \frac{\partial g}{\partial z}\mathbf{u}_z$$

and

$$\nabla^2 g = \frac{1}{a^2}(\cos(v) - \cos(u))^2\left[\frac{\partial^2 g}{\partial u^2} + \frac{\partial^2 g}{\partial v^2} + \frac{\partial}{\partial z}\left(\frac{a^2}{(\cosh(v) - \cos(u))^2}\frac{\partial g}{\partial z}\right)\right].$$

And, if $\mathbf{F}(u, v, z)$ is a vector field, then

$$\begin{aligned}\nabla \cdot \mathbf{F} &= \frac{1}{a^2}(\cosh(v) - \cos(u))^2\left[\frac{\partial}{\partial u}\left(\frac{a}{\cosh(v) - \cos(u)}f_1\right)\right. \\ &\left. + \frac{\partial}{\partial v}\left(\frac{a}{\cosh(v) - \cos(u)}f_2\right) + \frac{\partial}{\partial z}\left(\left(\frac{a}{\cosh(v) - \cos(u)}\right)^2 f_3\right)\right]\end{aligned}$$

and

$$\nabla \times \mathbf{F} = \frac{(\cosh(v) - \cos(u))^2}{a^2} \begin{vmatrix} h_1 \mathbf{u}_u & h_2 \mathbf{u}_v & \mathbf{u}_z \\ \partial/\partial u & \partial/\partial v & \partial/\partial z \\ h_u f_1 & h_v f_2 & f_3 \end{vmatrix}.$$

Chapter 13

Fourier Series

13.1 Why Fourier Series?

1. Suppose p is a polynomial of degree k and

$$p(x) = \sum_{n=1}^{N} b_n \sin(nx) \text{ for } 0 \le x \le \pi.$$

The $k+1$ order derivative of $p(x)$ is identically zero, while the $k+1$ derivative of the right side is a sum of constants times functions $\sin(nx)$ or a sum of constants times functions $\cos(nx)$, depending on whether k is even or odd. The only way such a linear combination can be identically zero for all x in $[0, \pi]$ is for each of the coefficients to be zero. But then each $b_n = 0$, so $p(x)$ is identically zero.

3. Figure 13.1 shows graphs of $f(x)$ and $S_2(x)$, while Figure 13.2 has $f(x)$ and $S_{10}(x)$. $S_2(x)$ is not very close to $f(x)$, while the function and $S_{10}(x)$ appear indistinguishable in the scale of the graphs. As $N \to \infty$, $S_N(x) \to f(x)$ for all x in $[0, \pi]$, as is shown in Section 13.2.

In general, convergence of Fourier series can be slow, and it might take large numbers of terms before the partial sums of the series close in on the function.

13.2 The Fourier Series of a Function

1. The Fourier series is

$$\frac{\sin(3)}{3} + 6\sin(3)\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2\pi^2 - 9} \cos\left(\frac{n\pi x}{3}\right),$$

converging to $\cos(x)$ on $[-3, 3]$.

Figure 13.3 compares the function to the fifth partial sum of this series.

3. The Fourier series of $f(x) = 4$ on $[-3, 3]$ has the form

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos(n\pi x/3) + b_n \sin(n\pi x/3)).$$

We must compute the coefficients. First, because f is an even function, each $b_n = 0$. Next,

$$a_0 = \frac{2}{3}\int_0^3 4\,dx = 8,$$

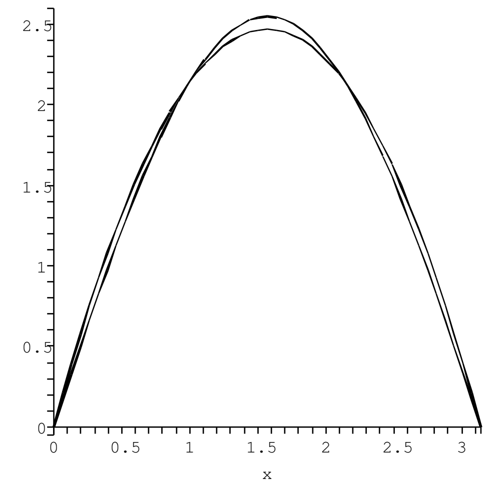

Figure 13.1: $f(x)$ and $S_2(x)$ in Problem 3, Section 13.1.

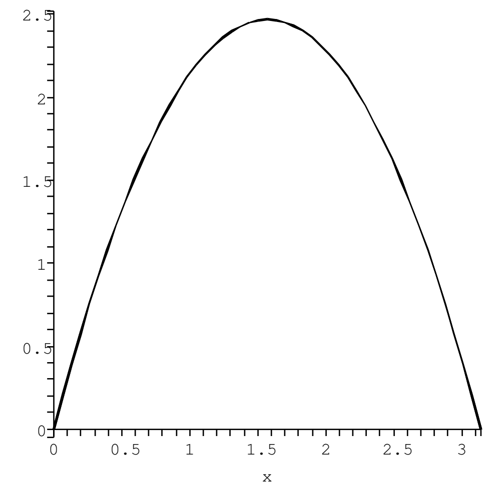

Figure 13.2: $f(x)$ and $S_{10}(x)$ in Problem 3, Section 13.1.

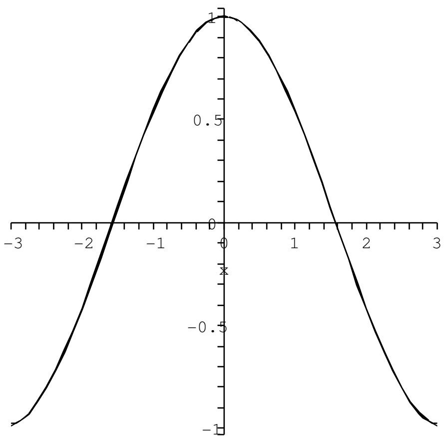

Figure 13.3: Fifth partial sum of the Fourier series in Problem 1.

and, for $n = 1, 2, \cdots$,

$$a_n = \frac{2}{3}\int_0^3 4\cos(n\pi x/3)\,dx = 0.$$

This Fourier series has just one term, 4 itself. Of course, this converges to 4 on $[-3, 3]$.

5. Since $f(x) = \cosh(\pi x)$ is an even function, each $b_n = 0$. Further,

$$a_0 = \int_0^1 \cosh(\pi x)\,dx = \frac{1}{\pi}\sinh(\pi)$$

and, for $n = 1, 2, \cdots$,,

$$a_n = 2\int_0^1 \cosh(\pi x)\cos(n\pi x)\,dx = \frac{2\sinh(\pi)}{\pi}\frac{(-1)^n}{1+n^2}.$$

The Fourier series is

$$\frac{1}{\pi}\sinh(\pi) + \frac{2}{\pi}\sinh(\pi)\sum_{n=1}^{\infty}\frac{(-1)^n}{1+n^2}\cos(n\pi x).$$

This series converges to $\cosh(\pi x)$ for $-1 \le x \le 1$. Figure 13.4 shows the function and eighth partial sum of this Fourier series.

7. The series is

$$\frac{16}{\pi}\sum_{n=1}^{\infty}\frac{1}{2n-1}\sin((2n-1)x),$$

converging to

$$\begin{cases} -4 & \text{for } -\pi < x < 0, \\ 4 & \text{for } 0 < x < 4, \\ 0 & \text{for } 0, \pi, -\pi. \end{cases}$$

Figure 13.5 shows a graph of this function and the twentieth partial sum of its Fourier series.

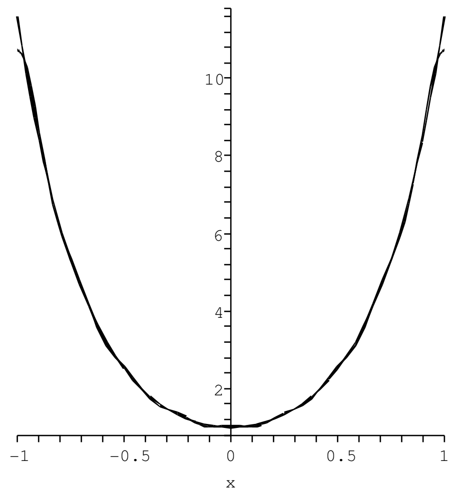

Figure 13.4: Eighth partial sum of the Fourier series in Problem 5.

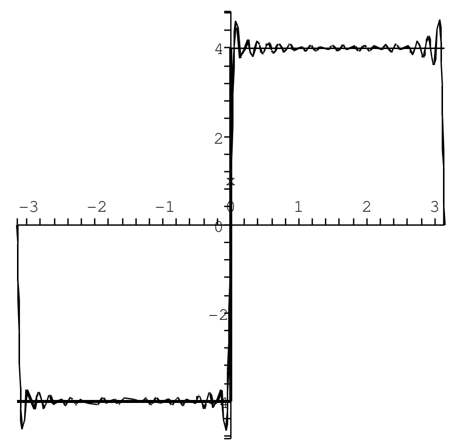

Figure 13.5: Twentieth partial sum of the Fourier series in Problem 7.

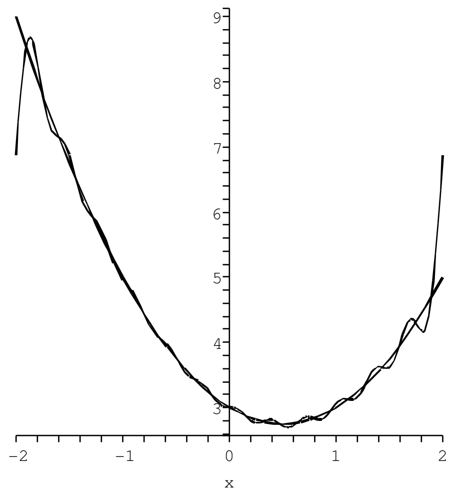

Figure 13.6: Twelfth partial sum of the Fourier series in Problem 9.

9. The Fourier series is

$$\frac{13}{3} + \sum_{n=1}^{\infty}(-1)^n \left[\frac{16}{(n\pi)^2}\cos(n\pi x/2) + \frac{4}{n\pi}\sin(n\pi x/2)\right],$$

converging to $f(x)$ for $-2 < x < 2$, and, at 2 and at -2, to

$$\frac{1}{2}(f(2+) + f(2-)) = \frac{1}{2}(9+5) = 7.$$

A graph of $f(x)$ and the twelfth partial sum of this Fourier series is shown in Figure 13.6.

11. The Fourier series of $f(x)$ on $[-\pi, \pi]$ is

$$\frac{3}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{2n-1}\sin((2n-1)x).$$

This converges to

$$\begin{cases} 1 & \text{for } -\pi < x < 0, \\ 2 & \text{for } 0 < x < \pi, \\ 3/2 & \text{for } x = 0, \pi, \text{ and } -\pi. \end{cases}$$

Figure 13.7 shows the function and the thirtieth partial sum of this Fourier series.

For each of Problems 13, 15, 17, and 19, the convergence theorem is used to determine the sum of the Fourier series of the function on the interval. It is not necessary (nor is it of any relevance) to write the Fourier series to obtain this information.

13. The Fourier series converges to

$$\begin{cases} -1 & \text{for } -4 < x < 0, \\ 0 & \text{for } x = \pm 4 \text{ and for } x = 0, \\ 1 & \text{for } 0 < x < 4. \end{cases}$$

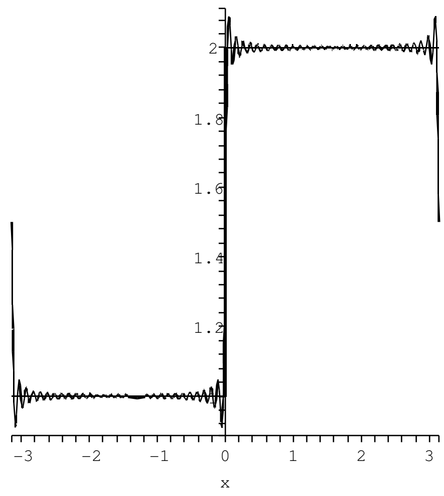

Figure 13.7: Thirtieth partial sum of the Fourier series in Problem 11.

15. The Fourier series converges to

$$\begin{cases} -1 & \text{for } x = -4, 4, \\ 3/2 & \text{for } x = -2, \\ 5/2 & \text{for } x = 2, \\ f(x) & \text{for all other } x \text{ in } [-4, 4]. \end{cases}$$

17. The Fourier series converges to

$$\begin{cases} (2+\pi^2)/2 & \text{for } x = \pm\pi, \\ x^2 & \text{for } -\pi < x < 0, \\ 1 & \text{for } x = 0, \\ 2 & \text{for } 0 < x < \pi. \end{cases}$$

19. The Fourier series of $f(x)$ on this interval converges to

$$\begin{cases} 3/2 & \text{for } x = \pm 3, \\ 2x & \text{for } -3 < x < -2, \\ -2 & \text{for } x = -2, \\ 0 & \text{for } -2 < x < 1, \\ 1/2 & \text{for } x = 1, \\ x^2 & \text{for } 1 < x < 3. \end{cases}$$

13.3 Sine and Cosine Series

1. The cosine expansion is

$$\frac{5}{6} + \frac{16}{\pi^2}\sum_{n=1}^{\infty}\left[\frac{1}{n^2}\cos\left(\frac{n\pi}{4}\right) - \frac{4}{n^3\pi}\sin\left(\frac{n\pi}{4}\right)\right]\cos\left(\frac{n\pi x}{4}\right)$$

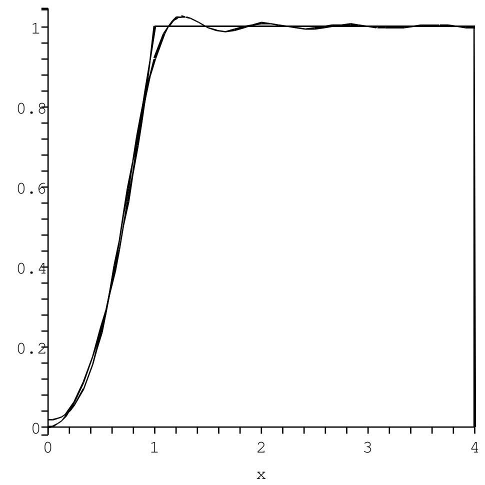

Figure 13.8: Tenth partial sum of the cosine expansion in Problem 1, Section 13.3.

and this converges to x^2 if $0 \le x \le 1$ and to 1 if $1 < x \le 4$. Figure 13.8 shows the tenth partial sum of this cosine expansion, compared to a graph of the function.

The sine expansion is

$$\sum_{n=1}^{\infty}\left[\frac{16}{n^2\pi^2}\sin\left(\frac{n\pi}{4}\right)+\frac{64}{n^3\pi^3}\left(\cos\left(\frac{n\pi}{4}\right)-1\right)-\frac{2(-1)^n}{n\pi}\right]\sin\left(\frac{n\pi x}{4}\right),$$

converging to x^2 for $0 \le x \le 1$, to 1 if $1 < x < 4$, and to 0 if $x = 4$. Figure 13.9 shows a graph of the twentieth partial sum, compared to the function.

3. The cosine series is

$$\frac{4}{3}+\frac{16}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos(n\pi x/2),$$

converging to x^2 for $0 \le x \le 2$. Figure 13.10 compares $f(x)$ to the tenth partial sum of this cosine expansion.

The sine expansion is

$$-\frac{8}{\pi}\sum_{n=1}^{\infty}\left[\frac{(-1)^n}{n}+\frac{2(1-(-1)^n)}{n^3\pi^2}\right]\sin(n\pi x/2),$$

converging to x^2 for $0 \le x < 2$ and to 0 for $x = 2$. Figure 13.11 shows the fiftieth partial sum of this sine expansion.

5. The cosine expansion is

$$\frac{1}{2}+\sum_{n=1}^{\infty}\left[\frac{4}{n\pi}\sin(2n\pi/3)+\frac{12}{n^2\pi^2}\cos(2n\pi/3)-\frac{6}{n^2\pi^2}(1+(-1)^n)\right]\cos(n\pi x/3),$$

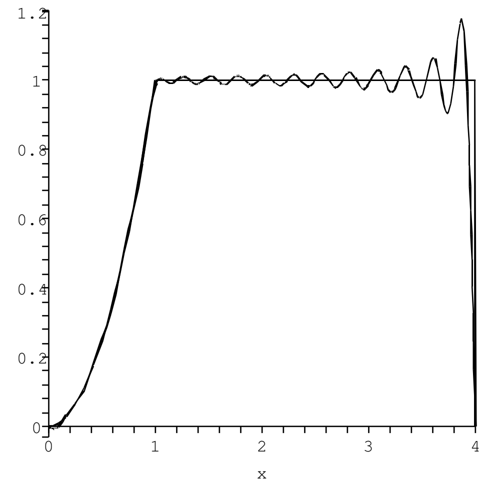

Figure 13.9: Twentieth partial sum of the sine expansion in Problem 1, Section 13.3.

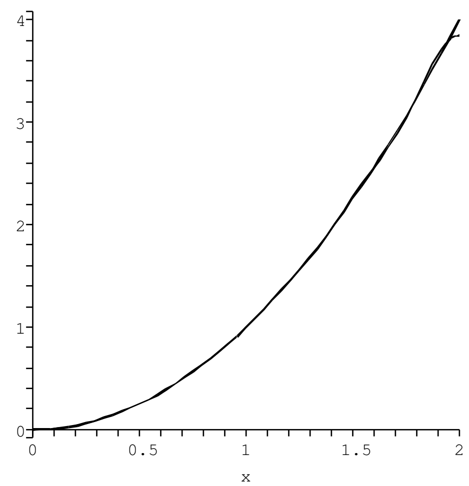

Figure 13.10: Tenth partial sum of the cosine expansion in Problem 3, Section 13.3.

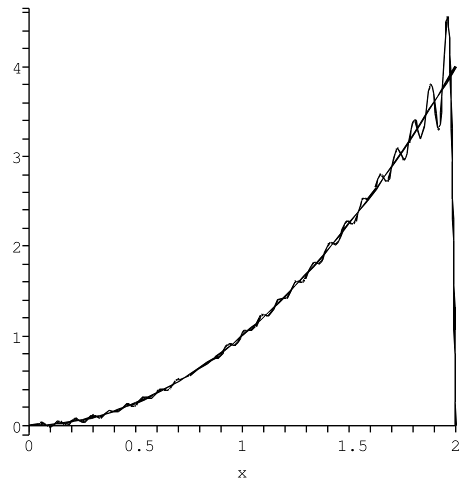

Figure 13.11: Fiftieth partial sum of the sine expansion in Problem 3, Section 13.3.

converging to

$$\begin{cases} x & \text{for } 0 \leq x < 2, \\ 1 & \text{for } x = 2 \\ 2-x & \text{for } 2 < x \leq 3. \end{cases}$$

Figure 13.12 compares $f(x)$ with the fortieth partial sum of this cosine series.

The sine expansion is

$$\sum_{n=1}^{\infty} \left[\frac{12}{n^2\pi^2} \sin(2n\pi/3) - \frac{4}{n\pi} \cos(2n\pi/3) + \frac{2}{n\pi}(-1)^n \right] \sin(n\pi x/3),$$

converging to

$$\begin{cases} x & \text{for } 0 \leq x < 2, \\ 1 & \text{for } x = 2, \\ 2-x & \text{for } 2 < x < 3, \\ 0 & \text{for } x = 3. \end{cases}$$

Figure 13.13 shows the fifty-fifth partial sum of this sine series.

7. The cosine expansion is 4, just the constant term. The sine expansion is

$$\frac{16}{\pi} \sum_{n=1}^{\infty} \frac{1}{2n-1} \sin((2n-1)\pi x/3),$$

converging to 4 if $x = 0, 3$ and to 4 if $0 < x < 3$. Figure 13.14 shows the tenth and twenty-fifth partial sums of this series compared to the function.

9. The cosine series is

$$\frac{1}{2}\cos(x) - \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n(2n-1)}{(2n-3)(2n+1)} \cos((2n-1)x/2),$$

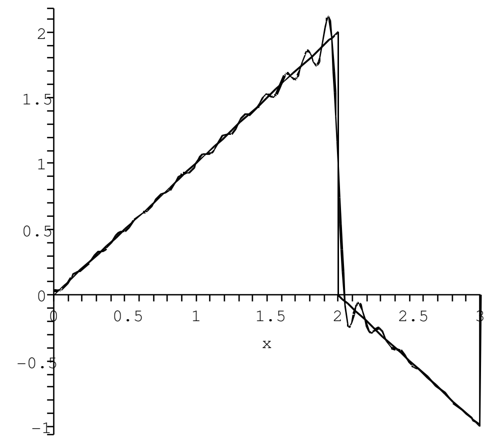

Figure 13.12: Fortieth partial sum of the cosine expansion in Problem 5, Section 13.3.

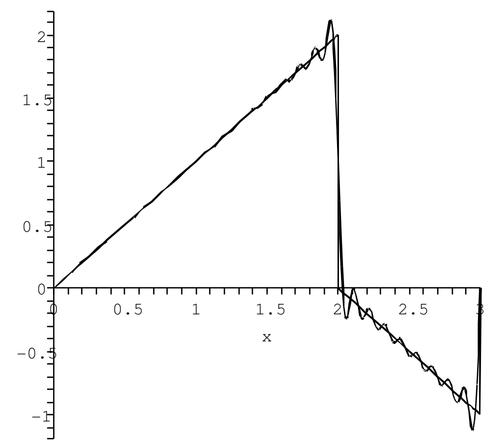

Figure 13.13: Fifty-fifth partial sum of the sine expansion in Problem 5, Section 13.3.

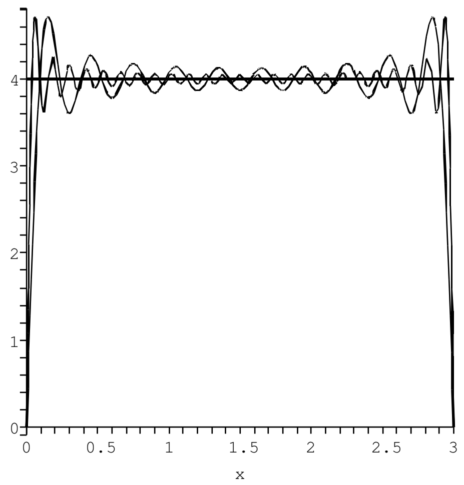

Figure 13.14: Partial sums of the sine series in Problem 7, Section 13.3.

converging to

$$\begin{cases} 0 & \text{for } 0 \leq x < \pi, \\ -1/2 & \text{for } x = \pi, \\ \cos(x) & \text{for } \pi < x < 2\pi, \\ 0 & \text{for } x = 2\pi. \end{cases}$$

Figure 13.15 shows a graph of the function and the fifteenth partial sum of this cosine expansion.

The sine series is

$$-\frac{2}{3\pi}\sin(x/2) - \sum_{n=3}^{\infty} \frac{2n}{(n^2-4)\pi}((-1)^n + \cos(n\pi/2))\sin(nx/2),$$

converging to

$$\begin{cases} 0 & \text{for } 0 \leq x < \pi, \\ -1/2 & \text{for } x = \pi, \\ \cos(x) & \text{for } \pi < x < 2\pi, \\ 0 & \text{for } x = 2\pi. \end{cases}$$

Figure 13.16 is a graph of the function and the fortieth partial sum of this sine series.

11. Suppose f is both even and odd on $[-L, L]$. Then, for any x in this interval,

$$f(x) = f(-x) = -f(x)$$

so $f(x) = 0$. To be both even and odd, the function must be identically zero on the interval.

12. **Hint** Consider the functions

$$f_e(x) = \frac{f(x) + f(-x)}{2}$$

and

$$f_o(x) = \frac{f(x) - f(-x)}{2}.$$

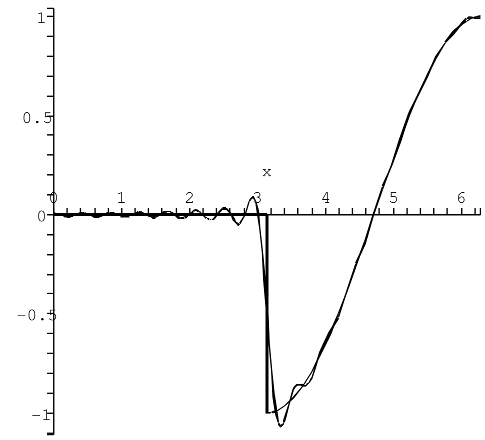

Figure 13.15: Partial sum of the cosine series in Problem 9, Section 13.3.

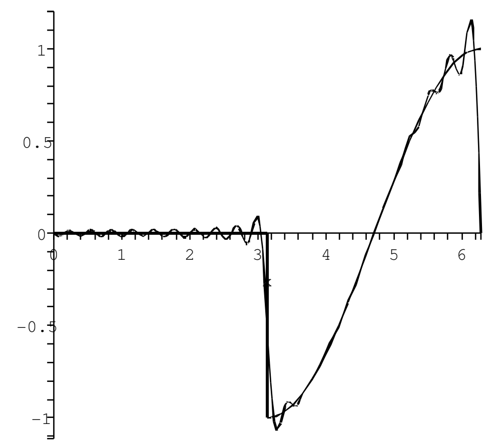

Figure 13.16: Partial sum of the sine series in Problem 9, Section 13.3.

13. The Fourier cosine expansion of $\sin(x)$ on $[0,\pi]$ is

$$\frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{4n^2-1}\cos(2nx).$$

This converges to $\sin(x)$ for $0 \le x \le \pi$. Put $x = \pi/2$ in this series to obtain

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{4n^2-1} = \frac{\pi}{4}\left(\frac{2}{\pi}-1\right) = \frac{1}{2}-\frac{\pi}{4}.$$

13.4 Integration and Differentiation of Fourier Series

1. The Fourier expansion of $f(x)$ on $[-\pi,\pi]$ is

$$1 - \frac{1}{2}\cos(x) - 2\sum_{n=2}^{\infty}\frac{(-1)^n}{n^2-1}\cos(nx).$$

This converges to $x\sin(x)$ for $-\pi \le x \le \pi$. Note that f is continuous on $[-\pi,\pi]$, that $f(\pi) = f(-\pi)$, and that $f'(x)$ is continuous (hence piecewise continuous) on this interval. We can differentiate the Fourier series expansion to write, for $-\pi < x < \pi$,

$$\begin{aligned} f'(x) &= \sin(x) + x\cos(x) \\ &= \frac{1}{2}\sin(x) + 2\sum_{n=2}^{\infty}\frac{n(-1)^n}{n^2-1}\sin(nx). \end{aligned}$$

It is routine to check that the Fourier expansion of $g(x) = \sin(x) + x\cos(x)$ agrees with this result.

3. Let the Fourier coefficients of f on $[-L,L]$ be a_n, b_n, as usual. From Bessel's inequality, the series

$$\sum_{n=0}^{\infty} a_n^2 \text{ and } \sum_{n=1}^{\infty} b_n^2$$

both converge. As with any convergent series, the general term has limit zero as $n \to \infty$, so

$$\lim_{n\to\infty} a_n^2 = \lim_{n\to\infty} b_n^2 = 0.$$

This means that a_n^2 and b_n^2 can be made as close to zero as we like, by choosing n sufficiently large. But then this will hold also for a_n and b_n, so

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = 0.$$

Inserting the integrals for the Fourier coefficients, we have

$$\lim_{n\to\infty}\frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right) = \lim_{n\to\infty}\frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right) = 0.$$

The positive factor of $1/L$ does not affect this limit, so

$$\lim_{n\to\infty}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right) = \lim_{n\to\infty}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right) = 0.$$

4. **Hint** We will outline a proof of Theorem 13.8. Let the Fourier coefficients of f on $[-L, L]$ be a_n, b_n, and the Fourier coefficients of f', A_n, B_n. Show that $A_0 = 0$ and use integration by parts and properties of $f(x)$ to show that

$$A_n = \frac{n\pi}{L} a_n \text{ and } B_n = -\frac{n\pi}{L} a_n$$

for $n = 1, 2, 3, \cdots$. Check that

$$0 \leq \left(|A_n| - \frac{1}{n}\right)^2 = A_n^2 - \frac{2}{n}|A_n| + \frac{1}{n^2}$$

with a similar inequality for B_n, and add these inequalities to obtain

$$\frac{2}{n}\left(|A_n| + |B_n|\right) \leq A_n^2 + B_n^2 + \frac{2}{n^2}.$$

Thus show that

$$|a_n| + |b_n| \leq \frac{L}{2\pi}(A_n^2 + B_n^2) + \frac{L}{\pi}\frac{1}{n^2}.$$

Use Bessel inequality to show that

$$\sum_{n=1}^{\infty} A_n^2 \text{ and } \sum_{n=1}^{\infty} B_n^2$$

converge. From this, show that

$$\sum_{n=1}^{\infty}(|a_n| + |b_n|)$$

converges. Finally, observe that, on $[-L, L]$,

$$|a_n \cos(n\pi x/L) + b_n \sin(n\pi x/L)| \leq |a_n| + |b_n|$$

and use a theorem of Weierstrass on uniform convergence to conclude the proof.

5. Since f is continuous on $[-\pi, \pi]$ and piecewise smooth on this interval. By the Fourier convergence theorem,

$$f(x) = \frac{\pi}{4} + \sum_{n=1}^{\infty}\left(\frac{1}{n^2\pi}((-1)^n - 1)\cos(nx) - \frac{(-1)^n}{n}\sin(nx)\right)$$

for $-\pi < x < \pi$. For $-\pi \leq x \leq \pi$, we can integrate the Fourier series term by term to obtain

$$\begin{aligned}\int_{\pi}^{x} f(t)\,dt &= \frac{\pi}{4}(x + \pi) \\ &+ \sum_{n=1}^{\infty}\left(\frac{1}{n^3\pi}((-1)^n - 1)\sin(nx) + \frac{(-1)^n}{n^2}\cos(nx) - \frac{1}{n^2}\right).\end{aligned}$$

13.5 Phase Angle Form

1.

$$\begin{aligned}f'(t + p) &= \lim_{h\to 0}\frac{f(t + p + h) - f(t + p)}{h} \\ &= \lim_{h\to 0}\frac{f(t + h) - f(t)}{h} = f'(t).\end{aligned}$$

3. For any t,

$$\begin{aligned}(\alpha f+\beta g)(t+p) &= \alpha f(t+p)+\beta g(t+p)\\ &= \alpha f(t)+\beta g(t) = (\alpha f+\beta g)(t).\end{aligned}$$

5. The Fourier series of f is

$$\frac{19}{8}+\sum_{n=1}^{\infty}\frac{2}{n^2\pi^2}\left[n\pi\sin\left(\frac{3n\pi}{2}\right)+\cos\left(\frac{3n\pi}{2}\right)-1\right]\cos\left(\frac{n\pi x}{2}\right)$$
$$+\left[\sin\left(\frac{3n\pi}{2}\right)-\frac{n\pi}{2}-n\pi\cos\left(\frac{3n\pi}{2}\right)\right]\sin\left(\frac{n\pi x}{2}\right).$$

The phase angle form is

$$\frac{19}{8}+\frac{1}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}d_n\cos\left(\frac{n\pi x}{2}+\delta_n\right),$$

where

$$d_n=\sqrt{8+5n^2\pi^2-12n\pi\sin(3n\pi/2)+4(n^2\pi^2-2)\cos(3n\pi/2)}$$

and

$$\delta_n=\arctan\left(\frac{n\pi/2+n\pi\cos(3n\pi/2)-\sin(3n\pi/2)}{n\pi\sin(3n\pi/2)+\cos(3n\pi/2)-1}\right).$$

7. The Fourier series of f is

$$\frac{1}{2}+\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{2n-1}\sin((2n-1)\pi x).$$

The phase angle form of this series is

$$1+\frac{2}{\pi}\sum_{n=1}^{\infty}\cos\left((2n-1)\pi x-\frac{\pi}{2}\right).$$

Points of the amplitude spectrum are

$$(0,1),\ (n\pi, 1/((2n-1)\pi)).$$

9. Write

$$f(x)=\begin{cases}1 & \text{for } 0\le x<1,\\ 2 & \text{for } 1<x<3,\\ 1 & \text{for } 3<x<4,\end{cases}$$

with $f(x+4)=f(x)$. The Fourier series of this function is

$$\frac{3}{2}+\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{2n-1}\cos\left(\frac{(2n-1)\pi x}{2}\right).$$

This has phase angle form

$$\frac{3}{2}+\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{2n-1}\cos\left((2n-1)\frac{\pi x}{2}+\frac{\pi}{2}(1-(-1)^n)\right).$$

11. We can write

$$f(x) = \begin{cases} x & \text{for } 0 \leq x < 1, \\ x-2 & \text{for } 1 < x \leq 2, \end{cases}$$

and $f(x+2) = f(x)$, so f has period 2. The Fourier series is

$$\frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin(n\pi x).$$

The phase angle form is

$$\frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \cos\left(n\pi x + (-1)^{n+1}\frac{\pi}{2}\right).$$

13.6 Complex Fourier Series

1. The complex Fourier series of f is

$$\frac{1}{2} - \frac{2}{\pi^2} \sum_{n=-\infty, n\neq 0}^{\infty} \frac{1}{(2n-1)^2} e^{(2n-1)\pi i x},$$

converging to $f(x)$ for $0 \leq x \leq 2$.

Points of the frequency spectrum are

$$(0, 1/2), \left(n\pi, \frac{2}{\pi^2}\frac{1}{(2n-1)^2}\right).$$

3. The complex Fourier series is

$$\frac{1}{2} + \frac{3i}{\pi} \sum_{n=-\infty, n\neq 0}^{\infty} e^{(2n-1)\pi i x/2},$$

converging to

$$\begin{cases} 1/2 & \text{for } x = 0, 2, 4, \\ -1 & \text{for } 0 < x < 2, \\ 2 & \text{for } 2 < x < 4. \end{cases}$$

Points of the frequency spectrum are

$$(0, 1/2), \left(\frac{n\pi}{2}, \frac{3}{(2n-1)\pi}\right).$$

5. The complex Fourier expansion of $f(x)$ is

$$\frac{3}{4} - \frac{1}{2\pi} \sum_{n=-\infty, n\neq 0}^{\infty} \frac{1}{n}\left(\sin(n\pi/2) + (\cos(n\pi/2) - 1)i\right) e^{n\pi i x/2}.$$

This converges to

$$\begin{cases} 1/2 & \text{for } x = 0 \text{ or } x = 1 \text{ or } x = 4, \\ 0 & \text{for } 0 < x < 1, \\ 1 & \text{for } 1 < x < 4. \end{cases}$$

Points of the frequency spectrum are

$$(0, 3/4), \left(\frac{n\pi}{2}, \frac{1}{2n\pi}\sqrt{\sin^2(n\pi/2) + (\cos(n\pi/2) - 1)^2}\right).$$

7. Compute

$$d_0 = \frac{1}{3}\int_0^3 2t\,dt = 3$$

and, for $n \neq 0$,

$$d_n = \frac{1}{3}\int_0^3 2te^{-2n\pi it/3}\,dt = \frac{3}{n\pi}i.$$

The complex Fourier series expansion of $f(x)$ is

$$\begin{aligned} &3 + \frac{3i}{\pi}\sum_{n=-\infty, n\neq 0}^{\infty} \frac{1}{n}e^{2n\pi ix/3} \\ &= \begin{cases} 3 & \text{for } x = 0 \text{ or } x = 3, \\ 2x & \text{for } 0 < x < 3. \end{cases} \end{aligned}$$

Points of the frequency spectrum are

$$(0, 3), \left(\frac{2n\pi}{3}, \frac{3}{n\pi}\right).$$

13.7 Filtering of Signals

1. We find the partial sums

$$S_N(t) = \frac{17}{4} + \sum_{n=1}^{\infty}\left[\frac{1-(-1)^n}{n^2\pi^2}\cos(n\pi t) + \frac{5-6(-1)^n}{n\pi}\sin(n\pi t)\right],$$

and

$$\begin{aligned} \sigma_N(t) = &\frac{17}{4} \\ &+ \frac{17}{4} + \sum_{n=1}^{\infty}\left(1 - \frac{n}{N}\right)\left[\frac{1-(-1)^n}{n^2\pi^2}\cos(n\pi t) + \frac{5-6(-1)^n}{n\pi}\sin(n\pi t)\right]. \end{aligned}$$

Figures 13.17, 13.18, and 13.19 compare the fifth, tenth and twenty-fifth partial sums of these sums with $f(t)$.

3. The complex Fourier coefficients of f are $d_0 = 0$ and, for nonzero n,

$$d_n = \frac{1}{4}\left[\int_{-2}^{0} -e^{-n\pi it/2}\,dt + \int_0^2 e^{-n\pi it/2}\,dt\right] = \frac{i}{\pi n}[(-1)^n - 1].$$

The complex Fourier series is

$$\sum_{n=-\infty, n\neq 0}^{\infty} \frac{i}{n\pi}[(-1)^n - 1]e^{n\pi it/2}.$$

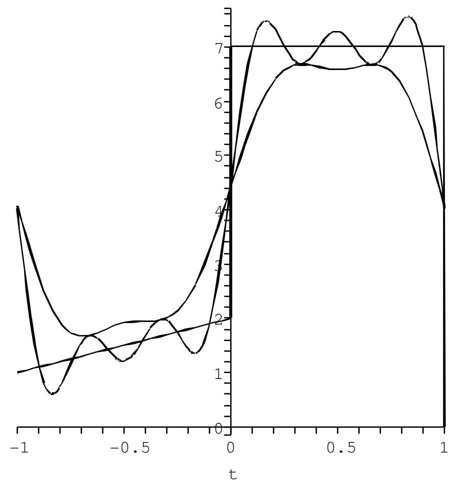

Figure 13.17: Fifth partial sum and Cesáro sum in Problem 1, Section 13.7.

If we carry out a calculation like that of Example 13.17, we obtain the Fourier series

$$\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{2n-1}\sin\left(\frac{(2n-1)\pi t}{2}\right).$$

The Nth partial sum is therefore

$$S_N(t)=\frac{4}{\pi}\sum_{n=1}^{N}\frac{1}{2n-1}\sin\left(\frac{(2n-1)\pi t}{2}\right).$$

The Nth Cesáro sum is formed by inserting factors $1-|n|/N$:

$$\sigma_N(t)=\frac{4}{\pi}\sum_{n=1}^{N}\left(1-\frac{2n-1}{N}\right)\frac{1}{2n-1}\sin\left(\frac{(2n-1)\pi t}{2}\right).$$

Figures 13.20, 13.21, and 13.22 compare $f(t)$, $S_N(t)$ and $\sigma_N(t)$ for $N=5,10,25$, respectively. Notice that the Cesáro sums have the effect of smoothing the Gibbs effect seem at 0 and the ends of the interval.

5. We find that

$$S_N(t)=\sum_{n=1}^{\infty}\frac{2}{n\pi}\left[\cos(n\pi/2)-(-1)^n\right]\sin(n\pi t)$$

and

$$\sigma_N(t)=\sum_{n=1}^{\infty}\left(1-\frac{n}{N}\frac{2}{n\pi}\right)\left[\cos(n\pi/2)-(-1)^n\right]\sin(n\pi t)$$

Figures 13.23, 13.24, and 13.25 compare the fifth, tenth and twenty-fifth partial sums of these sums with $f(t)$.

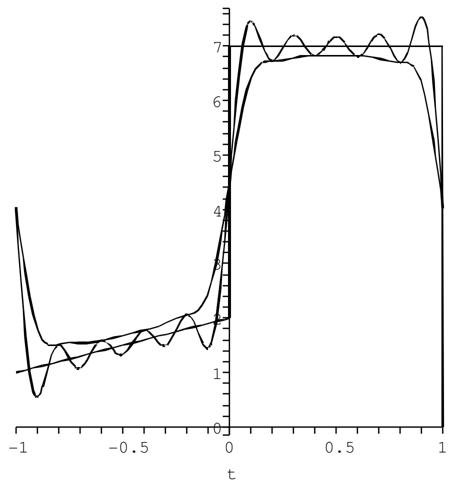

Figure 13.18: Tenth partial sum and Cesáro sum in Problem 1, Section 13.7.

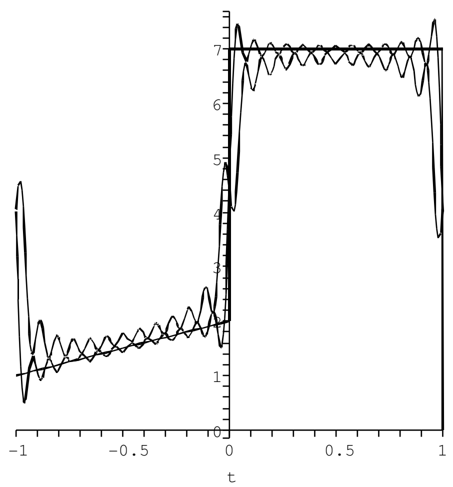

Figure 13.19: Twenty-fifth partial sum and Cesáro sum in Problem 1, Section 13.7.

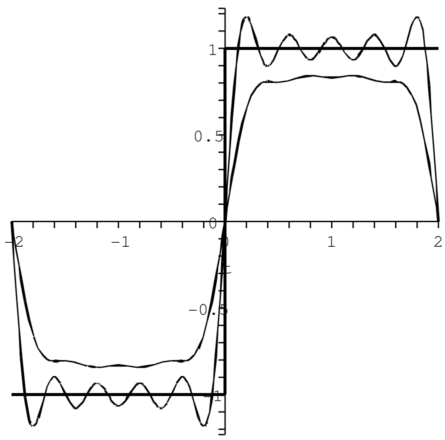

Figure 13.20: Fifth partial sum and Cesáro sum in Problem 3, Section 13.7.

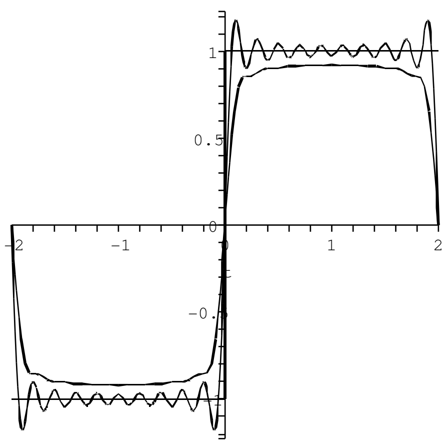

Figure 13.21: Tenth partial sum and Cesáro sum in Problem 3, Section 13.7.

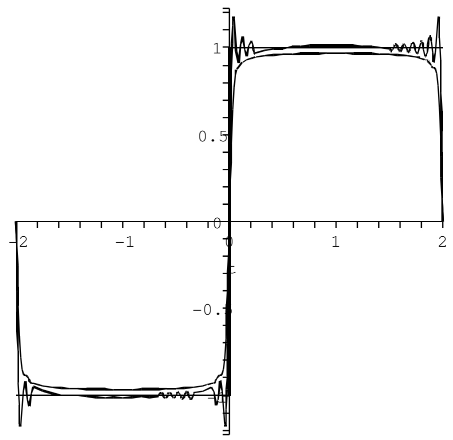

Figure 13.22: Twenty-fifth partial sum and Cesáro sum in Problem 3, Section 13.7.

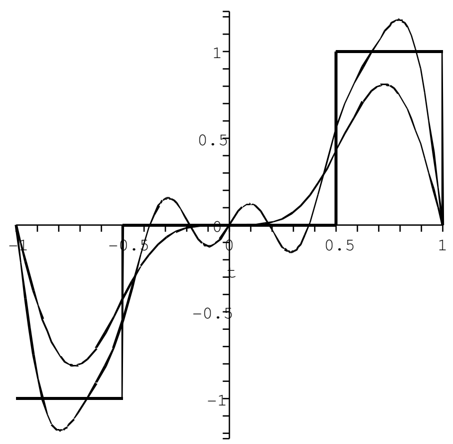

Figure 13.23: Fifth partial sum and Cesáro sum in Problem 5, Section 13.7.

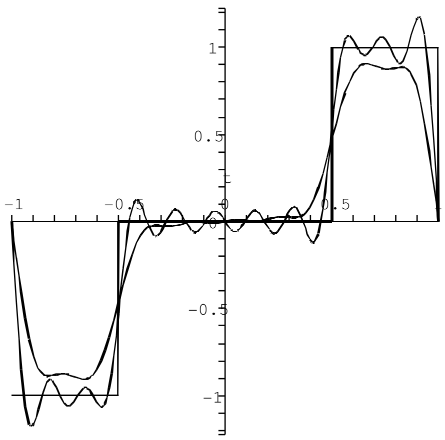

Figure 13.24: Tenth partial sum and Cesáro sum in Problem 5, Section 13.7.

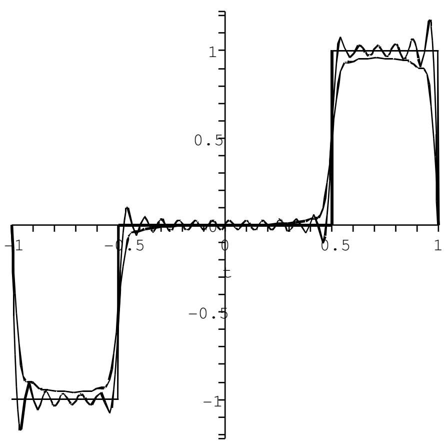

Figure 13.25: Twenty-fifth partial sum and Cesáro sum in Problem 5, Section 13.7.

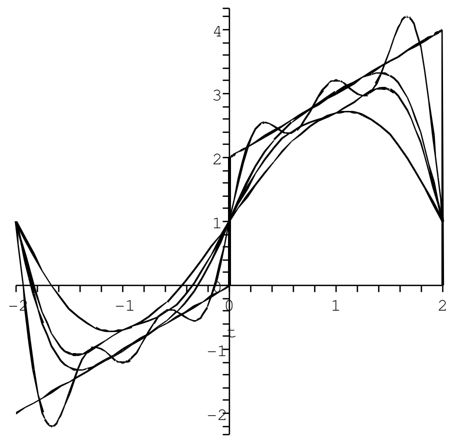

Figure 13.26: Fourier, Cesáro, Hamming, and Gauss partial sums, $N = 5$, in Problem 7, Section 13.7.

7. The partial sums are

$$S_N(t) = 1 + \sum_{n=1}^{\infty} \frac{2}{n\pi}(1 - 3(-1)^n)\sin\left(\frac{n\pi t}{2}\right),$$

$$\sigma_N(t) = 1 + \sum_{n=1}^{\infty} \frac{2}{n\pi}\left(1 - \frac{n}{N}\right)(1 - 3(-1)^n)\sin\left(\frac{n\pi t}{2}\right),$$

$$H_N(t) = 1 + \sum_{n=1}^{\infty} \frac{2}{n\pi}(0.54 + 0.46\cos(\pi n/N))(1 - 3(-1)^n)\sin\left(\frac{n\pi t}{2}\right),$$

$$G_N(t) = 1 + \sum_{n=1}^{\infty} \frac{2}{n\pi}e^{-n^2\pi^2/N^2}(1 - 3(-1)^n)\sin\left(\frac{n\pi t}{2}\right).$$

Graphs of these partial sums are given for $N = 5, 10, 25$, respectively, in Figures 13.26, 13.27, and 13.28.

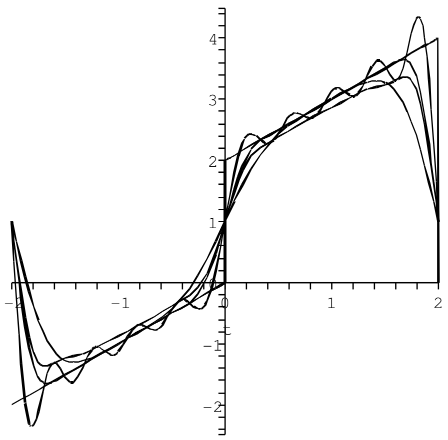

Figure 13.27: Fourier, Cesáro, Hamming, and Gauss partial sums, $N = 10$, in Problem 7, Section 13.7.

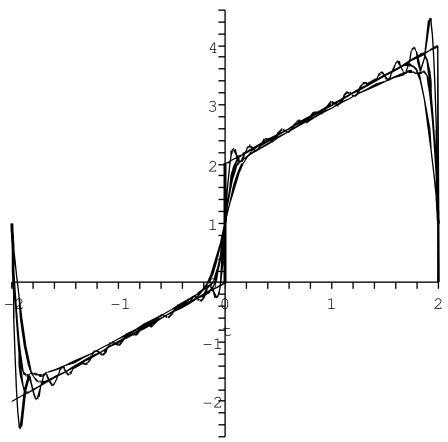

Figure 13.28: Fourier, Cesáro, Hamming, and Gauss partial sums, $N = 25$, in Problem 7, Section 13.7.

Chapter 14

The Fourier Integral and Transforms

14.1 The Fourier Integral

1. Clearly $\int_{-\infty}^{\infty} |f(x)|\,dx$ converges. Since $f(x)$ is even, $B_\omega = 0$. Compute

$$\begin{aligned} A_\omega &= \frac{1}{\pi}\int_{-100}^{100} t^2\cos(\omega t)\,dt = \frac{2}{\pi}\int_0^{100} t^2\cos(\omega t)\,dt \\ &= \frac{2}{\pi}\left[\frac{t^2\sin(\omega t)}{\omega} + \frac{2t\cos(\omega t)}{\omega^2} - \frac{2\sin(\omega t)}{\omega^3}\right]_0^{100} \\ &= \frac{20000\sin(100\omega)}{\pi\omega} - \frac{4\sin(100\omega)}{\pi\omega^3} + \frac{400\cos(100\omega)}{\pi\omega^2}. \end{aligned}$$

The Fourier integral representation of $f(x)$ is

$$\int_0^\infty \left[\frac{400\cos(100\omega)}{\pi\omega^2} + \frac{20000\omega^2 - 4}{\pi\omega^3}\sin(100\omega)\right]\cos(\omega x)\,d\omega.$$

This converges to

$$\begin{cases} x^2 & \text{for } -100 < x < 100, \\ 0 & \text{for } |x| > 100, \\ 5000 & \text{for } x = 100 \text{ and for } x = -100. \end{cases}$$

3. First,

$$\int_{-\infty}^{\infty} |f(x)|\,dx = \int_{-\pi}^{\pi} |x|\,dx = 2\int_0^\pi x\,dx = \pi^2.$$

Now $\xi\cos(\omega\xi)$ is an odd function of ξ, so each $A_\omega = 0$. Further,

$$\begin{aligned} B_\omega &= \frac{1}{\pi}\int_{-\infty}^{\infty} \xi\sin(\omega\xi)\,d\xi \\ &= \frac{1}{\pi}\int_{-\pi}^{\pi} \xi\sin(\omega\xi)\,d\xi = \frac{2}{\pi}\left[\frac{\sin(\pi\omega)}{\omega^2} - \frac{\pi}{\omega}\cos(\pi\omega)\right]. \end{aligned}$$

The Fourier integral representation of $f(x)$ is

$$\int_0^\infty \left[\frac{2\sin(\pi\omega)}{\pi\omega^2} - \frac{2\cos(\pi\omega)}{\omega}\right]\sin(\omega x)\,d\omega.$$

This representation converges to

$$\begin{cases} -\pi/2 & \text{for } x = -\pi, \\ x & \text{for } -\pi < x < \pi, \\ \pi/2 & \text{for } x = \pi, \\ 0 & \text{for } |x| > \pi. \end{cases}$$

5. With $f(x) = e^{-|x|}$, integrations yield the Fourier integral representation

$$\int_0^\infty \frac{1}{\pi(\omega^2+1)}\cos(\omega x)\,d\omega,$$

converging to $e^{-|x|}$ for all x.

7. Certainly $\int_{-\infty}^{\infty} |f(x)|\,dx$ converges, and each $A_\omega = 0$ because f is an odd function. Compute

$$B_\omega = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(\omega t)\,dt = \frac{2}{\pi\omega}(1-\cos(\pi\omega)).$$

The Fourier integral representation of $f(x)$ is

$$\int_0^\infty \frac{2}{\pi\omega}(1-\cos(\pi\omega))\sin(\omega x)\,d\omega.$$

This converges to

$$\begin{cases} -1/2 & \text{for } x = -\pi, \\ -1 & \text{for } -\pi < x < 0, \\ 0 & \text{for } x = 0 \text{ and for } |x| > \pi, \\ 1 & \text{for } 0 < x < \pi, \\ 1/2 & \text{for } x = \pi. \end{cases}$$

9. Certainly $\int_{-\infty}^{\infty} |f(t)|\,dt$ converges. Compute

$$A_\omega = \frac{1}{\pi}\int_{-3\pi}^{\pi} \sin(t)\cos(\omega t)\,dt = \frac{4\cos(\pi\omega)(\cos^2(\pi\omega)-1)}{\pi(\omega^2-1)}$$

and

$$B_\omega = \frac{1}{\pi}\int_{-3\pi}^{\pi} \sin(t)\sin(\omega t)\,dt = -\frac{4\sin(\pi\omega)\cos^2(\pi\omega)}{\pi(\omega^2-1)}.$$

The Fourier integral representation is

$$\int_0^\infty (A_\omega\cos(\omega x) + B_\omega\sin(\omega x))\,d\omega.$$

This converges to

$$\begin{cases} \sin(x) & \text{for } -3\pi \le x \le \pi, \\ 0 & \text{for } x < -3\pi \text{ and for } x > \pi. \end{cases}$$

11. First, we can write the Fourier integral representation of $f(x)$ as

$$\frac{1}{\pi}\int_0^\infty \int_{-\infty}^\infty f(t)\cos(\omega(t-x))\,dt\,d\omega.$$

Interchange the order of integration and use the fact that $f(t)\cos(\omega(t-x))$ is an even function of ω to write this integral representation as

$$\frac{1}{2\pi}\int_{-\infty}^\infty \int_{-\infty}^\infty f(t)\cos(\omega(t-x))\,d\omega\,dt.$$

Now $f(t)\sin(\omega(t-x))$ is a odd function of ω, so

$$\frac{1}{2\pi}\int_{-\infty}^\infty f(t)\sin(\omega(t-x))\,d\omega = 0.$$

We can therefore write the integral representation of $f(x)$ as

$$\begin{aligned}
&\frac{1}{2\pi}\int_{-\infty}^\infty \int_{-\infty}^\infty f(t)[\cos(\omega(t-x)) + i\sin(\omega(t-x))]\,d\omega\,dt \\
&= \frac{1}{2\pi}\int_{-\infty}^\infty f(t)\left[\lim_{r\to\infty}\int_{-r}^r e^{i\omega(t-x)}\,d\omega\right]\,dt \\
&= \frac{1}{\pi}\int_{-\infty}^\infty f(t)\left[\lim_{r\to\infty}\frac{e^{ir(t-x)} - e^{-ir(t-x)}}{2i(t-x)}\right]\,dt \\
&= \frac{1}{\pi}\lim_{\omega\to\infty}\int_{-\infty}^\infty f(t)\frac{\sin(\omega(t-x))}{t-x}\,dt.
\end{aligned}$$

14.2 Fourier Cosine and Sine Integrals

For Problems 1 - 10 we will give the cosine and sine integral representations without all of the details of the integrations for the coefficients.

1. The cosine integral is

$$\int_0^\infty \left[\frac{2}{\pi\omega}((2\pi-1)\sin(\pi\omega) + 2\sin(3\pi\omega)) + \frac{4}{\pi\omega^2}(\cos(\pi\omega)-1)\right]\cos(\omega x)\,d\omega,$$

converging to

$$\begin{cases} 1+2x & \text{for } 0<x<\pi, \\ (3+2\pi)/2 & \text{for } x=\pi, \\ 2 & \text{for } \pi<x<3\pi, \\ 1 & \text{for } x=3\pi \text{ and for } x=0, \\ 0 & \text{for } x>3\pi. \end{cases}$$

The sine integral is

$$\int_0^\infty \left[\frac{2}{\pi\omega}(1+(1-2\pi)\cos(\pi\omega) - 2\cos(3\pi\omega)) + \frac{4}{\pi\omega^2}\sin(\pi\omega)\right]\sin(\omega x)\,d\omega,$$

converging to

$$\begin{cases} 1+2x & \text{for } 0 < x < \pi, \\ (3+2\pi)/2 & \text{for } x = \pi, \\ 2 & \text{for } \pi < x < 3\pi, \\ 1 & \text{for } x = 3\pi, \\ 0 & \text{for } x > 3\pi \text{ and for } x = 0. \end{cases}$$

3. The cosine integral is

$$\int_0^\infty \frac{2}{\pi}\left(\frac{2+\omega^2}{4+\omega^4}\right)\cos(\omega x)\,d\omega,$$

converging to

$$\begin{cases} e^{-x}\cos(x) & \text{for } x > 0, \\ 1 & \text{for } x = 0. \end{cases}$$

The sine integral is

$$\int_0^\infty \frac{2}{\pi}\left(\frac{\omega^3}{4+\omega^4}\right)\sin(\omega x)\,d\omega,$$

converging to

$$\begin{cases} e^{-x}\cos(x) & \text{for } x > 0, \\ 0 & \text{for } x = 0. \end{cases}$$

5. The cosine integral is

$$\int_0^\infty \frac{2}{\pi\omega}(2\sin(4\omega) - \sin(\omega))\cos(\omega x)\,d\omega,$$

converging to

$$\begin{cases} 1 & \text{for } 0 < x < 1, \\ 3/2 & \text{for } x = 1, \\ 2 & \text{for } 1 < x < 4, \\ 1 & \text{for } x = 0 \text{ and for } x = 4, \\ 0 & \text{for } x > 4. \end{cases}$$

The sine integral is

$$\int_0^\infty \frac{2}{\pi\omega}(1 + \cos(\omega) - 2\cos(4\omega))\sin(\omega x)\,d\omega,$$

converging to

$$\begin{cases} 1 & \text{for } 0 < x < 1, \\ 3/2 & \text{for } x = 1, \\ 2 & \text{for } 1 < x < 4, \\ 1 & \text{for } x = 4, \\ 0 & \text{for } x = 0 \text{ and for } x > 4. \end{cases}$$

7. The cosine representation is

$$\int_0^\infty \frac{2k}{\pi\omega}\sin(c\omega)\cos(\omega x)\,d\omega.$$

The sine integral representation is

$$\int_0^\infty \frac{2k}{\pi\omega}(1-\cos(c\omega))\sin(\omega x)\,d\omega.$$

Both integrals converge to

$$\begin{cases} k & \text{for } 0 < x < c, \\ k/2 & \text{for } x = c, \\ 0 & \text{for } x > c, \end{cases}$$

while the cosine expansion converges to k at 0, and the sine expansion converges to 0 at 0.

9. The Fourier cosine integral representation of $f(x)$ is

$$\int_0^\infty \frac{4}{\pi\omega^3}(10\omega\cos(10\omega) - (50\omega^2 - 1)\sin(10\omega))\cos(\omega x)\,d\omega.$$

The sine integral representation is

$$\int_0^\infty \frac{4}{\pi\omega^3}(10\omega\sin(10\omega) - (50\omega^2 - 1)\cos(10\omega) - 1)\sin(\omega x)\,d\omega.$$

Both integrals converge to

$$\begin{cases} x^2 & \text{for } 0 \le x < 10, \\ 0 & \text{for } x > 10, \\ 50 & \text{for } x = 10. \end{cases}$$

11. From the Laplace integrals and the convergence theorem, we can write

$$e^{-kx} = \frac{2k}{\pi}\int_0^\infty \frac{1}{k^2+\omega^2}\cos(\omega x) \text{ for } x \ge 0$$

and

$$e^{-kx} = \frac{2}{\pi}\int_0^\infty \frac{\omega}{k^2+\omega^2}\sin(\omega x)\,d\omega \text{ for } x > 0.$$

Put $k = 1$ and interchange the symbols x and ω to obtain

$$A_\omega = \frac{\pi e^{-\omega}}{2} = \int_0^\infty \frac{1}{1+x^2}\cos(\omega x)\,dx$$

and

$$B_\omega = \frac{\pi e^{-\omega}}{2} = \int_0^\infty \frac{x}{1+x^2}\sin(\omega x)\,dx.$$

From these it follows that the Fourier cosine integral representation of $1/(1+x^2)$ is

$$C(x) = \int_0^\infty e^{-\omega}\cos(\omega x)\,d\omega = \frac{1}{1+x^2} \text{ for } x \ge 0$$

and the Fourier sine integral for $x/(1+x^2)$ is

$$S(x) = \int_0^\infty e^{-\omega}\sin(\omega x)\,d\omega = \frac{x}{1+x^2} \text{ for } x > 0.$$

By direct computation, we also have $S(0) = 0$.

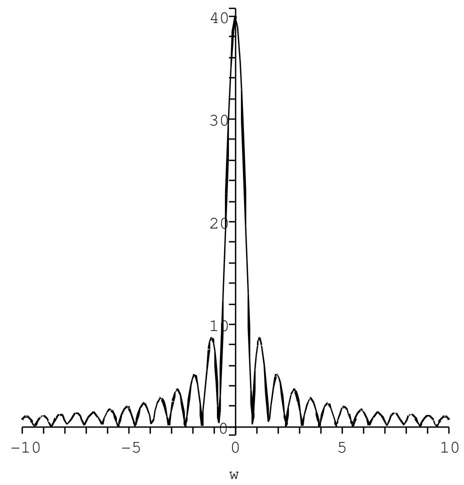

Figure 14.1: Amplitude spectrum in Problem 1, Section 14.3.

14.3 The Fourier Transform

1. $f(t) = 5(H(t-3) - H(t-11)) = 5(H(t+4-7) - H(t+4+7))$, so

$$\hat{f}(\omega) = 5e^{-7i\omega}\left(\frac{2\sin(4\omega)}{\omega}\right) = \frac{10}{\omega}e^{-7i\omega}\sin(4\omega).$$

The amplitude spectrum is the graph of

$$|\hat{f}(\omega)|(\omega) = \left|\frac{10}{\omega}\sin(4\omega)\right|,$$

shown in Figure 14.1.

3.
$$\hat{f}(\omega) = \int_{-1}^{0} -e^{-i\omega t}\,dt + \int_{0}^{1} e^{-i\omega t}\,dt = \frac{2i}{\omega}(\cos(\omega) - 1).$$

The amplitude spectrum is the graph of

$$|\hat{f}(\omega)| = \left|\frac{2}{\omega}(\cos(\omega) - 1)\right|,$$

shown in Figure 14.2.

5.
$$\hat{f}(\omega) = \frac{24}{16+\omega^2}e^{2i\omega}$$

The amplitude spectrum is the graph of

$$|\hat{f}(\omega)| = \frac{24}{16+\omega^2},$$

shown in Figure 14.3.

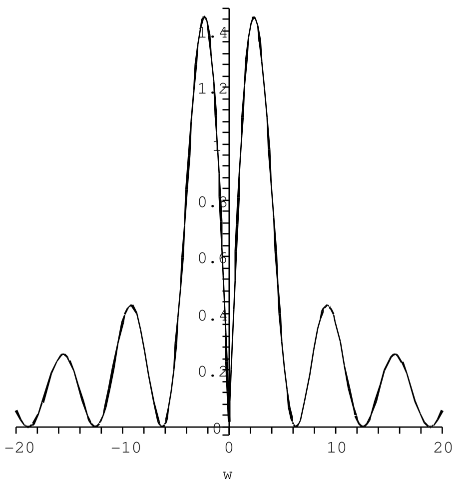

Figure 14.2: Amplitude spectrum in Problem 3, Section 14.3.

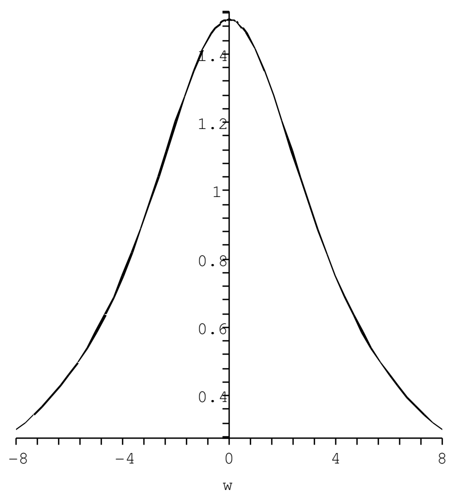

Figure 14.3: Amplitude spectrum in Problem 5, Section 14.3.

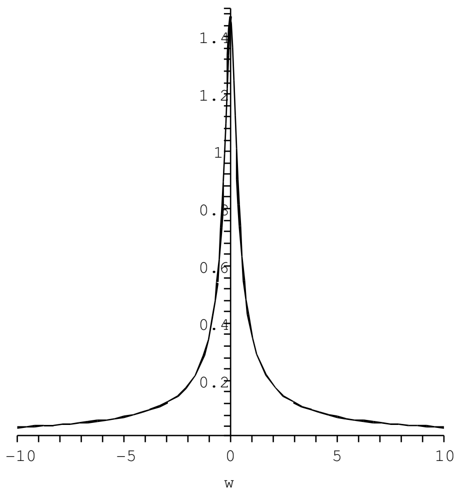

Figure 14.4: Amplitude spectrum in Problem 7, Section 14.3.

7.

$$\hat{f}(\omega) = \int_k^{\infty} e^{-t/4} e^{-i\omega t}\, dt$$
$$= \left. \frac{e^{-(i\omega+1/4)t}}{-(i\omega+1/4)} \right]_k^{\infty} = \frac{4e^{-(i\omega+1/4)k}}{1+4i\omega}.$$

Then

$$|\hat{f}(\omega)|(\omega) = \frac{4e^{-k/4}}{\sqrt{1+16\omega^2}}.$$

The amplitude spectrum is shown in Figure 14.4 for $k = 4$.

9.

$$\hat{f}(\omega) = \pi e^{-|\omega|}.$$

The amplitude spectrum is shown in Figure 14.5.

11. Write

$$\hat{f}(\omega) = \frac{1+i\omega}{(3+i\omega)(2+i\omega)} = \frac{2}{3+i\omega} - \frac{1}{2+i\omega}.$$

Then

$$f(t) = H(t)(2e^{-3t} - e^{-2t}).$$

13.

$$f(t) = 18\sqrt{\frac{2}{\pi}} e^{-4it} e^{-8t^2}$$

15. Write

$$\hat{f}(\omega) = \frac{e^{2(\omega-3)i}}{5+(\omega-3)i},$$

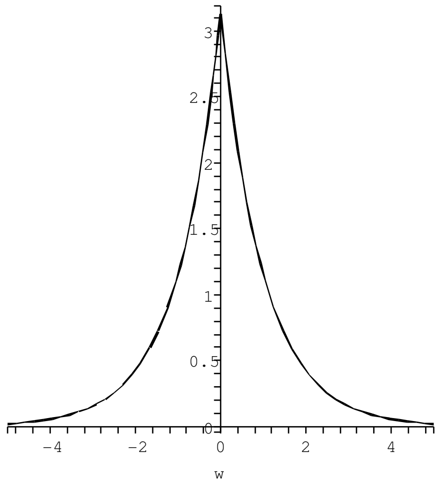

Figure 14.5: Amplitude spectrum in Problem 9, Section 14.3.

so

$$\begin{aligned} f(t) &= e^{3it}\hat{f}^{-1}\left[\frac{e^{2i\omega}}{5+i\omega}\right] \\ &= e^{3it}H(t+2)e^{-5(t+2)} = H(t+2)e^{-(10+(5-3i)t)}. \end{aligned}$$

17.

$$\begin{aligned} \hat{f}^{-1}\left(\frac{1}{(1+i\omega)^2}\right) &= H(t)e^{-t} * H(t)e^{-t} \\ &= \int_{-\infty}^{\infty} H(\tau)e^{-\tau}H(t-\tau)e^{-(t-\tau)}\,d\tau \\ &= H(t)e^{-t}\int_0^t d\tau = H(t)te^{-t}. \end{aligned}$$

19. Compute

$$\int_{-\infty}^{\infty} |f(t)|^2\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\omega)\overline{\hat{f}(\omega)}\,d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\hat{f}(\omega)|^2\,d\omega.$$

21. Begin with

$$\begin{aligned} \hat{f}\left[\frac{H(t+3)-H(t-3)}{2}\right](\omega) &= \frac{1}{2}\int_{-3}^{3} e^{-i\omega t}\,dt \\ &= \frac{e^{3i\omega}-e^{-3i\omega}}{2i\omega} = \frac{\sin(3\omega)}{\omega}. \end{aligned}$$

Using the symmetry property of the transform,

$$\begin{aligned} \hat{f}\left[\frac{\sin(3t)}{t}\right](\omega) &= \pi[H(-\omega+3)-H(-\omega-3)] \\ &= \pi[H(\omega+3)-H(\omega-3)]. \end{aligned}$$

Now use Parseval's identity to write

$$\int_{-\infty}^{\infty} \left(\frac{\sin(3t)}{t}\right)^2 dt = \frac{1}{2\pi}\int_{-3}^{3} \pi^2\, d\omega = 3\pi.$$

22. One way to compute this energy is to start with

$$\hat{f}(\omega)[H(t)e^{-2t}](\omega) = \frac{1}{2+i\omega}.$$

By Parseval's theorem (Problem 19),

$$\begin{aligned}\int_{-\infty}^{\infty} |f(t)|^2\, dt &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \left|\frac{1}{2+i\omega}\right|^2 d\omega \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{4+\omega^2}\, d\omega \\ &= \frac{1}{4\pi}\arctan^{-1}\left(\frac{\omega}{2}\right)\Big]_{-\infty}^{\infty} = \frac{1}{4}.\end{aligned}$$

Another way to compute the same result is to proceed directly:

$$\int_{-\infty}^{\infty} (H(t)e^{-2t})^2\, dt = \int_{0}^{\infty} e^{-2t}\, dt = \frac{1}{4}.$$

In this example the direct computation is clearly simpler, but for some problems it is useful to be aware of this use of Parseval's theorem.

23. First,

$$\begin{aligned}\hat{f}_{\text{win}}(\omega) &= \int_{-2}^{2} (t+2)^2 e^{-i\omega t}\, dt \\ &= \frac{4}{\omega^3}\left((4\omega^2-1)\sin(2\omega) + 2\omega\cos(2\omega)\right) \\ &+ \frac{8i}{\omega^2}\left(2\omega\cos(2\omega) - \sin(2\omega)\right).\end{aligned}$$

With $w(t) = 1$ and support $[-2, 2]$, $t_C = 0$. Finally,

$$w_{\text{RMS}} = 2\left(\frac{\int_{-2}^{2} t^2\, dt}{\int_{-\infty}^{\infty} dt}\right)^{1/2} = \frac{4}{\sqrt{3}}.$$

25. Compute

$$\begin{aligned}\hat{f}_{\text{win}}(\omega) &= \int_{-5}^{5} t^2 e^{-i\omega t}\, dt \\ &= \frac{2}{\omega^3}(25\omega^2\sin(5\omega) + 10\omega\cos(5\omega) - 2\sin(5\omega)).\end{aligned}$$

Since $w(t) = 1$ and the support of g is $[-5, 5]$, then $t_C = 0$. For the RMS bandwidth of the window function,

$$w_{\text{RMS}} = 2\left(\frac{\int_{-5}^{5} t^2\, dt}{\int_{-5}^{5} dt}\right)^{1/2} = \frac{10}{\sqrt{3}}.$$

27. Compute

$$\begin{aligned}
\hat{f}_{\text{win}}(\omega) &= \int_0^4 e^{-t}e^{-i\omega t}\,dt = \frac{1}{1+i\omega}(1-e^{-4(1+i\omega)}) \\
&= \frac{1}{1+\omega^2}(1-e^{-4}(\cos(4\omega)-i\sin(4\omega))(1-i\omega) \\
&= \frac{1-e^{-4}\cos(4\omega)+e^{-4}\sin(4\omega)}{1+\omega^2} \\
&\quad + i\left[\frac{e^{-4}\sin(4\omega)+(e^{-4}\cos(4\omega)-1)\omega}{1+\omega^2}\right].
\end{aligned}$$

We also have

$$t_C = \frac{\int_0^4 t\,dt}{\int_0^4\,dt} = 2$$

and

$$w_{\text{RMS}} = 2\left(\frac{\int_0^4 (t-2)^2\,dt}{\int_0^4\,dt}\right)^{1/2} = \frac{4}{\sqrt{3}}.$$

14.4 Fourier Cosine and Sine Transforms

In these problems the integrations are straightforward and are omitted.

1.

$$\hat{f}_C(\omega) = \frac{1}{2}\left[\frac{1}{1+(\omega+1)^2}+\frac{1}{1+(\omega-1)^2}\right]$$

$$\hat{f}_S(\omega) = \frac{1}{2}\left[\frac{\omega+1}{1+(\omega+1)^2}+\frac{\omega-1}{1+(\omega-1)^2}\right]$$

3.

$$\hat{f}_C(\omega) = \int_0^\infty e^{-t}\cos(\omega t)\,dt = \frac{1}{1+\omega^2}$$

$$\hat{f}_S(\omega) = \int_0^\infty e^{-t}\sin(\omega t)\,dt = \frac{\omega}{1+\omega^2}$$

5.

$$\hat{f}_C(\omega) = \frac{1}{2}\left[\frac{\sin(K(\omega+1))}{\omega+1}+\frac{\sin(K(\omega-1))}{\omega-1}\right] \text{ for } \omega \neq \pm 1$$

$$\hat{f}_C(1) = \hat{f}_C(-1) = \frac{K}{2}+\frac{1}{2}\sin(2K)$$

$$\hat{f}_S(\omega) = \frac{\omega}{\omega^2-1}-\frac{1}{2}\left[\frac{\cos((\omega+1)K)}{\omega+1}+\frac{\cos((\omega-1)K)}{\omega-1}\right] \text{ for } \omega \neq \pm 1$$

$$\hat{f}_S(1) = \frac{1}{4}(1-\cos(2K)), \hat{f}_S(-1) = -\frac{1}{4}(1-\cos(2K))$$

7. Suppose for each $L > 0$, $f^{(4)}(t)$ is piecewise continuous on $[0, L]$, $f^{(3)}(t)$ is continuous, and, as $t \to \infty$, $f^{(3)}(t) \to 0$, $f''(t) \to 0$ and $f(t) \to 0$. Then we can integrate by parts four times to obtain

$$\begin{aligned}
&\mathcal{F}_S[f^{(4)}(t)](\omega) = \int_0^\infty f^{(4)}(t)\sin(\omega t)\,dt \\
&= \left[f^{(3)}(t)\sin(\omega t) - \omega f''(t)\cos(\omega t) - \omega^2 f'(t)\sin(\omega t) + \omega^3\cos(\omega t)f(t)\right]_0^\infty \\
&+ \omega^4 \int_0^\infty \sin(\omega t)f(t)\,dt \\
&= \omega^4 \hat{f}_S(\omega) - \omega^3 f(0) + \omega f''(0).
\end{aligned}$$

8. Under the same conditions as in the solution to Problem 7, four integrations by parts give us

$$\begin{aligned}
&\mathcal{F}_C[f^{(4)}(t)](\omega) = \int_0^\infty f^{(4)}(t)\cos(\omega t)\,dt \\
&= \left[f^{(3)}(t)\cos(\omega t) + \omega f''(t)\sin(\omega t) - \omega^2 f'(t)\cos(\omega t) - \omega^3 f(t)\sin(\omega t)\right]_0^\infty \\
&+ \omega^4 \int_0^\infty f(t)\cos(\omega t)\,dt \\
&= \omega^4 \hat{f}_C(\omega) + \omega^2 f'(0) - f^{(3)}(0).
\end{aligned}$$

14.5 The Discrete Fourier Transform

The six point discrete Fourier transform of $u(j)$ is calculated by

$$\mathcal{D}[u](k) = \sum_{j=0}^{5} u(j)e^{-\pi kji/3}$$

for $k = -4, -3, -2, -1, 0, 1, 2, 3, 4$. For Problems 1 through 6, these values were computed using MAPLE and rounded to the five decimal places.

1.

$$\begin{aligned}
\mathcal{D}[u](-4) &\approx -14.00000 + 10.39230i, \\
\mathcal{D}[u](-3) &\approx -15.00000 + 0.22023(10^{-7})i, \\
\mathcal{D}[u](-2) &\approx -14.00000 - 10.39230i, \\
\mathcal{D}[u](-1) &\approx -6.00000 - 31.17691i, \\
\mathcal{D}[u](0) &\approx 55.00000 + 0i, \\
\mathcal{D}[u](1) &\approx -6.00000 + 31.17691i, \\
\mathcal{D}[u](2) &\approx -14.00000 + 10.39230i, \\
\mathcal{D}[u](3) &\approx -15.00000 - 0.22023(10^{-7})i, \\
\mathcal{D}[u](4) &\approx -14.00000 - 10.39230i
\end{aligned}$$

3.

$$\begin{aligned}
\mathcal{D}[u](-4) &\approx 1.3292 - 0.01658i,\\
\mathcal{D}[u](-3) &\approx 0.09624 + 0.72830(10^{-9})i,\\
\mathcal{D}[u](-2) &\approx 0.13292 + 0.01658i,\\
\mathcal{D}[u](-1) &\approx 2.93687 + 0.42794i,\\
\mathcal{D}[u](0) &\approx 1.82396 + 0i,\\
\mathcal{D}[u](1) &\approx 2.93687 - 0.42794i,\\
\mathcal{D}[u](2) &\approx 0.13292 - 0.01658i,\\
\mathcal{D}[u](3) &\approx 0.09624 - 0.72830(10^{-9})i,\\
\mathcal{D}[u](4) &\approx 0.13292 + 0.01658i
\end{aligned}$$

5.

$$\begin{aligned}
\mathcal{D}[u](-4) &\approx 0.65000 - 0.17321i,\\
\mathcal{D}[u](-3) &\approx 0.61667 - 0.25346(10^{-9})i,\\
\mathcal{D}[u](-2) &\approx 0.65000 + 0.17321i,\\
\mathcal{D}[u](-1) &\approx 0.81667 + 0.40415i,\\
\mathcal{D}[u](0) &\approx 2.45000 + 0i,\\
\mathcal{D}[u](1) &\approx 0.81667 - 0.40415i,\\
\mathcal{D}[u](2) &\approx 0.65000 - 0.17321i,\\
\mathcal{D}[u](3) &\approx 0.61667 + 0.25346(10^{-9})i,\\
\mathcal{D}[u](4) &\approx 0.65000 + 0.17321i
\end{aligned}$$

For Problems 7, 9, and 11, the $N-$ point inverse discrete Fourier transform of the sequence $[U_j]_{j=0}^{N-1}$ is the sequence computed by

$$u_j = \frac{1}{N}\sum_{k=0}^{N-1} U_k e^{2\pi ijk/N}.$$

Values were computed using MAPLE to nine decimal places, with results recorded below to six places.

7. $N = 7$ and

$$u_j = \frac{1}{7}\sum_{k=0}^{6}(e^{-ik})e^{2n\pi ijk/7}.$$

Approximate values are

$$\begin{aligned}
u_0 &\approx 0.103479 + 0.014751i,\\
u_1 &\approx 0.933313 - 0.296094,\\
u_2 &\approx -0.094163 + 0.088785i,\\
u_3 &\approx -0.023947 + 0.062482i,\\
u_4 &\approx 0.004307 + 0.051899i,\\
u_5 &\approx 0.025788 + 0.043852i,\\
u_6 &\approx 0.051222 + 0.034325i.
\end{aligned}$$

9. $N = 5$ and

$$u_j = \frac{1}{5}\sum_{k=0}^{4}(\cos(k))e^{2\pi ijk/5}.$$

Approximate values are

$$\begin{aligned}
u_0 &\approx -0.103896, \\
u_1 &\approx 0.420513 + 0.294562i, \\
u_2 &\approx 0.131434 + 0.031205i, \\
u_3 &\approx 0.131434 - 0.031205i, \\
u_4 &\approx 0.420513 - 0.294562i.
\end{aligned}$$

11. For the given sequence, $N = 6$ and

$$u_j = \frac{1}{6}\sum_{k=0}^{5}(1+i)^k e^{2\pi ijk/6}.$$

We obtain

$$\begin{aligned}
u_0 &\approx -1.333333 + 0.166667i, \\
u_1 &\approx -0.427030 + 0.549038i, \\
u_2 &\approx -0.016346 + 0.561004i, \\
u_3 &\approx 0.333333 + 0.5000000i, \\
u_4 &\approx 0.849679 + 0.272329i, \\
u_5 &\approx 1.593696 - 2.049038i.
\end{aligned}$$

For Problems 13 and 15, the complex Fourier coefficients of the function $f(t)$ having period p are calculated by

$$d_k = \frac{1}{p}\int_0^p f(t)e^{-2\pi ikt}\,dt, k = -3, -2, \cdots, 2, 3.$$

The DFT $N = 2^7 = 128$ is used to approximate these coefficients, using

$$f_k = \frac{1}{128}\sum_{j=0}^{127} f\left(\frac{jp}{128}\right)e^{-2\pi ijk/128}$$

for $k = -3, -2, -1, 0, 1, 2, 3$. These values were computed using MAPLE to nine decimal places and are given below rounded to six places.

13. $f(t) = t^2$, $p = 1$ and

$$f_k = \int_0^1 t^2 e^{-2\pi ikt}\,dt = \frac{1}{2k^2\pi^2} + \frac{1}{2k\pi}i.$$

Table 14.1 lists DFT approximate values.

15. $f(t) = \cos(t)$, $p = 2$, and

$$\begin{aligned}
d_k &= \frac{1}{2}\int_0^2 \cos(t)e^{-i\pi kt}\,dt \\
&= -\frac{\sin(2)}{2(\pi^2k^2-1)} + \frac{ki(\cos(2)-1)}{2(\pi^2k^2-1)}i.
\end{aligned}$$

DFT approximate values are given in Table 14.2.

k	d_k	f_k
-3	$0.005629 - 0.053051i$	$0.001733 - 0.052956i$
-2	$0.012665 - 0.078577i$	$0.008769 - 0.079514i$
-1	$0.050661 - 0.159155i$	$0.046765 - 0.159123i$
0	0.333333	0.329437
1	$0.050661 + 0.159155i$	$0.046765 + 0.159123i$
2	$0.012665 + 0.079577i$	$0.008769 + 0.079514i$
3	$0.005629 + 0.053052i$	$0.001733 + 0.052956i$

Table 14.1: Approximate values in Problem 13, Section 14.5.

k	d_k	f_k
-3	$-0.005177 + 0.075984i$	$0.000346 + 0.075849i$
-2	$-0.011816 + 0.115622i$	$-0.006293 + 0.115532i$
-1	$-0.051259 + 0.250780i$	$-0.045737 + 0.250753i$
0	0.454649	0.460171
1	$-0.051259 - 0.250798i$	$-0.045737 - 0.250753i$
2	$-0.011816 - 0.115622i$	$-0.006293 - 0.115532i$
3	$-0.005177 - 0.075984i$	$0.000346 - 0.075849i$

Table 14.2: Approximate values in Problem 15, Section 14.5.

14.6 Sampled Fourier Series

In Problems 1, 3, and 5, the complex Fourier coefficients of $f(t)$, a function of period p, are computed using

$$d_n = \frac{1}{p}\int_0^p f(t)e^{-2k\pi it/p}\,dt.$$

The $10th$ partial sum of the series is formed and evaluated at t_0 to yield $S_{10}(t_0)$. Next, using $N = 128$, the DFT approximation is $S_{10}(t_0)$ requires the values $U_{n}{}_{n=0}^{10}$ computed by

$$U_n = \sum_{j=0}^{127} f\left(\frac{jp}{128}\right)e^{-2\pi ijn/128}.$$

Then, with

$$V_n = U_n \text{ for } n = 0, 1, \cdots, 10, 118, 119, \cdots, 127$$

and

$$V_n = 0 \text{ for } 11 \le n \le 117,$$

we obtain the DFT approximation

$$w = \frac{1}{128}\sum_{k=0}^{127} V_k e^{2\pi ikt_0/p}.$$

The nonzero values if U_n (to six decimal places) are recorded below for each problem, followed by the DFT approximation w and the difference

$$\text{var} = |S_{10}(t_0) - w|$$

$\mathbf{U}_0$	58.901925		
$\mathbf{U}_1$	$-5.854287 - 32.096339i$	$\mathbf{U}_{118}$	$0.647851 + 2.829713i$
$\mathbf{U}_2$	$-0.805518 - 14.788044i$	$\mathbf{U}_{119}$	$0.633992 + 3.157208i$
$\mathbf{U}_3$	$0.044274 - 9.708611i$	$\mathbf{U}_{120}$	$0.614603 + 3.565443i$
$\mathbf{U}_4$	$0.336014 - 7.235154i$	$\mathbf{U}_{121}$	$0.586989 + 4.089267i$
$\mathbf{U}_5$	$0.470070 - 5.764387i$	$\mathbf{U}_{122}$	$0.542633 + 4.787014i$
$\mathbf{U}_6$	$0.542633 - 4.787014i$	$\mathbf{U}_{123}$	$0.470070 + 5.764387i$
$\mathbf{U}_7$	$0.586299 - 4.089267$	$\mathbf{U}_{124}$	$0.336014 + 7.235154i$
$\mathbf{U}_8$	$-3.565442i0.614603$	$\mathbf{U}_{125}$	$0.044274 + 9.708611i$
$\mathbf{U}_9$	$0.63391 - 3.157208i$	$\mathbf{U}_{126}$	$-0.805518 + 14.788044i$
$\mathbf{U}_{10}$	$0.647851 - 2.829712i$	$\mathbf{U}_{127}$	$-5.854287 + 32.096339i$

Table 14.3: $\mathbf{U}_n$ values in Problem 1, Section 14.6.

between the actual value and the DFT approximate value.

1. Compute
$$d_0 = \frac{1}{2}\sin(2), d_n = \frac{-\sin(2) + n\pi(\cos(2) - 1)i}{2(n^2\pi^2 - 1)}.$$
The complex Fourier series of $f(t)$ is
$$\frac{1}{2}\sin(2) + \frac{1}{2}\sum_{n=-\infty, n\neq 0}^{\infty} \frac{-\sin(2) + n\pi(\cos(2) - 1)i}{2(n^2\pi^2 - 1)} e^{n\pi i t}.$$
Using this, compute
$$S_{10}(1/8) \approx 1.067161.$$
For the DFT approximation, compute Compute
$$w \approx 1.042757 - 0.267410(10^{-9})i$$
and
$$\text{var} \approx 0.024403.$$
Approximate values for $\mathbf{U}_n$ are given in Table 14.3.

3. Compute
$$d_0 = \frac{1}{4}, d_n = \frac{3n\pi + (2n^2\pi^2 - 3)i}{4n^3\pi^3}.$$
The complex Fourier series is
$$\frac{1}{4} + \sum_{n=-\infty, n\neq 0}^{\infty} \frac{3n\pi + (2n^2\pi^2 - 3)i}{4n^3\pi^3} e^{2n\pi i t}.$$
Then
$$S_{10}(1/4) \approx -0.000729.$$
For the DFT calculations, we need
From these obtain
$$w \approx 0.003483 - 0.781250(10^{-10})i$$

$\mathbf{U}_0$	31.501953		
$\mathbf{U}_1$	$9.228787 + 17.271595i$	$\mathbf{U}_{118}$	$-0.400755 - 1.993017i$
$\mathbf{U}_2$	$1.933662 + 9.790716i$	$\mathbf{U}_{119}$	$-0.377943 - 2.222355i$
$\mathbf{U}_3$	$0.582715 + 6.663663i$	$\mathbf{U}_{120}$	$-0.346050 - 2.507623i$
$\mathbf{U}_4$	$0.109884 + 5.028208i$	$\mathbf{U}_{121}$	$-0.299528 - 2.872545i$
$\mathbf{U}_5$	$-0.108968 + 4.029124i$	$\mathbf{U}_{122}$	$-0.227849 - 3.356393i$
$\mathbf{U}_6$	$-0.227849 + 3.356393i$	$\mathbf{U}_{123}$	$-0.108968 - 4.029124i$
$\mathbf{U}_7$	$-0.299528 + 2.872544i$	$\mathbf{U}_{124}$	$0.109884 - 5.028208i$
$\mathbf{U}_8$	$-0.346050 + 2.507623i$	$\mathbf{U}_{125}$	$0.582715 - 6.663663i$
$\mathbf{U}_9$	$-0.377943 + 2.222355i$	$\mathbf{U}_{126}$	$1.933662 - 9.790715i$
$\mathbf{U}_{10}$	$-0.400755 + 1.993017i$	$\mathbf{U}_{127}$	$9.228787 - 17.271595i$

Table 14.4: $\mathbf{U}_n$ values in Problem 3, Section 14.6.

$\mathbf{U}_0$	255		
$\mathbf{U}_1$	$-1 + 40.735481i$	$\mathbf{U}_{118}$	$-1 - 3.992224i$
$\mathbf{U}_2$	$-1 + 20.355468i$	$\mathbf{U}_{119}$	$-1 - 4.453202i$
$\mathbf{U}_3$	$-1 + 13.556669i$	$\mathbf{U}_{120}$	$-1 - 5.027339i$
$\mathbf{U}_4$	$-1 + 10.153170i$	$\mathbf{U}_{121}$	$-1 - 5.763142i$
$\mathbf{U}_5$	$-1 + 8.107786i$	$\mathbf{U}_{122}$	$-1 - 6.741452i$
$\mathbf{U}_6$	$-1 + 6.74152i$	$\mathbf{U}_{123}$	$-1 - 8.107786i$
$\mathbf{U}_7$	$-1 + 5.763142i$	$\mathbf{U}_{124}$	$-1 - 10.153170i$
$\mathbf{U}_8$	$-1 + 5.027339i$	$\mathbf{U}_{125}$	$-1 - 13.556670i$
$\mathbf{U}_9$	$-1 + 4.453202i$	$\mathbf{U}_{126}$	$-1 - 20.355468i$
$\mathbf{U}_{10}$	$-1 + 3.992224i$	$\mathbf{U}_{127}$	$-1 - 40.735484i$

Table 14.5: $\mathbf{U}_n$ values in Problem 5, Section 14.6.

and

$$\text{var} \approx 0.004212.$$

Table 14.4 lists the approximate values of $\mathbf{U}_n$.

5. Compute $d_0 = 2$ and, for $n \neq 0$m

$$d_n = \frac{1}{2}\int_0^2 (1+t)e^{-n\pi it}\,dt = \frac{1-2n\pi i}{n^2\pi^2}.$$

The complex Fourier expansion of $f(t)$ is

$$2 + \sum_{n=-\infty, n\neq 0}^{\infty} \frac{1-2n\pi i}{n^2\pi^2}e^{n\pi it}.$$

The tenth partial sum at $1/8$ is

$$S_{10}(1/8) \approx 1.020712.$$

For the DFT approximation we have Using these, compute

$$w \approx 1.055233 + 0.278759(10^{-9})i.$$

Finally,

$$\text{var} \approx |S_{10}(1/8) - w| \approx 0.034520.$$

Values of $\mathbf{U}_n$ are given in Table 14.5.

14.7 DFT Approximation of the Fourier Transform

1. With $ft) = te^{-2t}$, compute

$$\hat{f}(\omega) = \frac{4-\omega^2}{(\omega^2+4)^2} - \frac{4\omega}{(\omega^2+4)^2}i.$$

Then

$$\hat{f}(12) \approx -0.006392 - 0.002191i.$$

The DFT approximation is

$$\frac{3\pi}{256}\sum_{j=0}^{511} f\left(\frac{3\pi j}{256}\right)e^{-9\pi ij/64} \approx -0.006506 - 0.002191i.$$

The error in the approximation is approximately 0.000114.

3. With $f(t) = e^{-4t}$,

$$\hat{f}(\omega) = \int_0^\infty e^{-4t}e^{-i\omega t}\,dt = \frac{4-i\omega}{\omega^2+16}.$$

Then

$$\hat{f}(4) = \frac{1}{8}(1-i).$$

The DFT approximation to $\hat{f}(4)$ with $L = 3$ and $N = 512$ is

$$\frac{3\pi}{256}\sum_{j=0}^{511} f\left(\frac{3\pi j}{256}\right)e^{-3\pi ij/64} \approx 0.143860 - 0.124549i.$$

The error in the DFT approximation is approximately 0.018887.

Chapter 15

Special Functions and Eigenfunction Expansions

15.1 Eigenfunction Expansions

1. This is a regular problem on $[0, 1]$. Eigenvalues are positive solutions of

$$\tan(\sqrt{\lambda}) = \frac{1}{2\sqrt{\lambda}}.$$

There are infinitely many such eigenvalues (examine graphs, a strategy suggested for Problem 4). The first four are

$$\lambda_1 \approx 0.42676, \lambda_2 \approx 10.8393, \lambda_3 \approx 40.4702, \lambda_4 \approx 89.8227.$$

Eigenfunctions are

$$\varphi_n(x) = 2\sqrt{\lambda_n}\cos(\sqrt{\lambda_n}x) + \sin(\sqrt{\lambda_n}x).$$

3. The problem is regular on $[0, \pi]$. The differential equation can be written

$$y'' + 2y' + \lambda y = 0$$

and the characteristic equation has roots

$$-1 \pm \sqrt{1 - \lambda}.$$

Consider cases on λ.

Case 1: $1 - \lambda = a^2 > 0$. The general solution is

$$y(x) = c_1 e^{(-1+a)x} + c_2 e^{(-1-a)x}.$$

Now

$$y(0) = c_1 + c_2 = 0$$

so $c_2 = -c_1$. Next

$$y(\pi) = c_1\left(e^{-\pi}e^{a\pi} - e^{-\pi}e^{-a\pi}\right).$$

Assuming that $c_1 \neq 0$ to avoid the trivial solution, this implies that

$$e^{a\pi} - e^{-a\pi} = 0,$$

so $e^{2a\pi} = 1$. But then $2a\pi = 0$, impossible since $a \neq 0$. Case 1 produces no eigenvalue for this problem.

Case 2: $1 - \lambda = 0$, so $\lambda = 1$. Now the general solution is

$$y(x) = c_1 e^{-x} + c_2 x e^{-x}.$$

Then $y(0) = c_1 = 0$, and then

$$y(\pi) = c_2 \pi e^{-\pi} = 0,$$

impossible unless $c_2 = 0$, resulting again in the trivial solution. This case yields no eigenvalue.

Case 3: $1 - \lambda = -a^2$, where $a > 0$. Now the general solution is

$$y = c_1 e^{-x}\cos(ax) + c_2 e^{-x}\sin(ax).$$

Then $y(0) = c_1 = 0$, and

$$y(\pi) = c_2 e^{-\pi}\sin(a\pi) = 0.$$

Again, to avoid the trivial solution, we must have $c_2 \neq 0$, so $\sin(a\pi) = 0$, so

$$a = \sqrt{\lambda - 1} = n,$$

a positive integer. Then $\lambda - 1 = n^2$, so the eigenvalues are

$$\lambda_n = 1 + n^2$$

for $n = 1, 2, \cdots$. Corresponding eigenfunctions are

$$\varphi_n(x) = e^{-x}\sin(nx).$$

5. The problem is regular on $[0, 4]$. Eigenvalues are

$$\lambda_n = \left[\left(n - \frac{1}{2}\right)\frac{\pi}{4}\right]^2$$

for $n = 1, 2, \cdots$. Corresponding eigenfunctions are

$$\varphi_n(x) = \cos((n - 1/2)\pi x/4).$$

7. This problem is regular on $[0, L]$. The differential equation has characteristic equation

$$r^2 + \lambda^2 = 0,$$

with roots $r = \pm\sqrt{\lambda}$. We must take cases on λ.

Case 1. If $\lambda = 0$, the differential equation is $y'' = 0$, with general solution

$$y = a + bx$$

for constants a and b. Now $y(0) = 0 = a$ and $y'(L) = b = 0$, so the problem has only the trivial solution if $\lambda = 0$. Therefore 0 is not an eigenvalue of this problem.

Case 2. λ is positive, say $\lambda = \alpha^2$, with $\alpha > 0$. Then $\sqrt{\lambda} = \pm\alpha$, so the general solution of the differential equation is

$$y = c_1 e^{\alpha x} + c_2 e^{-\alpha x}.$$

Now $y(0) = c_1 + c_2 = 0$, so

$$y = c_1 e^{\alpha x} - c_1 e^{-\alpha x} = 2c_1 \sinh(\alpha x).$$

Next,

$$y'(L) = 2c_1 \alpha \cosh(\alpha L) = 0.$$

But $\cosh(\alpha L) > 0$, and $\alpha > 0$, so $c_1 = 0$ and the problem has only the trivial solution for $\lambda > 0$. This problem has no positive eigenvalue.

Case 3. $\lambda < 0$, say $\lambda = -\alpha^2$, with $\alpha > 0$. Now the differential equation has the general solution

$$y = c_1 \cos(\alpha x) + c_2 \sin(\alpha x).$$

Now $y(0) = c_1 = 0$, so

$$y = c_2 \sin(\alpha x).$$

Next,

$$y'(L) = c_2 \alpha \cos(\alpha L) = 0.$$

To have a nontrivial solution we want to be able to choose $c_2 \neq 0$, so we must have $\cos(\alpha L) = 0$. Then αL is a positive zero of the cosine function,

$$\alpha L = \frac{(2n-1)\pi}{2},$$

in which n can be any positive integer. Then

$$\alpha = \sqrt{\lambda} = \frac{(2n-1)\pi}{2L}.$$

Since $\lambda = \alpha^2$, the eigenvalues of this problem, indexed by n, are

$$\lambda_n = \left(\frac{(2n-1)\pi}{2L}\right)^2$$

for $n = 1, 2, 3, \cdots$. Corresponding to each such eigenvalue we have the eigenfunction

$$\varphi_n(x) = \sin\left(\frac{(2n-1)\pi}{2L} x\right).$$

Of course, any nonzero constant multiple of this eigenfunction is also an eigenfunction.

9. The problem is periodic on $[-3\pi, 3\pi]$. Eigenvalues are

$$\lambda_0 = 0 \text{ and } \lambda_n = \frac{n^2}{9}.$$

Eigenfunctions are

$$\varphi_n(x) = a_n \cos(nx/3) + b_n \sin(nx/3)$$

for $n = 0, 1, 2, \cdots$, with a_n and b_n not both zero.

11. From Problem 5, the eigenfunctions are

$$\varphi(x) = \cos((2n-1)\pi x/8).$$

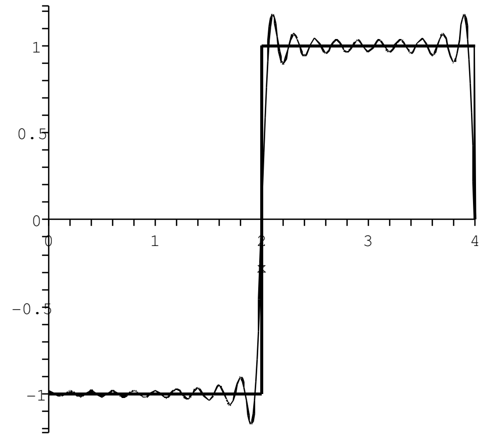

Figure 15.1: Partial sum in Problem 11, Section 15.1.

The coefficients in the expansion are

$$\begin{aligned} c_n &= \frac{1}{2}\int_0^2 -\cos((2n-1)\pi\xi/8)\,d\xi + \frac{1}{2}\int_2^4 \cos((2n-1)\pi\xi/8)\,d\xi \\ &= \frac{4}{(2n-1)\pi}\left[(-1)^{n+1} + \sqrt{2}(\cos(n\pi/2) - \sin(n\pi/2))\right] \end{aligned}$$

for $n = 1, 2, \cdots$. The expansion is

$$\sum_{n=1}^{\infty} \frac{4}{(2n-1)\pi}\left[(-1)^{n+1} + \sqrt{2}(\cos(n\pi/2) - \sin(n\pi/2))\right]\cos((2n-1)\pi x/8)$$

and this converges to

$$\begin{cases} -1 & \text{for } 0 < x < 2, \\ 0 & \text{for } x = 0, 2, 4, \\ 1 & \text{for } 2 < x < 4. \end{cases}$$

Figure 15.1 shows a graph of the function compared to the fortieth partial sum of this eigenfunction expansion.

12. The eigenfunctions are $\varphi_n(x) = e^{-x}\sin(n\pi x)$ for $n = 1, 2, \cdots$. Notice that the Sturm-Liouville form of the differential equation is

$$(e^{2x}y')' + e^{2x}(1+\lambda)y = 0.$$

Therefore the weight function in this Sturm-Liouville problem is $p(x) = e^{2x}$. The coefficients in the eigenfunction expansion are

$$\begin{aligned} c_n &= \frac{\int_{1/2}^1 e^x \sin(n\pi x)\,dx}{\int_0^1 \sin^2(n\pi x)\,dx} \\ &= \frac{2e^{1/2}(n\pi\cos(n\pi/2) - \sin(n\pi/2)) - 2en\pi(-1)^n}{1 + n^2\pi^2}. \end{aligned}$$

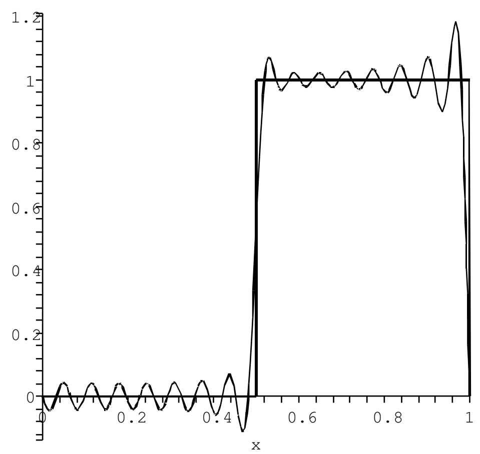

Figure 15.2: Thirtieth partial sum in Problem 12, Section 15.1.

The eigenfunction expansion is

$$\sum_{n=1}^{\infty} c_n e^{-x} \sin(n\pi x)$$

and this converges to

$$\begin{cases} 0 & \text{for } 0 < x < 1/2, \\ 1/2 & \text{for } x = 0, 1/2, 1, \\ 1 & \text{for } 1/2 < x < 1. \end{cases}$$

Figure 15.2 shows a graph of $f(x)$ compared to the thirtieth partial sum of this expansion. This partial sum is not a very good fit to the function. Figure 15.3 shows a graph of the function and the ninetieth partial sum, a better fit. For improved accuracy we would have to take more terms in the partial sum.

13. The eigenfunctions are

$$\varphi_0(x) = 1, \varphi_n(x) = a_n \cos(nx/3) + b_n \sin(nx/3) \text{ for } n = 1, 2, \cdots .$$

The coefficients in the eigenfunction expansion x^2 on $[-3\pi, 3\pi]$ are

$$c_0 = \frac{1}{6\pi} \int_{-3\pi}^{3\pi} \xi^2 \, d\xi = 3\pi^2,$$

$$a_n = \frac{1}{3\pi} \int_{-3\pi}^{3\pi} \xi^2 \cos(n\xi/3) \, d\xi = \frac{36}{n^2}(-1)^n \text{ for } n = 1, 2, \cdots ,$$

and

$$b_n = \frac{1}{3\pi} \int_{-3\pi}^{3\pi} \xi^2 \sin(n\xi/3) \, d\xi = 0 \text{ for } n = 1, 2, \cdots .$$

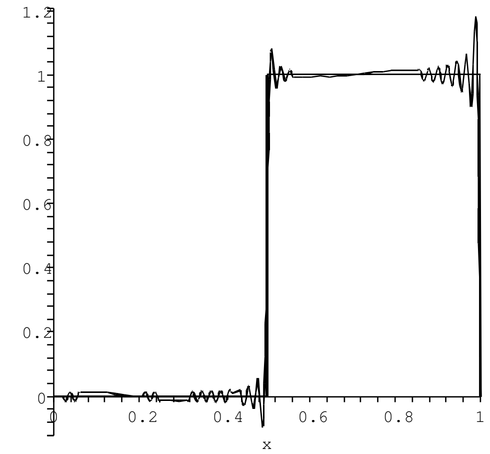

Figure 15.3: Ninetieth partial sum in Problem 12, Section 15.1.

The expansion is

$$3\pi^2 + 36 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos(nx/3) = x^2$$

for $-3\pi < x < 3\pi$. Figure 15.4 is a graph of this function compared to the tenth partial sum of this expansion.

15. The eigenfunctions are $\varphi_n(x) = \sin(n\pi x/L)$. The coefficients in the eigenfunction expansion are

$$c_n = \frac{2}{L} \int_0^L (1 - \xi) \sin(n\pi\xi/L)\, d\xi = \frac{2}{n\pi}(1 + (-1)^n(L - 1))$$

for $n = 1, 2, \cdots$. The expansion is

$$1 - x = \sum_{n=1}^{\infty} \frac{2}{n\pi}(1 + (-1)^n(L - 1)) \sin(n\pi x/L)$$

for $0 < x < L$. The fortieth partial sum of this series is compared to the function in Figure 15.5 for $L = 1$.

17. Normalized eigenfunctions for Problem 5 are obtained by dividing each eigenfunction by its length, whose square is the dot product of this eigenfunction with itself.

$$\int_0^4 \cos^2\left(\frac{(2n - 1)\pi x}{8}\right) dx = 2.$$

Therefore the normalized eigenfunctions are

$$\varphi_n(x) = \frac{1}{\sqrt{2}} \cos\left(\frac{(2n - 1)\pi x}{8}\right),$$

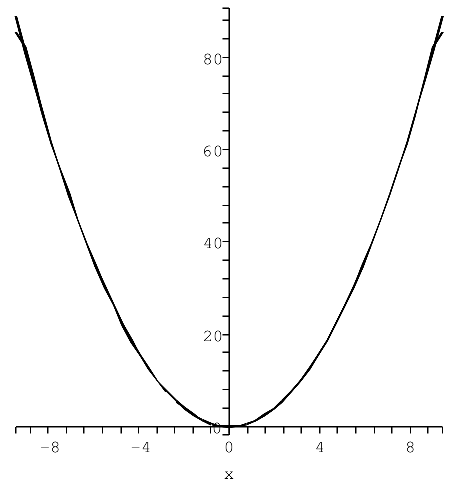

Figure 15.4: Partial sum in Problem 13, Section 15.1.

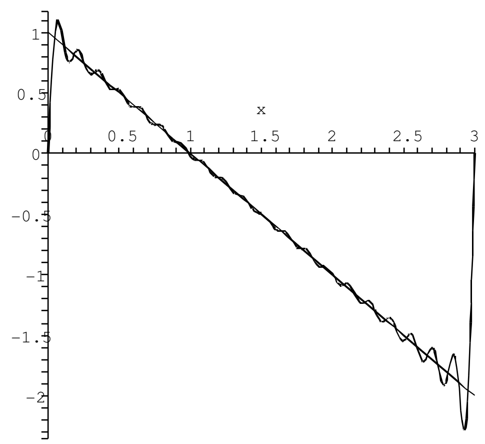

Figure 15.5: Partial sum in Problem 15, Section 15.1.

for $n = 1, 2, 3, \cdots$. Now calculate the dot product

$$\begin{aligned}\varphi_n \cdot f &= \frac{1}{\sqrt{2}}\int_0^4 x(4 - x\cos\left(\frac{(2n-1)\pi x}{8}\right)\,dx\\ &= -\frac{256}{\sqrt{2}}\frac{4(-1)^n + (2n-1)\pi}{(2n-1)^3\pi^3}.\end{aligned}$$

Since

$$f \cdot f = \int_0^4 x^2(4-x)^2\,dx = \frac{512}{15},$$

then Bessel's inequality yields (after some simplification),

$$\sum_{n=1}^{\infty}\left(\frac{4(-1)^n + (2n-1)\pi}{(2n-1)^3\pi^3}\right)^2 \le \frac{512}{15}\frac{2}{(256)^2} = \frac{1}{960}.$$

18. Eigenfunctions are functions $\sin(\sqrt{\lambda}x)$, where λ are solutions of the transcendental equation

$$\sin(\sqrt{\lambda}\pi) + 2\sqrt{\lambda}\cos(\sqrt{\lambda}\pi) = 0.$$

Each λ is a positive solution of the equation

$$\tan(\sqrt{\lambda}\pi) = -2\sqrt{\lambda}.$$

We first need to normalize these eigenfunctions. Compute

$$\begin{aligned}\int_0^\pi \sin^2(\sqrt{\lambda}x)\,dx &= \frac{1}{2}\int_0^\pi (1-\cos(\sqrt{\lambda}x))\,dx\\ &= \frac{1}{2}\left[\pi - \frac{2\sin(\sqrt{\lambda}\pi)\cos(\sqrt{\lambda}\pi)}{2\sqrt{\lambda}}\right]\\ &= \frac{1}{2}\left(\pi + 2\cos^2(\sqrt{\lambda}\pi)\right).\end{aligned}$$

The normalized eigenfunctions are

$$\varphi_n(x) = \left(\frac{2}{\pi + 2\cos^2(\sqrt{\lambda_n}\pi)}\right)^{1/2}\sin(\sqrt{\lambda_n}x),$$

in which we have assigned subscripts to λ to indicate that this is the nth eigenfunction, associated with the nth eigenvalue. Now compute

$$\begin{aligned}\varphi_n \cdot f &= \sqrt{\frac{2}{\pi + 2\cos^2(\sqrt{\lambda_n}\pi)}}\int_0^\pi e^{-x}\sin(\sqrt{\lambda_n}x)\,dx\\ &= \sqrt{\frac{2}{\pi + 2\cos^2(\sqrt{\lambda_n}\pi)}}\left[\frac{e^{-\pi}}{1+\lambda_n}\left(-\sin(\sqrt{\lambda_n}\pi) - \sqrt{\lambda_n}\cos(\sqrt{\lambda_n}\pi)\right) + \frac{\sqrt{\lambda_n}}{1+\sqrt{\lambda_n}}\right]\\ &= \sqrt{\frac{2}{\pi + 2\cos^2(\sqrt{\lambda_n}\pi)}}\frac{\lambda_n}{1+\lambda_n}(1 + e^{-\pi}\cos(\sqrt{\lambda_n}\pi)).\end{aligned}$$

Further,

$$f \cdot f = \int_0^\pi e^{-2x}\,dx = \frac{1}{2}(1 - e^{-2\pi}) = e^{-\pi}\sinh(\pi).$$

Now Bessel's inequality gives us

$$\sum_{n=1}^{\infty} \frac{2\lambda_n^2}{(1+\lambda_n)^2(\pi + 2\cos^2(\sqrt{\lambda_n}\pi))} \left(1 + e^{-\pi}\cos(\sqrt{\lambda_n}\pi)\right)^2$$
$$\le e^{-\pi}\sinh(\pi).$$

15.2 Legendre Polynomials

1. From the diagram and the law of cosines,

$$R^2 = r^2 + d^2 - 2rd\cos(\theta)$$

so

$$\frac{R^2}{d^2} = 1 - 2\frac{r}{d}\cos(\theta) + \frac{r^2}{d^2}.$$

Then

$$\varphi(x,y,z) = \frac{1}{R} = \frac{1}{d}\frac{d}{R} = \frac{1}{d}\frac{1}{\sqrt{1 - 2\frac{r}{d}\cos(\theta) + \frac{r^2}{d^2}}}.$$

This concludes part (a). For (b), suppose $r/d < 1$. By comparing the result of (a) with the generating function for Legendre polynomials (with $x = \cos(\theta)$ and $t = r/d$), we have

$$\varphi(r) = \frac{1}{d}\sum_{n=0}^{\infty} P_n(\cos(\theta))\left(\frac{r^n}{d^n}\right),$$

which is equivalent to

$$\varphi(r) = \sum_{n=0}^{\infty} \frac{1}{d^{n+1}} P_n(\cos(\theta)) r^n.$$

For (c), suppose $r/d < 1$. Now write

$$\frac{R^2}{r^2} = 1 - 2\frac{d}{r}\cos(\theta) + \frac{d^2}{r^2}.$$

Then

$$\frac{r}{R} = \frac{1}{\sqrt{1 - 2\frac{d}{r}\cos(\theta) + \frac{d^2}{r^2}}}.$$

Again comparing with the generating function, we have

$$\varphi(r) = \frac{1}{r}\sum_{n=0}^{\infty} P_n(\cos(\theta))\left(\frac{d^n}{r^n}\right).$$

This is equivalent to

$$\varphi(r) = \frac{1}{r}\sum_{n=0}^{\infty} d^n P_n(\cos(\theta)) r^{-n}.$$

2. Here we give just $P_2(x)$ and $P_4(x)$, derived from the Rodrigues formula:

$$\begin{aligned} P_2(x) &= \frac{1}{2^2 1!}\frac{d^2}{dx^2}((x^2-1)^2) \\ &= \frac{1}{8}\frac{d^2}{dx^2}(x^4-2x^2+1) = \frac{1}{2}(3x^2-1), \\ P_4(x) &= \frac{1}{2^4 4!}\frac{d^4}{dx^4}((x^2-1)^4) \\ &= \frac{1}{384}\frac{d^4}{dx^4}(x^8-4x^6+6x^4-4x^2+1) = \frac{1}{8}(35x^4-30x^2+3). \end{aligned}$$

3. For this problem, use the recurrence relation to write $P_{n+1}(x)$ in terms of $P_n(x)$ and $P_{n-1}(x)$:

$$P_{n+1}(x) = \frac{2n+1}{n+1}xP_n(x) + \frac{n}{n+1}P_{n-1}(x)$$

for $n = 1, 2, \cdots$. Since we know $P_0(x)$ through $P_5(x)$, it is routine to derive the following:

$$\begin{aligned} P_6(x) &= \frac{1}{16}(231x^6 - 315x^4 + 105x^2 - 5) \\ P_7(x) &= \frac{1}{16}(429x^7 - 693x^5 + 315x^3 - 35x) \\ P_8(x) &= \frac{1}{128}(6435x^8 - 12012x^6 + 6930x^4 - 1260x^2 + 35). \end{aligned}$$

4. **Hint** We can do these expansions by straightforward algebraic manipulation. However, we can also do this efficiently using matrices. Since the highest power of x occurring in the polynomials of (a) through (c) is 4, we need only use $P_0(x)$ through $P_4(x)$. Write

$$\begin{pmatrix} P_0(x) \\ P_1(x) \\ P_2(x) \\ P_3(x) \\ P_4(x) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ -1/2 & 0 & 3/2 & 0 & 0 \\ 0 & -3/2 & 0 & 5/2 & 0 \\ 3/8 & 0 & -30/8 & 0 & 35/8 \end{pmatrix} \begin{pmatrix} 1 \\ x \\ x^2 \\ x^3 \\ x^4 \end{pmatrix}.$$

By inverting the coefficient matrix in this equation, write 1, x, x^2, x^3 and x^4 as linear combinations of Legendre polynomials. From these, any polynomial of degree ≤ 4 can be written in terms of $P_0(x), \cdots, P_4(x)$.

5. Using the given formula, obtain:

$$\begin{aligned} P_0(x) &= (-1)^0 x^0 = 1, \\ P_1(x) &= \frac{(-1)^0 2!}{2}x = x, \\ P_2(x) &= (-1)^0\frac{4!}{2!2!}x^2 + \frac{(-1)2!}{2^2}x^0 = \frac{1}{2}(3x^2-1); \\ P_3(x) &= \frac{(-1)^0 6!}{2^3 3!3!}x^3 + \frac{(-1)4!}{2^3 2!}x = \frac{1}{2}(5x^3-3x), \\ P_4(x) &= \frac{(-1)^0 8!}{2^4 4!4!}x^4 - \frac{6!}{2^4 3!2!}x^2 + \frac{4!}{2^4 2!2!} = \frac{1}{8}(35x^4-30x^2+3), \\ P_5(x) &= \frac{(-1)^0 10!}{2^5 5!5!}x^5 - \frac{8!}{2^5 4!3!}x^3 + \frac{6!}{2^5 2!3!}x = \frac{1}{8}(63x^5-70x^3+15x). \end{aligned}$$

6. **Hint** Attempt to find a second solution of the form $Q_n(x) = z(x)P_n(x)$. Substitute this into Legendre's equation to show that $z(x)$ satisfies

$$\frac{z''}{z'} - \frac{2x}{1-x^2} + 2\frac{P_n'}{P_n} = 0.$$

Integrate this and solve for $z'(x)$ to obtain

$$z'(x) = \frac{K}{(1-x^2)(P_n(x))^2},$$

in which K is constant. Integrate

$$z(x) = K\int \frac{1}{(1-x^2)(P_n(x))^2}\,dx.$$

Finally, choose $K = 1$ to obtain a second solution

$$Q_n(x) = P_n(x)\int \frac{1}{P_n(x)^2(1-x^2)}\,dx.$$

Using this expression, obtain

$$Q_0(x)\frac{1}{2}\ln\left|\frac{1+x}{1-x}\right| = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right),$$

$$Q_1(x) = -1 + \frac{x}{2}\ln\left(\frac{1+x}{1-x}\right),$$

and

$$Q_2(x) = \frac{1}{4}(3x^2-1)\ln\left(\frac{1+x}{1-x}\right) - \frac{3}{2}x$$

for $-1 < x < 1$.

7. One way to derive these results is to use the expression for $P_n(x)$ given in Problem 5. First,

$$\left[\frac{2n+1}{2}\right] = \left[n + \frac{1}{2}\right] = n,$$

so

$$P_{2n+1}(x) = \sum_{k=0}^{n}(-1)^k \frac{(4n+2-2k)!}{2^{2n+1}k!(2n+1-k)!(2n+1-2k)!}x^{2n+1-2k}.$$

Then $P_{2n+1}(0) = 0$, because there is a positive power of x in every term of $P_{2n+1}(x)$.

Next,

$$\left[\frac{2n}{2}\right] = n,$$

so

$$P_{2n}(x) = \sum_{k=0}^{\infty}(-1)^k \frac{(4n-2k)!}{2^{2n}k!(2n-k)!(2n-2k)!}x^{2n-2k}.$$

The constant term in this polynomial occurs when $k = n$, so

$$P_{2n}(0) = \frac{(-1)^n(2n)!}{2^{2n}(n!)^2}.$$

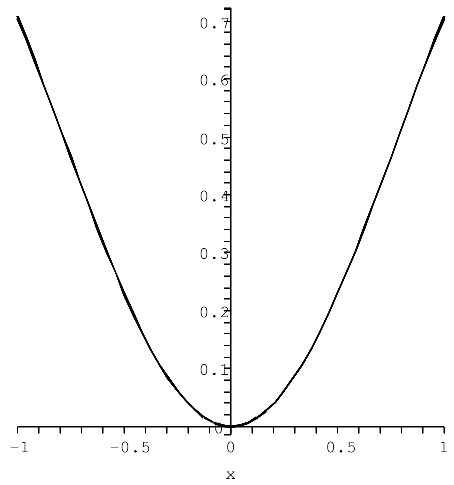

Figure 15.6: Partial sum in Problem 9, Section 15.2.

8. **Hint** Put $x = t = 1/2$ in the generating function for the Legendre polynomials.

In Problems 9, 11, and 13, use MAPLE to compute Fourier-Legendre coefficients, which require integrals of the form $\int_{-1}^{1} f(x)P_n(x)\,dx$. This type of computation is reviewed in the MAPLE primer of the seventh edition of Advanced Engineering Mathematics. Recall that $P_n(x)$ is denoted in MAPLE as *LegendreP(n,x)*.

In some examples a "small" partial sum provides an approximation to the function with an error that is nearly undetectable in the scale of the graph. This is not to be expected in general, however, and sometimes many terms of a partial sum must be used to approximate a function with a partial sum of a Fourier-Legendre expansion.

9. Compute

$$c_0 = 0.2726756433, c_1 = 0, c_2 = 0.4961198722,$$
$$c_3 = 0, c_4 = -0.06335726400.$$

Figure 15.6 shows a graph of this partial sum and the function.

11. Compute

$$c_0 = 0, c_1 = 1.500000000, c_2 = 0,$$
$$c_3 = -0.8750000000, c_4 = 0.$$

Figure 15.7 shows a graph of this partial sum and the function. In this case, many more terms of the eigenfunction expansion are needed to approximate the function with any accuracy. Figure 15.8 shows the function and the fortieth partial sum of this expansion. This is a much better fit.

12. Compute

$$c_0 = 0.8411909850, c_1 = 0.7174008810, c_2 = -0.3101752600,$$
$$c_3 = -0.1820611100, c_4 = -0.00909900000.$$

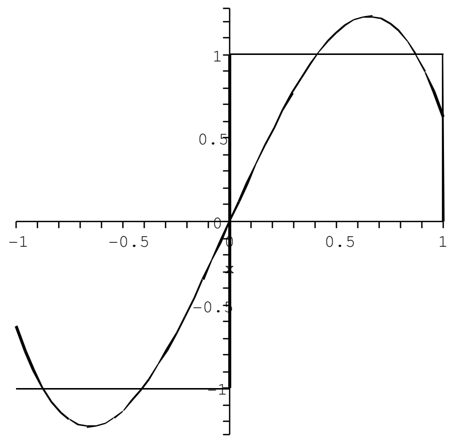

Figure 15.7: Fifth partial sum in Problem 11, Section 15.2.

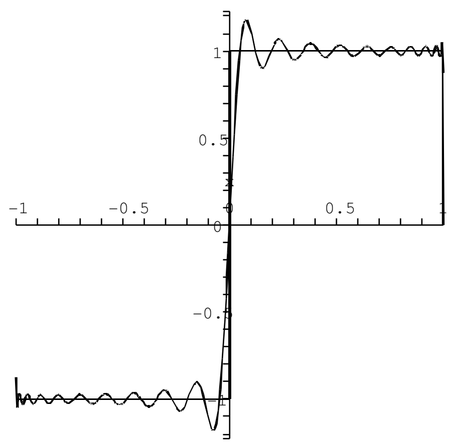

Figure 15.8: Fortieth partial sum in Problem 11, Section 15.2.

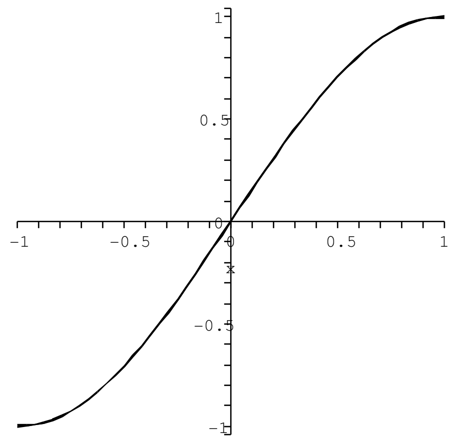

Figure 15.9: Partial sum in Problem 13, Section 15.2.

13. The function to be expanded is $f(x) = \sin(\pi x/2)$. Again approximating the integrals yielding the coefficients, we obtain

$$c_0 = c_2 = c_4 = 0, c_1 = 1.215854203, c_3 = -0.2248913308.$$

Figure 15.9 shows a graph of $\sin(\pi x/2)$ and this partial sum. These graphs are nearly identical in the scale of the graphics.

15.3 Bessel Functions

Computations involving Bessel functions (values of Bessel functions at specific points, zeros, graphs, and integrals involving Bessel functions) require use of computational software. This is reviewed for MAPLE in the MAPLE primer of edition seven. Recall that $J_n(x)$ is denoted in MAPLE as BesselJ(n,x), and the kth zero of $J_n(x)$ is BesselJZeros(n,k). For a decimal value of this zero, use the evalf command.

1. Let $y = x^a J_\nu(bx^c)$. First compute

$$y' = ax^{a-1}J_\nu(bx^c) + x^a bcx^{c-1}J'_\nu(bx^c)$$

and

$$\begin{aligned} y'' &= a(a-1)x^{a-2}J_\nu(bx^c) \\ &\quad + [2ax^{a-1}bcx^{c-1} + x^a bc(c-1)x^{c-2}]J'_\nu(bx^c) \\ &\quad + x^a b^2 c^2 x^{2c-2} J''_\nu(bx^c). \end{aligned}$$

Substitute these into the differential equation and simplify to obtain

$$c^2 x^{a-2}[(bx^c)^2 J''_\nu(bx^c) + bx^c J'_\nu(bx^c) + ((bx^c)^2 - \nu^2)J_\nu(bx^c)] = 0.$$

3. $a = -1, c = 2, b = 2, \nu = 3/4$ and the general solution is

$$y = c_1 \frac{1}{x} J_{3/4}(2x^2) + c_2 \frac{1}{x} J_{-3/4}(2x^2).$$

4. **Hint** $a = b = 0$, so this method produces only the trivial solution. However, observe that the differential equation is an Euler equation.

5. $a = 4, c = 3, b = 2, \nu = 3/4$, so

$$y = c_2 x^4 J_{3/4}(2x^3) + c_2 x^4 J_{-3/4}(2x^3).$$

7. $a = -2, c = 3, b = 3, \nu = 1/2$, so the general solution is

$$y = c_1 \frac{1}{x^2} J_{1/2}(3x^3) + c_2 \frac{1}{x^2} J_{-1/2}(3x^3).$$

9. We need

$$1 - 2a = 1, b^2c^2 = 4, 2c - 2 = 2, a^2 - \nu^2 c^2 = -\frac{4}{9},$$

so

$$a = 0, c = 2, b = 1, \nu = \frac{1}{3}.$$

The general solution is

$$y = c_1 J_{1/3} x^2 + c_2 J_{-1/3}(x^2).$$

11. With $z = 2x^{1/3}$, the transformed equation is

$$9\left(\frac{z}{2}\right)^6 \left[\frac{4}{9}\left(\frac{z}{2}\right)^{-4} \frac{d^2y}{dz^2} - \frac{4}{9}\left(\frac{z}{2}\right)^{-5} \frac{dy}{dz}\right]$$
$$+ 9\left(\frac{z}{2}\right)^3 \left[\frac{2}{3}\left(\frac{z}{2}\right)^{-2} \frac{dy}{dz}\right] + \left(4\left(\frac{z}{2}\right)^2 - 16\right) y = 0.$$

This simplifies to

$$z^2 y'' + zy' + (z^2 - 16)y = 0,$$

with general solution

$$y(z) = c_1 J_4(z) + c_2 Y_4(z).$$

Then

$$y(x) = c_1 J_4(2x^{1/3}) + c_2 Y_4(2x^{1/3}).$$

13. With $u = x^{-2/3} y$, we have $y = x^{2/3} u$. The transformed equation is

$$36x^2 \left[x^{2/3} u'' + \frac{4}{3} x^{-1/3} u' - \frac{2}{9} x^{-4/3} u\right]$$
$$- 12x \left[x^{2/3} u' + \frac{2}{3} x^{-1/3} u\right] + (36x^2 + 7) x^{2/3} u = 0.$$

Collect terms and divide by $36x^{2/3}$ to obtain

$$x^2 u'' + xu' + (x^2 - 1/4)u = 0.$$

This has general solution

$$u(x) = c_1 J_{1/2}(x) + c_2 Y_{1/2}(x).$$

Then

$$y(x) = c_1 x^{2/3} J_{1/2}(x) + c_2 x^{2/3} Y_{1/2}(x).$$

15. With $z = x^{1/2}$, compute

$$\frac{dy}{dx} = \frac{1}{2}x^{-1/2}\frac{dy}{dz},$$
$$\frac{d^2y}{dx^2} = -\frac{1}{4}x^{-3/2}\frac{dy}{dz} + \frac{1}{4}x^{-1}\frac{d^2y}{dz^2}.$$

The transformed differential equation is

$$4z^4\left(\frac{1}{4}z^{-2}\frac{d^2y}{dz^2} - \frac{1}{4}z^{-3}\frac{dy}{dz}\right) + 4z^2\left(\frac{1}{2}z^{-1}\frac{dy}{dz}\right) + (z^2-9)y = 0.$$

Upon simplifying, this is

$$z^2y'' + zy' + (z^2-9)y = 0.$$

This fits the template (15.8) and has general solution

$$y(z) = c_1J_3(z) + c_2Y_3(z).$$

Then

$$y(x) = c_1J_3(\sqrt{x}) + c_2Y_3(\sqrt{x}).$$

17. By equation (15.20),

$$(x^nJ_n(x))' = x^nJ_{n-1}(x)$$

and integrating both sides yields

$$\int x^nJ_{n-1}(x)\,dx = x^nJ_n(x).$$

By equation (15.21),

$$(x^{-n}J_n(x))' = -x^{-n}J_{n+1}(x).$$

Again, by integrating, we obtain immediately that

$$\int x^{-n}J_{n+1}(x)\,dx = -x^{-n}J_n(x),$$

and this is equivalent to what we want to show.

19. Let α be a positive zero of J_0. Then $J_0(\alpha) = 0$. We want to show that

$$\int_0^1 J_1(\alpha x)\,dx = \frac{1}{\alpha}.$$

First, recall that $J_0'(x) = -J_1(x)$. Then

$$\int_0^\alpha J_1(s)\,ds = -J_0(s)]_0^\alpha = J_0(0) - J_0(\alpha) = 1,$$

since $J_0(0) = 1$. Now make the change of variables $s = \alpha x$ in the integral to obtain

$$\alpha\int_0^1 J_1(\alpha x)\,dx = 1,$$

and this is equivalent to what we want to show.

21. Define

$$I_{n,k} = \int_0^1 (1-x^2)^k x^{n+1} J_n(\alpha x)\,dx.$$

For (a), begin with a result from Problem 17:

$$\int s^n J_{n-1}(s)\,ds = s^n J_n(s).$$

Replace n with $n+1$:

$$\int s^{n+1} J_n(s)\,ds = s^{n+1} J_{n+1}(s).$$

Then

$$\int_0^\alpha s^{n+1} J_n(s)\,ds = \left[s^{n+1} J_{n+1}(s)\right]_0{}^\alpha = \alpha^{n+1} J_{n+1}(\alpha).$$

Now let $s = \alpha x$ to obtain

$$\int_0^1 \alpha^{n+1} x^{n+1} J_n(\alpha x)\alpha, dx = \alpha^{n+1} J_{n+1}(\alpha).$$

Then

$$\int_0^1 x^{n+1} J_n(\alpha x)\,dx = \frac{1}{\alpha} J_{n+1}(\alpha).$$

But

$$I_{n,0} = \int_0^1 x^{n+1} J_n(\alpha x)\,dx.$$

This proves that

$$I_{n,0} = \frac{1}{\alpha} J_{n+1}(\alpha).$$

Now use the first integral in Problem 18, with $n+1$ in place of n, to write

$$x^{n+1} J_n(\alpha x) = \frac{d}{dx}\left(\frac{1}{\alpha} x^{n+1} J_{n+1}(\alpha x)\right).$$

Substitute this into the definition of $I_{n,k}$ to write

$$I_{n,k} = \int_0^1 (1-x^2)^k \frac{d}{dx}\left(\frac{1}{\alpha} x^{n+1} J_{n+1}(\alpha x)\right)\,dx.$$

This completes part (b). Now, for (c), integrate the expression of (b) by parts:

$$\begin{aligned}
I_{n,k} &= \int_0^1 (1-x^2)^k \frac{d}{dx}\left(\frac{1}{\alpha} x^{n+1} J_{n+1}(\alpha x)\right)\,dx \\
&= (1-x^2)^k \frac{x^{n+1}}{\alpha} J_{n+1}(\alpha x)\Big]_0^1 \\
&- \int_0^1 \frac{1}{\alpha} x^{n+1} J_{n+1}(\alpha x) k (1-x^2)^{k-1}(-2x)\,dx \\
&= \frac{2k}{\alpha}\int_0^1 (1-x^2)^{k-1} x^{n+2} J_{n+1}(\alpha x)\,dx \\
&= \frac{2k}{\alpha} I_{n+1,k-1}.
\end{aligned}$$

This relates $I_{n,k}$ to the value of this integral when n is increased by 1 and k decreased by 1. In particular, if we carry out k repetitions of this operation, eventually increasing n by k, and decreasing k by k, we obtain

$$\begin{aligned} I_{n,k} &= \frac{2k}{\alpha} I_{n+1,k-1} \\ &= \frac{2k}{\alpha}\left[\frac{2(k-1)}{\alpha} I_{n+2,k-2}\right] \\ &= \frac{2^2 k(k-1)}{\alpha^2} I_{n-2,k-2} \\ &= \frac{2^2 k(k-1)}{\alpha^2}\left[\frac{2(k-2)}{\alpha} I_{n+3,k-2}\right] \\ &= \frac{2^3 k(k-1)(k-2)}{\alpha^3} I_{n+3,k-3} \\ &= \cdots = \frac{2^k k!}{\alpha^k} I_{n+k,0}. \end{aligned}$$

Since k is a positive integer, we can write $k! = \Gamma(k+1)$, as in the statement of the problem. This gives us the result of part (d).

Finally, for part (e), combine the conclusions of parts (a) and (d) to write

$$\int_0^1 (1-x^2)^k x^{n+1} J_n(\alpha x)\,dx = \frac{2^k \Gamma(k+1)}{\alpha^{k+1}} J_{n+k+1}(\alpha).$$

To obtain the result of part (f), write the last equation as

$$J_{n+k+1}(\alpha) = \frac{\alpha^{k+1}}{2^k \Gamma(k+1)} \int_0^1 (1-x^2)^k x^{n+1} J_n(\alpha x)\,dx.$$

The rest is just notation to obtain a different perspective. Rewrite the last equation by writing x in place of α and t in place of x to obtain

$$J_{n+k+1}(x) = \frac{x^{k+1}}{2^k \Gamma(k+1)} \int_0^1 t^{n+1}(1-t^2)^k J_n(xt)\,dt.$$

Finally, for (g), let $m - n = k + 1$ to write

$$J_m(x) = \frac{2x^{m-n}}{2^{m-n}\Gamma(m-n)} \int_0^1 t^{n+1}(1-t^2)^{m-n-1} J_n(xt)\,dt.$$

It is important to observe that these results do not require that k be an integer, since $k!$ has been replaced by $\Gamma(k+1)$, which is well defined if $k+1 > 0$. In the conclusions derived in this problem, it is enough to have $n > -1$, $k > -1$ and, in (g), $m > n > -1$.

23. Start with the following result from Problem 22:

$$J_m(x) = \frac{x^m}{2^{m-1}\Gamma(m+1/2)} \int_0^1 (1-t^2)^{m-1/2} \cos(xt)\,dt.$$

Now make the change of variables $t = \sin(\theta)$ in the integral. When $t = 0$, $\theta = 0$ and when $t = 1$, $\theta = \pi/2$. Further, $dt = \cos(t)\,dt$ and

$$1 - t^2 = 1 - \sin^2(t) = \cos^2(t).$$

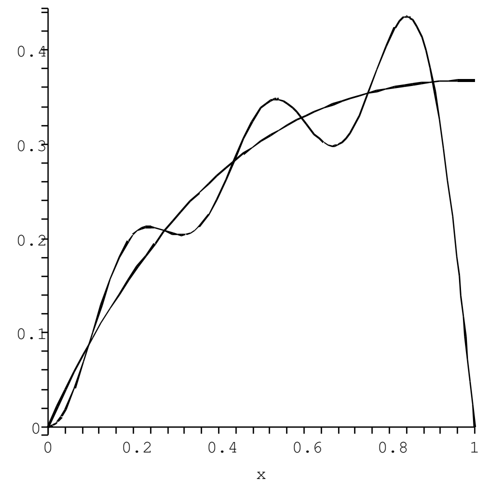

Figure 15.10: Fifth partial sum in Problem 25, Section 15.3.

Then

$$\begin{aligned}J_m(x) &= \frac{x^m}{2^{m-1}\Gamma(m+1/2)}\int_0^{\pi/2}(\cos^2(t))^{m-1/2}\cos(x\sin(\theta))\cos(t)\,dt\\ &= \frac{x^m}{2^{m-1}\Gamma(m+1/2)}\int_0^{\pi/2}\cos^{2m}(\theta)\cos(x\sin(\theta))\,d\theta.\end{aligned}$$

For Problems 25, 27, and 29, we expand the functions of Problems 30 through 35, respectively, except now we use a Fourier-Bessel expansion in terms of Bessel functions of the first kind of order 2. This series will have the form

$$\sum_{n=1}^{\infty} c_n J_2(j_n x),$$

where j_n is the nth positive zero of $J_2(x)$ and

$$c_n = \frac{2\int_0^1 xf(x)J_2(j_n x)\,dx}{J_3(j_n)^2}.$$

For each problem, we graph the fifth and the twenty-fifth partial sum, compared to the function.

25. The fifth and twenty-fifth partial sums are shown in Figures 15.10 and 15.11.
27. Figures 15.12 and 15.13 show the fifth and twenty-fifth partial sums of this expansion.
29. The fifth and twenty-fifth partial sums are shown in Figures 15.14 and 15.15.

In Problems 31, 33, and 35, we want a Fourier-Bessel expansion

$$\sum_{n=1}^{\infty} c_n J_1(j_n x),$$

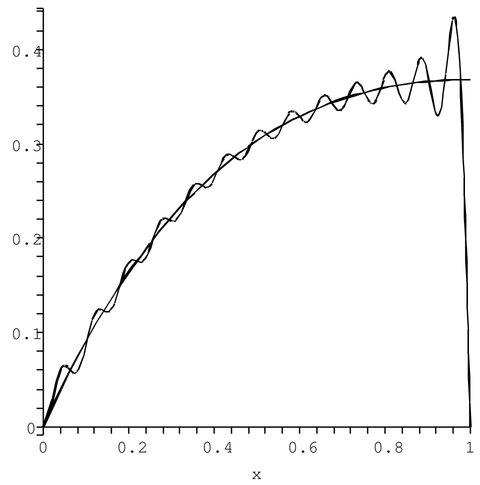

Figure 15.11: Twenty-fifth partial sum in Problem 25, Section 15.3.

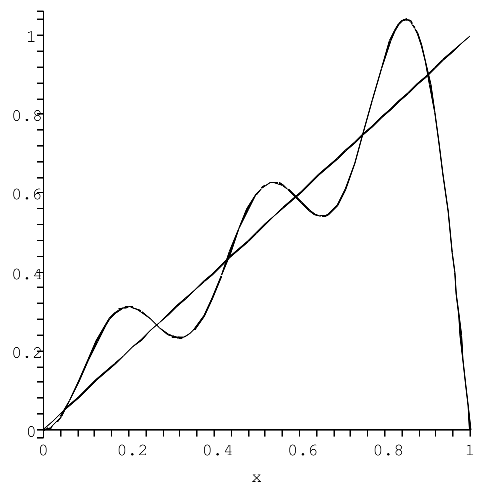

Figure 15.12: Fifth partial sum in Problem 27, Section 15.3.

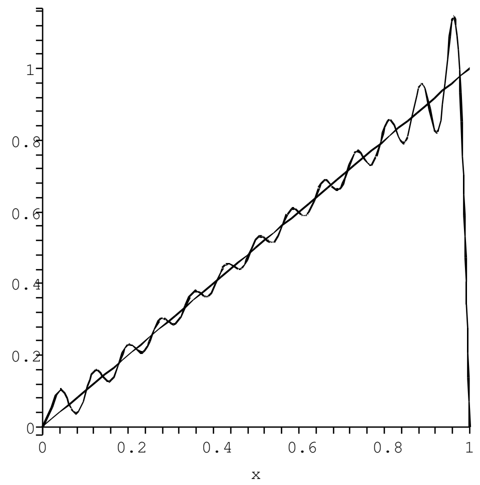

Figure 15.13: Twenty-fifth partial sum in Problem 27, Section 15.3.

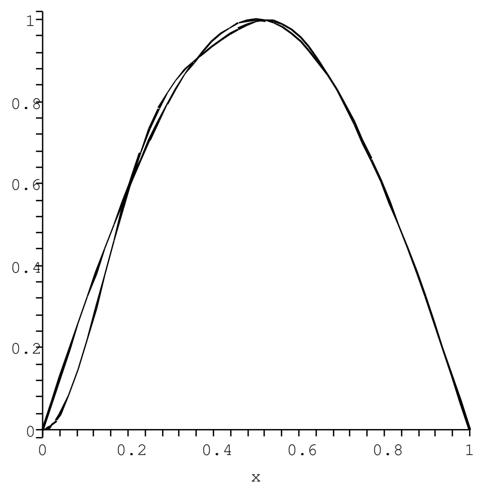

Figure 15.14: Fifth partial sum in Problem 29, Section 15.3.

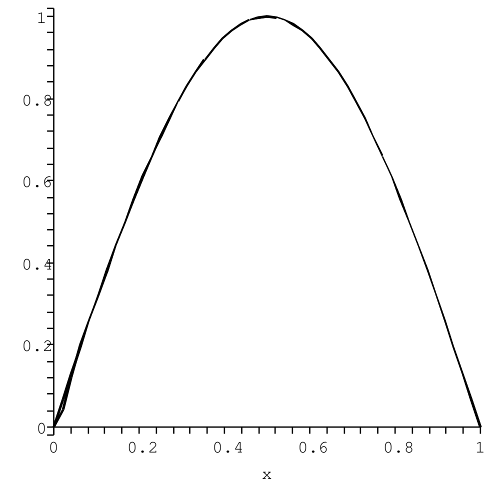

Figure 15.15: Twenty-fifth partial sum in Problem 29, Section 15.3.

where j_n is the nth positive zero of $J_1(x)$ and

$$c_n = \frac{2\int_0^1 xf(x)J_1(j_n x)\,dx}{J_2(j_n)^2}.$$

Computation of these coefficients requires the use of MAPLE or some other computational software package. In constructing the graphs of partial sums, we wrote a general MAPLE expression for c_n as a function of n, and then, using the sum command, wrote the MAPLE expression for the Nth partial sum

$$\sum_{n=1}^{N} c_n J_1(j_n x).$$

This could then be computed and graphed for any N we wanted to insert into the code. One nice feature of this process is that it does not require that we actually write out a numerical value for each c_n, although these values were computed by the program in the course of computing the partial sum to construct the graph.

31. With $f(x) = x$, then nth Fourier-Bessel coefficient for expanding in a series of eigenfunctions $J_1(j_n x)$ is

$$c_n = \frac{2}{J_2(j_n)^2}\int_0^1 xJ_1(j_n x)\,dx.$$

Figure 15.16 shows the fifth partial sum, compared to the function in this expansion. Clearly this fifth partial sum does not approximate the function at all well. Figure 15.17 shows the function and the thirty-fifth partial sum, suggesting convergence of the Fourier-Bessel expansion to the function as more terms are included in the expansion.

33. Figures 15.18 and 15.19 show the fifth and thirty-fifth partial sums of this Fourier-Bessel expansion of $f(x) = xe^{-x}$.

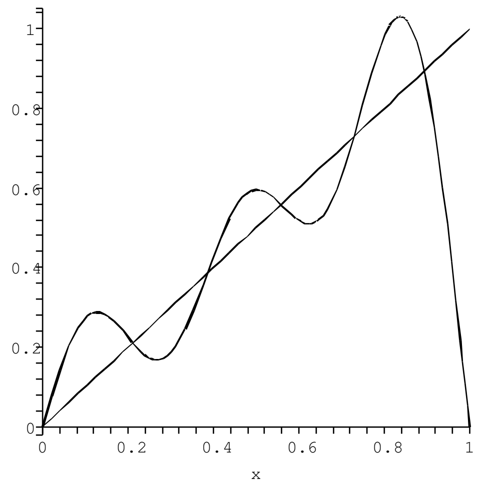

Figure 15.16: Fifth partial sum in Problem 31, Section 15.3.

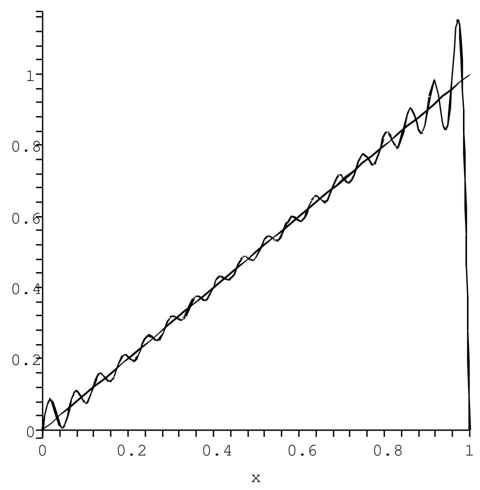

Figure 15.17: Thirty-fifth partial sum in Problem 31, Section 15.3.

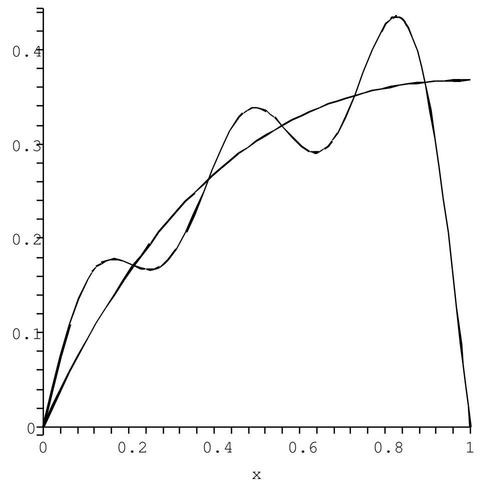

Figure 15.18: Fifth partial sum in Problem 33, Section 15.3.

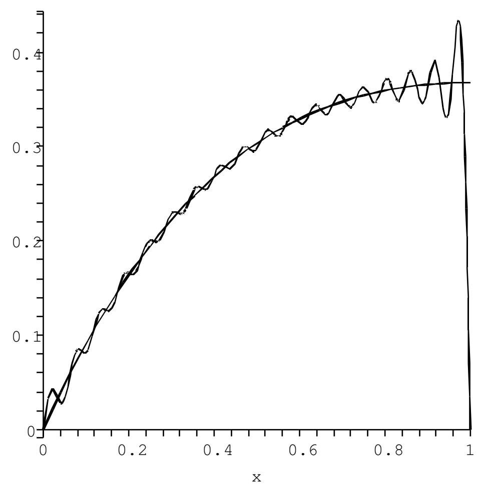

Figure 15.19: Thirty-fifth partial sum in Problem 33, Section 15.3.

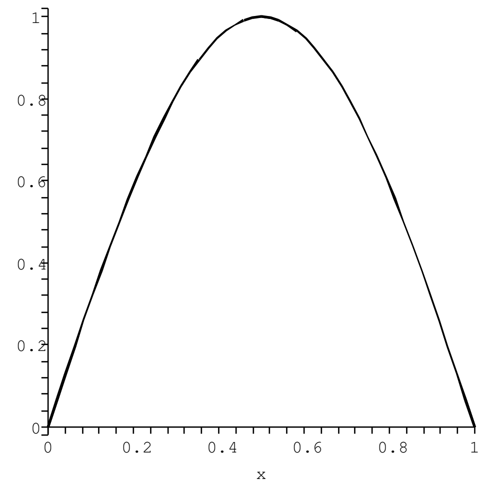

Figure 15.20: Fifth partial sum in Problem 35, Section 15.3.

35. Figure 15.20 shows the fifth partial sum of the Fourier-Bessel expansion of $f(x) = \sin(\pi x)$. This partial sum appears to be a good approximation to the function.

37. Let $t = u/(1+u)$ in the definition of the beta function. Then $u \to \infty$ as $t \to 1$, and

$$dt = \frac{1}{(1+u)^2}\,du.$$

We obtain

$$\begin{aligned} B(x,y) &= \int_0^1 t^{x-1}(1-t)^{y-1}\,dt \\ &= \int_0^\infty \left(\frac{u}{1+u}\right)^{x-1}\left(1-\frac{u}{1+u}\right)^{y-1}\frac{1}{(1+u)^2}\,du \\ &= \int_0^\infty \frac{u^{x-1}}{(1+u)^{x+y}}\,du, \end{aligned}$$

and this is what we wanted to show.

38. **Hint** Let x and y be positive numbers. We want to show that

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

This can be done by using double integrals, but here is an outline of a proof using the convolution operation of the Laplace transform. First, it is routine to check that

$$\mathcal{L}[t^{x-1}](s) = \frac{\Gamma(x)}{s^x}.$$

From this,

$$\mathcal{L}^{-1}\left[\frac{1}{s^x}\right] = \frac{t^{x-1}}{\Gamma(x)}.$$

Now compute

$$\mathcal{L}^{-1}\left[\frac{1}{s^{x+y}}\right](t)$$

in two ways, first using the convolution theorem, and then directly. Comparing the results of these two computations yields the result that is to be proved.

39. Let $t = ry$ in the integral defining the gamma function to write

$$\begin{aligned}\Gamma(x) &= \int_0^\infty t^{x-1}e^{-t}\,dt \\ &= \int_0^\infty (ry)^{x-1}e^{-ry}r\,dy \\ &= r^x\int_0^\infty y^{x-1}e^{-ry}\,dy\end{aligned}$$

and this is the integral to be derived with the variable of integration denoted y instead of t.

Chapter 16

The Wave Equation

16.1 Derivation of the Wave Equation

1. Chain-rule differentiations yield

$$\frac{\partial^2 y}{\partial t^2} = \frac{1}{2}(c^2 f''(x+ct) + c^2 f''(x-ct))$$

and

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{2}(f''(x+ct) + f''(x-ct)).$$

Then

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}.$$

3. Let $z(x, y, t)$ be the vertical displacement of the point of the membrane located at (x, y) at time $t > 0$. Then $z(x, y, t)$ satisfies the two-dimensional wave equation

$$\frac{\partial^2 z}{\partial t^2} = c^2 \left(\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} \right).$$

Because the membrane occupies the region $0 \le x \le a, 0 \le y \le b$ and is fastened at all the points of its rectangular boundary, we have the boundary conditions

$$z(0, y, t) = z(a, y, t) = z(x, 0, t) = z(x, b, t) = 0$$

for $0 < x < a$, $0 < y < b$ and $t > 0$. Finally, the initial conditions are

$$z(x, y, 0) = f(x, y), \frac{\partial z}{\partial t}(x, y, 0) = g(x, y).$$

5. Compute

$$\frac{\partial^2 y}{\partial t^2} = -\frac{n^2 \pi^2 c^2}{L^2} \sin(n\pi x/L) \cos(n\pi ct/L)$$

and

$$\frac{\partial^2 y}{\partial x^2} = -\frac{n^2 \pi^2}{L^2} \sin(n\pi x/L) \cos(n\pi ct/L).$$

Therefore

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}.$$

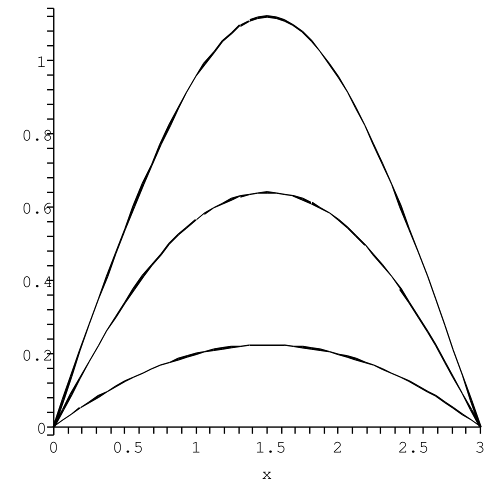

Figure 16.1: Waves in Problem 1, Section 16.2.

16.2 Wave Motion on an Interval

For each of Problems 1, 3, 5, and 7, separation of variables in the wave equation with the fixed end conditions at $x = 0$ and $x = L$ yields the general solution

$$y(x,t) = \sum_{n=1}^{\infty} \left[a_n \cos(n\pi ct/L) + \frac{L}{n\pi c} b_n \sin(n\pi ct/L) \right] \sin(n\pi x/L),$$

where

$$a_n = \frac{2}{L} \int_0^L f(\xi) \sin(n\pi\xi/L)\, d\xi$$

for $n = 1, 2, \cdots$ and

$$b_n = \frac{2}{L} \int_0^L g(\xi) \sin(n\pi\xi/L)\, d\xi$$

for $n = 1, 2, \cdots$. f is the initial position function and g is the initial velocity function.

To write the solution in specific instances, we need only identify c and L and compute the coefficients for the particular initial position and velocity functions.

1. The solution is

$$y(x,t) = \frac{108}{\pi^4} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^4} \sin((2n-1)\pi x/3) \sin((2(2n-1)\pi t/3)).$$

Figure 16.1 shows the solution waves increasing in amplitude over times $t = 0.1, 0.3$ and 0.7.

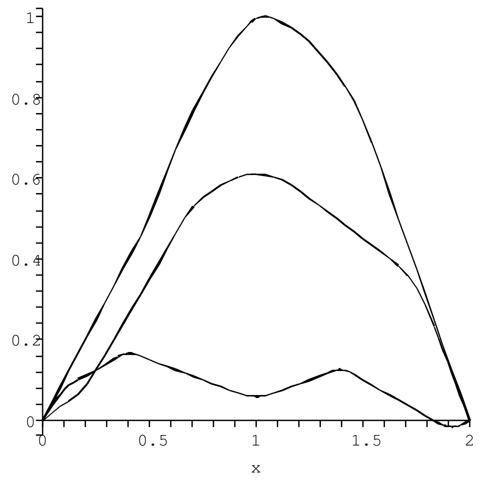

Figure 16.2: Profiles of the solution in Problem 3, Section 16.2.

3. The solution is

$$y(x,t) = -\frac{32}{\pi^3}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^3}\sin((2n-1)\pi x/2)\cos(3(2n-1)\pi t/2)$$
$$+\frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\left[\cos(n\pi/4)-\cos(n\pi/2)\right]\sin(n\pi x/2)\sin(3n\pi t/2)$$

Waves for this solution are shown in Figure 16.2, increasing in amplitude for times $t = 0.3$, 0.4 and 0.5.

5. $L = 2$, $f(x) = 0$ and $g(x) = 2x(1 - H(x-1))$, with H the Heaviside function. Then

$$a_n = 0 \text{ and } b_n = \frac{4}{n^2\pi^2}\left[2\sin(n\pi/2) - n\pi\cos(n\pi/2)\right]$$

for $n = 1, 2, \cdots$. Then

$$y(x,t) = \sum_{n=1}^{\infty}\frac{8}{n^3\pi^3 c}\left[2\sin(n\pi/2) - n\pi\cos(n\pi/2)\right]\sin(n\pi x/2)\sin(n\pi ct/2).$$

Figure 16.3 shows wave profiles increasing in amplitude at times $t = 1/10, 1/3$ and $1/2$, with the wave at $t = 1/2$ achieving its highest point near $x = 1.3$.

7. The solution is

$$y(x,t) = \frac{24}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n-1)^2}\sin((2n-1)\pi x/2)\cos((2n-1)\sqrt{2}t).$$

Figure 16.4 shows the waves moving downward through times $t = 0.3$, 0.5, 0.9 and 1.4.

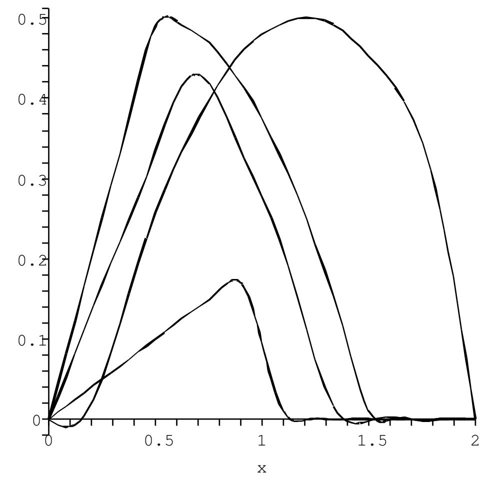

Figure 16.3: Waves in Problem 5, Section 16.2.

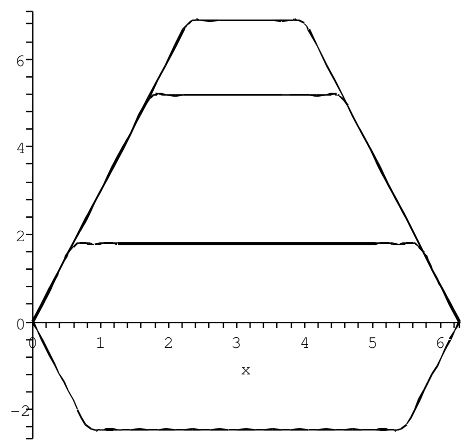

Figure 16.4: Profiles of the solution in Problem 7, Section 16.2.

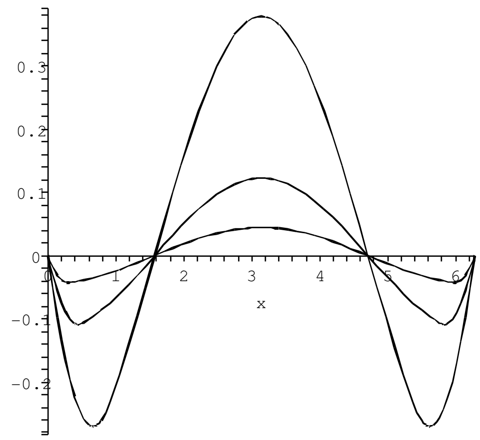

Figure 16.5: Profiles of the solution in Problem 9, Section 16.2.

9. Let $y(x,t) = Y(x,t) - h(x)$. Substitute $y(x,t)$ into the wave equation and use the boundary conditions to obtain a simpler problem for Y (that is, one we have already solved). This occurs if

$$-h''(x) - \cos(x) = 0, h(0) = h(2\pi) = 0.$$

Then $h(x) = \cos(x) - 1$ and $Y(x,t)$ satisfies the wave equation with $c = 1$ and the conditions

$$Y(0,t) = Y(2\pi,t) = 0, Y(x,0) = \cos(x) - 1, \frac{\partial Y}{\partial t}(x,0) = 0.$$

Solve this familiar problem for Y to obtain

$$\begin{aligned} y(x,t) &= 1 - \cos(x) \\ &\quad + \frac{16}{\pi}\sum_{n=1}^{\infty} \frac{1}{(2n-1)((2n-1)^2-4)} \sin((2n-1)x/2)\cos((2n-1)t/2). \end{aligned}$$

Graphs of solutions in Figure 16.5 move upward (nearest the origin) through times $t = 0.3$, 0.5 and 0.9.

11. Separation of variables gives us

$$X'' + \lambda X = 0, X(0) = X(L) = 0,$$

and

$$T'' + AT' + (B + c^2\lambda)T = 0, T'(0) = 0.$$

Eigenvalues and eigenfunctions for X are

$$\lambda_n = \frac{n^2\pi^2}{L^2}, X_n(t) = \sin(n\pi x/L).$$

With these eigenvalues, the characteristic equation of the differential equation for T is

$$r^2 + Ar + (B + c^2n^2\pi^2/L^2) = 0,$$

with roots

$$r = \frac{-A}{2} \pm \frac{1}{2}\sqrt{A^2 - 4\left(B + \frac{c^2 n^2 \pi^2}{L^2}\right)}.$$

The given condition $A^2L^2 < 4(BL^2 + c^2\pi^2)$ ensures that these roots are complex. Let

$$r_n^2 = 4(BL^2 + c^2n^2\pi^2) - A^2L^2.$$

The roots are then

$$r = -\frac{A}{2} \pm \frac{r_n}{2L} i.$$

Then

$$T_n(t) = e^{-At/2}\left[a_n \cos(r_n t/2L) + b_n \sin(r_n t/2L)\right].$$

Then $T'(0) = 0$ gives $-Aa_n/2 + b_n r_n/2L = 0$, hence

$$b_n = \frac{AL}{r_n} a_n.$$

By superposition,

$$u(x,t) = e^{-At/2} \sum_{n=1}^{\infty} a_n \sin(n\pi x/L)\left[\cos(r_n t/2L) + \frac{AL}{r_n}\sin(r_n t/2L)\right].$$

To satisfy $u(x,0) = f(x)$, choose

$$a_n = \frac{2}{L}\int_0^L f(\xi)\sin(n\pi\xi/L)\,d\xi.$$

13. Set $y(x,t) = Y(x,t) + \psi(x)$. To simplify the problem for $Y(x,t)$, choose $\psi(x)$ to satisfy

$$\psi''(x) = -\frac{1}{9}\cos(\pi x), \psi(0) = \psi(4) = 0.$$

By integrating we find that

$$\psi(x) = \frac{1}{9\pi^2}(\cos(\pi x) - 1).$$

The solution $Y(x,t)$ has the form

$$Y(x,t) = \sum_{n=1}^{\infty} a_n \sin(n\pi x/4)\cos(3n\pi t/4),$$

where

$$\begin{aligned} a_n &= \frac{2}{4}\int_0^4 \left[x(4-x) - \frac{1}{9\pi^2}(\cos(\pi x) - 1)\right]\sin(n\pi x/4)\,dx \\ &= \begin{cases} \frac{-32(1-(-1)^n)(288-17n^2)}{9n^3\pi^3(n^2-16)} & \text{for } n \neq 4, \\ 0 & \text{for } n = 4. \end{cases} \end{aligned}$$

The solution for the forced motion is

$$y(x,t) = \sum_{n=1}^{\infty} a_n \sin(n\pi x/4)\cos(3n\pi t/4) + \frac{1}{9\pi^2}(\cos(\pi x) - 1).$$

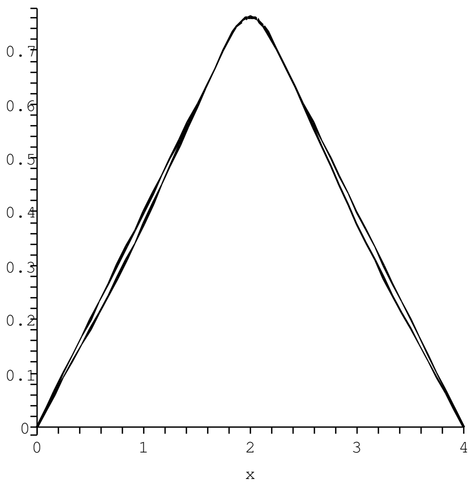

Figure 16.6: Forced and unforced motion in Problem 13, Section 16.2, at $t = 0.6$.

Without the forcing term, the solution has the form

$$y(x,t) = \sum_{n=1}^{\infty} \alpha_n \sin(n\pi x/4)\cos(3n\pi t/4),$$

where

$$\alpha_n = \frac{1}{2}\int_0^4 x(4-x)\sin(n\pi x/4)\,dx = \frac{64(1-(-1)^n)}{n^3\pi^3}.$$

Thus the solution for the unforced motion is

$$y(x,t) = \sum_{n=1}^{\infty} \frac{64(1-(-1)^n)}{n^3\pi^3}\sin(n\pi x/4)\cos(3n\pi t/4).$$

Forced and unforced solutions at $t = 0.6$, 1 and 1.4 are shown, respectively, in Figures 16.6, 16.7, and 16.8. In this example the forced motion is very similar to the unforced motion.

15. The differential equation is not separable, due to the $2x$ forcing term. Let $y(x,t) = Y(x,t) - h(x)$ and choose h to obtain a problem for Y that we have solved. Substitute y into the partial differential equation to obtain

$$\frac{\partial^2 Y}{\partial t^2} = 3\left(\frac{\partial^2 Y}{\partial x^2} - h''\right) + 2x.$$

This is the standard wave equation for Y if $3h''(x) = 2x$. For homogeneous boundary conditions at 0 and 2, we need $h(0) = h(2) = 0$. Solve for $h(x)$ by two integrations to obtain

$$h(x) = \frac{1}{9}(x^3 - 4x).$$

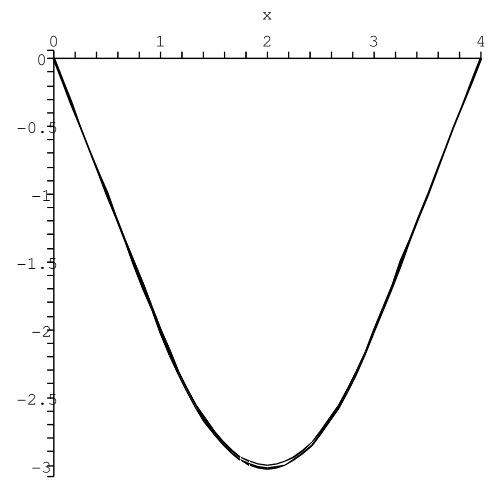

Figure 16.7: Forced and unforced motion in Problem 13, Section 16.2, at $t = 1$.

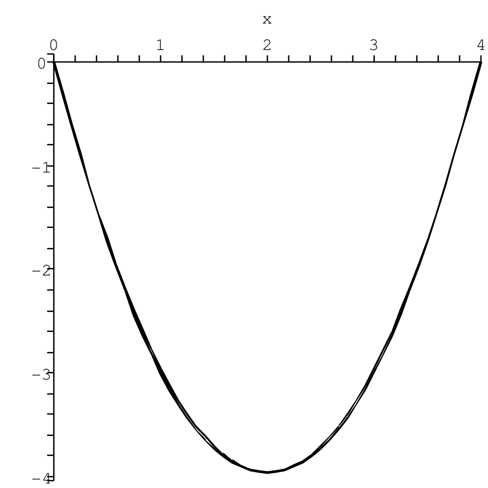

Figure 16.8: Forced and unforced motion in Problem 13, Section 16.2, at $t = 1.4$.

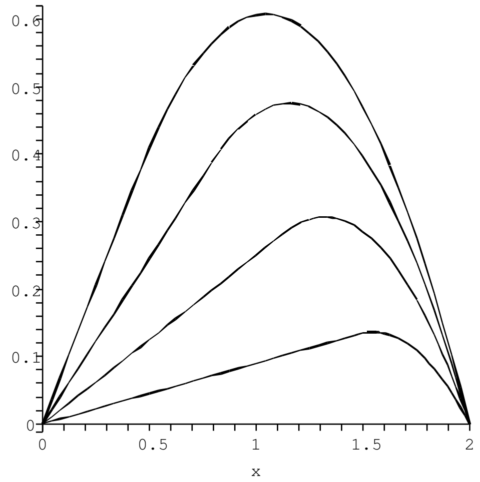

Figure 16.9: Profiles of the solution in Problem 15, Section 16.2.

Then $Y(x,t)$ satisfies the standard problem

$$\begin{aligned}
\frac{\partial^2 Y}{\partial t^2} &= 3\frac{\partial^2 Y}{\partial x^2},\\
Y(0,t) &= Y(2,t)0,\\
Y(x,0) &= h(x) = \frac{1}{9}(x^3-4x), \frac{\partial Y}{\partial t}(x,0) = 0.
\end{aligned}$$

Write the solution $Y(x,t)$ and then

$$\begin{aligned}
y(x,t) &= -h(x) + Y(x,t)\\
&= -\frac{1}{9}(x^3-4x) + \frac{32}{3\pi^3}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}\sin(n\pi x/3)\cos(\sqrt{3}n\pi t/2).
\end{aligned}$$

The waves increase in amplitude in Figure 16.9 through times $t = 0.3$, 0.5, 0.7 and 1.4.

16.3 Wave Motion in an Infinite Medium

In each of Problems 1, 3, and 5, the Fourier integral on $-\infty < x < \infty$ yields a solution of the wave equation with initial condition $f(x)$ and initial velocity $g(x)$, and having the form

$$\begin{aligned}
y(x,t) = \int_0^{\infty} \Big[&(a_\omega \cos(\omega x) + b_\omega \sin(\omega x))\cos(c\omega t)\\
&+ \int_0^{\infty} (\alpha_\omega \cos(\omega x) + \beta_\omega \sin(\omega x))\sin(c\omega t)\Big]\, d\omega,
\end{aligned}$$

where

$$\begin{aligned}
a_\omega &= \frac{1}{\pi}\int_{-\infty}^{\infty} f(\xi)\cos(\omega\xi)\,d\xi,\\
b_\omega &= \frac{1}{\pi}\int_{-\infty}^{\infty} f(\xi)\sin(\omega\xi)\,d\xi,\\
\alpha_\omega &= \frac{1}{\pi\omega c}\int_{-\infty}^{\infty} g(\xi)\cos(\omega\xi)\,d\xi,\\
\beta_\omega &= \frac{1}{\pi\omega c}\int_{-\infty}^{\infty} g(\xi)\sin(\omega\xi)\,d\xi.
\end{aligned}$$

1. For the Fourier integral solution, calculate the coefficients

$$\frac{1}{4\pi\omega}\int_{-\pi}^{\pi}\sin(\xi)\cos(\omega\xi)\,d\xi = 0$$

and

$$\frac{1}{4\pi\omega}\int_{-\pi}^{\pi}\sin(\xi)\sin(\omega\xi)\,d\xi = -\frac{\sin(\pi\omega)}{2\pi\omega(\omega^2-1)}.$$

The solution is

$$y(x,t) = \int_0^\infty \left(-\frac{\sin(\pi\omega)}{2\pi\omega(\omega^2-1)}\right)\sin(\omega x)\sin(4\omega t)\,d\omega.$$

To apply the Fourier transform, first transform the initial-boundary value problem to obtain

$$\begin{aligned}
&\hat{y}'' + 16\omega^2\hat{y} = 0;\\
&\hat{y}(\omega,0) = 0;\\
&\hat{y}(\omega,0) = \int_{-\pi}^{\pi}\sin(\xi)e^{-i\omega\xi}\,d\xi = \frac{2i\sin(\pi\omega)}{\omega^2-1}.
\end{aligned}$$

The solution of this transformed problem is

$$\hat{y}(\omega,t) = \frac{2i\sin(\pi\omega)}{4\omega(\omega^2-1)}\sin(4\omega t).$$

Invert this to obtain the solution

$$y(x,t) = \text{Re}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{i\sin(\pi\omega)}{2\omega(\omega^2-1)}\sin(4\omega t)e^{i\omega x}\,d\omega\right).$$

3. Compute the coefficients

$$a_\omega = \frac{1}{3\pi\omega}\int_1^\infty e^{-2\xi}\cos(\omega\xi)\,d\xi = \frac{e^{-2}}{3\pi\omega}\,\frac{2\cos(\omega)-\omega\sin(\omega)}{4+\omega^2}$$

and

$$b_\omega = \frac{1}{3\pi\omega}\int_1^\infty e^{-2\xi}\sin(\omega\xi)\,d\xi = \frac{e^{-2}}{3\pi\omega}\,\frac{2\sin(\omega)+\omega\cos(\omega)}{4+\omega^2}.$$

The solution is

$$y(x,t) = \int_0^\infty (a_\omega\cos(\omega x) + b_\omega\sin(\omega x))\sin(3\omega t).$$

To obtain the solution using the Fourier transform, first transform the problem to obtain

$$\hat{y}'' + 9\omega^2 \hat{y} = 0;$$
$$\hat{y}(\omega, 0) = 0;$$
$$\hat{y}'(\omega, 0) = \mathcal{F}(e^{-2x}H(x-1)) = \frac{(2-i\omega)e^{-(2+i\omega)}}{4+\omega^2}.$$

This problem has solution

$$\hat{y}(\omega, t) = \frac{(2-i\omega)e^{-(2+i\omega)}}{3\omega(4+\omega^2)}\sin(3\omega t).$$

Invert this to obtain the solution

$$y(x,t) = \text{Re}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{(2-i\omega)e^{-(2+i\omega)}}{3\omega(4+\omega^2)}\sin(3\omega t)e^{i\omega x}\,d\omega\right).$$

5. Compute

$$a_\omega = \frac{1}{\pi}\int_{-\infty}^{\infty} e^{-5|\xi|}\cos(\omega\xi)\,d\xi = \frac{10}{(25+\omega^2)\pi}.$$

Immediately $b_\omega = 0$ because $e^{-5|\omega|}\sin(\omega\omega)$ is an odd function. With the zero initial velocity condition, these are all the coefficients and the solution is

$$y(x,t) = \frac{10}{\pi}\int_0^{\infty}\left(\frac{1}{25+\omega^2}\right)\cos(\omega x)\cos(12\omega t)\,d\omega.$$

If we use the Fourier transform in x, take the transform of the wave equation to obtain

$$\hat{y}'' + 144\omega^2\hat{y} = 0;$$
$$\hat{y}(\omega, 0) = \int_{-\infty}^{\infty} e^{-5|\xi|}e^{-i\omega\xi}\,d\xi = \frac{10}{25+\omega^2},$$
$$\hat{y}'(\omega, 0) = 0.$$

The solution of this problem is

$$\hat{y}(\omega, t) = \frac{10}{25+\omega^2}\cos(12\omega t).$$

Invert this to obtain the solution

$$y(x,t) = \text{Re}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{10}{25+\omega^2}\cos(12\omega t)e^{i\omega x}\,d\omega\right].$$

Since $e^{i\omega x} = \cos(\omega x) + i\sin(\omega x)$, it is easy to extract the real part of this integral and verify that the solution obtained by using the transform agrees with that obtained using the Fourier integral.

16.4 Wave Motion in a Semi-Infinite Medium

For each of these problems, separation of variables and the Fourier sine integral yields a solution of the form

$$y(x,t) = \int_0^{\infty}\sin(\omega x)(a_\omega\cos(c\omega t) + b_\omega\sin(\omega ct))\,d\omega,$$

where

$$a_\omega = \frac{2}{\pi}\int_0^\infty f(\xi)\sin(\omega\xi)\,d\xi$$

and

$$b_\omega = \frac{2}{\pi c\omega}\int_0^\infty g(\xi)\sin(\omega\xi)\,d\xi.$$

1. To solve the problem using the Fourier sine integral, compute $a_\omega = 0$ and

$$b_\omega = \frac{2}{2\pi\omega}\int_{\pi/2}^{5\pi/2}\cos(\xi)\sin(\omega\xi),d\xi = \frac{\sin(\omega\pi/2)-\sin(5\omega\pi/2)}{\pi\omega(\omega^2-1)}.$$

   This gives us the solution

$$y(x,t) = \int_0^\infty \frac{\sin(\omega\pi/2)-\sin(5\omega\pi/2)}{\pi\omega(\omega^2-1)}\sin(\omega x)\sin(2\omega t)\,d\omega.$$

   For the Fourier sine transform solution, transform the problem to obtain

$$\begin{aligned}
&\hat{y}_S'' + 4\omega^2\hat{y}_S = 0;\\
&\hat{y}_S(\omega,0) = 0;\\
&\hat{y}_S'(\omega,0) = \int_{\pi/2}^{5\pi/2}\cos(\xi)\sin(\omega\xi)\,d\xi = \frac{\sin(\omega\pi/2)-\sin(5\omega\pi/2)}{\omega^2-1}.
\end{aligned}$$

   The solution of the transformed problem is

$$\hat{y}_S(\omega,t) = \frac{\sin(\omega\pi/2)-\sin(5\omega\pi/2)}{2\omega(\omega^2-1)}\sin(2\omega t).$$

   Invert this to obtain the solution

$$y(x,t) = \frac{2}{\pi}\int_0^\infty \frac{\sin(\omega\pi/2)-\sin(5\omega\pi/2)}{2\omega(\omega^2-1)}\sin(\omega x)\sin(2\omega t)\,d\omega.$$

3. To use the Fourier sine integral, compute $a_\omega = 0$ and

$$\begin{aligned}
b_\omega &= \frac{2}{14\pi\omega}\int_0^3 \xi^2(3-\xi)\sin(\omega\xi)\,d\xi\\
&= \frac{3}{7\pi\omega^5}(2\sin(3\omega) - 4\omega\cos(3\omega) - 3\omega^2\sin(3\omega) - 2\omega).
\end{aligned}$$

   This yields the solution

$$y(x,t) = \int_0^\infty b_\omega\sin(\omega x)\sin(14\omega t)\,d\omega.$$

   To use the Fourier sine transform, first transform the problem to obtain

$$\begin{aligned}
\hat{y}_S'' + 196\omega^2\hat{y}_S &= 0;\\
\hat{y}_S(\omega,0) &= 0;\\
\hat{y}_S'(\omega,0) &= \int_0^3 \xi^2(3-\xi)\sin(\omega\xi)\,d\xi\\
&= \frac{3}{\omega^4}\left(2\sin(3\omega) - 4\omega\cos(3\omega) - 3\omega^2\sin(3\omega) - 2\omega\right).
\end{aligned}$$

This transformed problem has the solution

$$\hat{y}_S(\omega, t) = \frac{3}{14\omega^5}(2\sin(3\omega) - 4\omega\cos(3\omega) - 3\omega^2\sin(3\omega) - 2\omega)\sin(14\omega t).$$

Invert this to obtain the solution

$$y(x,t) = \\ \frac{2}{\pi}\int_0^\infty \frac{3}{14\omega^5}(2\sin(3\omega) - 4\omega\cos(3\omega) - 3\omega^2\sin(3\omega) - 2\omega)\sin(\omega x)\sin(14\omega t)\,d\omega.$$

5. For a Fourier sine integral solution, calculate

$$a_\omega = \frac{2}{\pi}\int_0^1 \xi(1-\xi)\sin(\omega\xi)\,d\xi = \frac{2}{\pi}\left[\frac{2}{\omega^3}(1-\cos(\omega)) - \frac{\sin(\omega)}{\omega^2}\right].$$

and $b_\omega = 0$. The solution is

$$y(x,t) = \frac{2}{\pi}\int_0^\infty \left[\frac{2}{\omega^3}(1-\cos(\omega)) - \frac{\sin(\omega)}{\omega^2}\right]\sin(\omega x)\cos(3\omega t)\,d\omega.$$

To solve the problem using the Fourier sine transform, first take the transform of the initial-boundary value problem to obtain

$$\hat{y}_S'' + 9\omega^2\hat{y}_S = 0;$$
$$\hat{y}_S(\omega, 0) = \int_0^1 \xi(1-\xi)\sin(\omega\xi)\,d\xi = \frac{2(1-\cos(\omega)) - \omega\sin(\omega)}{\omega^3};$$
$$\hat{y}_S'(\omega, 0) = 0.$$

The solution of this transformed problem is

$$\hat{y}_S(\omega, t) = \left[\frac{2(1-\cos(\omega)) - \omega\sin(\omega)}{\omega^3}\right]\cos(3\omega t).$$

Invert this to obtain the solution

$$y(x,t) = \frac{2}{\pi}\int_0^\infty \left[\frac{2(1-\cos(\omega)) - \omega\sin(\omega)}{\omega^3}\right]\sin(\omega x)\cos(3\omega t)\,d\omega.$$

16.5 Laplace Transform Techniques

1. Transforming the partial differential equation yields

$$s^2Y(x,s) = c^2Y''(x,s) - \frac{Ax}{s^2}.$$

Then

$$Y'' - \frac{s^2}{c^2}Y = \frac{Ax}{c^2s^2}.$$

This has general solution

$$Y(x,s) = c_1e^{sx/c} + c_2e^{-sx/c} - \frac{Ax}{s^4}.$$

Now $c_1 = 0$ from the condition that $\lim_{x\to\infty} y(x,t) = 0$. Then

$$Y(x,s) = c_2 e^{-sx/c} - \frac{Ax}{s^4}.$$

Then

$$\mathcal{L}[y(0,t)](s) = Y(0,s) = c_2 = F(s),$$

so

$$Y(x,s) = e^{-sx/c}F(s) - \frac{Ax}{s^4}.$$

Invert this to obtain the solution

$$y(x,t) = f\left(t - \frac{x}{c}\right) H\left(t - \frac{x}{c}\right) - \frac{1}{6}Axt^4.$$

3. Apply the Laplace transform (with respect to t) of the partial differential equation to obtain

$$s^2 Y(x,s) = c^2 Y''(x,s) + \frac{K}{s}.$$

Here primes denote differentiation with respect to x, and the initial conditions have been inserted through the operational formula for the transform of $\partial^2 y/\partial t^2$. Write this equation as

$$Y'' - \frac{s^2}{c^2}Y = -\frac{K}{c^2 s}.$$

Think of this as a linear second-order differential equation in x, with s carried along as a parameter. The general solution is

$$Y(x,s) = c_1 e^{sx/c} + c_2 e^{-sx/c} + \frac{K}{s^3}.$$

Here c_1 and c_2 are "constant" in the sense of having no dependence on x, but they may be functions of s. Now

$$Y(0,s) = [y(0,t)](s) = F(s) = c_1 + c_2 + \frac{K}{s^3}.$$

We want $\lim_{x\to\infty} y(x,t) = 0$, so $\lim_{s\to\infty} Y(x,s) = 0$, hence $c_1 = 0$. Therefore

$$c_2 = F(s) - \frac{K}{s^3}.$$

Then

$$Y(x,s) = \left(F(s) - \frac{K}{s^3}\right) e^{-sx/c} + \frac{K}{s^3}.$$

The solution is obtained by applying the inverse transform (in s) to the last equation. Recalling equation (3.6) for the inverse Laplace transform of a function of the form $e^{-as}F(s)$, we obtain

$$y(x,t) = \left[f\left(t - \frac{x}{c}\right) - \frac{K}{2}\left(t - \frac{x}{c}\right)^2\right] H\left(t - \frac{x}{c}\right) + \frac{1}{2}Kt^2,$$

in which H is the Heaviside function.

5. From the partial differential equation and the initial conditions,

$$s^2Y(x,s) = c^2Y'' - \frac{A}{s^2}.$$

Then

$$Y'' - \frac{s^2}{c^2}Y = \frac{A}{s^2}.$$

This has general solution

$$Y(x,s) = c_1e^{sx/c} + c_2e^{-sx/c} - \frac{A}{s^4}.$$

Because $\lim_{x\to\infty} y(x,t) = 0$, we must also have

$$\lim_{s\to\infty} Y(x,s) = 0.$$

This requires that $c_1 = 0$, so

$$Y(x,s) = c_2e^{-sx/c} - \frac{A}{s^4}.$$

Next, $y(0,t) = 0$, so

$$Y(0,s) = c_2 - \frac{A}{s^4},$$

so

$$c_2 = \frac{A}{s^4}.$$

Finally we have

$$Y(x,s) = \frac{A}{s^4}e^{-sx/c} - \frac{A}{s^4}.$$

Then

$$y(x,t) = \frac{A}{6}\left(t - \frac{x}{c}\right)^3 H\left(t - \frac{x}{c}\right) - \frac{A}{6}t^3.$$

16.6 Characteristics and d'Alembert's Solution

1.

$$\begin{aligned} u(x,t) &= \frac{1}{2}(f(x-3t) + f(x+3t)) + \frac{1}{6}\int_{x-3t}^{x+3t} d\xi \\ &+ \int_0^t \int_{x-3t+3\eta}^{x+3t-3\eta} 3\xi\eta^3 \, d\xi \, d\eta \\ &= \frac{1}{2}(\cosh(x-3t) + \cosh(x+3t)) + t + \frac{9}{10}xt^5 \end{aligned}$$

3.

$$\begin{aligned} u(x,t) &= \frac{1}{2}(f(x-4t) + f(x+4t)) + \frac{1}{8}\int_{x-4t}^{x+4t} e^{-\xi} \, d\xi \\ &+ \frac{1}{8}\int_0^t \int_{x-4t+4\eta}^{x+4t-4\eta} (\xi+\eta) \, d\xi \, d\eta \\ &= x + \frac{1}{4}e^{-x}\sinh(4t) + \frac{1}{2}xt^2 + \frac{1}{6}t^3 \end{aligned}$$

5.

$$\begin{aligned}
u(x,t) &= \frac{1}{2}(f(x-8t)+f(x+8t)) + \frac{1}{16}\int_{x-8t}^{x+8t}\cos(2\xi)\,d\xi \\
&+ \frac{1}{16}\int_0^t \int_{x-8t+8\eta}^{x+8t-8\eta} \eta\cos(\xi)\,d\xi\,d\eta \\
&= x^2 + 64t^2 - x + \frac{1}{32}(\sin(2(x+8t)) - \sin(2(x-8t))) + \frac{1}{12}xt^4
\end{aligned}$$

7. With $c = 1$, characteristics are $x - t = k_1$ and $x + t = k_2$. The solution by d'Alembert's formula is

$$\begin{aligned}
u(x,t) &= \frac{1}{2}(f(x-t)+f(x+t)) - \frac{1}{2}\int_{x-t}^{x+t}\xi\,d\xi \\
&= \frac{1}{2}\left((x-t)^2 + (x+t)^2\right) - \left[\frac{\xi^2}{4}\right]_{x-t}^{x+t} \\
&= x^2 - xt + t^2.
\end{aligned}$$

9.

$$u(x,t) = \frac{1}{2}\left(e^{x-14t} + e^{x+14t}\right) + xt = e^x\cosh(14t) + xt$$

11.

$$\begin{aligned}
u(x,t) &= \frac{1}{2}(\cos(\pi(x-7t)) + \cos(\pi(x+7t))) \\
&+ t - x^2t - \frac{49}{3}t^3 \\
&= \frac{1}{2}\cos(\pi x)\cos(7\pi t) + t - x^2t - \frac{49}{3}t^3
\end{aligned}$$

For each of Problems 13 and 17, we give graphs of the wave position at selected times.

13. Figures 16.10 through 16.13 show graphs of the solution at times $t = 1/2$, $t = 0.9$, $t = 1.3$ and $t = 1.8$.

15. Figures 16.14 through 16.16 show wave positions at times $t = 1, 1.4, 1.7$.

17. In Figures 16.17 through 16.21, the wave is shown at times $t = 1/2, 1, 2, 3$ and 4.

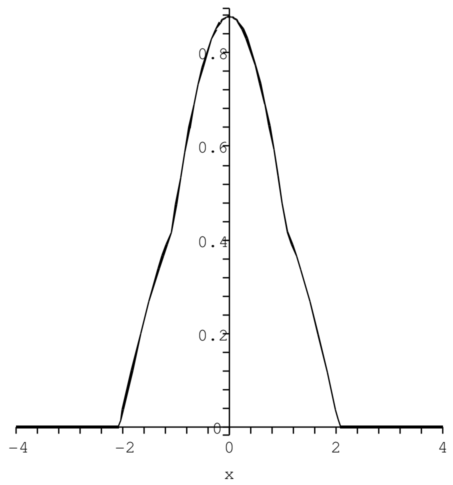

Figure 16.10: Problem 13, Section 16.6, at $t = 1/2$.

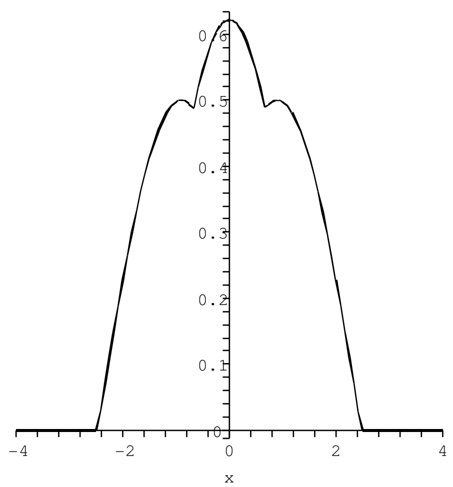

Figure 16.11: Problem 13, Section 16.6, at $t = 0.9$.

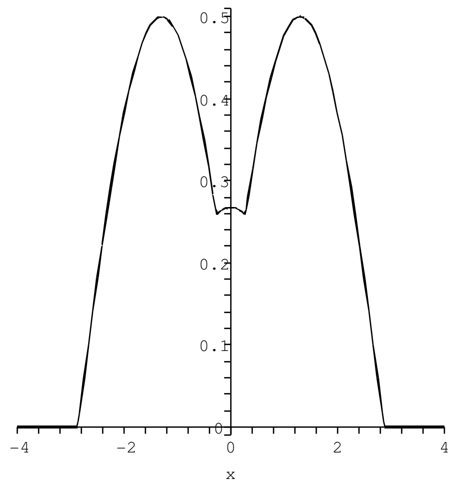

Figure 16.12: Problem 13, Section 16.6, at $t = 1.3$.

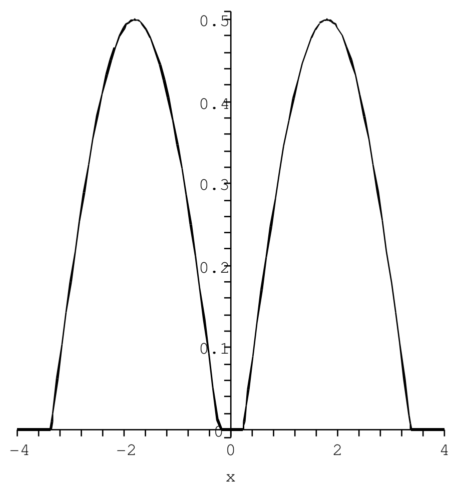

Figure 16.13: Problem 13, Section 16.6, at $t = 1.8$.

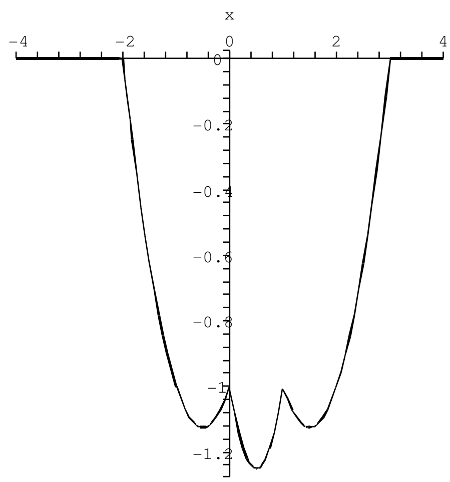

Figure 16.14: Problem 15, Section 16.6, at $t = 1$.

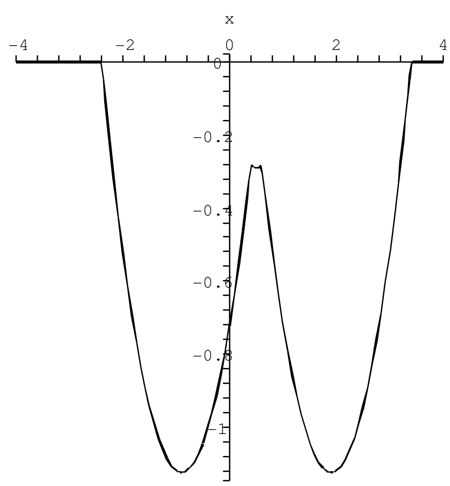

Figure 16.15: Problem 15, Section 16.6, at $t = 1.4$.

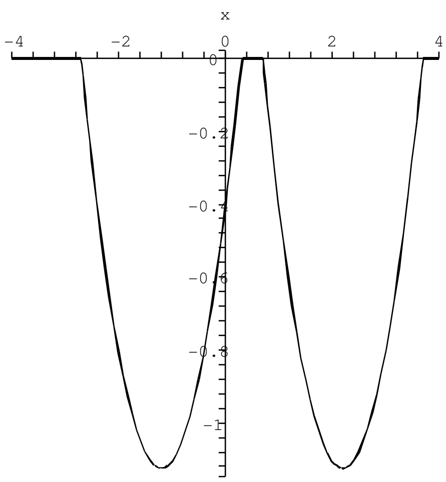

Figure 16.16: Problem 15, Section 16.6, at $t = 1.7$.

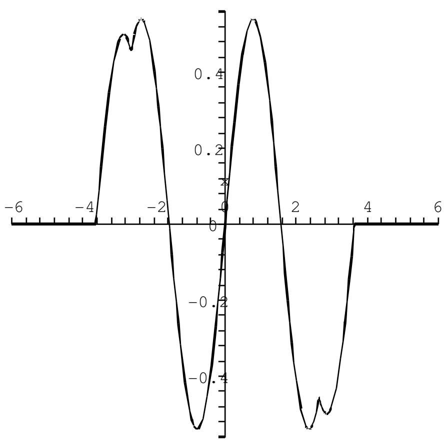

Figure 16.17: Wave position in Problem 17, Section 16.6, at $t = 1/2$.

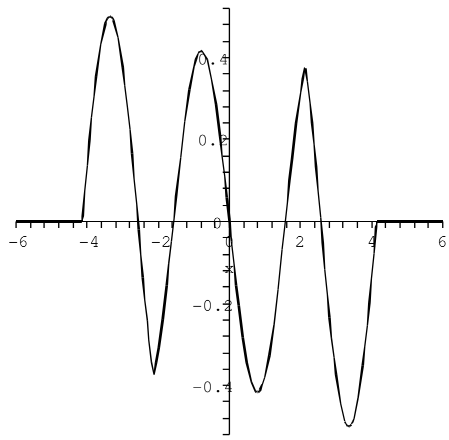

Figure 16.18: Problem 17, Section 16.6, $t = 1$.

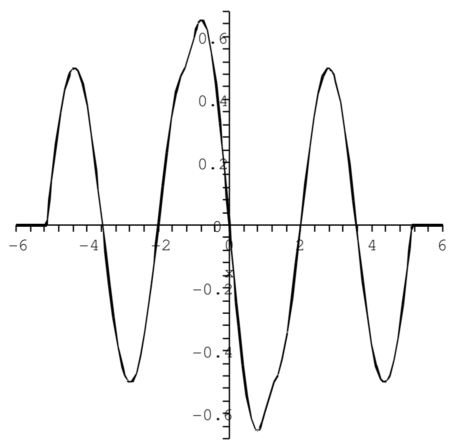

Figure 16.19: Problem 17, Section 16.6, $t = 2$.

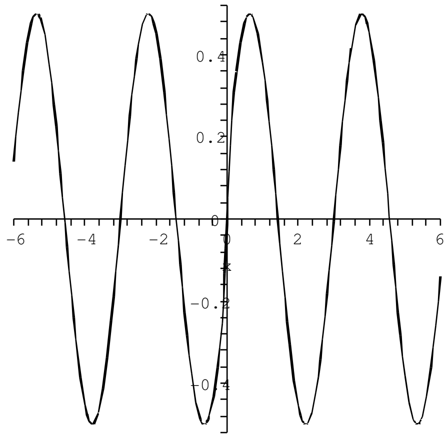

Figure 16.20: Problem 17, Section 16.6, $t = 3$.

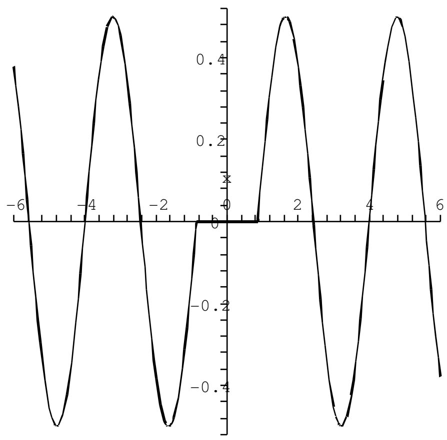

Figure 16.21: Problem 17, Section 16.6, $t = 4$.

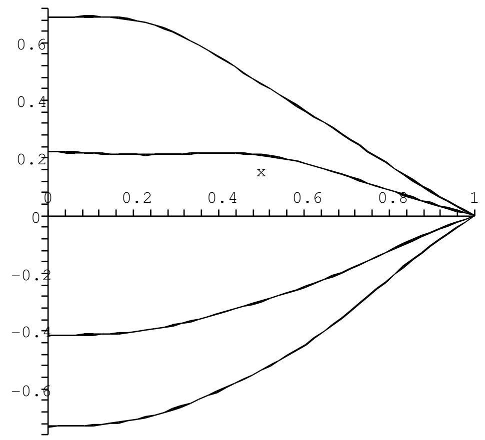

Figure 16.22: Solution positions in Problem 1, Section 16.7.

16.7 Vibrations in a Circular Membrane I

In each of these problems, the solution has the form

$$z(r,t) = \sum_{n=1}^{\infty} c_n J_0(j_n r)\cos(j_n t),$$

where j_n is the nth zero of $J_0(x)$. For a given initial displacement $f(r)$, the coefficients are

$$z(r,t) = \frac{2}{J_1(j_n)^2}\int_0^1 sf(s)J_0(j_n s)$$

for $n = 1, 2, \cdots$.

1. For $f(r) = 1 - r$, these coefficients are approximately

$$a_1 = 0.78542, a_2 = 0.06869, a_3 = 0.05311, a_4 = 0.01736, a_5 = 0.01698.$$

   Figure 16.22 shows the displacement at times $t = 0.05, 0.25, 0.5, 0.75$ and 1.25.

3. For $f(r) = \sin(\pi r)$, the coefficients are approximately

$$a_1 = 1.25335, a_2 = -0.80469, a_3 = -0.11615, a_4 = -0.09814, a_5 = -0.03740.$$

   Figure 16.23 shows the displacement at times $t = 0.05, 0.25, 0.5, 0.75$ and 1.25.

16.8 Vibrations in a Circular Membrane II

1. With zero initial velocity the solution will have the appearance

$$z(r,\theta,t) = \sum_{n=0}^{\infty}\sum_{k=1}^{\infty}[a_{nk}\cos(n\theta) + b_{nk}\sin(n\theta)]J_n\left(\frac{j_{nk}}{2}r\right)\cos(j_{nk}t).$$

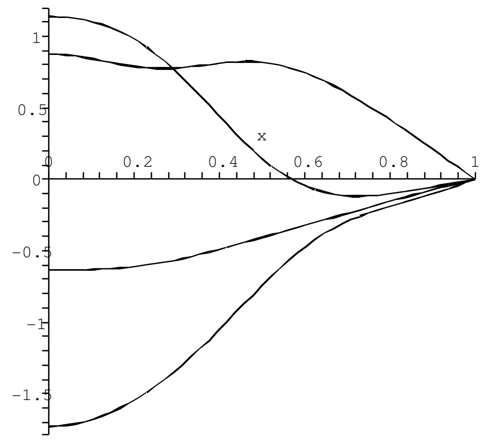

Figure 16.23: Solution positions in Problem 3, Section 16.7.

We need to choose the coefficients to satisfy the initial condition that

$$z(r,\theta,0) = f(r,\theta) = (4-r^2)\sin^2(\theta).$$

Putting $t = 0$ into the series, we need

$$(4-r^2)\sin(\theta) = \sum_{n=0}^{\infty}\sum_{k=1}^{\infty}[a_{nk}\cos(n\theta) + b_{nk}\sin(n\theta)]J_n\left(\frac{j_{nk}}{2}r\right).$$

Write $\sin(\theta) = (1-\cos(2\theta))/2$ and exploit the simple nature of the θ dependence in $f(r,\theta)$ to conclude, by matching coefficients of the $\cos(n\theta)$ terms for $n = 0$ and $n = 2$, that

$$\frac{4-r^2}{2} = \frac{1}{2}\alpha_0(r) = \sum_{k=1}^{\infty} a_{0k}J_0\left(\frac{j_{0k}}{2}r\right)$$

and

$$-\frac{4-r^2}{2} = \alpha_2(r) = \sum_{k=1}^{\infty} a_{2k}J_2\left(\frac{j_{2k}}{2}r\right).$$

Further, $\alpha_n(r) = 0$ for $n \neq 2$ and $\beta_n(r) = 0$ for $n \geq 0$, from which it follows that

$$a_{nk} = 0 \text{ for } n \neq 0, n \neq 2, k \geq 1$$

and

$$b_{nk} = 0 \text{ for } n \geq 0, k \geq 1.$$

Finally, using the orthogonality of the Bessel functions $J_0(j_{0k}r/2)$ for $k = 1, 2, \cdots$, and $J_2(j_{0k}r/2)$, we can calculate the coefficients as

$$a_{0k} = \frac{2}{[J_1(j_{0k})]^2}\int_0^1 \xi(1-\xi^2)J_0(j_{0k}\xi)\,d\xi \text{ for } k \geq 1$$

and

$$a_{2k} = \frac{4}{[J_3(j_{2k})]^2}\int_0^1 \xi(\xi^2-1)J_2(j_{2k}\xi)\,d\xi \text{ for } k \geq 1.$$

Using MAPLE, we can carry out numerical approximations of coefficients in the solution. Some of the terms are

$$\begin{aligned}z(r,\theta,t) &\approx 1.108022J_0(1.202413r)\cos(2.404826t) - 0.139778J_0(2.760039r)\cos(5.520078t)\\ &+ 0.045476J_0(4.326864r)\cos(8.653728t) + \cdots\\ &- 2.976777J_2(2.567811r)\cos(5.135622t)\cos(2\theta)\\ &- 1.434294J_2(4.208622r)\cos(8.417244t)\cos(2\theta)\\ &- 1.140494J_2(5.809921r)\cos(11.619841t)\cos(2\theta) + \cdots .\end{aligned}$$

16.9 Vibrations in a Rectangular Membrane

1. Separate variables in the wave equation by setting $z(x,y,t) = X(x)Y(y)T(t)$ to obtain

$$\frac{T''}{T} - \frac{X''}{X} = \frac{Y''}{Y} = -\alpha,$$

in which α is the separation constant. Separate again to get

$$\frac{T''}{T} + \alpha = \frac{X''}{X} = -\lambda.$$

This gives us the separated problems for X, Y and Z:

$$\begin{aligned}X'' + \lambda X &= 0; X(0) = X(2\pi) = 0,\\ Y'' + \alpha Y &= 0; Y(0) = Y(2\pi) = 0,\\ T'' + (\alpha+\lambda)T &= 0; T'(0) = 0.\end{aligned}$$

The eigenvalues and eigenfunctions are, respectively,

$$\begin{aligned}\lambda_n &= n^2/4, X_n(x) = \sin(nx/2),\\ \alpha_m &= m^2/4, Y_m(y) = \sin(my/2),\\ T_{nm}(t) &= \cos(\sqrt{n^2+m^2}t/2).\end{aligned}$$

The solution has the form

$$z(x,y,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} c_{nm}\sin(nx/2)\sin(my/2)\cos(\sqrt{n^2+m^2}t/2).$$

We need

$$z(x,y,0) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} c_{nm}\sin(nx/2)\sin(my/2) = x^2\sin(y).$$

Choose the coefficients

$$\begin{aligned}c_{nm} &= \frac{1}{\pi^2}\int_0^{2\pi}\int_0^{2\pi} \xi^2\sin(\eta)\sin(n\xi/2)\sin(m\eta/2)\,d\xi\,d\eta\\ &= \frac{1}{\pi^2}\int_0^{2\pi}\xi^2\sin(n\xi/2)\,d\xi\int_0^{2\pi}\sin(\eta)\sin(m\eta/2)\,d\eta\\ &= -\frac{8}{\pi n^3}\left(2(1-(-1)^n) + n^2\pi^2(-1)^n\right).\end{aligned}$$

The solution is

$$z(x,y,t)$$
$$= \sum_{n=1}^{\infty} \frac{-8}{\pi} \left(\frac{1}{n^3}(2(1-(-1)^n) + n^2\pi^2(-1)^n) \right) \sin(nx/2)\sin(y)\cos(\sqrt{4+n^2}t/2).$$

This is a single sum because the integrals

$$\int_0^{2\pi} \sin(\eta)\sin(m\eta/2)\, d\eta$$

are zero except for $m = 2$.

3. Separation of variables gives us the eigenfunctions $X_n(x) = \sin(nx/2)$ and $Y_m(y) = \sin(my/2)$, and we find that

$$T_{nm}(t) = a_{nm}\cos(\sqrt{n^2+m^2}t) + b_{nm}\sin(\sqrt{n^2+m^2}t).$$

The solution has the form

$$z(x,y,t) =$$
$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} (a_{nm}\cos(\sqrt{n^2+m^2}t) + b_{nm}\sin(\sqrt{n^2+m^2}t))\sin(nx/2)\sin(my/2).$$

The condition that $z(x,y,0) = 0$ is satisfied if all $a_{nm} = 0$. Thus the solution has the form

$$z(x,y,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} b_{nm}\sin(nx/2)\sin(my/2)\sin(\sqrt{n^2+m^2}t).$$

Now we need to choose the coefficients b_{nm} so that

$$\frac{\partial z}{\partial t}(x,y,0) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} b_{nm}\sqrt{n^2+m^2}\sin(nx/2)\sin(my/2) = 1.$$

Then

$$\begin{aligned} b_{nm} &= \frac{1}{\sqrt{n^2+m^2}}\frac{1}{\pi}\int_0^{2\pi}\sin(nx/2)\,dx\,\frac{1}{\pi}\int_0^{2\pi}\sin(my/2)\,dy \\ &= \frac{1}{\pi\sqrt{n^2+m^2}}\left(\frac{2(1-(-1)^n)}{n}\right)\left(\frac{2(1-(-1)^m)}{m}\right). \end{aligned}$$

Notice that $b_{nm} = 0$ if either n or m is even. Thus in the double summation we need only retain the terms in which both n abd m are odd. We can therefore write the solution

$$z(x,y,t) =$$
$$+\frac{16}{\pi^2}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} c_{nm}\sin((2n-1)x/2)\sin((2m-1)y/2)\sin(\sqrt{\alpha_{nm}}t),$$

where

$$c_{nm} = \frac{1}{(2n-1)(2m-1)\sqrt{\alpha_{nm}}}$$

and

$$\alpha_{nm} = (2n-1)^2 + (2m-1)^2.$$

Chapter 17

The Heat Equation

17.1 Initial and Boundary Conditions

1. $u(x,t)$ satisfies the conditions

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} \text{ for } t > 0, 0 < x < L,$$

with

$$\frac{\partial u}{\partial x}(0,t) = 0, u(L,t) = \beta(t) \text{ for } t > 0,$$

$$u(x,0) = f(x) \text{ for } 0 < x < L.$$

3. Let $u(x,t)$ be the temperature at time t of the cross section at x. Then u satisfies

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} \text{ for } t > 0, 0 < x < L.$$

The boundary conditions are

$$u(0,t) = 0, \frac{\partial u}{\partial x}(L,t) = 0 \text{ for } t > 0.$$

The initial condition is

$$u(x,0) = f(x) \text{ for } 0 < x < L.$$

17.2 The Heat Equation on $[0, L]$

For Problems 1 and 5, separation of variables and the given boundary conditions $u(0,t) = u(L,t) = 0$ yield the eigenvalues and eigenfunctions

$$\lambda_n = \frac{n^2\pi^2}{L^2}, X_n(x) = \sin(n\pi x/L).$$

The corresponding time solutions are

$$T_n(t) = e^{-kn^2\pi^2 t/L^2}.$$

Solutions have the form

$$u(x,t) = \sum_{n=1}^{\infty} c_n \sin(n\pi x/L) e^{-kn^2\pi^2 t/L^2}.$$

The coefficients are determined by

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} c_n \sin(n\pi x/L),$$

hence

$$c_n = \frac{2}{L}\int_0^{\infty} f(\xi)\sin(n\pi\xi/L)\,d\xi.$$

1. The coefficients are given by

$$\begin{aligned} c_n &= \frac{2}{L}\int_0^L L(1-\cos(2\pi\xi/L))\sin(n\pi\xi/L)\,d\xi \\ &= \begin{cases} \frac{8L((-1)^n-1)}{n\pi(n^2-4)} & \text{for } n \neq 2, \\ 0 & \text{for } n = 2. \end{cases} \end{aligned}$$

In addition, since $(-1)^n - 1 = 0$ if n is even, we have

$$c_4 = c_6 = \cdots = c_{\text{even}} = 0.$$

Therefore the solution is

$$u(x,t) = \\ -\frac{16L}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)((2n-1)^2-4)}\sin((2n-1)\pi x/L)e^{-3(2n-1)^2\pi^2 t/L^2}.$$

Figure 17.1 shows the temperature function at times $t = 0.2, 0.5$ and 1.1.

In Problems 3 and 7, separation of variables and the insulated end conditions $\partial u/\partial x(0,t) = \partial u/\partial x(L,t) = 0$ yield the eigenvalue $\lambda_0 = 1$ with eigenfunction $X_0(x) = 1$, and eigenvalues and eigenfunctions

$$\lambda_n = \frac{n^2\pi^2}{L^2}, X_n(x) = \cos(n\pi x/L).$$

The associated time functions are

$$T_n(t) = e^{-kn^2\pi^2 t/L^2}.$$

The solution has the form

$$u(x,t) = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n \cos(n\pi x/L)e^{-kn^2\pi^2 t/L^2},$$

where

$$c_n = \frac{2}{L}\int_0^L f(\xi)\cos(n\pi\xi/L)\,d\xi \text{ for } n = 0, 1, 2, \cdots.$$

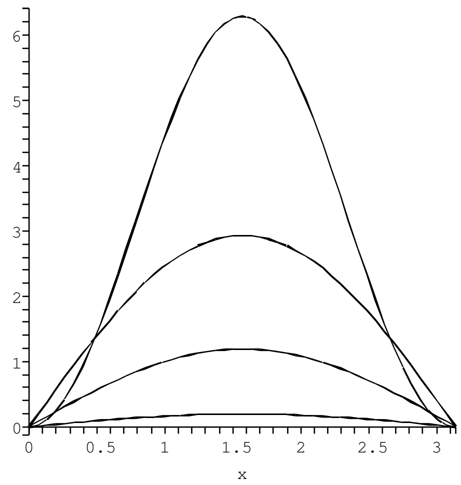

Figure 17.1: Problem 1, Section 17.2.

3. Compute

$$c_0 = \frac{2}{6}\int_0^6 e^{-\xi}\,d\xi = \frac{1}{3}\left(1-e^{-6}\right),$$

and, for $n = 1, 2, \cdots$,

$$\begin{aligned} c_n &= \frac{1}{3}\int_0^6 e^{-\xi}\cos(n\pi\xi/6)\,d\xi \\ &= \frac{12}{36+n^2\pi^2}\left(1-(-1)^n e^{-6}\right). \end{aligned}$$

The solution is

$$u(x,t) = \frac{1}{6}\left(1-e^{-6}\right) + \sum_{n=1}^{\infty}\frac{12}{36+n^2\pi^2}\left(1-(-1)^n e^{-6}\right)\cos(n\pi x/6)e^{-n^2\pi^2 t/18}.$$

Figure 17.2 shows the temperature function at $t = 0.2$, 0.4 and 0.8.

5. With $f(x) = x(L-x)$,

$$c_n = \frac{2}{L}\int_0^L \xi(L-\xi)\sin(n\pi\xi/L)\,d\xi = \frac{4L^2}{n^3\pi^3}(1-(-1)^n).$$

Note that $c_{2n} = 0$ because $1-(-1)^{2n} = 0$. We therefore retain only the odd indices in the solution:

$$u(x,t) = \frac{8L^2}{\pi^3}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^3}\sin((2n-1)\pi x/L)e^{-k(2n-1)^2\pi^2 t/L^2}.$$

Figure 17.3 shows the temperature function (decreasing) at times $t = 0.2$, 0.4, 0.7 and 1.5.

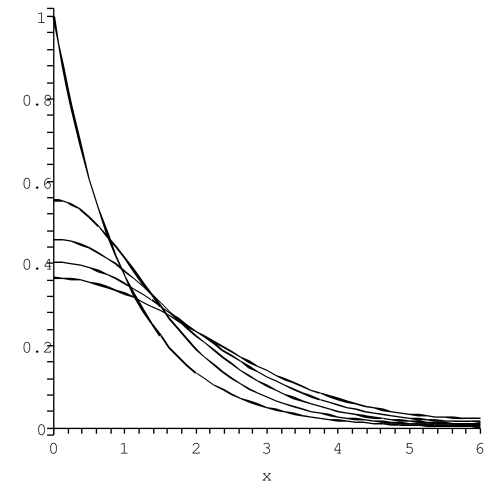

Figure 17.2: Problem 3, Section 17.2.

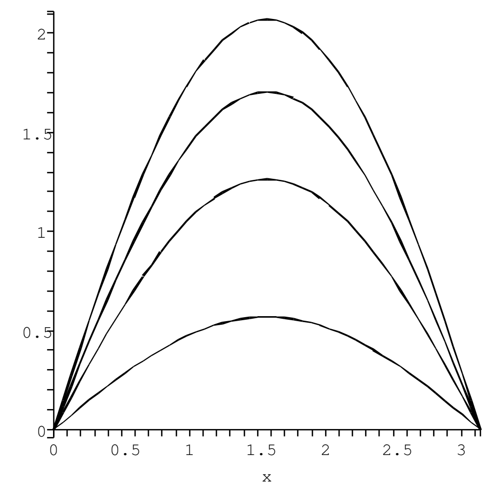

Figure 17.3: Temperature distribution in Problem 5, Section 17.2.

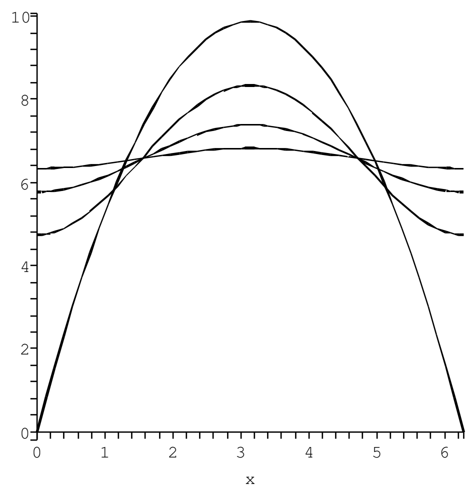

Figure 17.4: Problem 7, Section 17.2.

7. Compute

$$c_0 = \frac{1}{\pi}\int_0^{2\pi} \xi(2\pi-\xi)\,d\xi = \frac{4\pi^2}{3},$$

and

$$c_n = \frac{1}{\pi}\int_0^{2\pi} \xi(2\pi-\xi)\cos(nx)\,d\xi = -\frac{4}{n^2} \text{ for } n = 1, 2, \cdots .$$

The solution is

$$u(x,t) = \frac{2\pi^2}{3} - 4\sum_{n=1}^{\infty} \frac{1}{n^2}\cos(nx)e^{-4n^2 t}.$$

Figure 17.4 shows the solution at times $t = 0.2$, 0.4 and 0.7.

9. Let $u(x,t) = e^{\alpha x + \beta t} v(x,t)$ to transform the given problem. Substitute this into the heat equation and divide out the common exponential factor to obtain

$$\beta v + \frac{\partial v}{\partial t} = k\left(\alpha^2 v + 2\alpha\frac{\partial v}{\partial x} + \frac{\partial^2 v}{\partial x^2} + A\alpha v + A\frac{\partial v}{\partial x} + Bv\right).$$

The idea is to choose α and β to obtain a standard heat equation for v. To do this, we must eliminate terms containing v or $\partial v/\partial x$. Thus choose

$$2\alpha + A = 0,$$
$$k(\alpha^2 + A\alpha + B) - \beta = 0.$$

Then

$$\alpha = -\frac{A}{2} \text{ and } \beta = k\left(B - \frac{A^2}{4}\right).$$

With these choices, v satisfies

$$\frac{\partial v}{\partial t} = k\frac{\partial^2 v}{\partial x^2}$$
$$v(0,t) = v(L,t) = 0$$
$$v(x,0) = e^{-\alpha x}u(x,0).$$

11. The initial-boundary value problem for the temperature function is

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2},$$
$$u(0,t) = \frac{\partial u}{\partial x}(L,t) = 0,$$
$$u(x,0) = B.$$

Separate variables in the heat equation by putting $u(x,t) = X(x)T(t)$. We obtain the two problems:

$$X'' + \lambda X = 0, X(0) = 0, X'(L) = 0$$

and

$$T' + \lambda kT = 0.$$

The problem for $X(x)$ is routine and we obtain the eigenvalues and eigenfunctions

$$\lambda_n = \frac{(2n-1)^2\pi^2}{4L^2} \text{ and } X_n(x) = \sin((2n-1)\pi x/2L).$$

Then

$$T_n(t) = e^{-k(2n-1)^2\pi^2 kt/4L^2}.$$

By superposition, the solution has the form

$$u(x,t) = \sum_{n=1}^{\infty} c_n \sin((2n-1)\pi x/2L)e^{-k(2n-1)^2\pi^2 t/4L^2}.$$

The coefficients are given by

$$c_n = \frac{2}{L}\int_0^L B\sin((2n-1)\pi\xi/2L)\,d\xi = \frac{4B}{(2n-1)\pi}.$$

The solution is

$$u(x,t) = \frac{4B}{\pi}\sum_{n=1}^{\infty}\frac{1}{2n-1}\sin((2n-1)\pi x/2L)e^{-k(2n-1)^2\pi^2 t/4L^2}.$$

13. If we attempt a separation of variables in this problem, we find that this method fails because of the nonhomogeneous boundary conditions $u(0,t) = 2$ and $u(1,t) = 5$. We address this issue by transforming the problem. Let $u(x,t) = v(x,t) + L(x)$, where the idea is to choose $L(x)$ to obtain a problem for v that we know how to solve. Substituting u into the given initial-boundary value problem, we obtain a problem for v:

$$\frac{\partial v}{\partial t} = 16\frac{\partial^2 v}{\partial x^2} + 16L''(x),$$
$$v(0,t) + L(0) = 2, v(1,t) + L(1) = 5,$$
$$v(x,0) + L(x) = x^2.$$

To simplify the partial differential equation, make $L''(x) = 0$. To make the boundary conditions homogeneous, also choose L so that $L(0) = 2$ and $L(1) = 5$. Thus, we want

$$L''(x) = 0; L(0) = 2, L(1) = 5.$$

Routine integrations yield $L(x) = 3x + 2$. Now the problem for $v(x,t)$ is standard:

$$\begin{aligned}\frac{\partial v}{\partial t} &= 16\frac{\partial^2 v}{\partial x^2}\\ v(0,t) &= 0, v(1,t) = 0,\\ v(x,0) &= x^2 - 3x - 2.\end{aligned}$$

This problem has the solution

$$v(x,t) = \sum_{n=1}^{\infty} c_n \sin(n\pi x)e^{-16n^2\pi^2 t},$$

where

$$\begin{aligned}c_n &= 2\int_0^1 (\xi^2 - 3\xi - 2)\sin(n\pi\xi)\,d\xi\\ &= \frac{4}{n^3\pi^3}\left((-1)^n(1 + 2n^2\pi^2) - (1 + n^2\pi^2)\right).\end{aligned}$$

The original problem has the solution

$$u(x,t) = 3x + 2 + \sum_{n=1}^{\infty} c_n \sin(n\pi x)e^{-16n^2\pi^2 t}.$$

14. Let $u(x,t) = v(x,t) + h(x)$. Substitute this into the problem for u and choose h to obtain homogeneous boundary conditions. This gives us

$$h(x) = T\left(1 - \frac{x}{L}\right).$$

The problem for v is

$$\begin{aligned}\frac{\partial v}{\partial t} &= 9\frac{\partial^2 v}{\partial x^2},\\ v(0,t) &= v(L,t) = 0,\\ v(x,0) &= -T\left(1 - \frac{x}{L}\right).\end{aligned}$$

By separation of variables we obtain the solution

$$v(x,t) = \sum_{n=1}^{\infty} a_n \sin(n\pi x/L)e^{-9n^2\pi^2 t/L^2},$$

where

$$a_n = \frac{2}{L}\int_0^L -T\left(1 - \frac{\xi}{L}\right)\sin(n\pi\xi/L)\,d\xi = -\frac{2T}{n\pi}.$$

Then

$$u(x,t) = T\left(1 - \frac{x}{L}\right) - \frac{2T}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin(n\pi x/L)e^{-9n^2\pi^2 t/L^2}.$$

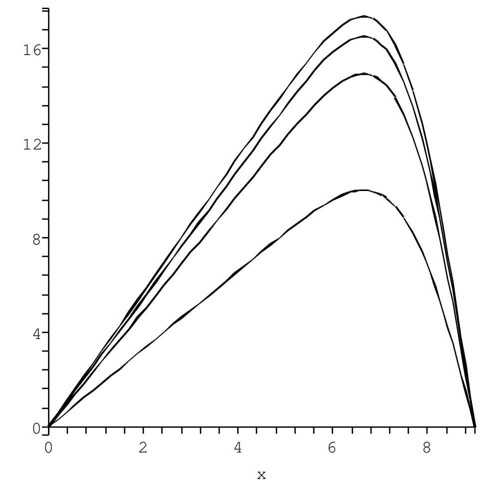

Figure 17.5: Problem 15, Section 17.2, with $t = 0.2$.

15. Let $u(x,t) = e^{-\alpha t}w(x,t)$ and substitute into the heat equation, choosing α to eliminate the $-Aw$ term. This requires that

$$-\alpha w e^{-\alpha t} + \frac{\partial w}{\partial t}e^{-\alpha t} = 4\frac{\partial^2 w}{\partial x^2}e^{-\alpha t} - Awe^{-\alpha t}.$$

Thus choose $\alpha = A$. Then w satisfies

$$\frac{\partial w}{\partial t} = 4\frac{\partial^2 w}{\partial x^2},$$
$$w(0,t) = w(9,t) = 0,$$
$$w(x,0) = 3x.$$

By separation of variables we obtain the solution

$$w(x,t) = \frac{54}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin(n\pi x/9)e^{-4n^2\pi^2 t/81}.$$

The solution of the problem for u is

$$u(x,t) = e^{-At}w(x,t).$$

The diagrams show the solution at various times for $A = 1/4$, $1/2$, 1 and 3. Figure 17.5 is for $t = 0.2$, Figure 17.6 for $t = 0.7$, and Figure 17.7 for $t = 1.4$.

17. Follow the idea of Problem 9 with $A = 6$, $B = 0$ and $k = 1$. Then $\alpha = -3$ and $\beta = -9$ to make the transformation

$$u(x,t) = e^{-3x-9t}v(x,t).$$

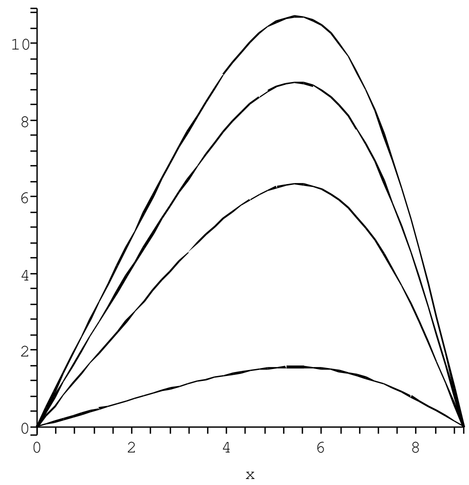

Figure 17.6: Problem 15, Section 17.2, for $t = 0.7$.

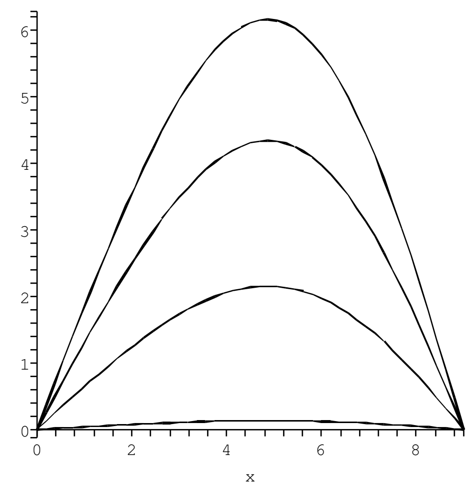

Figure 17.7: Problem 15, Section 17.2, at $t = 1.4$.

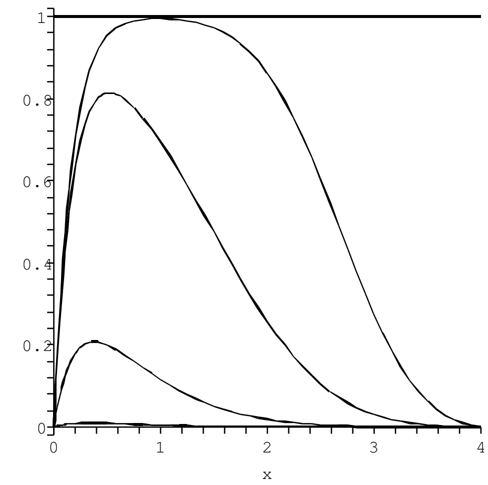

Figure 17.8: Problem 17, Section 17.2.

Then v satisfies the standard problem

$$\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2},$$
$$v(0,t) = v(4,t) = 0,$$
$$v(x,0) = e^{3x}u(x,0) = e^{3x}.$$

This problem has a solution of the form

$$v(x,y) = \sum_{n=1}^{\infty} c_n \sin(n\pi x/4) e^{-n^2\pi^2 t/16},$$

where

$$c_n = \frac{1}{2}\int_0^4 e^{3\xi}\sin(n\pi\xi/4)\,d\xi$$
$$= \frac{2n\pi}{144 + n^2\pi^2}(1 - e^{12}(-1)^n).$$

The solution for the original problem for u is

$$u(x,t) = e^{-3x-9t}\sum_{n=1}^{\infty} c_n \sin(n\pi x/4) e^{-n^2\pi^2 t/16}.$$

Figure 17.8 shows the solution at times $t = 0.2$, 0.4, 0.7 and 1.1.

In each of Problems 19, 21, and 23, obtain a solution of the form

$$u(x,t) = \sum_{n=1}^{\infty} T_n(t)\sin(n\pi x/L) + \sum_{n=1}^{\infty} b_n \sin(n\pi x/L) e^{-kn^2\pi^2 t/L^2},$$

where

$$b_n = \frac{2}{L}\int_0^L f(\xi)\sin(n\pi\xi/L)\,d\xi$$

for $n = 1, 2, \cdots$ and $T_n(t)$ is the solution of

$$T_n'(t) + k\frac{n^2\pi^2}{L^2}T_n(t) = B_n(t); T_n(0) = b_n,$$

with

$$B_n(t) = \frac{2}{L}\int_0^L F(\xi,t)\sin(n\pi\xi/L)\,d\xi$$

for $n = 1, 2, \cdots$.

Note that the second term in this solution for $u(x,t)$ is the solution to the problem without the forcing term.

19. Compute

$$\begin{aligned}B_n(t) &= \frac{2}{5}\int_0^5 t\cos(\xi)\sin(n\pi\xi/5)\,d\xi\\ &= \frac{2t}{n^2\pi^2 - 25}((-1)^{n+1}(5+n\pi)+n\pi),\end{aligned}$$

$$b_n = \frac{2}{5}\int_0^5 \xi^2(5-\xi)\sin(n\pi\xi/5)\,d\xi = \frac{500}{n^3\pi^3}((-1)^{n+1}-1),$$

and

$$T_n(t) = \frac{50(1-\cos(5)(-1)^n)}{n^3\pi^3(n^2\pi^2-25)}(n^2\pi^2 t - 25 + 25e^{-n^2\pi^2 t/25}).$$

The solution is

$$\begin{aligned}u(x,t) &= \sum_{n=1}^{\infty}\frac{50(1-\cos(5)(-1)^n)}{n^3\pi^3(n^2\pi^2-25)}(n^2\pi^2 t - 25 + 25e^{-n^2\pi^2 t/25})\sin(n\pi x/5)\\ &+ \sum_{n=1}^{\infty}\frac{500}{n^3\pi^3}((-1)^{n+1}-1)\sin(n\pi x/5)e^{-n^2\pi^2 t/25}.\end{aligned}$$

Figures 17.9, 17.10, and 17.11 show the solution, with and without source term, at times $t = 1.5$, 2.5 and 2.9, respectively.

21. Compute

$$B_n(t) = \frac{2}{3}\int_0^3 \xi t\sin(n\pi\xi/3)\,d\xi = \frac{6t}{n\pi}(-1)^{n+1},$$

$$b_n = \frac{2}{3}\int_0^3 K\sin(n\pi\xi/3)\,d\xi = \frac{2K}{n\pi}(1-(-1)^n),$$

$$T_n(t) = \frac{27(-1)^{n+1}}{128n^5\pi^5}(16n^2\pi^2 - 9 + 9e^{-16n^2\pi^2 t/9}),$$

and the solution is

$$\begin{aligned}u(x,t) &= \sum_{n=1}^{\infty}\frac{27(-1)^{n+1}}{128n^5\pi^5}(16n^2\pi^2 - 9 + 9e^{-16n^2\pi^2 t/9})\sin(n\pi x/3)\\ &+ \sum_{n=1}^{\infty}\frac{2K}{n\pi}(1-(-1)^n)\sin(n\pi x/3)e^{-16n^2\pi^2 t/9}.\end{aligned}$$

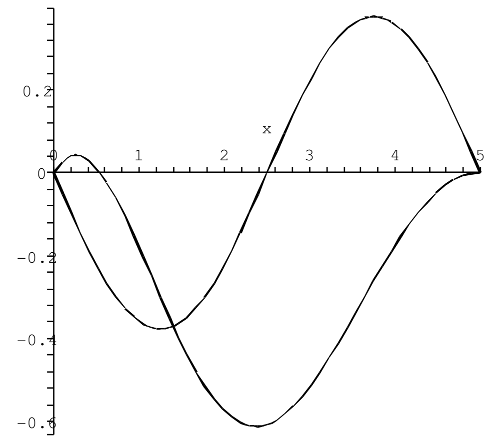

Figure 17.9: Problem 19, Section 17.2, with and without source term, at $t = 1.5$.

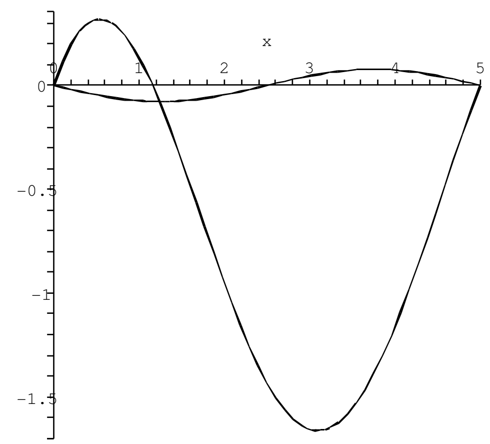

Figure 17.10: Problem 19, Section 17.2, at $t = 2.5$.

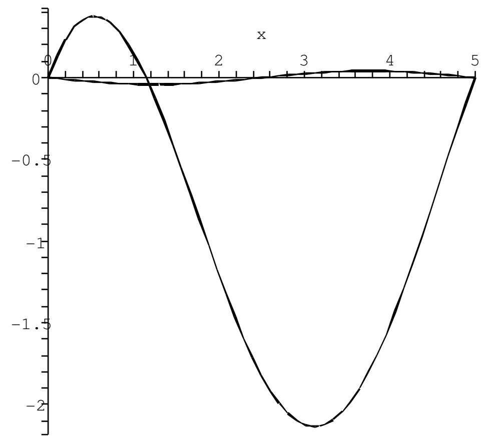

Figure 17.11: Problem 19, Section 17.2, at $t = 2.9$.

Figures 17.12, 17.13, and 17.14 show the solution, with and without source term, at times $t = 0.1$, 0.2 and 0.3, respectively. $K = 4$ is used in these graphs.

23. With $k = 4$, $L = \pi$, $f(x) = x(\pi - x)$ and $F(x,t) = t$, compute

$$B_n(t) = \frac{2}{\pi}\int_0^\pi t\sin(n\xi)\,d\xi = \frac{2t}{n\pi}(1-(-1)^n),$$

$$b_n = \frac{4}{\pi n^3}(1-(-1)^n),$$

and

$$T_n(t) = \frac{1}{8\pi n^5}(1-(-1)^n)(-1+4n^2t+e^{-4n^2t}).$$

The solution is

$$u(x,t) = \sum_{n=1}^{\infty}\frac{1}{8\pi n^5}(1-(-1)^n)(-1+4n^2t+e^{-4n^2t})\sin(nx) + \sum_{n=1}^{\infty}\frac{4}{\pi n^3}(1-(-1)^n)\sin(nx)e^{-4n^2t}.$$

Figure 17.15 shows the solution with and without the source term, at time $t = 0.2$. Figure 17.16 is at $t = 0.5$, and Figure 17.17 at $t = 1.1$.

17.3 Solutions in an Infinite Medium

In Problems 1 and 3, the boundary value problems are stated on the half line and are solved by separation of variables.

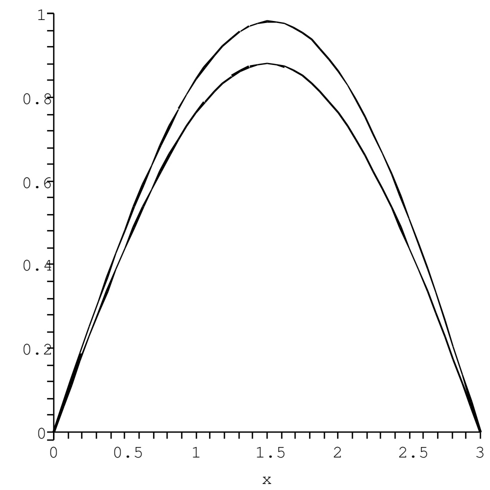

Figure 17.12: Problem 21, Section 17.2, with and without source term, at $t = 0.1$.

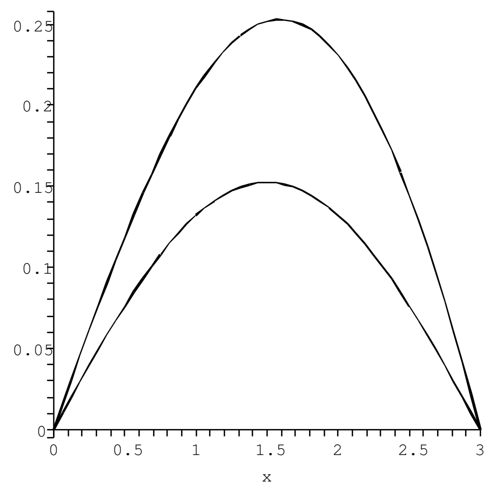

Figure 17.13: Problem 21, Section 17.2, at $t = 0.2$.

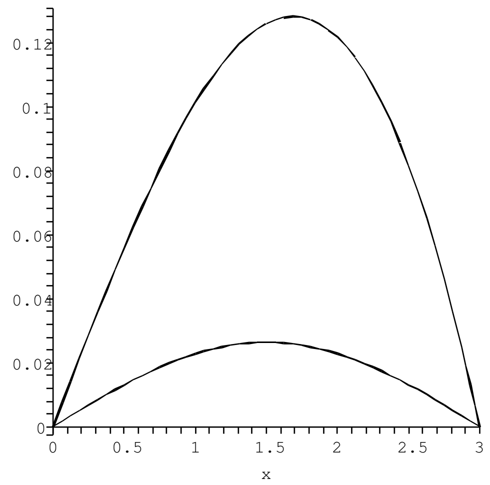

Figure 17.14: Problem 21, Section 17.2, at $t = 0.3$.

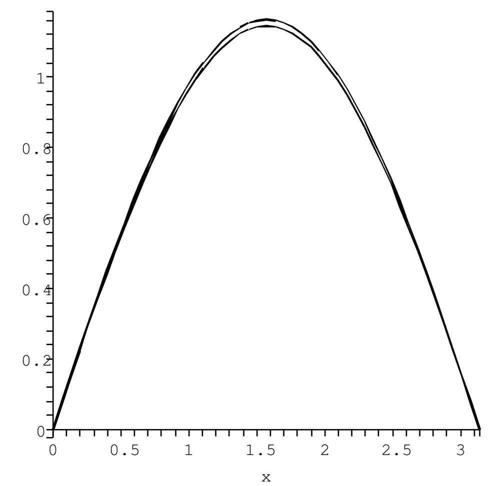

Figure 17.15: Problem 23, Section 17.2, at $t = 0.2$, with and without the source term.

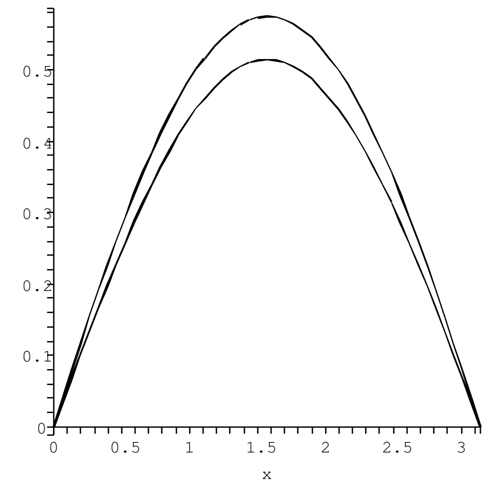

Figure 17.16: Problem 23, Section 17.2, at $t = 0.5$, with and without source term.

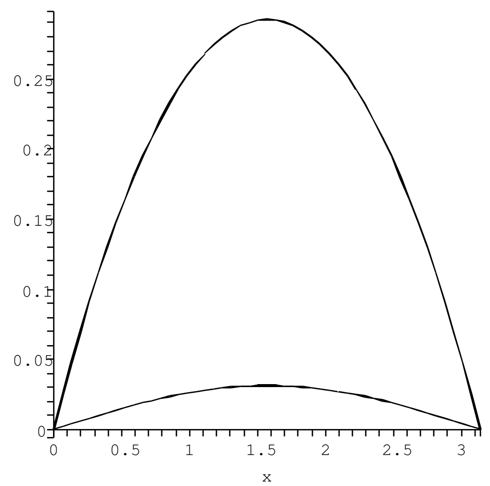

Figure 17.17: Problem 23, Section 17.2, at $t = 1.1$, with and without source term.

1. Compute
$$b_\omega = \frac{2}{\pi}\int_0^\infty e^{-\alpha\xi}\sin(\omega\xi)\,d\xi = \frac{2}{\pi}\frac{\omega}{\omega^2+\alpha^2}.$$
The solution is
$$u(x,t) = \frac{2}{\pi}\int_0^\infty \left(\frac{\omega}{\omega^2+\alpha^2}\right)\sin(\omega x)e^{-k\omega^2 t}\,d\omega.$$

3. The coefficients are
$$b_\omega = \frac{2}{\pi}\int_0^h \sin(\omega\xi)\,d\xi = \frac{2}{\pi}\frac{1-\cos(h\omega)}{\omega}.$$
The solution is
$$u(x,t) = \frac{2}{\pi}\int_0^\infty \left(\frac{1-\cos(h\omega)}{\omega}\right)\sin(\omega x)e^{-k\omega^2 t}\,d\omega.$$

In Problems 5 and 7, separation of variables and the requirement of a bounded solution yield a solution of the form
$$u(x,t) = \int_0^\infty (a_\omega\cos(\omega x) + b_\omega\sin(\omega x))e^{-\omega^2 kt}\,d\omega,$$
where
$$a_\omega = \frac{1}{\pi}\int_{-\infty}^\infty f(\xi)\cos(\omega\xi)\,d\xi \text{ and } b_\omega = \frac{1}{\pi}\int_{-\infty}^\infty f(\xi)\sin(\omega\xi)\,d\xi.$$

To write the solution of this problem using the Fourier transform, first transform the problem to obtain
$$\frac{d\hat{u}}{dt} + k\omega^2\hat{u} = 0;\ \hat{u}(\omega,0) = \hat{f}(\omega).$$
The solution of this transformed problem is
$$\hat{u}(\omega,t) = \hat{f}(\omega)e^{-k\omega^2 t}.$$
Recover the solution $u(x,t)$ of the original problem by taking the Fourier transform of this solution of the transformed problem. Use the result that
$$\mathcal{F}^{-1}\left[e^{-k\omega^2 t}\right] = \frac{1}{2\sqrt{\pi kt}}e^{-x^2/4kt},$$
together with the convolution theorem, to obtain
$$u(x,t) = \mathcal{F}^{-1}(\hat{f}(\omega)e^{-k\omega^2 t}) = \frac{1}{2\sqrt{\pi kt}}\int_{-\infty}^\infty f(\xi)e^{-(x-\xi)^2/4kt}\,d\xi.$$

5. Compute
$$a_\omega = \frac{1}{\pi}\int_0^4 \xi\cos(\omega\xi)\,d\xi = \frac{1}{\pi}\frac{4\omega\sin(4\omega)+\cos(4\omega)-1}{\omega^2}$$
and
$$b_\omega = \frac{1}{\pi}\int_0^4 \xi\sin(\omega\xi)\,d\xi = \frac{1}{\pi}\frac{\sin(4\omega)-4\omega\cos(\omega)}{\omega^2}.$$
The solution is
$$u(x,t) = \int_0^\infty (a_\omega\cos(\omega x) + b_\omega\sin(\omega x))e^{-\omega^2 kt}\,d\omega.$$

If we want to solve the problem using the Fourier transform, write

$$f(x) = x(H(x) - H(x-4))$$

to obtain

$$\begin{aligned} u(x,t) &= \frac{1}{2\sqrt{\pi k t}} \int_{-\infty}^{\infty} \xi(H(\xi) - H(\xi-4)) e^{-(x-\xi)^2/4kt} \, d\xi \\ &= \frac{1}{2\sqrt{\pi k t}} \int_{0}^{4} \xi e^{-(x-\xi)^2/4kt} \, d\xi. \end{aligned}$$

7. Compute

$$a_\omega = \frac{1}{\pi} \int_{-\infty}^{\infty} e^{-4|\xi|} \cos(\omega\xi) \, d\xi = \frac{8}{\pi} \frac{1}{16+\omega^2} \text{ and } b_\omega = 0.$$

The solution is

$$u(x,t) = \frac{8}{\pi} \int_0^\infty \frac{1}{16+\omega^2} \cos(\omega x) e^{-\omega^2 kt} \, d\omega.$$

Using the Fourier transform we obtain the form of the solution

$$u(x,t) = \frac{1}{2\sqrt{\pi k t}} \int_{-\infty}^{\infty} e^{-4|\xi|} e^{-(x-\xi)^2/4kt} \, d\xi.$$

9. Apply the Fourier sine transform with respect to x to the given problem to obtain:

$$\begin{aligned} &\hat{U}_S' + \omega^2 \hat{U}_S + t\hat{U}_S = 0, \\ &\hat{U}_S(\omega, 0) = \mathcal{F}(xe^{-x}) = \frac{2\omega}{(1+\omega^2)^2}. \end{aligned}$$

This problem has solution

$$\hat{U}_S(\omega,t) = \frac{2\omega}{(1+\omega^2)^2} e^{-(\omega^2 t + t^2/2)}.$$

Now use the inversion formula to obtain the solution

$$u(x,t) = \frac{4}{\pi} \int_0^\infty \frac{\omega}{(1+\omega^2)^2} e^{-\omega^2 t - t^2/2} \sin(\omega x) \, d\omega.$$

11. Let

$$F(x) = \int_0^\infty e^{-\zeta^2} \cos(x\zeta) \, d\zeta.$$

Think of this as a function of x. Differentiate under the integral sign to obtain

$$F'(x) = \int_0^\infty -\zeta e^{-\zeta^2} \sin(x\zeta) \, d\zeta.$$

Integrate by parts to obtain

$$F'(x) = \left[\frac{1}{2} e^{-\zeta^2} \sin(x\zeta)\right]_0^\infty - \frac{1}{2} \int_0^\infty \frac{1}{2} e^{-\zeta^2} x \cos(x\zeta) \, d\zeta.$$

Now observe that

$$F'(x) = -\frac{x}{2} F(x).$$

This is a separable first order differential equation. Write it as

$$\frac{F'(x)}{F(x)} = -\frac{x}{2}$$

and integrate to obtain

$$\ln|F(x)| = -\frac{1}{4}x^2 + c.$$

Then

$$F(x) = ke^{-x^2/4},$$

where $k = e^c$ is a constant to be determined. But,

$$F(0) = k = \int_0^\infty e^{-\zeta^2}\, d\zeta = \frac{1}{2}\sqrt{\pi},$$

an integral that is well known (for example, it is widely used in statistics). Therefore

$$F(x) = \frac{\sqrt{\pi}}{2}e^{-x^2/4}.$$

Upon letting $x = \alpha/\beta$, we have

$$F(\alpha/\beta) = \int_0^\infty e^{-\zeta^2}\cos\left(\frac{\alpha}{\beta}\zeta\right)\, d\zeta = \frac{\sqrt{\pi}}{2}e^{-\alpha^2/4\beta^2}.$$

Finally, since $e^{-\zeta^2}\cos(x\zeta$ is an even function in ζ, then

$$\int_{-\infty}^\infty e^{-\zeta^2}\cos\left(\frac{\alpha}{\beta}\zeta\right)\, d\zeta = \sqrt{\pi}e^{-\alpha^2/4\beta^2}.$$

17.4 Laplace Transform Techniques

1. Take the Laplace transform, with respect to t, of the heat equation to obtain

$$sU(x,s) - e^{-x} = kU''(x,s).$$

Then

$$U'' - \frac{s}{k}U = -\frac{1}{k}e^{-x}.$$

This is a linear, second-order, nonhomogeneous differential equation. The general solution of the associated homogeneous equation is

$$U_h(x,s) = c_1 e^{\sqrt{s/k}x} + c_2 e^{-\sqrt{s/k}x}.$$

Use undetermined coefficients to find a particular solution of the nonhomogeneous differential equation. Substitute $U_p(x,s) = Ae^{-x}$ into the differential equation to obtain

$$A - \frac{s}{k}A = -\frac{i}{k},$$

so

$$A = \frac{1}{s-k}.$$

We obtain

$$U(x,s) = U_h(x,s) + U_p(x,s) = c_1 e^{\sqrt{s/k}x} + c_2 e^{-\sqrt{s/k}x} + \frac{1}{s-k}e^{-x}.$$

Now $\lim_{x\to\infty} u(x,t) = 0$, so $c_1 = 0$. Then

$$U(x,s) = ce^{-\sqrt{s/k}x} + \frac{1}{s-k}e^{-x},$$

in which we have written c in place of c_2. Since $u(0,t) = 0$, then

$$U(0,s) = c + \frac{1}{s-k}.$$

Then $c = -1/(s-k)$, so

$$U(x,s) = -\frac{1}{s-k}e^{-\sqrt{s/k}x} + \frac{1}{s-k}e^{-x}.$$

Since

$$\mathcal{L}^{-1}\left[\frac{1}{s-k}\right](t) = e^{kt}$$

then we have, using the convolution theorem,

$$u(x,t) = -e^{kt} * \mathcal{L}^{-1}\left[e^{-\sqrt{s/k}x}\right](t) + e^{kt}e^{-x}.$$

We find using a table that

$$\mathcal{L}^{-1}\left[e^{-(x/\sqrt{k})s}\right](t) = \frac{x}{2\sqrt{\pi k t^3}}e^{-x^2/4kt}.$$

Therefore, more explicitly,

$$u(x,t) = -e^{kt} * \frac{x}{2\sqrt{\pi k t^3}}e^{-x^2/4kt} + e^{kt-x}.$$

3. Apply the Laplace transform (in t) to the partial differential equation, using the initial condition, to write

$$sU(x,s) - u(x,0) = kU''(x,s),$$

or, since $u(x,0) = 0$,

$$U''(x,s) - \frac{s}{k}U(x,s) = 0.$$

This has general solution

$$U(x,s) = c_1 e^{\sqrt{s/k}x} + c_2 e^{-\sqrt{s/k}x}.$$

Now $u(0,t) = 0$, so

$$U(0,s) = c_1 + c_2 = 0$$

so $c_2 = -c_1$. Then

$$U(x,s) = c_1\left(e^{\sqrt{s/k}x} - e^{-\sqrt{s/k}x}\right) = c\sinh\left(\sqrt{\frac{s}{k}}x\right).$$

Next, $u(L,t) = T_0$, so $U(L,s) = T_0/s$, so

$$c \sinh\left(\sqrt{\frac{s}{k}}L\right) = \frac{T_0}{s}$$

so

$$U(x,s) = \frac{T_0}{s}\frac{\sinh(\sqrt{s/k}x)}{\sinh(\sqrt{s/k}L)}.$$

The solution $u(x,t)$ is the inverse transform of $U(x,t)$. To compute this inverse, let $\alpha = \sqrt{s/k}$ and write

$$\begin{aligned}\frac{\sinh(\sqrt{s/k}x)}{s\sinh(\sqrt{s/k}L)} &= \frac{e^{\alpha x} - e^{-\alpha x}}{s(e^{\alpha L} - e^{-\alpha L})} \\ &= \frac{e^{\alpha(x-L)} - e^{-\alpha(x+L)}}{s(1 - e^{-2\alpha L})}.\end{aligned}$$

Now essentially duplicate the calculation done in the section (with $\cosh$ in place of $\sinh$) to obtain the solution

$$u(x,t) = T_0 \sum_{n=0}^{\infty}\left(\text{erfc}\left(\frac{(2n+1)L - x}{2\sqrt{kt}}\right) - \text{erfc}\left(\frac{(2n+1)L + x}{2\sqrt{kt}}\right)\right).$$

17.5 Heat Conduction in an Infinite Cylinder

In these problems the solution has the form

$$u(r,t) = \sum_{n=1}^{\infty} a_n J_0(j_n r/R) e^{j_n^2 kt/R^2},$$

where

$$a_n = \frac{2}{J_1(j_n)^2}\int_0^1 \xi f(R\xi) J_0(j_n\xi)\,d\xi,$$

with j_n the nth positive zero of $J_0(x)$.

1. With $R = 1$ and $f(r) = r$, the first five coefficients are approximately

$$a_1 = 0.8175, a_2 = -1.1335, a_3 = 0.7983, a_4 = -0.7470, a_5 = 0.6315.$$

The first five terms of the series solution are approximately

$$\begin{aligned}u(r,t) \approx{}& 0.8175 J_0(2.40483r)e^{-5.7832t} - 1.1335 J_0(5.5201r)e^{-30.5588t} \\ &+ 0.7983 J_0(8.6537r)e^{-74.8791t} - 0.74701 J_0(11.7914r)e^{-139.0402t} \\ &+ 0.6316 J_0(14.9309r)e^{-222.9324t}.\end{aligned}$$

Figure 17.18 shows the sum of these five terms for times $t = 0.001, 0.025, 0.1, 0.3$ and 0.5.

3. With $R = 3$ and $f(r) = 1 - r^2$, the first five coefficients are approximately

$$a_1 = 9.9722, a_2 = -1.2580, a_3 = 0.4093, a_4 = -0.1889, a_5 = 0.1047.$$

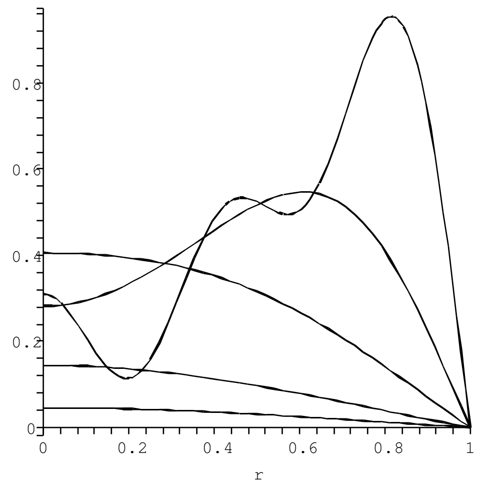

Figure 17.18: Problem 1, Section 17.5.

The first five terms of the solution are approximately

$$\begin{aligned}u(r,t) &\approx 9.9722J_0(0.8016r)e^{-0.3213t} - 1.2580J_0(1.8400r)e^{-1.6929t}\\&+ 0.4093J_0(2.8846r)e^{-4.1604t} - 0.1889J_0(3.9305r)e^{-7.7245t}\\&+ 0.1047J_0(4.9770r)e^{-12.3851t}.\end{aligned}$$

Figure 17.19 shows the sum of these five terms for times $t = 0.001, 0.05, 0.25, 0.5$ and 1.

17.6 Heat Conduction in a Rectangular Plate

1. Attempt a solution of the form $u(x,y,t) = X(x)Y(y)T(t)$. Substitution of this into the heat equation and separation of variables, coupled with the boundary conditions, yields the separated equations:

$$X'' + \lambda X = 0; X(0) = X(L) = 0,$$
$$Y'' + \mu Y = 0; Y(0) = Y(K) = 0,$$
$$T' + k(\lambda + \mu)T = 0.$$

The eigenvalues and eigenfunctions for the problems in X and Y are

$$\lambda_n = \frac{n^2\pi^2}{L^2}, X_n(x) = \sin(n\pi x/L),$$

and

$$\mu_m = \frac{m^2\pi^2}{K^2}, Y_m(y) = \sin(m\pi y/K).$$

The solution has the form of a double superposition

$$u(x,y,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} c_{nm}\sin(n\pi x/L)\sin(m\pi y/K)e^{-k\alpha_{nm}t},$$

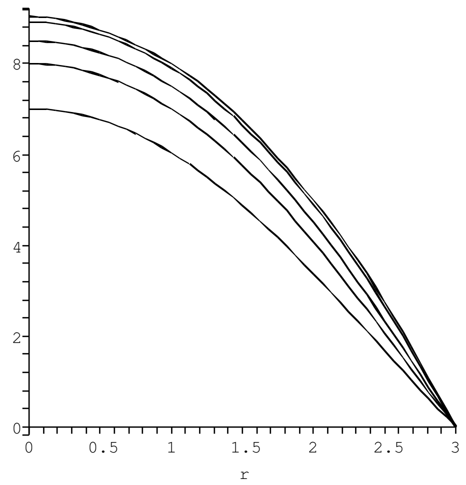

Figure 17.19: Problem 3, Section 17.5.

in which

$$\alpha_{nm} = \frac{n^2\pi^2}{L^2} + \frac{m^2\pi^2}{K^2}.$$

The coefficients must be chosen so that

$$u(x, y, 0) = f(x, y) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} c_{nm} \sin(n\pi x/L)\sin(m\pi y/K).$$

Thus choose

$$c_{nm} = \frac{4}{LK}\int_0^L \int_0^K f(\xi, \eta)\sin(n\pi\xi/L)\sin(m\pi\eta/K)\,d\xi\,d\eta.$$

3. The coefficients in the series solution are

$$c_{nm} = \frac{4}{\pi^2}\int_0^\pi \sin(\xi)\sin(n\xi)\,d\xi \int_0^\pi \cos(\eta/2)\sin(m\eta)\,d\eta.$$

Now

$$\int_0^\pi \sin(\eta)\sin(n\eta)\,d\eta = \begin{cases} 0 & \text{for } n \neq 1, \\ \pi/2 & \text{for } n = 1, \end{cases}$$

then the only nonzero coefficients are

$$c_{1m} = \frac{2}{\pi}\frac{4m}{4m^2 - 1}.$$

The solution is

$$u(x, y, t) = \frac{8}{\pi}\sin(x)\sum_{m=1}^{\infty}\left(\frac{m}{4m^2 - 1}\right)\sin(my)e^{-(1+m^2)t}.$$

Chapter 18

The Potential Equation

18.1 Laplace's Equation

1. If f and g are harmonic on D, then $f_{xx} + f_{yy} = 0$ and $g_{xx} + g_{yy} = 0$ on D. For any numbers α and β,

$$\begin{aligned}&(\alpha f + \beta g)_{xx} + (\alpha f + \beta g)_{yy}\\ &= \alpha(f_{xx} + f_{yy}) + \beta(g_{xx} + g_{yy}) = 0,\end{aligned}$$

so $\alpha f + \beta g$ is harmonic on D.

2. (a)

$$(x^3 - 3xy^2)_{xx} + (x^3 - 3xy^2)_{yy} = 6x - 6x = 0$$

(c)

$$\begin{aligned}&(x^4 - 6x^2y^2 - y^4)_{xx} + (x^4 - 6x^2y^2 + y^4)_{yy}\\ &= (12x^2 - 12y^2) + (-12x^2 + 12y^2) = 0.\end{aligned}$$

(e)

$$\begin{aligned}&(\sin(x)(e^y + e^{-y})_{xx} + (\sin(x)(e^y + e^{-y})_{yy}\\ &= -\sin(x)(e^y + e^{-y}) + \sin(x)(e^y + e^{-y}) = 0\end{aligned}$$

(g)

$$(e^{-x}\cos(y))_{xx} + (e^{-x}\cos(y))_{yy} = e^{-x}\cos(y) - e^{-x}\cos(y) = 0$$

18.2 Dirichlet Problem for a Rectangle

1. Separate the variables and use the homogeneous boundary conditions to derive the general form of the solution:

$$u(x, y) = \sum_{n=1}^{\infty} a_n \frac{\sinh(n\pi y)}{\sinh(4n\pi)} \sin(n\pi x).$$

The coefficients must be chosen so that

$$u(x, 4) = \sum_{n=1}^{\infty} a_n \sin(n\pi x) = x\cos(\pi x/2).$$

This is a Fourier sine expansion of $x\cos(\pi x/2)$ on $[0,1]$, so choose

$$a_n = 2\int_0^1 \xi\cos(\pi\xi/2)\sin(n\pi\xi)\,d\xi = \frac{32n(-1)^{n+1}}{\pi^2(4n^2-1)^2}.$$

3. There are nonhomogeneous boundary conditions on two edges, so write $u(x,y) = v(x,y) + w(x,y)$, where

$$\nabla^2 v = 0; v(0,y) = v(\pi,y) = v(x,0) = 0, v(x,\pi) = x\sin(\pi x),$$

and

$$\nabla^2 w = 0; w(x,0) = w(x,\pi) = w(0,y) = 0, w(2,y) = \sin(y).$$

These are defined on $0 < x < 2$, $0 < y < \pi$. The solution for w has the form

$$w(x,y) = \sum_{n=1}^{\infty} b_n \sin(ny)\frac{\sinh(nx)}{\sinh(2n)}.$$

We need

$$w(2,y) = \sum_{n=1}^{\infty} b_n \sin(ny) = \sin(y)$$

so choose $b_1 = 1$ and all other $b_n = 0$. Then

$$w(x,y) = \sin(y)\frac{\sinh(x)}{\sinh(2)}.$$

We find that v has the form

$$v(x,y) = \sum_{n=1}^{\infty} a_n \sin(n\pi x/2)\frac{\sinh(n\pi y/2)}{\sinh(n\pi^2/2)}.$$

We need

$$v(x,\pi) = x\sin(\pi x) = \sum_{n=1}^{\infty} a_n \sin(n\pi x/2).$$

This is the sine expansion of $x\sin(\pi x)$ on $[0,2]$, so choose

$$\begin{aligned} a_n &= \int_0^2 \xi\sin(\pi\xi)\sin(n\pi\xi/2)\,d\xi \\ &= \begin{cases} \frac{16n}{\pi^2((n^2-4)^2)}((-1)^n - 1) & \text{for } n = 1,3,4,\cdots, \\ 1 & \text{for } n = 2. \end{cases} \end{aligned}$$

Then

$$\begin{aligned} v(x,y) &= \sin(\pi x)\frac{\sinh(\pi y)}{\sinh(\pi^2)} \\ &\quad + \frac{16}{\pi^2}\sum_{n=1,n\neq 2}^{\infty} \frac{n}{(n^2-4)^2}((-1)^n - 1)\sin(n\pi x/2)\frac{\sinh(n\pi y/2)}{\sinh(n\pi^2/2)}. \end{aligned}$$

5. Substitute $u(x, y) = X(x)Y(y)$ into Laplace's equation and use the boundary conditions to obtain

$$X'' + \lambda X = 0; X(0) = X(1) = 0$$

and

$$Y'' - \lambda Y = 0; Y(\pi) = 0.$$

The regular Sturm-Liouville problem for X has been solved in connection with the heat and wave equations. The eigenvalues and eigenfunctions are

$$\lambda_n = n^2\pi^2, X_n(x) = \sin(n\pi x).$$

The problem for Y has solutions that are constant multiples of hyperbolic sines of the form $\sinh(n\pi(\pi - y))$. For each n, we have functions

$$u_n(x, y) = a_n \sin(n\pi x) \sinh(n\pi(\pi - y))$$

that are harmonic and satisfy the homogeneous boundary conditions on the edges $x = 0$, $x = 1$ and $y = \pi$. To satisfy a boundary condition $u(x, 0) = f(x)$, we would normally have to use a superposition

$$u(x, y) = \sum_{n=1}^{\infty} a_n \sin(n\pi x) \sinh(n\pi(\pi - y)).$$

However, in this simple problem in which $u(x, 0) = \sin(\pi x)$, we observe that we can get by with $n = 1$ and choose a_1 so that

$$u(x, 0) = a_1 \sin(\pi x) \sinh(\pi^2) = \sin(\pi x).$$

Thus choose $a_n = 0$ for $n = 2, 3, \cdots$, and

$$a_1 = \frac{1}{\sinh(\pi^2)}$$

to obtain the solution

$$u(x, y) = \frac{1}{\sinh(\pi^2)} \sin(\pi x) \sinh(\pi(\pi - y)).$$

7. Decompose the problem into two problems, in each of which the boundary data is homogeneous on three sides. Let $u(x, y) = v(x, y) + w(x, y)$, where

$$\nabla^2 v = 0; v(x, 0) = v(x, 1) = v(4, y) = 0, v(0, y) = \sin(\pi y)$$

and

$$\nabla^2 w = 0; w(x, 0) = w(x, 1) = w(0, y) = 0, w(4, y) = y(1 - y).$$

These problems are defined on $0 < x < 4$, $0 < y < 1$. The solution for v has the form

$$v(x, y) = \sum_{n=1}^{\infty} \sin(n\pi y) \frac{\sinh(n\pi(4 - x))}{\sinh(4n\pi)}.$$

Then

$$v(0, y) = \sin(\pi y) = \sum_{n=1}^{\infty} a_n \sin(n\pi y),$$

so $a_1 = 1$ and, for $n = 2, 3, \cdots, a_n = 0$. Then

$$v(x, y) = \sin(\pi y)\frac{\sinh(\pi(4 - x))}{\sinh(4\pi)}.$$

The solution for w has the form

$$w(x, y) = \sum_{n=1}^{\infty} b_n \sin(n\pi y)\sinh(n\pi x).$$

Then

$$w(4, y) = y(1 - y) = \sum_{n=1}^{\infty} b_n \sinh(4n\pi)\sin(n\pi y),$$

so

$$\begin{aligned} b_n &= \frac{2}{\sinh(4n\pi)}\int_0^1 \xi(1 - \xi)\sin(n\pi\xi)\,d\xi \\ &= \frac{4(1 - (-1)^n)}{n^3\pi^3\sinh(4n\pi)}. \end{aligned}$$

9. Separation of variables and the homogeneous boundary conditions on the sides $x = 0$, $x = a$ and $y = 0$ give us a solution of the form

$$u(x, y) = \sum_{n=1}^{\infty} c_n \sin((2n - 1)\pi x/2a)\frac{\sinh((2n - 1)\pi y/2a)}{\sinh((2n - 1)\pi b/2a)}.$$

We need

$$u(x, b) = f(x) = \sum_{n1}^{\infty} a_n \sin((2n - 1)\pi x/2a),$$

so choose

$$c_n = \frac{2}{a}\int_0^a f(\xi)\sin((2n - 1)\pi\xi/2a)\,d\xi.$$

18.3 Dirichlet Problem for a Disk

1. Letting $U(r, \theta) = u(r\cos(\theta, r\sin(\theta))$, this Dirichlet problem in polar coordinates is

$$\nabla^2 U(r, \theta) = 0, U(4, \theta) = 16\cos^2(\theta),$$

for $-\pi \le \theta \le \pi$ and $0 \le r < 3$. Write $16\cos^2(\theta) = 8(1 + \cos(2\theta))$ to recognize that

$$\frac{1}{2}a_0 = 8, a_2(4^2) = 8,$$

and all other $a_n = 0$. The solution is

$$U(r, \theta) = 8 + 8\left(\frac{r}{4}\right)^2 \cos(2\theta).$$

Convert this solution back to rectangular coordinates using $x = r\cos(\theta)$ and $y = r\sin(\theta)$ and the identity $\cos(2\theta) = 2\cos^2(\theta) - 1$ to obtain

$$u(x, y) = 8 + \frac{1}{2}(x^2 - y^2).$$

3. In polar coordinates the problem is

$$\nabla^2 U(r,\theta) = 0, U(2,\theta) = 4(\cos^2(\theta) - \sin^2(\theta)) = 4\cos(2\theta),$$

for $0 \le r < 2$ and $-\pi \le \theta \le \pi$. Identify $4 = a_2(2^2)$, with all other coefficients zero, to obtain

$$u(r,\theta) = r^2\cos(2\theta).$$

In rectangular coordinates, the solution is

$$u(x,y) = x^2 - y^2.$$

In each of Problems 5 through 12, the solution has the form

$$u(r,\theta) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(\frac{r}{R}\right)^n (a_n\cos(n\theta) + b_n\sin(n\theta)),$$

where

$$a_n = \frac{1}{\pi R^n}\int_{-\pi}^{\pi} f(\xi)\cos(n\xi)\,d\xi$$

for $n = 0, 1, 2, \cdots$ and

$$b_n = \frac{1}{\pi R^n}\int_{-\pi}^{\pi} f(\xi)\sin(n\xi)\,d\xi$$

for $n = 1, 2, \cdots$.

5. Calculate

$$\begin{aligned}
a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi}(\xi^2 - \xi)\,d\xi = \frac{2\pi^2}{3},\\
a_n &= \frac{1}{2^n\pi}\int_{-\pi}^{\pi}(\xi^2 - \xi)\cos(n\xi)\,d\xi = \frac{4(-1)^n}{n^2 2^n},\\
b_n &= \frac{1}{2^n\pi}\int_{-\pi}^{\pi}(\xi^2 - \xi)\sin(n\xi)\,d\xi = \frac{2(-1)^n}{n2^n}.
\end{aligned}$$

The solution is

$$u(r,\theta) = \frac{\pi^2}{3} + 2\sum_{n=1}^{\infty}\left(\frac{r}{2}\right)^n\frac{(-1)^n}{n^2}(2\cos(n\theta) + n\sin(n\theta)).$$

7. Compute

$$\begin{aligned}
a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} e^{-\xi}\,d\xi = \frac{2\sinh(\pi)}{\pi},\\
a_n &= \frac{1}{4^n\pi}\int_{-\pi}^{\pi} e^{-\xi}\cos(n\xi)\,d\xi = \frac{2\sinh(\pi)}{\pi}\frac{(-1)^n}{4^n(n^2+1)} \text{ for } n \ge 1\\
b_n &= \frac{1}{4^n\pi}\int_{-\pi}^{\pi} e^{-\xi}\sin(n\xi)\,d\xi = \frac{2\sinh(\pi)}{\pi}\frac{n(-1)^n}{4^n(n^2+1)} \text{ for } n \ge 1.
\end{aligned}$$

The solution is

$$u(r,\theta) = \frac{\sinh(\pi)}{\pi} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2+1}\left(\frac{r}{4}\right)^n\sinh(\pi)(\cos(n\theta) + n\sin(n\theta)).$$

9. Compute

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi}(1-\xi^2)\,d\xi = 2 - \frac{2}{3}\pi^2,$$
$$a_n = \frac{1}{8^n\pi}\int_{-\pi}^{\pi}(1-\xi^2)\cos(n\xi)\,d\xi = \frac{4(-1)^{n+1}}{8^n n^2}, n \geq 1,$$
$$b_n = \frac{1}{8^n\pi}\int_{-\pi}^{\pi}(1-\xi^2)\sin(n\xi)\,d\xi = 0, n \geq 1.$$

The solution is

$$u(r,\theta) = 1 - \frac{1}{3}\pi^2 + \sum_{n=1}^{\infty}\frac{4(-1)^{n+1}}{n^2}\left(\frac{r}{8}\right)^n\cos(n\theta).$$

11. With $f(\theta) = 1$ we can easily match coefficients to get $a_0 = 2$ and, for $n = 1, 2, \cdots$, $a_n = b_n = 0$. The solution is $u(r,\theta) = 1$.

18.4 Poisson's Integral Formula

1. With $R = 15$ and $f(\theta) = \theta^3 - \theta$, we obtain

$$u(4,\pi) \approx 0.837758(10)^{-12}, u(12, 3\pi/2) \approx -2.571176, u(8,\pi/4) \approx 0.59705,$$
$$u(7,0) \approx -0.628310(10^{-11}).$$

3. From Poisson's integral formula with $R = 1$ and $f(\theta) = \theta$ we obtain the integral

$$u(r,\theta) = \frac{1-r^2}{2\pi}\int_{-\pi}^{\pi}\frac{\xi}{1+r^2-2r\cos(\xi-\theta)}\,d\xi.$$

The requested numerical values are

$$u(1/2,\pi) \approx 0, u(3/4,\pi/3) \approx 0.882613, u(0.2,\pi/4) \approx 0.2465422.$$

5. First observe that $u(r,\theta) = r^n\sin(n\theta)$ is harmonic on the disk $r \leq 1$, hence is the solution of the Dirichlet problem

$$\nabla^2(r,\theta) = 0 \text{ for } 0 \leq r < 1, -\pi \leq \theta < \pi,$$
$$u(1,\theta) = \sin(n\theta) \text{ for } -\pi \leq \theta < \pi.$$

By the Poisson integral formula, this unique solution must be given by

$$r^n\sin(n\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{1-r^2}{1+r^2-2r\cos(\xi-\theta)}\sin(n\xi)\,d\xi.$$

18.5 Dirichlet Problem for Unbounded Regions

1. Use the integral formula for the solution on the upper half plane to obtain

$$\begin{aligned} u(x,y) &= \frac{y}{\pi}\left[\int_{-4}^{0}\frac{-1}{y^2+(\xi-x)^2}\,d\xi + \int_0^4\frac{1}{y^2+(\xi-x)^2}\,d\xi\right] \\ &= \frac{y}{\pi}\int_0^4\left[\frac{-1}{y^2+(\xi+x)^2} + \frac{1}{y^2+(\xi-x)^2}\right]d\xi \\ &= \frac{1}{\pi}\left[2\arctan\left(\frac{x}{y}\right) - \arctan\left(\frac{4+x}{y}\right) + \arctan\left(\frac{4-x}{y}\right)\right] \end{aligned}$$

for $-\infty < x < \infty, y > 0$.

3. To solve this problem, split it into two problems, in each of which there is a single nonhomogeneous boundary condition on one edge. One of the problems thus formed is exactly Problem 6. For the other, exchange x and y and the names of f and g. Finally, add these two solutions to obtain

$$u(x, y) = \frac{2}{\pi}\int_0^\infty \left(\int_0^\infty f(\xi)\sin(\omega\xi)\,d\xi\right)\sin(\omega x)e^{-\omega y}\,d\omega$$
$$+\frac{2}{\pi}\int_0^\infty \left(\int_0^\infty g(\xi)\sin(\omega\xi)\,d\xi\right)\sin(\omega y)e^{-\omega x}\,d\omega.$$

5. Because of the homogeneous boundary condition along $y = 0$ and the particular function specified along the $x = 0$ edge, we are led to try a Fourier sine transform in y. Let $\hat{u}_S(x, \omega)$ be the Fourier sine transform of $u(x, y)$ in y. The transformed problem is

$$\hat{u}_S'' - \omega^2\hat{u}_S = 0; \hat{u}_S(0, \omega) = \frac{1}{1+\omega^2}.$$

This problem is easily solved to obtain

$$\hat{u}_S(x, \omega) = \frac{e^{-\omega x}}{1+\omega^2}.$$

Invert this to obtain the solution

$$u(x, y) = \frac{2}{\pi}\int_0^\infty \frac{e^{-\omega x}}{1+\omega^2}\sin(\omega y)\,d\omega.$$

6. First separate variables by setting $u(x, y) = X(x)Y(y)$. We obtain

$$X'' - \omega^2 Y = 0, Y'' + \omega^2 Y = 0.$$

Use the condition $u(x, 0) = X(x)Y(0) = 0$ and the condition that $X(x)$ remains bounded as $x \to \infty$ to obtain

$$X(x)Y(y) = B_\omega e^{-\omega x}\sin(\omega y)$$

for each $\omega > 0$. Now attempt a superposition

$$u(x, y) = \int_0^\infty e^{-\omega x}\sin(\omega y)\,d\omega.$$

Now

$$u(x, 0) = g(y) = \int_0^\infty B_\omega \sin(\omega y)\,d\omega.$$

This is the Fourier sine expansion of $g(y)$, hence choose

$$B_\omega = \frac{2}{\pi}\int_0^\infty g(\eta)\sin(\omega\eta)\,d\eta.$$

Given g, this yields an integral formula for the solution $u(x, y)$.

We can also approach this problem using the Fourier sine transform in y. Let the Fourier sine transform of $u(x, y)$ be $\hat{u}_S(x, \omega)$. Transform the differential equation to obtain and boundary condition to obtain

$$\hat{u}_S'' - \omega^2\hat{u}_S = 0; \hat{u}_S(0, \omega) = \hat{g}_S(\omega).$$

We also require that $\hat{u}_S(x,\omega)$ remains bounded as x increases. This problem for the transformed function has solution

$$\hat{u}_S(x,\omega) = \hat{g}_S(\omega)e^{-\omega x}.$$

Invert this to obtain

$$u(x,y) = \frac{2}{\pi}\int_0^\infty \hat{g}_S(\omega)\sin(\omega y)e^{-\omega x}\,d\omega.$$

To see that this is the same solution obtained by separation of variables, replace $\hat{g}_S(\omega)$ by its integral from the definition of the sine transform to obtain

$$u(x,y) = \frac{2}{\pi}\int_0^\infty \left(\int_0^\infty g(\xi)\sin(\xi\omega)\,d\xi\right)\sin(\omega y)e^{-\omega x}\,d\omega.$$

7. From the formula derived for the solution in the right quarter plane, an integral solution of this problem is

$$u(x,y) = \frac{2}{\pi}\int_0^\infty \left[\frac{1}{y^2+(t-x)^2} - \frac{1}{y^2+(t+x)^2}\right] e^{-\xi}\cos(\xi)\,d\xi.$$

9. The solution for the right half plane can be obtained from the integral formula for the upper half plane by interchanging x and y. We obtain

$$\begin{aligned} u(x,y) &= \frac{x}{\pi}\int_{-1}^1 \frac{1}{x^2+(\eta-y)^2}\,d\eta \\ &= \frac{1}{\pi}\left[\arctan\left(\frac{1-y}{x}\right) + \arctan\left(\frac{1+y}{x}\right)\right] \end{aligned}$$

for $x > 0$ and $-\infty < y < \infty$.

11. The solution for the upper half plane is

$$\begin{aligned} u(x,y) &= \frac{y}{\pi}\int_4^8 \frac{A}{y^2+(\xi-x)^2}\,d\xi \\ &= \frac{A}{\pi}\left[\arctan\left(\frac{8-x}{y}\right) - \arctan\left(\frac{4-x}{y}\right)\right]. \end{aligned}$$

18.6 A Dirichlet Problem for a Cube

1. Write the solution as the sum of solutions of two simpler problems:

$$\begin{aligned} &\nabla^2 w = 0, \\ &w(0,y,z) = w(1,y,z) = w(x,0,z) = w(x,2\pi,z) = w(x,y,0) = 0, \\ &w(x,y,\pi) = 1, \end{aligned}$$

and

$$\begin{aligned} &\nabla^2 v = 0, \\ &v(0,y,z) = v(1,y,z) = v(x,y,0) = v(x,y,\pi)v(x,0,z) = 0, \\ &v(x,2\pi,z) = 1. \end{aligned}$$

Each of these problems is solved by a straightforward separation of variables. For the problem in w, we obtain

$$w(x,y,z) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} a_{nm}\sin(n\pi x)\sin(my/2)\sinh(\sqrt{4n^2\pi^2+m^2}z/2),$$

in which

$$\begin{aligned} a_{nm} &= \frac{1}{\sinh(\sqrt{4n^2\pi^2+m^2}\pi/2)}\int_0^1 2\sin(n\pi\xi)\,d\xi\int_0^{2\pi}\frac{1}{\pi}\sin(m\eta/2)\,d\eta \\ &= \frac{4}{\sinh(\sqrt{4n^2\pi^2+m^2}\pi/2)}\left(\frac{1-(-1)^n}{n\pi}\right)\left(\frac{1-(-1)^m}{m\pi}\right). \end{aligned}$$

Next, we obtain

$$v(x,y,z) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} b_{nm}\sin(n\pi x)\sin(mz)\sinh(\sqrt{n^2\pi^2+m^2}y),$$

where

$$\begin{aligned} b_{nm} &= \frac{1}{\sinh(2\sqrt{n^2\pi^2+m^2}\pi)}(2)\int_0^1 2\sin(n\pi\xi)\,d\xi\frac{2}{\pi}\int_0^{\pi}\sin(m\eta)\,d\eta \\ &= \frac{8}{\sinh(2\sqrt{n^2\pi^2+m^2}\pi)}\left(\frac{1-(-1)^n}{n\pi}\right)\left(\frac{1-(-1)^m}{m\pi}\right). \end{aligned}$$

3. Let $u(x,y,z) = X(x)Y(y)Z(z)$ and separate variables. The boundary conditions give us

$$X(0) = X(1) = Y(0) = Y(1) = Z(0) = 0.$$

We obtain solutions of the form

$$u_{nm}(x,y,z) = \sin(n\pi x)\sin(m\pi y)\sinh(\pi\sqrt{n^2+m^2}z).$$

Use a superposition

$$u(x,y,z) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} c_{nm}\sin(n\pi x)\sin(m\pi y)\sinh(\pi\sqrt{n^2+m^2}z).$$

We need

$$u(x,y,1) = xy = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} c_{nm}\sin(n\pi x)\sin(m\pi y)\sinh(\pi\sqrt{n^2+m^2}).$$

As we have done with wave and heat equations in two space variables, choose

$$\begin{aligned} c_{nm} &= \frac{4}{\sinh(\pi\sqrt{n^2+m^2})}\int_0^1 \xi\sin(n\pi\xi)\,d\xi\int_0^1 \eta\sin(m\pi\eta)\,d\eta \\ &= \frac{4(-1)^{n+m}}{nm\pi^2\sinh(\sqrt{n^2+m^2}\pi)}. \end{aligned}$$

The solution is

$$u(x,y,z) = \frac{4}{\pi^2}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{(-1)^{n+m}}{nm\sinh(\sqrt{n^2+m^2}\pi)}\sin(n\pi x)\sin(m\pi y)\sinh(\sqrt{n^2+m^2}\pi z).$$

18.7 Steady-State Equation for a Sphere

In Problems 1 through 4, the solution has the form

$$u(\rho,\varphi)=\sum_{n=0}^{\infty}\frac{2n+1}{2}\left(\int_{-1}^{1}f(\arccos(\xi))P_n(\xi)\,d\xi\right)\left(\frac{\rho}{R}\right)^n P_n(\cos(\varphi)).$$

In the following, we approximate the required integrals for numerical values of the coefficients of the first six terms in this series solution. In some cases, integrals can be seen to be zero by exploiting even-odd properties of Legendre polynomials and possibly of the function f.

1. The first six terms of the approximation are

$$\begin{aligned}u(\rho,\varphi)=&\frac{1}{2}12.15672076-\frac{3}{2}(6.573472)\left(\frac{\rho}{R}\right)\cos(\varphi)\\&+\frac{5}{4}(2.094395)\left(\frac{\rho}{R}\right)^2(3\cos^2(\varphi)-1)\\&-\frac{7}{4}(0.6869585)\left(\frac{\rho}{R}\right)^3(5\cos^3(\varphi)-3\cos(\varphi)\\&+\frac{9}{16}(-.33510322)\left(\frac{\rho}{R}\right)^3(35\cos^4(\varphi)-30\cos^3(\varphi)+3)\\&-\frac{11}{16}(0.17787555)\left(\frac{\rho}{R}\right)^5(63\cos^5(\varphi)-70\cos^3(\varphi)+15\cos(\varphi))+\cdots.\end{aligned}$$

3. For $f(\varphi)=A\varphi^2$, the integrals to be approximated are

$$I_n=\int_{-1}^{1}(\arccos(\xi))^2P_n(\xi)\,d\xi$$

for $n=0,1,\cdots,5$. We will insert A into the series after these integrals are computed. We have

$$I_0\approx 5.86960441, I_1\approx -2.46740110, I_2\approx 0.4444444,$$
$$I_3\approx -1.154212688, I_4\approx 0.07111111, I_5\approx -0.03855314.$$

The first six terms of the approximated series solution are

$$\begin{aligned}u(\rho,\varphi)\approx A&\left[\frac{1}{2}(5.86960441)-\frac{3}{2}(2.46740)\frac{\rho}{R}\cos(\varphi)\right.\\&\frac{5}{4}(0.44444)\left(\frac{\rho}{R}\right)^2(3\cos^2(\varphi)-1)-\frac{7}{4}(0.154212)\left(\frac{\rho}{R}\right)^3(5\cos^3(\varphi)-3\cos(\varphi))\\&+\frac{9}{16}(0.071111)\left(\frac{\rho}{R}\right)^4(35\cos^4(\varphi)-30\cos^3(\varphi)+3)\\&-\frac{11}{16}(0.03855314)\left(\frac{\rho}{R}\right)^5(63\cos^5(\varphi)-70\cos^3(\varphi)+15\cos(\varphi))+\cdots.\end{aligned}$$

5. The problem to be solved is

$$\begin{aligned}\frac{\partial^2u}{\partial\rho^2}+\frac{2}{\rho}\frac{\partial u}{\partial\rho}+\frac{1}{\rho^2}\frac{\partial^2u}{\partial\varphi^2}+\frac{\cot(\varphi)}{\rho^2}\frac{\partial u}{\partial\varphi}&=0,\\u(R_1,\varphi)&=T_1,\\u(R_2,\varphi)&=0.\end{aligned}$$

Here $-\pi/2 \leq \varphi \leq \pi/2$ and $R_1 \leq \rho \leq R_2$. We will solve this problem by separation of variables. Let $u(\rho,\varphi) = F(\rho)\Phi(\varphi)$ to obtain

$$F'' + \frac{2}{\rho}F' - \frac{\lambda}{\rho^2}F = 0$$

and

$$\Phi'' + \cot(\varphi)\Phi' + \lambda\Phi = 0$$

with λ the separation constant. The bounded solution for $\Phi(\varphi)$ on $[-\pi/2, \pi/2]$ is

$$\Phi_n(\varphi) = P_n(\cos(\varphi))$$

and this is the nth eigenfunction corresponding to the eigenvalue $\lambda_n = n(n+1)$ of Legendre's differential equation. Solutions for $F(\rho$ for $n = 0, 1, 2, \cdots$ have the form

$$F_n(\rho) = a_n\rho^n + b_n\rho^{-n-1}.$$

Attempt a superposition

$$u(\rho,\varphi) = \sum_{n=0}^{\infty}(a_n\rho^n + b_n\rho^{-n-1})P_n(\cos(\varphi)).$$

The condition specified at $\rho = R_1$ requires that

$$u(R_1,\varphi) = T_1 = \sum_{n=0}^{\infty}(a_nR_1^n + b_nR_1^{-n-1})P_n(\cos(\varphi)).$$

The condition at $\rho = R_2$ requires that

$$u(R_2,\varphi) = 0 = \sum_{n=0}^{\infty}(a_nR_2^n + b_nR_2^{-n-1})P_n\cos(\varphi).$$

From the orthogonality of the Legendre polynomials $P_n(x)$ on $[-1, 1]$, we conclude that

$$a_0 + \frac{1}{R_1}b_0 = T_1, a_0 + \frac{1}{R_2}b_0 = 0,$$

and, for $n = 1, 2, \cdots$,

$$a_nR_1^n + \frac{1}{R_1^{n+1}}b_n = 0, a_nR_2^n + \frac{1}{R_2^{n+1}}b_n = 0.$$

Solve these equations for the coefficients to obtain

$$a_0 = \frac{T_1R_1}{R_1 - R_2}, b_0 = -\frac{T_1R_1R_2}{R_1 - R_2},$$

and

$$a_n = b_n = 0 \text{ for } n = 1, 2, \cdots.$$

The solution is

$$u(\rho,\varphi) = \frac{T_1R_1}{R_1 - R_2}\left[\frac{R_2}{\rho} - 1\right].$$

18.8 The Neumann Problem

1. A solution may exist because $\int_0^\pi \cos(3x)\,dx = 0$. From the zero boundary conditions on edges $x = 0$ and $x = \pi$, separation of variables will yield a solution of the form

$$u(x,y) = c_0 + \sum_{n=1}^{\infty}[c_n\cosh(ny) + d_n\cosh(n(\pi - y))]\cos(nx).$$

Now

$$\frac{\partial u}{\partial y}(x,0) = \cos(3x) = \sum_{n=1}^{\infty} -nd_n\sinh(n\pi)\cos(nx),$$

so

$$d_3 = -\frac{1}{3\sinh(3\pi)} \text{ and } d_n = 0 \text{ for } n \neq 3.$$

The boundary condition at $y = \pi$ gives us

$$\frac{\partial u}{\partial y}(x,\pi) = 6x - 3\pi = \sum_{n=1}^{\infty} nc_n\sinh(n\pi)\cos(nx),$$

so

$$\begin{aligned} c_n &= \frac{1}{n\sinh(n\pi)}\frac{2}{\pi}\int_0^\pi (6x - 3\pi)\cos(nx)\,dx \\ &= \frac{1}{n\sinh(n\pi)}\frac{12}{n^2\pi}((-1)^n - 1) \end{aligned}$$

for $n = 1, 2, 3, \cdots$. The solution is

$$\begin{aligned} u(x,y) = c_0 &- \frac{\cosh(3(\pi - y))}{3\sinh(3\pi)}\cos(3x) \\ &+ \sum_{n=1}^{\infty}\frac{12(-1)^n - 1)}{n^3\pi\sinh(n\pi)}\cos(ny)\cos(nx). \end{aligned}$$

3. First,

$$\int_0^1 4\cos(\pi x)\,dx = 0,$$

so a solution may exist. (If this integral had been nonzero, there could not have been a solution). From the boundary conditions on the opposite edges $x = 0$ and $x = 1$, we find by separation of variables that there will be a solution of the form

$$u(x,y) = c_0 + \sum_{n=1}^{\infty}[c_n\cosh(n\pi y) + d_n\cosh(n\pi(1 - y))]\cos(n\pi x).$$

The boundary condition at $y = 1$ becomes

$$\frac{\partial u}{\partial y}(x,1) = 0 = \sum_{n=1}^{\infty} n\pi c_n\sinh(n\pi)\cos(n\pi x).$$

Therefore $c_n = 0$ for $n \geq 1$. On the edge $y = 0$,

$$\frac{\partial u}{\partial y}(x,0) = 4\cos(n\pi x) = \sum_{n=1}^{\infty} -n\pi d_n\sinh(n\pi)\cos(n\pi x).$$

Then $d_n = 0$ if $n \geq 2$, and

$$d_1 = -\frac{4}{\pi \sinh(\pi)}.$$

A solution is given by

$$u(x,y) = c_0 - \frac{4}{\pi \sinh(\pi)} \cosh(\pi(1-y)) \cos(\pi x).$$

Since c_0 is arbitrary, this solution is not unique.

5. We will apply a Fourier cosine transform (with respect to x) to this problem. Let

$$U_C(\omega, y) = \mathcal{F}_C[u(x,y)](\omega, y).$$

Using the operational formula for the cosine transform, we obtain

$$-\omega^2 U_C - \frac{\partial u}{\partial x}(0,y) + U_C'' = 0,$$

in which primes denote differentiation with respect to y. Since $u_x(0,y) = 0$, we obtain

$$U_C'' - \omega^2 U_C = 0,$$

with the general solution

$$U_C(\omega, y) = a_\omega e^{\omega y} + b_\omega e^{-\omega y}.$$

To have bounded solutions for $y > 0$, choose each $a_\omega = 0$, so

$$U_C(\omega, y) = b_\omega e^{-\omega y}.$$

Now invert the cosine transform, obtaining

$$u(x,y) = \int_0^\infty a_\omega \cos(\omega x) e^{-\omega y} \, d\omega.$$

From this, calculate

$$\frac{\partial u}{\partial y}(x,0) = \int_0^\infty -\omega a_\omega \cos(\omega x) \, d\omega = f(x)$$

to complete the solution by setting

$$a_\omega = -\frac{2}{\pi \omega} \int_0^\infty f(\xi) \cos(\omega \xi) \, d\xi.$$

7. With $u(x,y) = X(x)Y(y)$ we obtain

$$X'' - \lambda X = 0; Y'' + \lambda Y = 0, Y(0) = Y(1) = 0.$$

Then

$$Y_n(y) = \sin(n\pi y), X_n(x) = c_n \cosh(n\pi x) + d_n \cosh(n\pi(x-1)).$$

A solution will have the form

$$u(x,y) = \sum_{n=1}^{\infty} [c_n \cosh(n\pi x) + d_n \cosh(n\pi(1-x))] \sin(n\pi y).$$

The boundary conditions at $x = 1$ and $x = 0$ give us, respectively,

$$\frac{\partial u}{\partial x}(1, y) = 0 = \sum_{n=1}^{\infty} n\pi c_n \sinh(n\pi)\sin(n\pi y)$$

and

$$\frac{\partial u}{\partial x}(0, y) = 3y^2 - 2y = \sum_{n=1}^{\infty} -n\pi \sinh(n\pi)\sin(n\pi y).$$

Then each $c_n = 0$ and

$$\begin{aligned} d_n &= \frac{-2}{n\pi\sinh(n\pi)}\int_0^1 (3y^2 - 2y)\sin(n\pi y)\,dy \\ &= \frac{2}{n^4\pi^4\sinh(n\pi)}[n^2\pi^2(-1)^n + 6(1-(-1)^n)] \end{aligned}$$

for $n = 1, 2, \cdots$. This yields the solution

$$u(x, y) = \sum_{n=1}^{\infty} \frac{2}{n^4\pi^4\sinh(n\pi)}[n^2\pi^2(-1)^n + 6(1-(-1)^n)]\cosh(n\pi(1-x))\sin(n\pi y).$$

9. Check that $\int_{-\pi}^{\pi}\cos(2\theta)\,d\theta = 0$, so there may be a solution. Any such solution must have the form

$$r(r, \theta) = \frac{1}{2}a_0 + \sum_{=1}^{\infty}[a_n r^n \cos(n\theta) + b_n r^n \sin(n\theta)].$$

From the boundary condition on $r = R$ we have

$$\begin{aligned} \frac{\partial u}{\partial r}(R, \theta) &= \cos(2\theta) \\ &= \sum_{n=1}^{\infty}[na_n R^{n-1}\cos(n\theta) + nb_n R^{n-1}\sin(n\theta)]. \end{aligned}$$

As in the preceding problem, observe that we can choose each $b_n = 0$, $a_n = 0$ for $n \neq 2$, and $2a_2 R = 1$. The solution is

$$u(r, \theta) = \frac{1}{2}a_0 + \frac{R}{2}\left(\frac{r}{R}\right)^2\cos(2\theta).$$

11. Since $\int_{-\infty}^{\infty} e^{-|x|}\sin(x)\,dx = 0$, the necessary condition for a solution to exist is satisfied. Write the solution

$$u(x, y) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\ln(y^2 + (\xi - y)^2)e^{-|\xi|}\sin(\xi)\,d\xi.$$

Chapter 19

Complex Numbers and Functions

19.1 Geometry and Arithmetic of Complex Numbers

1. $|5-2i| = \sqrt{29}$ and an argument of $5-2i$ is $-\arctan(2/5)$, so the polar form of $5-2i$ is

$$5 - 2i = \sqrt{29}e^{-\arctan(2/5)i}.$$

3. Since $|-2+2i| = 2\sqrt{2}$ and $3\pi/4$ is an argument of $2+2i$, the polar form is

$$z = 2\sqrt{2}e^{3\pi i/4}.$$

Here there is no point in adding $2n\pi$ to the argument to obtain all arguments, since $e^{2n\pi i} = 1$ for any integer n.

5. $|8+i| = \sqrt{65}$ and an argument of $8+i$ is $\arctan(1/8)$, so the polar form is

$$8 + i = \sqrt{65}e^{\arctan(1/8)i}.$$

In Problems 7, 9, and 11, n denotes an arbitrary integer.

7.

$$|3i| = 3, \arg(3i) = \frac{\pi}{2} + 2n\pi$$

9.

$$|-4| = 4 \text{ and } \arg(-4) = (2n+1)\pi$$

11.

$$|-3+2i| = \sqrt{13} \text{ and } \arg(-3+2i) = -\arctan(2/3) + (2n+1)\pi$$

13.

$$(3-4i)(6+2i) = (18+8) + (-24+6)i = 26 - 18i$$

15.

$$\frac{2+i}{4-7i} = \frac{(2+i)(4+7i)}{(4-7i)(4+7i)} = \frac{1}{65}(1+18i)$$

17.

$$(17-6i)\overline{-4-12i} = (17-6i)(-4+12i) = 4 + 228i$$

19.

$$\left(\frac{-6+2i}{1-8i}\right)^2 = \left[\frac{(-6+2i)(1+8i)}{(1-8i)(1+8i)}\right]^2$$
$$= \frac{(-22-46i)^2}{65^2} = \frac{-1632+2024i}{4225}$$

21.

$$i^3 - 4i^2 + 2 = -i + 4 + 2 = 6 - i$$

23. M consists of all $x+iy$ with $y < 7$. This is the half-plane lying below the horizontal line $y = 7$. M is open and the boundary points are all complex numbers $x + 7i$ on the "edge" of M. None of the boundary points of M belong to M.

25. W consists of all points $x + iy$ with $x > y^2$. These are points "enclosed by" the parabola $x = y^2$. Boundary points are the points on this parabola, which are complex numbers $y^2 + iy$. Since W contains no boundary points, W is open. Since W does not contain all of its boundary points, W is not closed.

27. U consists of all points $x+iy$ with $1 < x \leq 3$. This is the vertical strip between the lines $x = 1$ and $x = 3$, including points on the line $x = 3$, but not those on $x = 1$. The boundary points are the points on these vertical lines. these are all points $1 + iy$ and $3 + iy$. Only the boundary points $3 + iy$ are in U. The boundary points $1 + iy$ are not in U. U is not open, since it contains some boundary points. U is not closed, because U does not contain all of its boundary points.

29. Since $i^2 = -1$, we have

$$\begin{aligned} i^{4n} &= (i^2)^{2n} = ((-1)^2)^n = 1, \\ i^{4n+1} &= i^{4n} i = i, \\ i^{4n+2} &= i^{4n} i^2 = i^2 = -1, \\ i^{4n+3} &= i^{4n} i^3 = i^{4n} i^2 i = -i. \end{aligned}$$

31. Suppose first that z, w, u form the vertices of a triangle, labeled in the clockwise order around the triangle. The sides of this triangle are vectors represented by the complex numbers $w - z$, $u - w$ and $z - u$. This triangle is equilateral if and only if

$$|w - z| = |u - w| = |z - u|$$

and each of the vector sides can be rotated by $\theta = 2\pi/3$ radians clockwise to align with the next side. This occurs exactly when

$$(u - w) = (w - z)e^{-2\pi i/3} \text{ and } (z - u) = (u - w)e^{-2\pi i/3}.$$

Dividing these equations gives us

$$\frac{u - w}{z - u} = \frac{w - z}{u - w}.$$

Then

$$(u - w)(u - w) = (w - z)(z - u),$$

or, equivalently,

$$w^2 - 2wu + u^2 = zw + uz - uv - z^2.$$

Finally, rearrange this equation to obtain

$$z^2 + w^2 + u^2 = zw + zu + wu.$$

33. Suppose first that $|z| = 1$. Then
$$|\overline{z}| = |z\overline{z}| = 1.$$
Then
$$\left|\frac{z-w}{1-\overline{z}w}\right| = \left|\frac{z-w}{z\overline{z}-\overline{z}w}\right| = \frac{|z-w|}{|\overline{z}||z-w|} = \frac{1}{|\overline{z}|} = 1.$$
If $|w| = 1$, argue as follows:
$$\left|\frac{z-w}{1-\overline{z}w}\right| = \left|\frac{z-w}{\overline{w}w-\overline{z}w}\right| = \frac{1}{w}\left|\frac{z-w}{\overline{w}-\overline{z}}\right| = 1.$$
because
$$|z-w| = |\overline{w-z}| = |\overline{w}-\overline{z}|.$$

34. **Hint** Use the fact that
$$|w|^2 = w\overline{w}$$
for any complex number w.

19.2 Complex Functions

1.
$$f(z) = -4z + \frac{1}{z} = -4x - 4iy + \frac{1}{x+iy} = -4x - 4yi + \frac{x-iy}{x^2+y^2}$$
for $z \neq 0$. Then
$$u(x,y) = -4x + \frac{x}{x^2+y^2}, v(x,y) = -4y - \frac{y}{x^2+y^2}.$$
Compute
$$\frac{\partial u}{\partial x} = -4 + \frac{y^2-x^2}{(x^2+y^2)^2}, \frac{\partial u}{\partial y} = \frac{-2xy}{(x2+y^2)^2},$$
$$\frac{\partial v}{\partial x} = \frac{2xy}{(x^2+y^2)^2}, \frac{\partial v}{\partial y} = -4 + \frac{y^2-x^2}{(x^2+y^2)^2}.$$
The Cauchy-Riemann equations hold for all $z \neq 0$, so f is differentiable at all z at which it is defined ($z \neq 0$).

3. Write
$$f(z) = z - i = x + iy - i = x + (y-1)i$$
so
$$u(x,y) = x, v(x,y) = y - 1.$$
The Cauchy-Riemann equations are satisfied because
$$\frac{\partial u}{\partial x} = 1 = \frac{\partial v}{\partial y}$$
and
$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = 0.$$
Since u, v and their first partial derivatives are continuous for all $x + iy$, f is differentiable for all z.

5.
$$f(z) = \frac{x+iy}{x} = 1 + \frac{y}{x}i,$$
so
$$u(x,y) = 1, v(x,y) = \frac{y}{x}$$
for $x \neq 0$. Then
$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = 0, \frac{\partial v}{\partial x} = -y/x^2, \frac{\partial v}{\partial y} = 1/x.$$
The Cauchy-Riemann equations do not hold at any point at which the function is defined, so f is not differentiable anywhere.

7.
$$f(z) = i|z|^2 = (x^2+y^2)i$$
so
$$u(x,y) = 0, v(x,y) = x^2 + y^2.$$
Compute
$$\begin{aligned}\frac{\partial u}{\partial x} &= \frac{\partial u}{\partial y} = 0,\\ \frac{\partial v}{\partial x} &= 2x, \frac{\partial v}{\partial y} = 2y.\end{aligned}$$
The Cauchy-Riemann equations hold only at $z = 0$, so f is certainly not differentiable if $z \neq 0$. To determine if f is differentiable at 0, consider
$$\lim_{h\to 0} \frac{f(h)-f(0)}{h} = \lim_{h\to 0} \frac{i|h|^2}{h} = \lim_{h\to 0} i\left(\frac{|h|}{h}\right)|h| = 0$$
because $||h|/h| = 1$. Therefore $f'(0) = 0$.

9. $f(z) = |x+iy| = \sqrt{x^2+y^2}$, so
$$u(x,y) = \sqrt{x^2+y^2}, v(x,y) = 0.$$
If x and y are not both zero, then the partial derivatives are
$$\begin{aligned}\frac{\partial u}{\partial x} &= \frac{x}{\sqrt{x^2+y^2}}, \frac{\partial u}{\partial y} = \frac{y}{\sqrt{x^2+y^2}},\\ \frac{\partial v}{\partial x} &= \frac{\partial v}{\partial y} = 0.\end{aligned}$$
The Cauchy-Riemann equations are not satisfied at any point with both $x \neq 0$ and $y \neq 0$. The only point left to check is $z = 0$, where $x = y = 0$. Now the above expressions for the partial derivatives of v are still valid, but those for the partial derivatives of u are not, and we must fall back on the definition of the partial derivatives:
$$\begin{aligned}\frac{\partial u}{\partial x}(0,0) &= \lim_{h\to 0} \frac{u(h,0)-u(0,0)}{h}\\ &= \lim_{h\to 0} \frac{\sqrt{h^2}}{h}\\ &= \lim_{h\to 0} \frac{|h|}{h}.\end{aligned}$$

This limit does not exist, because

$$\frac{|h|}{h} = \begin{cases} 1 & \text{if } h > 0, \\ -1 & \text{if } h < 0. \end{cases}$$

Similarly, $(\partial u/\partial y)(0, 0)$ does not exist. Thus the Cauchy-Riemann equations fail at every z, and this function is not differentiable anywhere.

11.
$$f(z) = (\overline{z})^2 = (x - iy)^2 = x^2 - y^2 - 2xyi.$$

Then
$$u(x, y) = x^2 - y^2, v(x, y) = -2xy.$$

Then
$$\frac{\partial u}{\partial x} = 2x, \frac{\partial u}{\partial y} = -2y,$$
$$\frac{\partial v}{\partial x} = -2y, \frac{\partial v}{\partial y} = -2x.$$

The Cauchy-Riemann equations hold only at $z = 0$. But

$$\begin{aligned} \lim_{h\to 0} \frac{f(h) - f(0)}{h} &= \lim_{h\to 0} \frac{(\overline{h})^2}{h} \\ &= \lim_{h\to 0} \left(\frac{\overline{h}}{h}\right) \overline{h} \\ &= 0 \end{aligned}$$

because $\overline{h}/h$ has magnitude 1, and $\overline{h} \to 0$ as $h \to 0$. Therefore $f'(0) = 0$.

13. Let $z_n = x_n + iy_n$ and $z_0 = x_0 + iy_0$. Write $f(z) = u(x, y) + iv(x, y)$. Since u and v are continuous at (x_0, y_0), then

$$f(z_n) = u(x_n, y_n) + uv(x_n, y_n) \to x(x_0, y_0) + iv(x_0, y_0) = f(z_0).$$

19.3 The Exponential and Trigonometric Functions

1. First,

$$\begin{aligned} e^{z^2} &= e^{(x+iy)^2} = e^{x^2-y^2+2ixy} \\ &= e^{x^2-y^2}[\cos(2xy) + i\sin(2xy)]. \end{aligned}$$

Then
$$u(x, y) = e^{x^2-y^2} \cos(2xy) \text{ and } v(x, y) = e^{x^2-y^2} \sin(2xy).$$

Compute

$$\frac{\partial u}{\partial x} = e^{x^2-y^2}[2x\cos(2xy) - 2y\sin(2xy)],$$
$$\frac{\partial u}{\partial y} = e^{x^2-y^2}[-2y\cos(2xy) - 2x\sin(2xy)],$$
$$\frac{\partial v}{\partial x} = e^{x^2-y^2}[2x\sin(2xy) + 2y\cos(2xy)],$$
$$\frac{\partial v}{\partial y} = e^{x^2-y^2}[-2y\sin(2xy) + 2x\cos(2xy)].$$

The Cauchy-Riemann equations are satisfied for all $x + iy$.

3.

$$\begin{aligned} f(z) = ze^z &= (x+iy)e^x(\cos(y) + i\sin(y)) \\ &= xe^x\cos(y) - ye^x\sin(y) + i(ye^x\cos(y) + xe^x\sin(y)), \end{aligned}$$

so

$$u(x,y) = xe^x\cos(y) - ye^x\sin(y), v(x,y) = ye^x\cos(y) + xe^x\sin(y).$$

Then

$$\frac{\partial u}{\partial x} = e^x(\cos(y) + x\cos(y) - y\sin(y)) = \frac{\partial v}{\partial y}$$

and

$$\frac{\partial u}{\partial y} = e^x(-x\sin(y) - \sin(y) - y\cos(y)) = -\frac{\partial v}{\partial x}.$$

The Cauchy-Riemann equations hold for all z.

5. If $e^z = e^{x+iy} = 2i$, then

$$e^x[\cos(y) + i\sin(y)] = 2i.$$

Equating real and imaginary parts, we obtain

$$e^x\cos(y) = 0, e^x\sin(y) = 2.$$

Since $e^x \neq 0$ for all real x, then $\cos(y) = 0$, so

$$y = \frac{2n+1}{2}\pi$$

in which n can be any integer. This means that we need

$$e^x\sin\left(\frac{2n+1}{2}\pi\right) = 2.$$

Now $e^x > 0$ for all real x, so we must have

$$\sin\left(\frac{2n+1}{2}\pi\right) > 0.$$

But the quantity on the left is equal to 1 if n is even, and to -1 if n is odd. This means that n must be an even integer, say $n = 2m$, with m any integer. Therefore

$$y = \frac{4m+1}{2}\pi.$$

Then $\sin(y) = 1$ and we are left with $e^x = 2$, so $x = \ln(2)$. All the solutions of $e^z = 2i$ are

$$\ln(2) + \frac{4m+1}{2}\pi i,$$

with m any integer.

7. It is convenient to use the polar form of the given equation:

$$e^z = e^r e^{i\theta} = -2.$$

Since $|e^{i\theta)}| = 1$ if θ is real, then

$$|e^z| = e^r = |-2| = 2,$$

so $r = \ln(2)$. Further, we must have

$$e^{i\theta} = -1 = \cos(\theta) + i\sin(\theta),$$

so $\sin(\theta) = 0$ and $\cos(\theta) = -1$. Then $\theta = (2n+1)\pi$, for any integer n. The solutions are therefore

$$z = \ln(2) + (2n+1)\pi, \text{ with } n \text{ any integer.}$$

8. **Hint** We want all solutions of $\sin(z) = i$. Write this equation as

$$\sin(x+iy) = \sin(x)\cosh(y) + i\cos(x)\sinh(y) = i.$$

Now consider all possible simultaneous solutions of the equations

$$\sin(x)\cosh(y) = 0, \cos(x)\sinh(y) = 1.$$

9.

$$e^i = e^{0+i} = e^0(\cos(1) + i\sin(1)) = \cos(1) + i\sin(1).$$

10. There are several ways we can proceed. Perhaps the easiest is to use the fact that

$$\sin(x+iy) = \sin(x)\cosh(y) + i\cos(x)\sinh(y).$$

Then

$$\sin(1-4i) = \sin(1)\cosh(4) - i\cos(1)\sinh(4).$$

Here we have also used the fact that $\cosh(-4) = \cosh(4)$ and $\sinh(-4) = -\sinh(4)$.

Another approach is to begin with the definition of $\sin(z)$ and use Euler's formula:

$$\begin{aligned}
\sin(1-4i) &= \frac{1}{2i}\left(e^{i(1-4i)} - e^{-i(1-4i)}\right) \\
&= \frac{1}{2i}\left(e^{4+i} - e^{-4-i}\right) \\
&= \frac{1}{2i}\left(e^4 e^i - e^{-4}e^{-i}\right) \\
&= \frac{1}{2i}\left(e^4(\cos(1) + i\sin(1)) - e^{-4}(\cos(1) - i\sin(1))\right) \\
&= \frac{1}{2i}(e^4 - e^{-4})\cos(1) + \frac{1}{2}(e^4 + e^{-4})\sin(1) \\
&= \cosh(4)\sin(1) + \frac{1}{i}\sinh(4)\cos(1) \\
&= \sin(1)\cosh(4) - i\cos(1)\sinh(4).
\end{aligned}$$

11.

$$e^{5+2i} = e^5(\cos(2) + i\sin(2))$$

13.

$$e^{\pi i/2} = \cos(\pi/2) + i\sin(\pi/2) = i$$

15.

$$\begin{aligned}
\sin^2(1+i) &= \frac{1}{2}[1 - \cos(2(1+i))] \\
&= \frac{1}{2}[1 - \cos(2)\cosh(2) + i\sin(2)\sinh(2)] \\
&= \frac{1}{2}[1 - \cos(2)\cosh(2)] + \frac{i}{2}[\sin(2)\sinh(2)].
\end{aligned}$$

17. Use the fact that
$$\cos(x+iy) = \cos(x)\cosh(y) - i\sin(x)\sinh(y)$$
to write
$$\cos(3+2i) = \cos(3)\cosh(2) - i\sin(3)\sinh(2).$$

19.4 The Complex Logarithm

In these problems, $\ln(x)$ denotes the real natural logarithm of x, if x is a positive number. The complex logarithm of z is denoted $\log(z)$.

1. Write
$$-9+2i = \sqrt{85}e^{(\arctan(-2/9)+\pi)i}$$
to obtain
$$\log(-9+2i) = \frac{1}{2}\ln(85) + (-\arctan(2/9) + (2n+1)\pi)i.$$

3. In polar form,
$$z = -4i = 4e^{3n\pi i/2}.$$
Then
$$\log(-4i) = \ln(4) + \left(\frac{3\pi}{2} + 2n\pi\right)i.$$

5. Since $-5 = 5e^{\pi i}$, then
$$\log(-5) = \ln(5) + (2n+1)\pi i.$$

7. Because the complex logarithm of a nonzero number has infinitely many values, we cannot expect to have $\log(zw)$ equal to $\log(z)+\log(w)$. What we claim is that each value of $\log(zw)$ is equal to some value of $\log(z)$ added to some value of $\log(w)$.

   To verify this, let z and w be nonzero numbers. Let θ_z be any argument of z and θ_w any argument of w. Then
$$z = |z|e^{(\theta_1+2n\pi)i}, w = |w|e^{(\theta_2+2m\pi)i}$$
and
$$zw = |zw|e^{(\theta_1+\theta_2+2k\pi)i}.$$
Thus
$$\log(zw) = \ln(|zw|) + (\theta_1+\theta_2+2k\pi)i,$$
while
$$\log(z)+\log(w) = ln(|z|) + \ln(|w|) + i(\theta_1+\theta_2+2(n+m)\pi)i.$$
This means that, for any choice of n and m, we can choose $k = n+m$ to obtain a value of $\log(zw)$ that is equal to $\log(z)+\log(w)$.

19.5 Powers

In these problems, n always denotes an arbitrary integer.

1.

$$\begin{aligned}(-4)^{2-i} &= e^{(2-i)\log(-4)} = e^{(2-i)(\ln(4)+i(\pi+2n\pi))} \\ &= e^{2\ln(4)+\pi+2n\pi}e^{i(2\pi+4n\pi-\ln(4))} \\ &= 16e^{(2n+1)\pi}[\cos(\ln(4)) - i\sin(\ln(4))]\end{aligned}$$

3.

$$(-16)^{1/4} = \left(16e^{i(\pi+2n\pi)}\right)^{1/4} = 2e^{i(\pi/4+n\pi/2)},$$

for $n = 0, 1, 2, 3$. These values are

$$\sqrt{2}(1+i), \sqrt{2}(-1+i), \sqrt{2}(-1-i), \sqrt{2}(1-i).$$

5. These are the sixth roots of unity:

$$1^{1/6} = \left(e^{2n\pi i}\right)^{1/6} = e^{n\pi i/3}$$

for $n = 0, 1, 2, 3, 4, 5$. These values are

$$1, \frac{1}{2}(1+\sqrt{3}i), \frac{1}{2}(-1+\sqrt{3}i), -1, \frac{1}{2}(-1-\sqrt{3}i), \frac{1}{2}(1-\sqrt{3}i).$$

7.

$$\begin{aligned}i^{1+i} &= e^{(1+i)\log(i)} = e^{(1+i)((\pi/2+2n\pi)i)} \\ &= e^{-(\pi/2+2n\pi)}\left[\cos\left(\frac{\pi}{2}+2n\pi\right) + i\sin\left(\frac{\pi}{2}+2n\pi\right)\right] \\ &= ie^{-(\pi/2+2n\pi)},\end{aligned}$$

since $\sin(2n\pi + \pi/2) = 1$ for each integer n

9.

$$i^i = e^{i\log(i)} = e^{i(i(\pi/2+2n\pi))} = e^{-(\pi/2+2n\pi)}$$

This is consistent with Problem 7, since $i^{1+i} = i(i^i)$.

11.

$$\begin{aligned}(-1+i)^{-3i} &= e^{-3i\log(-1+i)} \\ &= e^{-3i(\ln(\sqrt{2})+i(3\pi/4+2n\pi))} \\ &= e^{(9\pi/4+6n\pi)}[\cos(3\ln(\sqrt{2})) - i\sin(3\ln(\sqrt{2}))]\end{aligned}$$

13.

$$\begin{aligned}i^{1/4} &= \left(e^{i(\pi/2+2n\pi)}\right)^{1/4} \\ &= e^{i(\pi/8+n\pi/2)}\end{aligned}$$

We obtain distinct values only for $n = 0, 1, 2, 3$. Other choices of n repeat these values.

15. Let ω be any nth root of unity different from 1. The numbers ω^j, for $j = 0, 1, \cdots, n-1$, are distinct, hence are all the nth roots of unity. Thus it is enough to show that

$$\sum_{j=0}^{n-1} \omega^j = 0.$$

But,
$$\sum_{j=0}^{n-1} \omega^j = \frac{1-\omega^n}{1-\omega} = 0.$$

We could also reason as follows. Let $\omega_1, \cdots, \omega_n$ be the nth roots of unity. Suppose $\omega_1 \neq 1$ and let $S = \sum_{j=1}^{n} \omega_j$. Then
$$\omega_1 S = \sum_{j=1}^{n} \omega_1 \omega_j = S$$
because the n numbers $\omega_1\omega_1, \cdots, \omega_1\omega_n$ are also the nth roots of unity. But then
$$S(1-\omega_1) = 0.$$
Since $\omega_1 \neq 1$, then $S = 0$.

Chapter 20

Complex Integration

20.1 The Integral of a Complex Function

1. Since f has no antiderivative, write the curve as $\gamma(t) = (1+i)t - i$ for $0 \le t \le 1$. Then

$$\int_\gamma |z|^2\,dz = \int_0^1 [t^2 + (t-1)^2](1+i)\,dt = \frac{2}{3}(1+i)$$

3. $i\overline{z}$ has no antiderivative, so proceed by parametrizing γ. One way is to write $\gamma(t) = (-4+3i)t$, $0 \le t \le 1$. Then

$$\begin{aligned}\int_\gamma i\overline{z}\,dz &= \int_0^1 i(-4t - 3ti)(-4+3i)\,dt \\ &= (-4+3i)\left(\frac{3}{2} - 2i\right) = \frac{25}{2}i\end{aligned}$$

5. An antiderivative is $F(z) = (z-i)^4/4$, so

$$\int_\gamma (z-i)^3\,dz = F(2-4i) - F(0) = 10 + 210i$$

7. An antiderivative is $F(z) = -i\sin(z)$, so

$$\begin{aligned}&\int_\gamma -i\cos(z)\,dz = -\sin(z)]_0^{2+i} \\ &= -i\sin(2+i) = -i[\sin(2)\cosh(1) + i\cos(2)\sinh(1)] \\ &\qquad = -\cos(2)\sinh(1) - i\sin(2)\cosh(1).\end{aligned}$$

9. An antiderivative is $F(z) = -\frac{1}{2}\cos(2z)$, so

$$\begin{aligned}&\int_\gamma \sin(2z)\,dz = -\frac{1}{2}\cos(2z)\Big]_{-i}^{-4i} \\ &= -\frac{1}{2}(\cos(-4i) - \cos(-i)) = -\frac{1}{2}[\cosh(8) - \cosh(2)].\end{aligned}$$

11. An antiderivative is $F(z) = (z-1)^2/2$, so

$$\int_\gamma f(z)\,dz = F(1-4i) - F(2i) = -\frac{13}{2} + 2i.$$

13. $f(z) = \text{Re}(z)$ does not have an antiderivative because f is not differentiable, so we must proceed by parametrizing C. This can be done in many ways, but one is to write $\gamma(t) = 1 + (1+i)t$ for $0 \le t \le 1$. Then, on the curve, $\text{Re}(z) = 1 + t$ and

$$\int_\gamma f(z)\,dz = \int_0^1 (1+t)(1+i)\,dt = \frac{3}{2}(1+i).$$

15. Since $f(z) = 1$ is differentiable for all z, we can also write the antiderivative $F(z) = z$. Since $\gamma(1) = 1 - i$ and $\gamma(3) = 9 - 3i$,

$$\int_\gamma f(z)dz = F(9-3i) - F(1-i) = (9-3i) - (1-i) = 8 - 2i.$$

Alternatively, we can use the parametric equations of γ to obtain

$$\begin{aligned} \int_\gamma dz &= \int_1^3 \gamma'(t)\,dt \\ &= \int_1^3 (2t - i)\,dt = \left[t^2 - ti\right]_1^3 \\ &= (9-3i) - (1-i) = 8 - 2i \end{aligned}$$

17. The length of γ is $\sqrt{5}$. Now we need a number M so that

$$\left|\frac{1}{1+z}\right| \le M \text{ for } z \text{ on } \gamma.$$

Now, the point on γ closest to $z = -1$ is $2 + i$, so for z on γ,

$$|z+1| = |z - (-1)| \ge |2 + i + 1| = \sqrt{10}.$$

Then

$$\left|\frac{1}{z+1}\right| = \frac{1}{|z+1|} \le \frac{1}{\sqrt{10}}$$

and we can choose $M = 1/\sqrt{10}$. Then

$$\left|\int_\gamma \frac{1}{1+z}\,, dz\right| \le \frac{\sqrt{5}}{\sqrt{10}} = \frac{1}{\sqrt{2}}.$$

20.2 Cauchy's Theorem

1. $f(z) = \text{Re}(z)$ is not differentiable, so parametrize $\gamma(t) = 2e^{it}$ for $0 \le t \le 2\pi$. Then

$$\begin{aligned} \oint_\gamma \text{Re}(z)\,dz &= \int_0^{2\pi} 2\cos(t)(2ie^{it})\,dt \\ &= \int_0^{2\pi} [4i\cos^2(t) - 4\cos(t)\sin(t)]\,dt \\ &= 4\pi i. \end{aligned}$$

3. $f(z) = |z|^2$ is not differentiable, so Cauchy's theorem does not apply. Parametrize $\gamma(t) = 7e^{it}$ for $0 \leq t \leq 2\pi$. Since $|z| = 7$ on the curve, then

$$\oint_\gamma |z|^2\, dz = \int_0^{2\pi} 49(7)ie^{it}\, dt = 0.$$

5. Since f is differentiable on the circle and at all of the points it encloses, then

$$\oint_\gamma ze^z\, dz = 0$$

by Cauchy's theorem.

7. γ encloses $2i$, at which the function is not defined, hence not differentiable. Parametrize γ by

$$\gamma(t) = 2i + 2e^{it} \text{ for } 0 \leq t \leq 2\pi.$$

Then

$$\begin{aligned}\oint_\gamma \frac{1}{(z-2i)^3}\, dt &= \int_0^{2\pi} \frac{1}{(2e^{it})^2} 2ie^{it}\, dt \\ &= \frac{i}{4}\int_0^{2\pi} e^{-2it}\, dt = 0.\end{aligned}$$

This integral turns out to be 0, but we could not have concluded this from Cauchy's theorem, which does not apply to this integral.

8. z^2 is differentiable for all z, so by Cauchy's theorem,

$$\oint_C z^2\, dz = 0.$$

We need only evaluate $\oint_C \mathrm{Im}(z)\, dz$. Cauchy's theorem does not apply to this integral because $\mathrm{Im}(z)$ is not a differentiable function. Parametrize each side of the square. Let S_1 bes the left side (0 to $-2i$), S_2 the lower side ($-2i$ to $2 - 2i$), S_3 the right side ($2 - 2i$ to 2), and S_4 the top side (2 to 0), oriented counterclockwise. We can parametrize

$$\begin{aligned}&S_1 : z = -2it, \\ &S_2 : z = 2t - 2i, \\ &S_3 : z = 2 - 2i(1-t), \\ &S_4 : z = 2(1-t),\end{aligned}$$

all preserving counterclockwise orientation as t increases from 0 to 1. Now

$$\begin{aligned}\oint_\gamma \mathrm{Im}(z))\, dz &= \int_0^1 (-2t)(-2i)\, dt \\ &+ \int_0^1 (-2)(2)\, dt + \int_0^2 -2(1-t)(2i)\, dt + \int_0^1 0\, dt \\ &= 2i - 4 - 2i = -4.\end{aligned}$$

Therefore

$$\oint_C f(z)\, dz = -4.$$

9. $f(z) = \overline{z}$ is not differentiable, so Cauchy's theorem does not apply. Write $\gamma(t) = e^{it}$ for $0 \leq t \leq 2\pi$. Then
$$\oint_\gamma \overline{z}\,dz = \int_0^{2\pi} e^{-it} ie^{it}\,dt = 2\pi i.$$

11. Since $\sin(3z)$ is differentiable everywhere, hence on the curve and at all points enclosed by γ, then
$$\oint_\gamma \sin(3z)\,dz = 0$$
by Cauchy's theorem.

20.3 Consequences of Cauchy's Theorem

For some of these problems, be on the alert to the possibility of using Cauchy's integral formula, or Cauchy's integral formula for derivatives.

1. Parametrize the curve by $\gamma(t) = 3 - t + (1 - 6t)i$. Then
$$\begin{aligned}\int_\gamma \text{Re}(z+4)\,dz &= \int_0^1 (7-t)(-1-6i)\,dt \\ &= (-1-6i)\frac{13}{2} = -\frac{13}{2} - 39i.\end{aligned}$$

3. We can use the Cauchy integral formula for the derivative of a function $(n = 1)$. With $f(z) = ie^z$, we have
$$\begin{aligned}\oint_\gamma \frac{ie^z}{(z-2+i)^2}\,dz &= 2\pi i f'(2-i) \\ &= 2\pi i(ie^{2-i}) = -2\pi e^2[\cos(1) - \sin(1)i].\end{aligned}$$

5. With $f(z) = z\sin(3z)$ and $n = 2$ in Cauchy's formula for derivatives,
$$\begin{aligned}\oint_\gamma \frac{z\sin(3z)}{(z+4)^3} &= \frac{2\pi i}{2} f''(-4) \\ &= \pi i[6\cos(12) - 36\sin(12)].\end{aligned}$$

6. γ is not a closed curve and the Cauchy integral formulas do not apply. Parametrize γ by $\gamma(t) = 1 - t - it$ for $0 \leq t \leq 1$. On the curve,
$$f(z) = 2i\overline{z}|z| = 2i[(1-t)+it]\sqrt{1-2t+2t^2}$$
so
$$\begin{aligned}\int_\gamma 2i\overline{z}|z|\,dz &= \int_0^1 2i[1-t+it]\sqrt{1-2t+2t^2}(-1-i)\,dt \\ &= 2\int_0^1 \sqrt{1-2t+2t^2}\,dt + 2i\int_0^1 (2t-1)\sqrt{1-2t+2t^2}\,dt \\ &= 1 + \frac{\sqrt{2}}{4}\ln\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right).\end{aligned}$$

These integrations can be done using MAPLE, or the indefinite integrals

$$\int \sqrt{1-2t+2t^2}\,dt = \frac{-1+4t}{8}\sqrt{1-2t+2t^2} + \frac{7\sqrt{2}}{32}\operatorname{arcsinh}\left(\frac{4}{\sqrt{7}}\left(t-\frac{1}{4}\right)\right)$$

and

$$\int (2t-1)\sqrt{1-2t+2t^2}\,dt = \frac{1}{3}(1-2t+2t^2)^{3/2}.$$

7. By Cauchy's integral formula, with $f(z) = z^2 - 5z + i$,

$$\begin{aligned}\oint_\gamma \frac{z^2-5z+i}{z-1+2i}\,dz &= 2\pi i f(1-2i) \\ = 2\pi i[(1-2i)^2 - 5(1-2i) + i] &= 2\pi i(-8+7i) \\ &= -14\pi - 16\pi i.\end{aligned}$$

9. Since $2i$ is the center of the circle γ, we can apply the Cauchy integral formula with $f(z) = z^4$ to write

$$\oint_\gamma \frac{z^4}{z-2i}\,dz = 2\pi i f(2i) = 2\pi i(2i)^4 = 32\pi i.$$

11.

$$\begin{aligned}\oint_\gamma \frac{-(2+i)\sin(z^4)}{(z+4)^2}\,dz &= -2\pi i(2+i)\frac{d}{dz}(\sin(z^4)\Big|_{z=-4} \\ &= 2\pi(1-2i)\left[4z^3\cos(z^4)\right]_{z=-4} \\ &= -512\pi(1-2i)\cos(256)\end{aligned}$$

13. First evaluate

$$\oint_\gamma \frac{e^z}{z}\,dz$$

by the Cauchy integral formula to obtain

$$\oint_\gamma \frac{e^z}{z}\,dz = 2\pi i\left[e^z\right]_{z=0} = 2\pi i.$$

Now evaluate this integral by parametrizing $\gamma(t) = e^{it}$ for $0 \le t \le 2\pi$. We obtain

$$\begin{aligned}\oint_\gamma \frac{e^z}{z}\,dz &= \int_0^{2\pi} \frac{e^{(\cos(t)+i\sin(t))}}{e^{it}} ie^{it}\,dt \\ &= i\int_0^{2\pi} e^{\cos(t)}\cos(\sin(t))\,dt - \int_0^{2\pi} e^{\cos(t)}\sin(\sin(t))\,dt \\ &= 2\pi i.\end{aligned}$$

Equate the real part of the left side of this equation to the real part of the right side to conclude that

$$\int_0^{2\pi} e^{\cos(t)}\cos(\sin(t))\,dt = 2\pi.$$

If we equate imaginary parts, we also obtain

$$\int_0^{2\pi} e^{\cos(t)} \sin(\sin(t))\, dt = 0.$$

However, we did not need this calculation to evaluate this integral, because this integral, from 0 to π, is the negative of the integral from π to 2π, hence the integral is zero.

14. **Hint** Begin by writing
$$f(z) = \frac{z-4i}{z^3+4z} = \frac{z-4i}{z(z-2i)(z+2i)}.$$
Let γ_1, γ_2 and γ_3 be nonintersecting circles enclosed by γ, which also do not intersect γ, and having centers, respectively, 0, $2i$ and $-2i$, and apply the extended deformation theorem.

15. We will use the notation of the theorem, Figure 20.14 of the text, and the hint outlined in the problem. In the text it was shown that it is sufficient to show that

$$\oint_{C^*} \frac{f(z)}{z-z_0}\, dz$$

can be made arbitrarily small by choosing b larger. Recall that, for some positive integer n and some positive number M $|z^n f(z)| \leq M$ for z sufficiently large. By choosing z far enough away from z_0, we can make
$$\left|\frac{z^n f(z)}{z-z_0}\right| \leq |z^n f(z)| \leq M.$$
Then, for z on C^*,
$$\left|\frac{f(z)}{z-z_0}\right| \leq \frac{M}{|z^n(z-z_0)}| = \frac{M}{|z^{n+1}||1-z_0/z|}.$$
But
$$\left|1-\frac{z_0}{z}\right| \geq 1 - \left|\frac{z_0}{z}\right| > \frac{1}{2}$$
if $|z_0/z| < 1/2$, that is, if $|z_0| < 2|z|$, so

$$\frac{1}{|1-z_0/z|} < 2$$

and then
$$\left|\frac{f(z)}{z-z_0}\right| \leq \frac{2M}{z^{n+1}} < \frac{2M}{b^{n+1}}.$$
Now, the length of C^* is

$$b + ib - (\sigma + ib) + 2b + (b - ib) - (\sigma - ib) = 4b - 2\sigma.$$

Then,
$$\left|\int_{C^*} \frac{f(z)}{z-z_0}\, dz\right| \leq \frac{2M}{b^{n+1}}\,(4b - 2\sigma)\,,$$
or, equivalently,
$$\left|\int_{C^*} \frac{f(z)}{z-z_0}\, dz\right| \leq \frac{2M}{b^{n}}\left(4 - \frac{2\sigma}{b}\right),$$
and this approaches 0 as $b \to \infty$.

Chapter 21

Series Representations of Functions

21.1 Power Series

In Problems 1 - 6, we use the ratio test. Take

$$\lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}(z-\zeta)\right|$$

and determine those values of z for which this limit is less than 1. The technique fails if infinitely many $c_n{}'s$ are zero, or if $|c_{n+1}/c_n|$ has no limit.

1.

$$\begin{aligned}
&\left|\frac{(n+1)^{n+1}/(n+2)^{n+1}}{n^n/(n+1)^n}(z-1+3i)\right|\\
&\frac{(n+1)^{2n+1}}{n^n(n+2)^{n+1}}|z-1+3i|\\
&=\left(\frac{n+1}{n}\right)^n\left(\frac{n+1}{n+2}\right)^{n+1}|z-1+2i|\\
&=\left(1+\frac{1}{n}\right)^n\left(\frac{1+1/n}{1+2/n}\right)^{n+1}|z-1+3i|\\
&\to e|z-1+3i|<1
\end{aligned}$$

if

$$|z-1+3i|<\frac{1}{e}.$$

The radius of convergence is $1/e$ and the open disk of convergence is $|z-1+3i|<1/e$.

3. Form

$$\left|\frac{i^{n+1}/2^{n+2}}{i^n/2^{n+1}}(z+8i)\right|\to\frac{1}{2}|z+8i|<1$$

if $|z+8i|<2$. This series has radius of convergence 2 and the open disk of convergence is $|z+8i|<2$, the open disk of radius 2 about the center $-8i$.

5.

$$\Big|\frac{(n+2)/2^{n+1}}{(n+1)/2^n}(z+3i)\Big| = \frac{1}{2}\frac{n+2}{n+1}|z+3i|$$
$$\to \frac{1}{2}|z+3i| < 1$$

if

$$|z+3i| < 2.$$

The radius of convergence of this series is 2 and the open disk of convergence is $|z+3i| < 2$, the open disk of radius 2 about $-3i$.

7. We know that

$$f(0) = 1, f'(0) = i.$$

Further,

$$f''(z) = 2f(z) + 1,$$

so

$$\begin{aligned} f''(0) &= 2f(0) + 1 = 3, \\ f^{(3)}(0) &= 2f'(0) = 2i, \\ f^{(4)}(0) &= 2f''(0) = 6, \\ f^{(5)}(0) &= 2f^{(3)}(0) = 4i. \end{aligned}$$

For the first six terms of the Maclaurin expansion, these numbers enable us to compute the Taylor coefficients $f^k(0)/k!$. We obtain

$$f(z) = 1 + iz + \frac{3}{2}z^2 + \frac{2i}{3!}z^3 + \frac{6}{4!}z^4 + \frac{4i}{5!}z^5 + \cdots .$$

In this problem we can write the entire Maclaurin expansion, since it is not difficult to show by induction that

$$f^{(2n)}(0) = 2^n + 2^{n-1} \text{ and } f^{(2n+1)}()) = 2^n i.$$

In Problems 9, 11, and 13, we attempt to use known series to derive the requested series.

9. This is just the rearrangement of a polynomial into powers of $z^2 - 3z + i$. This can be done algebraically, but it is also easy in this case to write the Taylor coefficients:

$$c_0 = f(2-i), c_1 = f'(2-i), c_2 = \frac{1}{2}f''(2-i),$$

in which $f(z) = z^2 - 3z + i$. We obtain

$$z^2 - 3z + i = -3 + (1-2i)(z-2+i) + (z-2+i)^2.$$

11. Like Problem 9, this is an algebraic rearrangement of a second degree polynomial in powers of $z - 1 - i$. If we use the Taylor coefficients, with $f(z) = (z-9)^2$, then

$$c_0 = f(1+i), c_1 = f'(1+i) \text{ and } c_2 = \frac{1}{2}f''(1+i).$$

We get

$$(z-9)^2 = (63-16i) + (-16+2i)(z-1-i) + (z-1-i)^2.$$

13. Since we know the series for $\cos(z)$ about 0, replace z with $2z$ to obtain

$$\cos(2z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}(2z)^{2n},$$

or

$$\cos(2z) = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n}}{(2n)!}(z)^{2n}.$$

This series has infinite radius of convergence, since the series for $\cos(z)$ as infinite radius of convergence. This means that both series are valid for all complex z.

15. No. The the power series has center $2i$. If the series converged at 0, it would converge also at the point i that is closer to $2i$ than 0 is.

17. Begin with z a given complex number, and consider the integral

$$\frac{1}{2\pi i}\oint_\gamma \frac{z^n}{n!w^{n+1}} e^{zw}\,dw$$

with γ the unit circle about the origin (oriented counterclockwise). First expand e^{zw} in its Maclaurin series and then parametrize γ by $\gamma(t) = e^{i\theta}$ to write

$$\begin{aligned}\oint_\gamma \frac{z^n}{n!w^{n+1}} e^{zw}\,dw &= \frac{1}{2\pi i}\oint_\gamma \frac{z^n}{n!w^{n+1}} \sum_{k=0}^{\infty} \frac{(zw)^k}{k!}\,dw \\ &= \frac{1}{2\pi i}\oint_\gamma \sum_{k=0}^{\infty} \frac{z^{n+k}w^{k-n-1}}{n!k!}\,dw \\ &= \frac{1}{2\pi i}\int_0^{2\pi} \sum_{k=0}^{\infty} \frac{z^{n+k}e^{i(k-n-1)\theta}}{n!k!} ie^{i\theta}\,d\theta \\ &= \sum_{k=0}^{\infty} \frac{1}{2\pi}\int_0^{2\pi} \frac{z^{n+k}}{n!k!} e^{i(k-n)\theta}\,d\theta.\end{aligned}$$

Now,

$$\int_0^{2\pi} e^{i(k-n)\theta}\,d\theta = \begin{cases} 0 & \text{if } k \neq n, \\ 2\pi & \text{if } k - n. \end{cases}$$

Therefore we have

$$\frac{1}{2\pi i}\oint_\gamma \frac{z^n}{n!w^{n+1}} e^{zw}\,dw = \frac{(z^n)^2}{(n!)^2}.$$

Finally, we can write

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{1}{(n!)^2}z^{2n} &= \sum_{n=0}^{\infty}\frac{1}{2\pi i}\frac{z^n}{n!w^{n+1}}e^{zw}\,dw\\ &= \frac{1}{2\pi i}\int_0^{2\pi}\sum_{n=0}^{\infty}\frac{z^n}{n!e^{i(n+1)\theta}}e^{ze^{i\theta}}e^{i\theta}\,d\theta\\ &= \frac{1}{2\pi}\left[\sum_{n=0}^{\infty}\frac{(ze^{-i\theta})^n}{n!}\right]e^{ze^{i\theta}}\,d\theta\\ &= \frac{1}{2\pi}\int_0^{2\pi}e^{ze^{-i\theta}}e^{ze^{i\theta}}\,d\theta\\ &= \frac{1}{2\pi}\int_0^{2\pi}e^{z(e^{-i\theta}+e^{i\theta})}\,d\theta\\ &= \frac{1}{2\pi}\int_0^{2\pi}e^{2\pi\cos(\theta)}\,d\theta.\end{aligned}$$

19. $f(z)$ has a zero of order 3 at $3\pi/2$ because $\cos(z)$ has a simple zero there.

21. $f(z)$ is not defined at zero. However, as we will see in the next section, we can write, for $z \neq 0$,

$$\begin{aligned}f(z) = \sin(z^4)/z^2 &= \frac{1}{z^2}\left(z^4 - \frac{1}{6}z^{12} + \cdots\right)\\ &= z^2 - \frac{1}{6}z^{10} + \cdots .\end{aligned}$$

Since the right side of this equation is a power series about 0, it defines a differentiable function $g(z)$ which is equal to $f(z)$ if $z \neq 0$. We can therefore extend $f(z)$ to a differentiable function by setting $f(z) = g(z)$ if $z \neq 0$, and $f(0) = g(0) = 0$. If we do this, then the extended function $g(z)$ has a zero of order 2 at 0. This is the idea behind a removable singularity, which is discussed in the next chapter.

23. $f(z)$ has a zero of order 4 at 0, because z^2 has a zero of order 2, and $\sin(z)$ has a zero of order 1 at 0, so $\sin^2(z)$ has a zero of order 2 there.

25. If we compute the kth derivative of $f(z)$ at z_0 by differentiating each series term by term, we obtain

$$\begin{aligned}f^{(k)}(z_0) &= \left[\sum_{n=0}^{\infty}a_n(n)(n-1)\cdots(n-k+1)(z-z_0)^{n-k}\right]_{z=z_0}\\ &= a_k k!\\ &= \left[\sum_{n=0}^{\infty}b_n(n)(n-1)\cdots(n-k+1)(z-z_0)^{n-k}\right]_{z=z_0}\\ &= b_k k!.\end{aligned}$$

Then

$$a_k = \frac{f^{(k)}(z_0)}{k!} = b_k.$$

The coefficients of the power series expansion of $f(z)$ about z_0 must be the Taylor coefficients at z_0.

21.2 The Laurent Expansion

For these problems, use known expansions, such as power series of exponential and trigonometric functions, and geometric series. This sometimes requires some ingenuity in rewriting functions in ways suited to the task at hand.

1. The denominator is already a power of $z - i$, so

$$\frac{z+i}{z-i} = \frac{2i + (z-i)}{z-i} = 1 + \frac{2i}{z-i}$$

for $0 < |z - i| < \infty$.

3. We want a series in powers of $z - 1$. The denominator is already a power of $-(z - 1)$, so all we need to do is fix the numerator:

$$\begin{aligned}\frac{z^2}{1-z} &= \frac{((z-1)+1)^2}{1-z} = -\frac{1 + 2(z-1) + (z-1)^2}{z-1} \\ &= -\frac{1}{z-1} - 2 - (z-1),\end{aligned}$$

for $0 < |z - 1| < \infty$ (the complex plane with 1 removed).

5. We want an expansion in powers of $z - i$. To this end, first write

$$\frac{2z}{1+z^2} = \frac{1}{z-i} + \frac{1}{z+i}.$$

The first term is a series (with just one term) in powers of $z - i$. For the second term, rearrange the denominator and use a geometric series:

$$\begin{aligned}\frac{1}{z+i} &= \frac{1}{2i + (z-i)} = \frac{1}{2i\left(1 + \frac{z-i}{2i}\right)} \\ &= \frac{1}{2i}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z-i}{2i}\right)^n \\ &= \sum_{n=0}^{\infty}\frac{(-1)^n}{(2i)^{n+1}}(z-i)^n.\end{aligned}$$

This expansion is valid for

$$\left|\frac{z-i}{2i}\right| < 1$$

or

$$|z - 2i| < 2.$$

The Laurent expansion of $2z/(1 + z^2)$ is therefore

$$\frac{1}{z-i} + \sum_{n=0}^{\infty}\frac{(-1)^n}{(2i)^{n+1}}(z-i)^n$$

and this is a valid representation of $f(z)$ in the annulus (punctured disk) $0 < |z - i| < 2$.

7. If $z \neq 0$, then

$$\frac{1-\cos(2z)}{z^2} = \frac{1}{z^2}\left[1-\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}(2z)^{2n}\right] = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}4^n}{(2n)!}z^{2n-2}.$$

This holds for $0 < |z| < \infty$.

9. Using the exponential series, we have

$$\frac{e^{z^2}}{z^2} = \frac{1}{z^2}\sum_{n=0}^{\infty}\frac{1}{n!}z^{2n} = \sum_{n=0}^{\infty}\frac{1}{n!}z^{2n-2}$$

for $0 < |z| < \infty$.

11. In the discussion, we will refer to Figures 21.2 and 21.3.

f is differentiable on and in the interior of Γ_1, so by Cauchy's integral formula, for any z enclosed by Γ_1.

$$f(z) = \frac{1}{2\pi i}\oint_{\Gamma_1}\frac{f(w)}{w-z}\,dw.$$

Since Γ_2 does not enclose z, then by Cauchy's theorem,

$$\frac{1}{2\pi i}\oint_{\Gamma_2}\frac{f(w)}{w-z}\,dw = 0,$$

in which the factor $1/2\pi i$ was introduced for the next part of the argument. Add these two equations to obtain

$$f(z) = \frac{1}{2\pi i}\left[\oint_{\Gamma_1}\frac{f(w)}{w-z}\,dz + \oint_{\Gamma_2}\frac{f(w)}{w-z}\,dw\right].$$

Orientation on both curves is counterclockwise. In this sum of integrals, each of L_1 and L_2 is traversed in both directions, so integrals over these segments cancel. This sum therefore gives integrals over γ_1 and γ_2, counterclockwise on γ_2 but clockwise on γ_1. Reversing this orientation on γ_1 so that all orientations are counterclockwise, we have

$$f(z) = \frac{1}{2\pi i}\left[\oint_{\gamma_2}\frac{f(w)}{w-z}\,dz - \oint_{\gamma_1}\frac{f(w)}{w-z}\,dw\right].$$

We will manipulate $1/(w-z)$ differently in each of these integrals. For the integral over γ_2, write

$$\frac{1}{w-z} = \frac{1}{w-z_0-(z-z_0)} = \frac{1}{w-z_0}\frac{1}{1-(z-z_0)/(w-z_0)} = \frac{1}{w-z_0}\sum_{n=0}^{\infty}\left(\frac{z-z_0}{w-z_0}\right)^n = \sum_{n=0}^{\infty}\frac{1}{(w-z_0)^{n+1}}(z-z_0)^n.$$

This geometric series is valid because, for w on γ_2,

$$\left|\frac{z-z_0}{w-z_0}\right| < 1.$$

For the integral over γ_1, we have, for w on γ_1,

$$\left|\frac{w-z_0}{z-z_0}\right| < 1,$$

so now write

$$\begin{aligned}
\frac{1}{w-z} &= \frac{1}{w-z_0-(z-z_0)} = \frac{-1}{z-z_0}\frac{1}{1-(w-z_0)/(z-z_0)} \\
&= -\frac{1}{z-z_0}\sum_{n=0}^{\infty}\left(\frac{w-z_0}{z-z_0}\right)^n \\
&= -\sum_{n=0}^{\infty}(w-z_0)^n\frac{1}{(z-z_0)^{n+1}}.
\end{aligned}$$

Substitute these into the integrals in the sum representing $f(z)$ and interchange the integrals and the summations to obtain

$$\begin{aligned}
f(z) &= \frac{1}{2\pi i}\oint_{\gamma_2}\left(\sum_{n=0}^{\infty}\frac{f(w)}{(w-z_0)^{n+1}}\,dw\right)(z-z_0)^n \\
&\quad + \frac{1}{2\pi i}\oint_{\gamma_1}\left(\sum_{n=0}^{\infty}f(w)(w-z_0)^n\,dw\right)\frac{1}{(z-z_0)}^{n+1} \\
&= \sum_{n=0}^{\infty}\left(\frac{1}{2\pi i}\oint_{\gamma_2}\frac{f(w)}{(w-z_0)^{n+1}}\,dw\right)(z-z_0)^n \\
&\quad + \sum_{n=0}^{\infty}\left(\frac{1}{2\pi i}\oint_{\gamma_1}f(w)(w-z_0)^n\,dw\right)\frac{1}{(z-z_0)^{n+1}}.
\end{aligned}$$

Put $n=-m-1$ in the last summation to obtain

$$\begin{aligned}
f(z) &= \sum_{n=0}^{\infty}\left(\frac{1}{2\pi i}\oint_{\gamma_2}\frac{f(w)}{(w-z_0)^{n+1}}\,dw\right)(z-z_0(^n \\
&\quad + \sum_{m=-1}^{-\infty}\left(\frac{1}{2\pi i}\oint_{\gamma_1}\frac{f(w)}{(w-z_0)^{m+1}}\right)(z-z_0)^m.
\end{aligned}$$

Finally, use the deformation theorem to replace these integrals over γ_1 and γ_2 with integrals over Γ, which is any path in the annulus and enclosing z_0. This gives us

$$f(z) = \sum_{n=-\infty}^{\infty}c_n(z-z_0)^n,$$

where

$$c_n = \frac{1}{2\pi i}\oint_{\Gamma}\frac{f(w)}{(w-z_0)^{n+1}}\,dw,$$

completing the proof.

Chapter 22

Singularities and the Residue Theorem

22.1 Singularities

1. Write
$$\frac{z-i}{z^2+1} = \frac{z-i}{(z+i)(z-i)} = \frac{1}{z+i}.$$
Then $f(z)$ has a removable singularity at i and a simple pole at $-i$.

3. Write
$$\frac{z}{z^4-1} = \frac{z}{(z-1)(z+1)(z-i)(z+i)}.$$
This function has simple poles at $\pm 1, \pm i$.

4. $e^{1/z(z+1)}$ has essential singularities at 0 and -1. One way to see this is to write
$$\frac{1}{z(z+1)} = \frac{1}{z} - \frac{1}{z+1},$$
so
$$e^{1/z(z+1)} = e^{1/z}e^{-1/(z+1)}.$$
Now look at $z = 0$, to be specific. We know that $e^{1/z}$ has an essential singularity at 0, and $e^{-1/(z+1)}$ is differentiable at 0, so the product will have an essential singularity there. Similar reasoning applies at -1, since $e^{1/z}$ is differentiable at -1.

5. $\sec(z) = 1/\cos(z)$ has simple poles at the zeros of $\cos(z)$. These are $(2n+1)\pi/2$, with n any integer.

6. To analyze singularities of $\sin(z)/\sinh(z)$, we must know the zeros of $\sinh(z)$. Of course, the real hyperbolic sine function $\sinh(x)$ is zero only for $x = 0$, but the complex hyperbolic sine function might have zeros in the complex plane. To answer this question, set
$$\sinh(z) = \frac{1}{2}\left(e^z - e^{-z}\right) = 0.$$
Then $e^z - e^{-z} = 0$, so
$$e^{2z} = 1.$$

We know where the complex exponential function equals 1. $e^{2z} = 1$ exactly when $2z = 2n\pi i$, or $z = n\pi i$, with n any integer. Since $\sin(n\pi i) \neq 0$ unless $n = 0$, then the numbers $n\pi i$ $n \neq 0$ are singularities of $\sin(z)/\sinh(z)$. Further, $\cosh(n\pi i) \neq 0$, so these are simple poles of $\sin(z)/\sinh(z)$. If $n = 0$, then we have the origin 0. But $\sin(z)$ and $\sinh(z)$ have simple zeros at 0, so 0 is a removable singularity of this function.

7. $e^{1/z}(z + 2i)$ has a essential singularity at $z = 0$. The factor $z + 2i$ makes no contribution to singularities of this function.
9. $\cos(z)/z^2$ has just one singularity, $z = 0$, and this is a pole of order 2 because $\cos(0) \neq 0$.
11. The only singularities are $1, i, -i$, and 1 is a double pole, while i and $-i$ are simple poles.
13. Suppose f is differentiable at z_0 and $f(z_0) \neq 0$, while g has a pole of order m at z_0. We want to show that fg has a pole of order m at z_0.

    Since g has a pole of order m at z_0, then $g(z)$ has a Laurent expansion about z_0 of the form

$$g(z) = \frac{k}{(z - z_0)^m} + \sum_{n=-m+1}^{\infty} c_n(z - z_0)^n,$$

with m the highest power of $1/(z - z_0)$ appearing in the series. Then $(z - z_0)^m g(z)$ is a series containing only nonnegative powers of $z - z_0$, hence is a power series expansion about z_0. If we denote this series as $h(z)$, then

$$g(z) = \frac{h(z)}{(z - z_0)^m},$$

with h differentiable at z_0 and $h(z_0) \neq 0$. We now have, in some annulus about z_0,

$$f(z)g(z) = \frac{f(z)h(z)}{(z - z_0)^m},$$

with $f(z_0)h(z_0) \neq 0$. Therefore fg has a pole of order m at z_0.

22.2 The Residue Theorem

1. The integrand has a simple pole at $z = -4i$, which is outside γ. Since f is differentiable on and in the region bounded by γ, then by Cauchy's theorem,

$$\oint_\gamma \frac{8z - 4i + 1}{z + 4i}\, dz = 0.$$

3. The function being integrated has simple poles at $\sqrt{6}i$ and $-\sqrt{6}i$, both enclosed by γ. Then

$$\begin{aligned}\oint_\gamma \frac{z + i}{z^2 + 6}\, dz &= 2\pi i \mathrm{Res}(f, \sqrt{6}i) + 2\pi i \mathrm{Res}(f, -\sqrt{6}i)\\ &= 2\pi i \frac{\sqrt{6} + 1}{2\sqrt{6}} + 2\pi i \frac{\sqrt{6} - 1}{2\sqrt{6}} = 2\pi i.\end{aligned}$$

5. $f(z)$ has a pole of order 2 at $z = 1$ and a simple pole at $z = -2i$. Both are enclosed by γ (recall that singularities not enclosed by γ are irrelevant for evaluating an integra of the function about γ). Then

$$\oint_\gamma \frac{1 + z^2}{(z - 1)^2(z + 2i)}\, dz = 2\pi i \mathrm{Res}(f, 1) + 2\pi i \mathrm{Res}(f, -2i).$$

Compute

$$\begin{aligned}\text{Res}(f,1) &= \lim_{z\to 1}\frac{d}{dz}\left(\frac{1+z^2}{z+2i}\right)\\ &= \lim_{z\to 1}\frac{(z+2i)(2z)-(1+z^2)}{(z+2i)^2}\\ &= \frac{4i}{-3+4i}\end{aligned}$$

and

$$\text{Res}(f,-2i) = \lim_{z\to -2i}\frac{1+z^2}{(z-1)^2} = \frac{-3}{-3+4i}.$$

Then

$$\oint_\gamma \frac{1+z^2}{(z-1)^2(z+2i)}\,dz = 2\pi i\left[\frac{4i}{-3+4i} - \frac{3}{-3+4i}\right] = 2\pi i.$$

7. $\coth(z) = \cosh(z)/\sinh(z)$, so singularities of $\coth(z)$ are zeros of $\sinh(z)$, which are $n\pi i$, with n any integer. Only the simple pole at $z=0$ is enclosed by γ, so

$$\oint_\gamma \coth(z)\,dz = 2\pi i\text{Res}(f,0) = 2\pi i\frac{\cosh(0)}{\cosh(0)} = 2\pi i.$$

9. 0 and $4i$ are both simple poles of $f(z)$, so

$$\begin{aligned}\oint_\gamma \frac{e^{2z}}{z(z-4i)}\,dz &= 2\pi i\left[\text{Res}(f,0)+\text{Res}(f,4i)\right]\\ &= 2\pi i\left[-\frac{1}{4i}+\frac{e^{8i}}{4i}\right]\\ &= \frac{\pi}{2}[\cos(8)-1+i\sin(8)].\end{aligned}$$

11. The integrand has simple poles at $i, 3i$ and $-3i$. Only the pole at $-3i$ is enclosed by γ, so

$$\oint_\gamma \frac{iz}{(z^2+9)(z-i)}\,dz = 2\pi i\text{Res}(f,-3i)$$
$$= 2\pi i\lim_{z\to -3i}\frac{iz}{(z-3i)(z-i)} = 2\pi i\left(-\frac{1}{8}\right) = -\frac{\pi i}{4}.$$

13. $z/\sinh^2(z)$ has a simple pole at $z=0$, and double poles at the nonzero zeros of $\sinh(z)$, which occur at $z=n\pi i$ for integer values of n. The only pole of $z/\sinh^2(z)$ enclosed by γ is $z=0$. Therefore

$$\oint_\gamma \frac{z}{\sinh^2(z)}\,dz = 2\pi i\text{Res}(f,0)$$

Compute this residue as

$$\begin{aligned}\text{Res}(f,0) &= \lim_{z\to 0}(zf(z)) = \lim_{z\to 0}\left(\frac{z^2}{\sinh^2(z)}\right)\\ &= \lim_{z\to 0}\frac{z^2}{z^2+\frac{1}{6}z^4+\cdots}\\ &= \lim_{z\to 0}\frac{1}{1+\frac{1}{6}z^2+\cdots} = 1.\end{aligned}$$

Therefore

$$\oint_\gamma \frac{z}{\sinh^2(z)}\,dz = 2\pi i.$$

15. γ does not enclose 0, the only singularity of $f(z)$, so by Cauchy's theorem,

$$\oint_\gamma \frac{e^z}{z}\,dz = 0.$$

17. First, $g(z) = (z+1)/(z^2+2z+4)$ has simple poles at $-1 \pm \sqrt{3}i$, enclosed by γ. Then

$$\begin{aligned}\oint_\gamma \frac{z+1}{z^2+2z+4}\,dz &= 2\pi i\left[\text{Res}(g, -1-\sqrt{3}i) + \text{Res}(g, -1+\sqrt{3}i)\right]\\ &= 2\pi i\left[\frac{1-1-\sqrt{3}i}{2(-1-\sqrt{3}i)+2} + \frac{-1+\sqrt{3}i+1}{2(-1+\sqrt{3}i)+2}\right]\\ &= 2\pi i\left(\frac{1}{2}+\frac{1}{2}\right) = 2\pi i.\end{aligned}$$

To use the argument principle, note that

$$\frac{z+1}{z^2+2z+4} = \frac{1}{2}\frac{f'(z)}{f(z)},$$

where $f(z) = z^2+2z+4$. f has $Z=2$ simple zeros enclosed by γ and no poles ($P=0$), so

$$\frac{1}{2}\oint_\gamma \frac{2z+2}{z^2+2z+4}\,dz = \pi i(Z-P) = 2\pi i.$$

19. We are supposing that z_0 is a zero of h of order 2, but is not a zero of g. We want to show that

$$\text{Res}(g/h, z_0) = \frac{2g'(z_0)}{h''(z_0)} - \frac{2}{3}\frac{g(z_0)h^{(3)}(z_0)}{(h''(z_0))^2}.$$

To do this, write

$$h(z) = (z-z_0)^2\varphi(z)$$

where $\varphi(z_0) \neq 0$. Then

$$\begin{aligned}\text{Res}(g/h, z_0) &= \lim_{z\to z_0}\frac{d}{dz}\left((z-z_0)^2\frac{g(z)}{h(z)}\right)\\ &= \lim_{z\to z_0}\frac{d}{dz}\left((z-z_0)^2\frac{g(z)}{(z-z_0)^2\varphi(z)}\right)\\ &= \lim_{z\to z_0}\frac{d}{dz}\left(\frac{g(z)}{\varphi(z)}\right)\\ &= \frac{\varphi(z_0)g'(z_0)-\varphi'(z_0)g(z_0)}{(\varphi(z_0))^2}.\end{aligned}$$

Now,

$$\begin{aligned}h'(z) &= 2(z-z_0)\varphi(z) + (z-z_0)^2\varphi'(z),\\ h''(z) &= 2\varphi(z) + 4(z-z_0)\varphi'(z) + (z-z_0)^2\varphi''(z).\end{aligned}$$

and

$$h^{(3)}(z) = 6\varphi'(z) + 6(z - z_0)\varphi''(z) + (z - z_0)^2\varphi^{(3)}(z).$$

Therefore

$$\varphi(z_0) = \frac{1}{2}h''(z_0) \text{ and } \varphi'(z_0) = \frac{1}{6}h^{(3)}(z_0).$$

Substituting these into the above expression for the residue, we obtain

$$\text{Res}(g/h, z_0) = \frac{2g'(z_0)}{h''(z_0)} - \frac{2}{3}\frac{g(z_0)h^{(3)}(z_0)}{(h''(z_0))^2}.$$

20. **Hint** Carry out two arguments.

First, show that, if f has a zero of order k at z_0 in G, then f'/f has a simple pole at z_0, and

$$\text{Res}(f'/f, z_0) = k.$$

Second, show that, if f has a pole of order m at z_1, then f'/f has a simple pole at z_1, and

$$\text{Res}(f'/f, z_1) = -m.$$

Now apply the residue theorem to evaluate

$$\oint_\gamma \frac{f'(z)}{f(z)}\,dz.$$

21. By the residue theorem, with $g(z) = z/(2 + z^2)$

$$\begin{aligned}\oint_\gamma \frac{z}{2+z^2}\,dz &= 2\pi i\left[\text{Res}(g, \sqrt{2}i) + \text{Res}(g, -\sqrt{2}i)\right]\\ &= 2\pi i\left[\frac{\sqrt{2}i}{2\sqrt{2}i} + \frac{-\sqrt{2}i}{-2\sqrt{2}i}\right]\\ &= 2\pi i\left(\frac{1}{2} + \frac{1}{2}\right) = 2\pi i.\end{aligned}$$

For the argument principle, we need to write

$$g(z) = \frac{f'(z)}{f(z)} = \frac{1}{2}\frac{2z}{2+z^2}$$

with $f(z) = 2 + z^2$. Then $f'/f = 2g$.

Now $f(z)$ has two simple zeros enclosed by γ, and no poles, so $Z = 2$ and $P = 0$. By the argument principle,

$$\begin{aligned}\oint_\gamma \frac{z}{2+z^2}\,dz &= \frac{1}{2}\oint_\gamma \frac{2z}{2+z^2}\,dz\\ &= \pi i(Z - P) = 2\pi i.\end{aligned}$$

It is important in this calculation of an integral to be clear on the difference between $\oint_\gamma g(z)\,dz$, and $\oint_\gamma (f'(z)/f(z))\,dz$. In this example $f'(z)/f(z) = 2g(z)$.

22.3 Evaluation of Real Integrals

Most of these problems are done using one of equations (22.3), (22.4) or (22.6). In problems involving rational functions of sine and cosine, γ always denotes the unit circle about the origin.

1. First use the identity
$$\cos^2(x) = \frac{1}{2}(1+\cos(2x))$$
to write the integral as
$$\int_{-\infty}^{\infty} \frac{\cos^2(x)}{(x^2+4)^2}\,dx = \frac{1}{2}\int_{-\infty}^{\infty} \frac{1+\cos(2x)}{(x^2+4)^2}\,dx.$$
Let
$$f(z) = \frac{1+e^{2iz}}{(z^2+4)^2}.$$
Then f has a pole of order 2 in the upper half-plane at $2i$, and
$$\mathrm{Re}(f,2i) = \lim_{z\to 2i}\frac{d}{dz}\left[\frac{1+e^{2iz}}{(z+2i)^2}\right] = \frac{1+5e^{-4}}{32i}.$$
Then
$$\begin{aligned}\int_{-\infty}^{\infty} \frac{\cos^2(x)}{(x^2+4)^2}\,dx &= \frac{1}{2}\mathrm{Re}\left[2\pi i\left(\frac{1+5e^{-4}}{32i}\right)\right]\\ &= \frac{\pi}{3}(1+5e^{-4}).\end{aligned}$$

3. Let
$$f(z) = \frac{ze^{2iz}}{z^4+16}.$$
Then f has simple poles in the upper half-plane at $z_1 = (1+i)\sqrt{2}$ and $z_2 = (-1+i)\sqrt{2}$. Compute
$$\mathrm{Res}(f,z_1) = \frac{e^{2\sqrt{2}(-1+i)}}{16i} \text{ and } \mathrm{Res}(f,z_2) = \frac{2^{2\sqrt{2}(-1-i)}}{-16i}$$
to obtain
$$\begin{aligned}\int_{-\infty}^{\infty} \frac{x\sin(2x)}{x^4+16}\,dx &= \mathrm{Im}\left[2\pi i\left(\frac{e^{-2\sqrt{2}}}{8}\right)\left(\frac{e^{2\sqrt{2}i}-e^{-2\sqrt{2}i}}{2i}\right)\right]\\ &= \frac{\pi e^{-2\sqrt{2}}}{4}\sin(2\sqrt{2}).\end{aligned}$$

5. Let $f(z) = z^2/(z^2+4)^2$. The only singularity of f in the upper half-plane is $2i$, which is a double pole. Compute
$$\mathrm{Res}(f,2i) = \lim_{z\to 2i}\frac{d}{dz}\left[\frac{z^2}{(z+2i)^2}\right] = -\frac{i}{8}.$$
Then
$$\int_{-\infty}^{\infty} \frac{x^2}{(x^2+4)^2}\,dx = 2\pi i\left(-\frac{i}{8}\right) = \frac{\pi}{4}.$$

7. $f(z) = 1/(1+z^6)$ has simple poles in the upper half-plane at $z_1 = i$, $z_2 = (\sqrt{3}+i)/2$ and $z_3 = (-\sqrt{3}+i)/2$. At each pole, compute

$$\operatorname{Res}(f, z_j) = \frac{1}{6z_j^5} = -\frac{1}{6}z_j,$$

so

$$\int_{-\infty}^{\infty} \frac{1}{1+x^6}\,dx = 2\pi i\left[\frac{1}{6}(z_1+z_2+z_3)\right] = \frac{2\pi}{3}.$$

Then

$$\int_0^{\infty} \frac{1}{1+x^6}\,dx = \frac{1}{2}\int_{-\infty}^{\infty} \frac{1}{1+x^6}\,dx = \frac{\pi}{3}.$$

9. With $z = e^{i\theta}$, we have

$$\cos(\theta) = \frac{1}{2}\left(z + \frac{1}{z}\right) \text{ and } d\theta = \frac{1}{iz}\,dz,$$

so

$$\begin{aligned}\int_0^{2\pi} \frac{1}{2-\cos(\theta)}\,d\theta &= \oint_\gamma \frac{1}{2-\frac{1}{2}(z+1/z)}\frac{1}{iz}\,dz\\ &= 2i\oint_\gamma \frac{1}{z^2-4z+1}\,dz.\end{aligned}$$

Now $f(z) = 1/(z^2-4z+1)$ has simple poles at $z_1 = 2-\sqrt{3}$ and $z_2 = 2+\sqrt{3}$. Only z_1 is enclosed by γ, and

$$\operatorname{Res}(f, 2-\sqrt{3}) = \frac{1}{2(2-\sqrt{3})-4} = -\frac{1}{2\sqrt{3}}.$$

Then

$$\int_0^{2\pi} \frac{1}{2-\cos(\theta)}\,d\theta = 2i(2\pi i)\left(\frac{-1}{2\sqrt{3}}\right) = \frac{2\pi}{\sqrt{3}}.$$

Note that the integral must be real and positive, since the integrand is positive, and this checks out.

11. Let Γ denote the suggested rectangular path. The four sides are

$$\begin{aligned}&\Gamma_1 : z = x, -R \le x \le R \text{ (lower side of the rectangle)},\\ &\Gamma_2 : z = R + it, 0 \le \beta \le R \text{ (right side)},\\ &\Gamma_3 : z = x + i\beta, x : R \to -R \text{ (top)},\\ &\Gamma_4 : z = -R + it, t : \beta \to 0 \text{ (right side)}.\end{aligned}$$

The intervals for the parameters on the sides are chosen to maintain counterclockwise orientation around Γ. Now observe that e^{-z^2} is differentiable on and in the region bounded by Γ, and use Cauchy's theorem and the parametrization on each side of the rectangle to write

$$\oint_\Gamma e^{-z^2}\,dz = 0 = \sum_{j=1}^{4}\int_{\Gamma_j} e^{-z^2}\,dz.$$

Look at each of the integrals on the right. First,

$$\int_{\Gamma_1} e^{-z^2}\,dz = \int_{-R}^{R} e^{-x^2}\,dx.$$

Next,

$$\int_{\Gamma_2} e^{-z^2}\,dz = \int_0^\beta e^{-(R^2+2Rti-t^2)} i\,dt = ie^{-R^2}\int_0^\beta e^{t^2}[\cos(2Rt) - i\sin(2Rt)]\,dt.$$

For the third side,

$$\int_{\Gamma_3} e^{-z^2}\,dz = \int_R^{-R} e^{-(x^2+2x\beta i-\beta^2)}\,dx = e^{-\beta^2}\int_{-R}^{R} e^{-x^2}[\cos(2\beta x) - i\sin(2\beta x)]\,dx.$$

Finally, on the fourth side,

$$\int_{\Gamma_4} e^{-z^2}\,dz = \int_\beta^0 e^{-(R^2-2Rti-t^2)} i\,dt = ie^{-R^2}\int_0^\beta e^{t^2}[-\cos(2Rt) - i\sin(2Rt)]\,dt.$$

The integrals having factors of e^{-R^2} tend to zero as $R \to \infty$. Thus, upon adding these four integrals and letting $R \to \infty$, we obtain

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx - e^{\beta^2}\int_{-\infty}^{\infty}[\cos(2\beta x) - i\sin(2\beta x)]\,dx = 0.$$

Now $e^{-x^2}\sin(2\beta x)$ is an odd function on the real line, so

$$\int_{-\infty}^{\infty} e^{-x^2}\sin(2\beta x)\,dx = 0.$$

Therefore,

$$e^{\beta^2}\int_{-\infty}^{\infty} e^{-x^2}\cos(2\beta x)\,dx = \int_0^{\infty} e^{-x^2}\,dx.$$

Finally, use the known result that

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$$

to obtain

$$\int_{-\infty}^{\infty} e^{-x^2}\cos(2\beta x)\,dx = \sqrt{\pi}e^{-\beta^2}.$$

Finally, because the integrand is an even function, then

$$\int_0^{\infty} e^{-x^2}\cos(2\beta x)\,dx = \frac{\sqrt{\pi}}{2}e^{-\beta^2}.$$

13. First observe that, because the integrand is an even function,

$$\int_0^\infty \frac{x\sin(\alpha x)}{x^4+\beta^4}\,dx = \frac{1}{2}\int_{-\infty}^{\infty}\frac{x\sin(\alpha x)}{x^4+\beta^4}\,dx.$$

Now

$$f(z) = \frac{ze^{i\alpha z}}{z^4+\beta^4}$$

has simple poles in the upper half-plane at $z_1 = \beta e^{i\pi/4}$ and $z_2 = \beta e^{3\pi i/4}$. Compute the residues of f at these poles. In general,

$$\operatorname{Res}(f, z_k) = \left[\frac{ze^{i\alpha z}}{4z^3}\right]_{z=z_k} = \frac{e^{i\alpha z_k}}{4z_k^2}.$$

Then,

$$\operatorname{Res}(f, z_1) = \frac{e^{i\alpha\beta e^{i\pi/4}}}{4\beta^2 i} \text{ and } \operatorname{Res}(f, z_2) = \frac{e^{i\alpha\beta e^{3\pi i/4}}}{-4\beta^2 i}.$$

Then

$$\int_0^\infty \frac{x\sin(\alpha x)}{x^4+\beta^4}\,dx = \frac{1}{2}\operatorname{Im}\left[\frac{2\pi i}{4\beta^2}\left(e^{i\alpha\beta(1+i)/\sqrt{2}} - e^{i\alpha\beta(-1+i)/\sqrt{2}}\right)\frac{1}{i}\right]$$
$$= \frac{\pi e^{-\alpha\beta/\sqrt{2}}}{2\beta^2}\sin\left(\frac{\alpha\beta}{\sqrt{2}}\right).$$

14. **Hint** First show that

$$\int_0^\pi \frac{1}{(\alpha+\beta\cos(\theta))^2}\,d\theta = \frac{1}{2}\int_0^{2\pi}\frac{1}{(\alpha+\beta\cos(\theta))^2}\,d\theta.$$

Now complex methods can be applied to the integral on the right.

15. Let

$$f(z) = \frac{e^{i\alpha z}}{z^2+1}.$$

The only singularity f has in the upper half-plane is a simple pole at i. Compute

$$\operatorname{Res}(f, i) = \frac{e^{-\alpha}}{2i}.$$

Then

$$\int_{-\infty}^\infty \frac{\cos(\alpha x)}{x^2+1}\,dx = 2\pi i\left(\frac{e^{-\alpha}}{2i}\right) = \pi e^{-\alpha}.$$

16. Let Γ be the path indicated in Figure 22.3 of the text. By Cauchy's theorem,

$$\oint_\Gamma e^{iz^2}\,dz = 0.$$

Now examine the integral on the left over the three pieces of Γ consisting of the segment on the $x-$ axis (Γ_1), the circular arc (Γ_2), then the segment from the end of this arc back to the origin (Γ_3).

On Γ_1, $z = x$ and

$$\int_{\Gamma_1} e^{iz^2}\,dz = \int_0^R e^{ix^2}\,dx = \int_0^R [\cos(x^2) + i\sin(x^2)]\,dx.$$

On Γ_2, $z = Re^{i\theta}$ and

$$\int_{\Gamma_2} e^{iz^2}\,dz = \int_0^{\pi/4} e^{iR^2}e^{2i\theta}\,d\theta.$$

On Γ_3, $z = re^{i\pi/4}$, so

$$\int_{\Gamma_3} e^{iz^2}\,dz = \int_R^0 e^{-r^2}e^{i\pi/4}\,dr.$$

Notice the integration from R to 0 here to maintain counterclockwise orientation on Γ.

We want to take the limit on these integrals as $R \to \infty$. The integral over Γ_3 clearly has limit zero, because of the factor e^{-r^2} in the integral. The integral over Γ only has R in the upper limit of integration. The integral over Γ_2 is less obvious. In this integral, first make the change of variable $u = 2\theta$ to obtain

$$\int_{\Gamma_2} e^{iz^2}\,dz = \frac{1}{2}\int_0^{\pi/2} e^{iR^2\cos(u) - R^2\sin(u)} iRe^{iu/2}\,du.$$

Then

$$\begin{aligned}
\Big|\int_{\Gamma_2} e^{iz^2}\,dz\Big| &\leq \frac{R}{2}\int_0^{\pi/2} |e^{iR^2\cos(u)}||e^{iu/2}|e^{-R^2\sin(u)}\,du \\
&= \frac{R}{2}\int_0^{\pi/2} e^{-R^2\sin(u)}\,du \\
&\leq \frac{R}{2}\left(\frac{\pi}{2\pi R^2}\right) \\
&= \frac{\pi}{4R} \to 0
\end{aligned}$$

as $R \to \infty$. Thus, when we form the sum of the integrals over Γ_1, Γ_2 and Γ_3, and take the limit as $R \to \infty$, we obtain

$$\int_0^\infty [\cos(x^2) + i\sin(x^2)]\,dx - e^{i\pi/4}\int_0^\infty e^{-r^2}\,dr = 0.$$

Since we know that

$$\int_0^\infty e^{-r^2}\,dr = \frac{\sqrt{\pi}}{2},$$

and

$$e^{i\pi/4} = \frac{1}{\sqrt{2}}(1+i),$$

we obtain

$$\int_0^\infty \cos(x^2)\,dx + i\int_0^\infty \sin(x^2)\,dx = \frac{\sqrt{\pi}}{2\sqrt{2}}(1+i).$$

Equating real parts and then imaginary parts on both sides of this equation, we have Fresnel's integrals,

$$\int_0^\infty \cos(x^2)\,dx = \int_0^\infty \sin(x^2)\,dx = \frac{1}{2}\sqrt{\frac{\pi}{2}}.$$

17. Begin with

$$\begin{aligned}
\int_0^{2\pi} \frac{1}{\alpha^2\cos^2(\theta) + \beta^2\sin^2(\theta)}\,d\theta &= \oint_\gamma \frac{1}{\alpha^2(z+1/z)^2/4 - \beta^2(z-1/z)^2/4}\frac{1}{iz}\,dz \\
&= \frac{4}{i}\oint_\gamma \frac{z}{(\alpha^2-\beta^2)z^4 + 2(\alpha^2+\beta^2)z^2 + (\alpha^2-\beta^2)}\,dz.
\end{aligned}$$

Solving for the zeros of the denominator of the integrand, we find that the singularities satisfy

$$z^2 = \frac{\beta-\alpha}{\beta+\alpha} \text{ or } z^2 = \frac{\beta+\alpha}{\beta-\alpha}.$$

Since $\alpha > 0$ and $\beta > 0$,

$$\Big|\frac{\beta-\alpha}{\beta+\alpha}\Big| < 1 \text{ and } \Big|\frac{\beta+\alpha}{\beta-\alpha}\Big| > 1.$$

The simple poles enclosed by the unit circle are the square roots z_1 and z_2 of $(\beta-\alpha)/(\beta+\alpha)$. The residue of the integrand at each of these poles are obtained by a straightforward computation using Corollary 22.1. After some computation, we obtain

$$\operatorname{Res}(f, z_j) = \frac{1}{8\alpha\beta}$$

for $j = 1, 2$. Then

$$\int_0^{2\pi} \frac{1}{\alpha^2\cos^2(\theta) + \beta^2\sin^2(\theta)}\,d\theta = \frac{4}{i}(2\pi i)\frac{2}{8\alpha\beta} = \frac{2\pi}{\alpha\beta}.$$

22.4 Residues and the Inverse Laplace Transform

1. Let $F(z) = z^2/(z-2)^3$. Then F has a pole of order 3 at 2. The residue is

$$\begin{aligned}\operatorname{Res}(e^{tz}F(z), 2) &= \lim_{z\to 2}\frac{1}{2}\frac{d^2}{dz^2}(z^2e^{tz})\\ &= \lim_{z\to 2}(2e^{2t} + 4tze^{2t} + t^2z^2e^{2t})\\ &= (1+4t+2t^2)e^{2t}.\end{aligned}$$

This is $\mathcal{L}^{-1}[F(s)](t)$.

3. $F(z) = z/(z^2+9)$ has simple poles at $\pm 3i$, so compute

$$\operatorname{Res}(e^{tz}F(z), 3i) = \frac{1}{2}e^{3i} \text{ and } \operatorname{Res}(e^{tz}F(z), -3i) = \frac{1}{2}e^{-3i}.$$

Then

$$\mathcal{L}^{-1}[F(s)](t) = \frac{1}{2}(e^{3i} + e^{-3i}) = \cos(3t).$$

5. Let

$$F(z) = \frac{1}{(z-2)^2(z+4)}.$$

F has a double pole at 2 and simple pole at -4. Compute

$$\begin{aligned}\operatorname{Res}(e^{tz}F(z), 2) &= \lim_{z\to 2}\frac{d}{dz}\left(\frac{e^{tz}}{z+4}\right)\\ &= \lim_{z\to 2}\left[\frac{te^{tz}}{z+4} - \frac{e^{tz}}{(z+4)^2}\right]\\ &= \frac{1}{6}te^{2t} - \frac{1}{36}e^{2t}.\end{aligned}$$

Next,

$$\operatorname{Res}(e^{tz}F(z), -4) = \frac{e^{-4t}}{36}.$$

Then

$$\mathcal{L}^{-1}[F(s)](t) = \left(\frac{1}{6}t - \frac{1}{36}\right)e^{2t} + \frac{1}{36}e^{-4t}.$$

7. Let $F(z) = 1/(z+5)^3$. Then F has a pole of order 3 at -5 and we compute

$$\operatorname{Res}(e^{tz}F(z), -5) = \frac{1}{2}t^2e^{-5t} = \mathcal{L}^{-1}[F(s)](t).$$

9. Let $F(z) = 1/(1+z^4)$. Then F has simple poles at the fourth roots of -1, which are

$$z_1 = \frac{1}{\sqrt{2}}(1+i), z_2 = \frac{1}{\sqrt{2}}(-1+i),$$
$$z_3 = \frac{1}{\sqrt{2}}(1-i), z_4 = \frac{1}{\sqrt{2}}(-1-i).$$

The residues are

$$\mathrm{Res}(e^{tz}F(z), z_1) = \frac{1}{2\sqrt{2}(-1+i)}e^{(1+i)t/\sqrt{2}},$$
$$\mathrm{Res}(e^{tz}F(z), z_2) = \frac{1}{2\sqrt{2}(1+i)}e^{(-1+i)t/\sqrt{2}},$$
$$\mathrm{Res}(e^{tz}F(z), z_3) = \frac{1}{2\sqrt{2}(-1-i)}e^{(1-i)t/\sqrt{2}},$$
$$\mathrm{Res}(e^{tz}F(z), z_4) = \frac{1}{2\sqrt{2}(1-i)}e^{(-1-i)t/\sqrt{2}}.$$

Upon rearranging the sum of these residues, we obtain

$$\mathcal{L}^{-1}[F(s)](t) = -\frac{1}{\sqrt{2}}\sinh\left(\frac{t}{\sqrt{2}}\right)\cos\left(\frac{t}{\sqrt{2}}\right) + \frac{1}{\sqrt{2}}\cosh\left(\frac{t}{\sqrt{2}}\right)\sin\left(\frac{t}{\sqrt{2}}\right).$$

Chapter 23

Conformal Mappings and Applications

23.1 Conformal Mappings

1. The analysis is like that of Problem 6. If $z = re^{i\theta}$, then $w = z^3 = r^3e^{3i\theta}$. If $\pi/6 \leq \theta \leq \pi/3$, then $\pi/2 \leq 3\theta \leq \pi$. The given sector is mapped to the second quadrant in the $w-$ plane.
2. Let $z = re^{i\theta}$. Then
$$w = u + iv = \frac{1}{2}\left(re^{i\theta} + \frac{1}{r}e^{-i\theta}\right).$$
Using Euler's formula for $e^{i\theta}$ and $e^{-i\theta}$, we obtain
$$u = \frac{1}{2}\left(r + \frac{1}{r}\right)\cos(\theta), v = \frac{1}{2}\left(r - \frac{1}{r}\right)\sin(\theta).$$

In each part of Problems 3 - 5, we use the MAPLE plotting routine *conformal* to generate the image of the given rectangle under the mapping. The rectangles themselves are not shown in this plot but are easily sketched separately.

3. The mapping is $w = e^z$, which was discussed in Example 23.2. The images of the rectangles of Parts (a) through (e) are shown in Figures 23.1 - 23.5, respectively.
4. **Hint** Write
$$w = u + iv = \cos(x + iy) = \cos(x)\cosh(y) - i\sin(x)\sinh(y)$$
so
$$u = \cos(x)\cosh(y), v = -\sin(x)\sinh(y).$$
Examine the image of an arbitrary horizontal line under this mapping, then consider vertical lines.
5. The mapping is $w = 4\sin(z)$, which was discussed in Example 23.2. The images of the specified rectangles are shown in Figures 23.6 through 23.10.
6. **Hint** Write $z = re^{i\theta}$ in polar form.

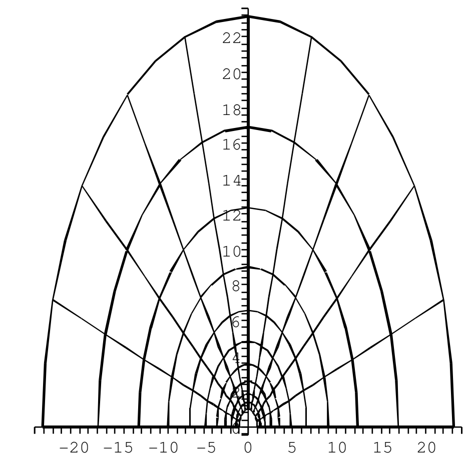

Figure 23.1: Problem 3(a).

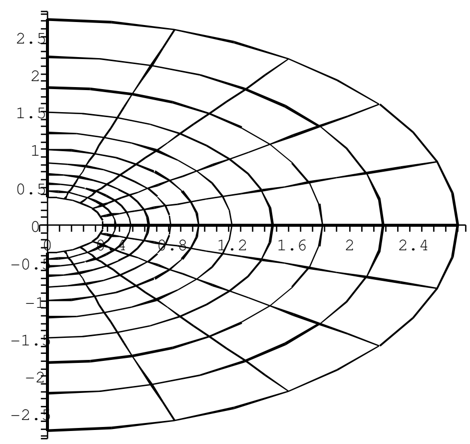

Figure 23.2: Problem 3(b).

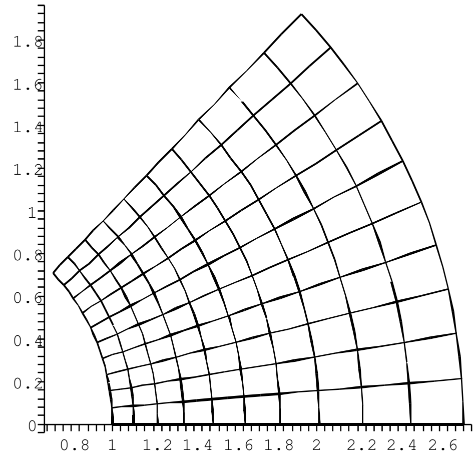

Figure 23.3: Problem 3(c).

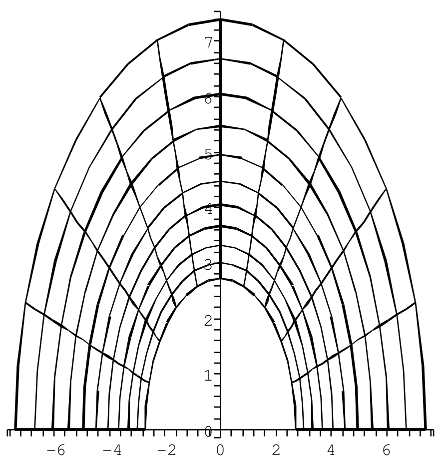

Figure 23.4: Problem 3(d).

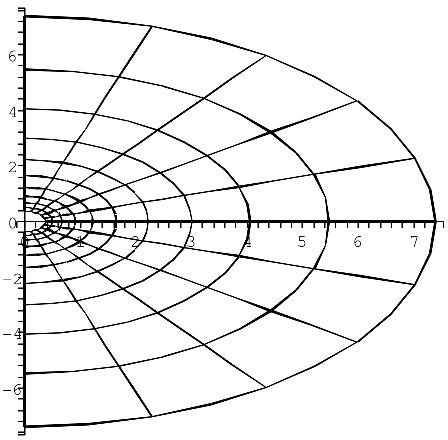

Figure 23.5: Problem 3(e).

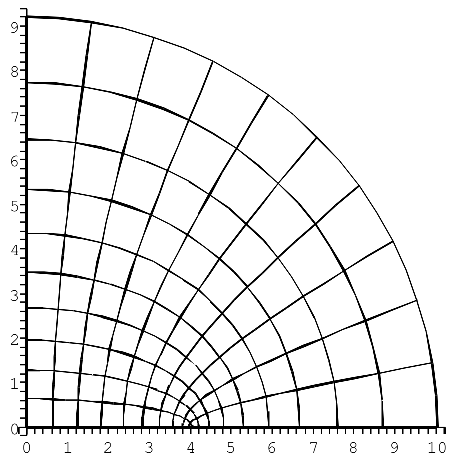

Figure 23.6: Problem 5(a).

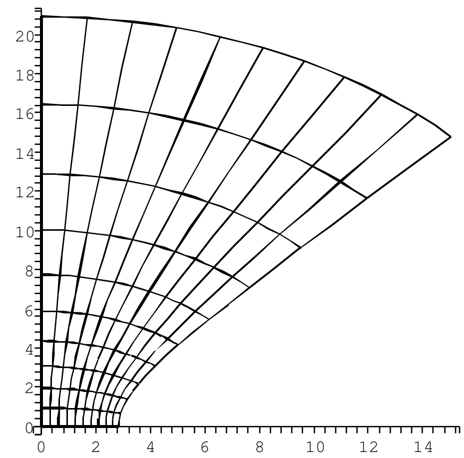

Figure 23.7: Problem 5(b).

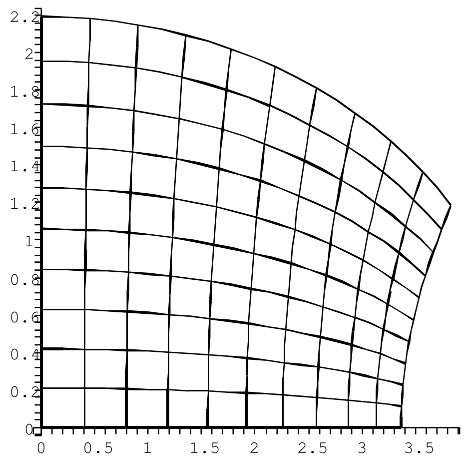

Figure 23.8: Problem 5(c).

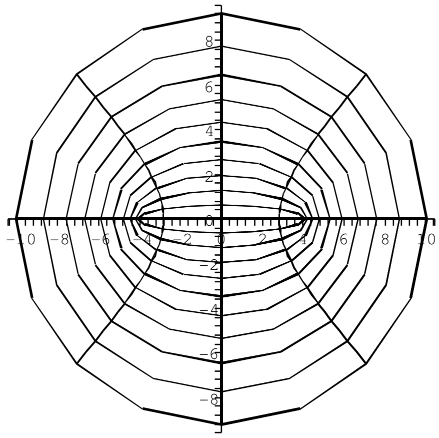

Figure 23.9: Problem 5(d).

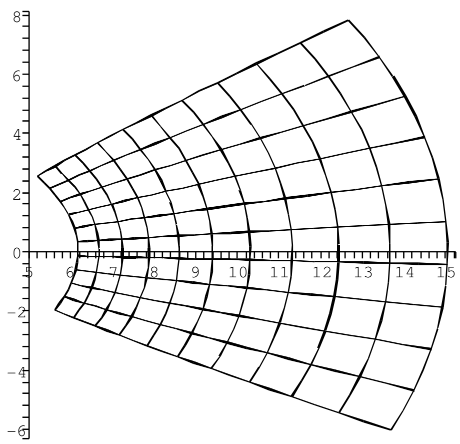

Figure 23.10: Problem 5(e).

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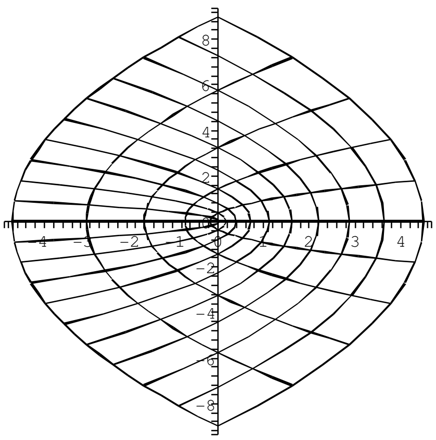

Figure 23.11: Problem 7.

7. Write
$$w = 2z^2 = 2(x+iy)^2 = 2(x^2-y^2) + 4ixy.$$
The vertical line $x = 0$ maps to $u = -2y^2, v = 0$, which is the negative $u-$ axis. Other vertical lines $x = a$ map to parabolas
$$u = 2a^2 - \frac{v^2}{8a^2}$$
having intercepts at $(2a^2, 0)$ and opening to the left. The horizontal line $y = 0$ maps to $u = 2y^2 \geq 0, v = 0$, the positive $u-$ axis. Other horizontal lines $y = b$ map onto the parabolas
$$u = \frac{v^2}{8b^2} - 2b^2$$
having intercepts $(-2b^2, 0)$ and opening right.

Figure 23.11 shows the image of the rectangle defined by $0 \leq x \leq 3/2, -3/2 \leq y \leq 3/2$.

9. Using some of the analysis done for Problem 2, a half-line $\theta = k$ maps to points $u + iv$ with
$$u = \frac{1}{2}\left(r + \frac{1}{r}\right)\cos(k), v = \frac{1}{2}\left(r - \frac{1}{r}\right)\sin(k).$$
Assuming that $\cos(k)$ and $\sin(k)$ are not zero, then a little algebraic manipulation gives us
$$\frac{u^2}{\cos^2(k)} - \frac{v^2}{\sin^2(k)} = 1$$
which is the equation of a hyperbola. The foci are $(\pm c, 0)$, where
$$c^2 = \cos^2(k) + \sin^2(k) = 1.$$
We must separately consider the cases the $\sin(k) = 0$, so $k = n\pi$, or $\cos(k) = 0$, so $k = (2n+1)\pi/2$, for n any integer.

The case $k = n\pi$ gives us

$$u = \frac{1}{2}\left(r + \frac{1}{r}\right)(-1)^n, v = 0,$$

which is the half-interval $u \geq 1, v = 0$ if n is even and the half-interval $u \leq -1, v = 0$ if n is odd.

The case $k = (2n+1)\pi/2$ gives us $u = 0, -\infty < v < \infty$, which is the imaginary axis in the $w-$ plane.

11. Invert the mapping to obtain

$$z = \frac{5 + iw}{2 - w}.$$

Then

$$\begin{aligned} z + \overline{z} = 2\text{Re}(z) &= 2\text{Re}\left(\frac{5 - v + iu}{2 - u - iv}\right) \\ &= \frac{2((5-v)(2-u) - uv)}{(u-2)^2 + v^2} = \frac{20 - 4v - 10u}{(u-2)^2 + v^2}. \end{aligned}$$

Next,

$$\frac{1}{2i}(z - \overline{z}) = \text{ Im}(z) = \frac{(2-u)u + (5-v)v}{(u-2)^2 + v^2}.$$

Substitute these into the equation of the given line and clear fractions to obtain

$$20 - 4v - 10u - 3(2u - u^2 + 5v - v^2) - 5(u^2 - 4u + 4 + v^2) = 0.$$

Simplify this expression and complete the square to write

$$(u-1)^2 + \left(v + \frac{19}{4}\right)^2 = \frac{377}{16}.$$

This is the equation of a circle of radius $\sqrt{377}/4$ and having center $(1, -19/4)$.

13. If Re$(z) = -4$, then $(z + \overline{z})2 = -4$, so $z + \overline{z} = -8$. Now, if $w = 2i/z$, then $z = 2i/w$, so

$$z + \overline{z} = \frac{2i}{w} - \frac{2i}{\overline{w}} = -8.$$

Multiply this by $w\overline{w}$ and rearrange terms to obtain

$$8w\overline{w} - 2i(w - \overline{w}) = 0.$$

Now put $w = u + iv$ to obtain

$$2(u^2 + v^2) + v = 0.$$

Complete the square to write

$$u^2 + \left(v + \frac{1}{4}\right)^2 = \frac{1}{4}.$$

This is the equation of a circle of radius $1/2$ centered at $(0, -1/4)$ in the $w-$ plane, and is the image of the vertical line $x = -4$ under the given mapping.

15. Solve the mapping for z in terms of w:

$$z = \frac{-1}{w+i}.$$

Substitute this into the given line to obtain

$$\frac{1}{2}\left(\frac{-1}{w+i} - \frac{1}{\overline{w}-i}\right) + \frac{1}{2i}\left(\frac{-1}{w+i} + \frac{1}{\overline{w}-i}\right) = 4.$$

After multiplying this by $2i(w+i)(\overline{w}-i)$ and rearranging terms, put $w = u+iv$ to obtain

$$4(u^2+v^2) + 7v + u = -3.$$

Complete the square in this equation to obtain

$$\left(u+\frac{1}{8}\right)^2 + \left(v+\frac{7}{8}\right)^2 = \frac{1}{32},$$

the equation of a circle with radius $\sqrt{2}/8$ and center $(-1/8, -7/8)$.

17. Substitute these values into equation (23.1) and solve for w to obtain

$$w = \frac{(3+22i)z + 4 - 75i}{(2+3i)z - (21-4i)}.$$

19. Substitute the given values into equation (3.1) to obtain

$$(1-w)(1+2i)(-1)(3-z) = (1-z)(1)(1+i)(1+i-w).$$

Solve for w:

$$w = \frac{(1+4i)z - (3+8i)}{(2+3i)z - (4+7i)}.$$

21. Since $w_3 = \infty$, substitute the given values into equation (23.1), but leave out the terms involving w_3 to obtain

$$(1+i-w)(1-2i)(4-z) = (1-z)(-2+2i)(4-2i).$$

Solve for w:

$$w = \frac{(33+i)z - (48+16i)}{5(z-4)}.$$

23. Given z_2, z_3, z_4, let P be the unique bilinear transformation that maps

$$z_2 \to 1, z_3 \to 0, z_4 \to \infty.$$

Then

$$[z_1, z_2, z_3, z_4] = P(z_1).$$

Now let T be any bilinear transformation. Then

$$[T(z_1), T(z_2), T(z_3), T(z_4)] = R(T(z_1)),$$

where R is the unique bilinear mapping that sends

$$T(z_2) \to 1, T(z_3) \to 0, T(z_4) \to \infty.$$

Then $R \circ T = P$. Then

$$[T(z_1), T(z_2), T(z_3), T(z_4)] = R(T(z_1)) = R(T(z_1))$$
$$= P(z_1) = [z_1, z_2, z_3, z_4].$$

25. In the definition of cross ratio, w_2, w_3, w_4 all lie on an (extended) line, the real axis. Since circles/lines map to circles/lines under bilinear transformations, then $[z_1, z_2, z_3, z_4]$ is real if and only if z_1, z_2, z_3, z_4 all lie on the same line or circle.

26. **Hint** Write
$$T(z) = \frac{az+b}{cz+d}$$
and consider the equation
$$T(z) = z = \frac{az+b}{cz+d}.$$

27. If we require that a conformal mapping be differentiable, then immediately $T(z) = \overline{z}$ is disqualified, because we have seen that this function is not differentiable. It is also easy to show that the conjugation mapping reverses sense of orientation. For example, let C_1 be the nonnegative real axis and C_2 the nonnegative imaginary axis in the $z-$ plane. The sense of rotation from C_1 to C_2 is counterclockwise, and the angle between these curves is $\pi/2$. However, T maps C_1 to C_1, and C_2 to the negative imaginary axis, a clockwise rotation. Therefore again T is not conformal.

29. Let
$$T(z) = \frac{az+b}{cz+d}.$$
If T is not a translation or the identity mapping, then by the argument used for Problem 26, T can have at most two fixed points. Therefore, if T has three fixed points, then either T is a translation or the identity mapping. But a translation has no fixed point, hence T is the identity mapping.

23.2 Construction of Conformal Mappings

There may be many conformal mappings between two given domains. In these solutions presented below, a conformal mapping is produced, together with some of the thought that went into its construction, but many other solutions are possible.

In particular, note that when circles and/or lines are involved as boundaries of domains, a bilinear transformation may serve in producing a conformal mapping. In other circumstances, we may have to construct a conformal mapping using other differentiable functions.

1. We can map the line $\text{Re}(z) = 0$ onto the circle $|w| = 4$ by a bilinear transformation. The domain $\text{Re}(z) < 0$ consists of all numbers to the left of the imaginary axis, which is the boundary. Choose three points on this axis, ordered upward so the region $\text{Re}(z) < 0$ is on the left as we walk up the line. Choose three points on the image circle $|w| = 4$, counterclockwise so the interior of this circle is on our left as we walk around it in this order. Convenient choices are
$$z_1 = -i, z_2 = 0, z_3 = i, w_1 = -4i, w_2 = 4, w_3 = 4i.$$
The bilinear transformation mapping $z_j \to w_j$
$$w = T(z) = 4\left(\frac{1+z}{1-z}\right).$$
As a check, $z = -1$, which has negative real part, maps to 0, interior to the circle $|w| < 4$. Thus w maps $\text{Re}(z) < 0$ to $|w| < 4$.

Of course, other choices for the $z_j\,'s$ and $w_j\,'s$ may result in different mappings between the two given domains.

3. Both domains are circles, having different radii (3 and 6) and centers. Thus map $|z| < 3$ onto $|w - 1 + i| < 6$ by using a scaling factor of 2 and then a translation to superimpose the center of the initial domain onto the center of the target domain. These two effects are achieved by the bilinear transformation
$$w = 2z + 1 - i.$$

5. We will need an inversion (at some stage) because we are mapping the interior of a disk to the exterior of a disk. First translate by using $w_1 = z + 2i$, so the image disk in the w_1- plane has center $(0, 0)$. Next invert by
$$w_2 = \frac{1}{z + 2i}.$$
Next scale by a factor of 2 to match radii of boundaries,
$$w_3 = 2w_2 = \frac{2}{z + 2i}.$$
Finally, translate the center by 3 to form
$$w = w_3 + 3 = \frac{2}{z + i} + 3 = \frac{3z + 2 + 6i}{z + 2i}.$$

7. The solution to this problem requires some familiarity with the gamma and beta functions, which are discussed in Section 15.3.

    To show that f maps the upper half-plane onto the given rectangle, we will evaluate the function at -1, 0, 1 and ∞ and then show that these are the vertices of that rectangle.

    First, it is obvious that $f(0) = 0$. Next,
$$\begin{aligned} f(1) &= 2i\int_0^1 (\xi^2 - 1)^{-1/2}\xi^{-1/2}\,d\xi \\ &= 2i\int_0^1 \frac{(1-\xi^2)^{-1/2}}{i}\xi^{-1/2}\,d\xi = 2\int_0^1 (1-\xi^2)^{-1/2}\xi^{-1/2}\,d\xi. \end{aligned}$$
Let $\xi = u^{1/2}$ to obtain (in terms of the beta and gamma functions)
$$\begin{aligned} f(1) &= \int_0^1 (1-u)^{-1/2}u^{-3/4}\,du \\ &= B(1/4, 1/2) = \frac{\Gamma(1/4)\Gamma(1/2)}{\Gamma(3/4)} = c. \end{aligned}$$
Next calculate
$$f(-1) = 2i\int_0^{-1} (\xi^2 - 1)^{-1/2}\xi^{-1/2}\,d\xi.$$
Let $\xi = -u$ to obtain
$$\begin{aligned} f(-1) &= 2i\int_0^1 (1-u^2)^{-1/2}u^{-1/2}\,du \\ &= iB(1/4, 1/2) = \frac{i\Gamma(1/4)\Gamma(1/2)}{\Gamma(3/4)} = ic. \end{aligned}$$

Finally, calculate

$$\begin{aligned} f(\infty) &= 2i\int_0^\infty (\xi+1)^{-1/2}(\xi-1)^{-1/2}\xi^{-1/2}\,d\xi \\ &= 2i\int_0^1 (\xi+1)^{-1/2}(\xi-1)^{-1/2}\xi^{-1/2}\,d\xi \\ &+ 2i\int_1^\infty (\xi+1)^{-1/2}(\xi-1)^{-1/2}\xi^{-1/2}\,d\xi. \end{aligned}$$

The first integral on the last line is $B(1/4, 1/2)$. In the second integral, put $\xi = 1/u$ to obtain

$$\begin{aligned} f(\infty) &= c + 2i\int_1^0 \left(\frac{1+u}{u}\right)^{-1/2}\left(\frac{1-u}{u}\right)^{-1/2} u^{1/2}\left(\frac{1}{u^2}\right) du \\ &= c + 2i\int_0^1 (1-u^2)^{-1/2}u^{-1/2}\,du = (1+i)c. \end{aligned}$$

9. Because the boundary of the wedge in the $w-$ plane is not a line or circle, we cannot construct a bilinear mapping to solve this problem. However, wedges suggest using polar representations. Let $z = re^{i\theta}$ for $0 < \theta < \pi$. These are points in the upper half-plane. Let

$$w = z^{1/3} = r^{1/3}e^{i\theta/3} = \rho e^{i\varphi},$$

where $\rho > 0$ and $0 < \varphi < \pi/3$. This mapping is conformal because

$$\frac{dw}{dz} = \frac{1}{3}z^{-2/3} \neq 0$$

for z in the upper half-plane, and the mapping takes the open upper half-plane onto the open wedge $0 < \theta < \pi/2$.

23.3 Conformal Mapping Solutions of Dirichlet Problems

In this section we use conformal mappings between domains to solve certain Dirichlet problems. In each case, one could use other conformal mappings than those used in these solutions.

1. Use Poisson's integral formula to obtain

$$u(r\cos(\theta), r\sin(\theta)) = \frac{1}{2\pi}\int_0^{2\pi} \frac{r(\cos(\varphi) - \sin(\varphi))(1-r^2)}{1 + r^2 - 2r\cos(\varphi - \theta)}\,d\varphi.$$

3. Begin by mapping the upper half-plane Im$(z) > 0$ to the unit disk $|w| < 1$. One such mapping is

$$w = T(z) = \frac{i-z}{i+z}.$$

The solution of this dirichlet problem for the upper half-plane is

$$u(x, y) = \text{Re}(f(z)),$$

where C is the boundary of the upper half-plane (the real line) and

$$f(z) = \frac{1}{2\pi i}\int_C g(\xi)\left(\frac{T(\xi) + T(z)}{T(\xi) - T(z)}\right)\frac{T'(\xi)}{T(\xi)}\,d\xi.$$

On C, parametrize $\xi = t$ for $-\infty < t < \infty$, going from left to right to preserve positive orientation. Now all we must do is compute the quantity to be integrated. First,

$$T'(z) = \frac{-2i}{(i+z)^2}.$$

Next,

$$\begin{aligned}&\frac{T(\xi)+T(z)}{T(\xi)-T(z)}\frac{T'(\xi)}{T(\xi)}\\ &= \frac{(i-t)/(i+t)+(i-z)/(i+z)}{(i-t)(i+t)-(i-z)(i+z)}\left(\frac{i+t}{i-t}\right)\left(\frac{-2i}{(i+t)^2}\right).\end{aligned}$$

After some algebra this simplifies to

$$\frac{-2(1+tz)}{(z-t)(1+t^2)}.$$

Put $z = x+iy$ to simplify this expression further to write it as

$$\frac{(1+tx)(x-t)-ty^2-iy(1+t^2)}{(x-t)^2+y^2}\frac{-2}{1+t^2}.$$

Substitute this into the integral and extract the real part, recalling that $g(t)$ is real-valued, to obtain

$$u(x,y) = \frac{y}{\pi}\int_{-\infty}^{\infty}\frac{g(t)}{(x-t)^2+y^2}\,dt.$$

This agrees with the solution of the Dirichlet problem for the upper half-plane obtained in Chapter 18.

5. The bilinear mapping

$$w = T(z) = \frac{1}{R}(z-z_0)$$

takes the disk $|z-z_0| < R$ to the unit disk $|w| < 1$. On C, the boundary of $|z-z_0| < R$, we can write $\xi = z_0 + Re^{it}$ as t varies from 0 to 2π. Compute

$$\frac{T(\xi)+T(z)}{T(\xi)-T(z)}\frac{T'(\xi)}{T(\xi)}\,d\xi = \frac{Re^{it}+(z-z_0)}{Re^{it}-(z-z_0)}\frac{ie^{it}}{e^{it}}\,dt.$$

Since $g(\xi) = g(z_0+Re^{it})$ is real-valued, we can write the solution

$$u(x,y) = \frac{1}{2\pi}\int_0^{2\pi} g(x_0+R\cos(t), y_0+R\sin(t))K(x,y,t)\,dt,$$

where

$$\begin{aligned}&K(x,y,t) = \text{Re}\left[\frac{R\cos(t)+x-x_0+i(R\sin(t)+y-y_0)}{R\cos(t)-x+x_0+i(R\sin(t)-y+y_0)}\right]\\ &= \frac{R^2-(x-x_0)^2-(y-y_0)^2}{R^2+(x-x_0)^2+(y-y_0)^2-2R(x-x_0)\cos(t)-2R(y-y_0)\sin(t)}.\end{aligned}$$

7. First construct a conformal mapping of the strip S onto the unit circle. Begin with $w_1 = \pi iz/2$, which rotates the strip $\pi/2$ radians counterclockwise and expands it to the strip $-\pi/2 \leq \text{Re}(w_1) \leq \pi/2$. The reason for doing this is to exploit the mapping of Example 23.3. From this, put $w_2 = \sin(w_1)$. This maps the w_1- strip onto the upper half-plane. Finally, find the bilinear mapping that maps

$$-1 \to -i, 0 \to 1, 1 \to i$$

to obtain

$$w = \frac{i - w_2}{i + w_2},$$

mapping the upper half-plane of the w_2- plane to the unit disk in the $w-$ plane. The end result of this sequence of mappings is

$$w = T(z) = \frac{i - \sin(\pi iz/2)}{i + \sin(\pi iz/2)}.$$

We can write this as

$$w = \frac{1 + \sinh(\pi z/2)}{1 - \sinh(\pi z/2)}.$$

The solution of the Dirichlet problem is

$$u(x, y) = \text{Re}\left[\frac{1}{2\pi i}\int_C g(\xi)\left(\frac{T(\xi) + T(z)}{T(\xi) - T(z)}\right)\frac{T'(\xi)}{T(\xi)}\,d\xi\right].$$

Since $g(\xi) = 0$ along the upper and lower edges of S, the solution simplifies to

$$u(x, y) = \text{Re}\left[\frac{1}{2\pi i}\int_K g(\xi)\left(\frac{T(\xi) + T(z)}{T(\xi) - T(z)}\right)\frac{T'(\xi)}{T(\xi)}\,d\xi\right],$$

where K is the segment of the imaginary axis from i to $-i$. On K,

$$g(\xi) = g(it) = g(0, t) = 1 - |t|$$

and

$$T(\xi) = T(it) = \frac{i + \sin(\pi t/2)}{i - \sin(\pi t/2)}.$$

We need to compute

$$\frac{T'(it)}{T(it)}\,d(it) = \frac{\pi\cos(\pi t/2)}{1 + \sin^2(\pi t/2)}\,dt,$$

and

$$\frac{T(\xi) + T(z)}{T(\xi) - T(z)} = \frac{|T(\xi)|^2 + 2i\text{Im}(T(z)\overline{T(\xi)}) + |T(z)|^2}{|T(\xi)|^2 - 2\text{Re}(T(z)\overline{T(\xi)}) + |T(z)|^2}.$$

Now $|T(\xi)|^2 = 1$, since T maps the boundary of S onto the unit circle $|w| = 1$. Finally, we can write the solution

$$u(x, y) = \int_1^{-1}\frac{(1 - |t|)\cos(\pi t/2)}{1 + \sin^2(\pi t/2)}\,\frac{\text{Im}(T(z)\overline{T(it)})}{1 - 2\text{Re}(T(z)\overline{T(it)}) + |T(z)|^2}\,dt.$$

23.4 Models of Plane Fluid Flow

1. Begin with

$$f(z) = \cos(z) = \cos(x)\cosh(y) - i\sin(x)\sinh(y) = \varphi(x, y) + i\psi(x, y).$$

Equipotential curves are graphs of $\cos(x)\cosh(y) = c$ (Figure 23.12) and streamlines (Figure 23.13) are graphs of $\sin(x)\sinh(y) = k$.

Since $f'(z) = -\sin(z) = 0$ if $z = n\pi$, with n any integer, this flow has infinitely many stagnation points.

The velocity is

$$\begin{aligned} V(x, y) &= \overline{f'(z)} = \overline{-\sin(z)} \\ &= -\sin(x)\cosh(y) + i\cos(x)\sinh(y) = u(x, y) + iv(x, y). \end{aligned}$$

Then

$$u(x, y) = -\sin(x)\cosh(y), v(x, y) = \cos(x)\sinh(y).$$

This has divergence zero. Further, using Green's theorem, it is routine to check that the flux of the flow across any closed path is zero, so the flow is solenoidal.

The circulation is also zero about any closed path, so there is no source or sink for this flow.

3. Write $a = Ke^{i\theta}$ and $z = x + iy$ to compute

$$\begin{aligned} f(z) &= az = Ke^{i\theta}(x + iy) \\ &= K[x\cos(\theta) - y\sin(\theta)] + iK[x\sin(\theta) + y\cos(\theta)]. \end{aligned}$$

With $f(z) = \varphi(x, y) + i\psi(x, y)$, we can identify equipotential curves as graphs of

$$\varphi(x, y) = K[x\cos(\theta) - y\sin(\theta)] = \text{ constant}$$

and streamlines as graphs of

$$\psi(x, y) = K[x\sin(\theta) + y\cos(\theta)] = \text{ cosntant.}$$

Since θ is a given constant, the equipotential lines are straight lines

$$y = \cot(\theta)x + b$$

having slope $\cot(\theta)$, while the streamlines are straight ines

$$y = -\tan(\theta)x + c,$$

of slope $-\tan(\theta)$. The equipotential lines and streamlines form orthogonal families, since $-\cot(\theta)\tan(\theta) = -1$, so the slopes of equipotential lines and streamlines are negative reciprocals of each other.

The velocity is

$$V(z) = V(x, y) = \overline{f'(z)} = \overline{a} = Ke^{-i\theta},$$

a constant velocity. Since $f'(z) \neq 0$, there are no stagnation points, hence no source or sink.

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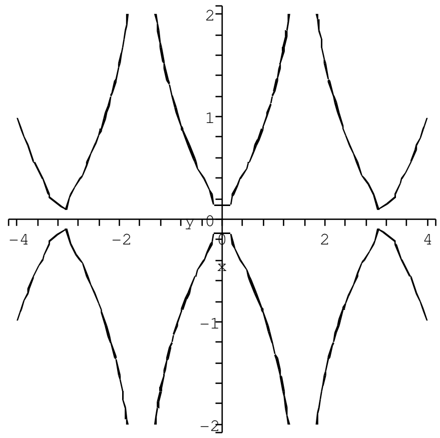

Figure 23.12: Equipotential curves in Problem 1.

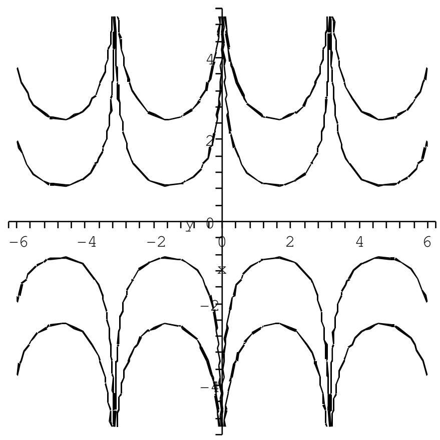

Figure 23.13: Streamlines in Problem 1.

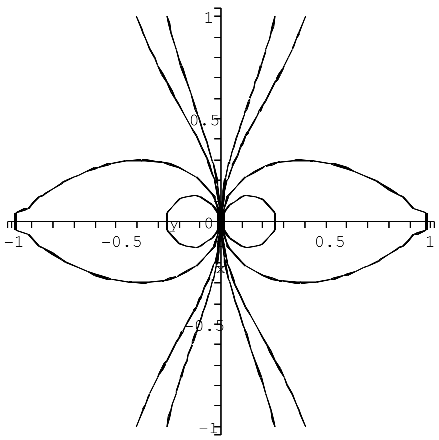

Figure 23.14: Equipotential curves in Problem 5.

5. Write

$$\begin{aligned} f(z) &= K\left(x+iy+\frac{1}{x+iy}\right) \\ &= \frac{Kx(x^2+y^2+1)}{x^2+y^2}+i\frac{Ky(x^2+y^2-1)}{x^2+y^2}. \end{aligned}$$

Equipotential curves are graphs of

$$\varphi(x,y)=\frac{Kx(x^2+y^2+1)}{x^2+y^2}=c_1$$

Some equipotential curves are shown in Figure 23.14 for $K=1$.

Streamlines are graphs of

$$\psi(x,y)=\frac{Ky(x^2+y^2-1)}{x^2+y^2}=c_2.$$

For $c_1=0$, we get the equipotential curve $x=0$, the imaginary axis. For $c_1\neq 0$, set $c_1=kb$ we can write

$$y^2=-\frac{x(x^2-bx+1)}{x-b}.$$

Figure 23.15 shows some streamlines for $K=1$.

The velocity of the flow is

$$\overline{f'(z)}=K\left(1-\frac{1}{\overline{z}^2}\right).$$

There is a stagnation point at $z=1$ and at $z=-1$.

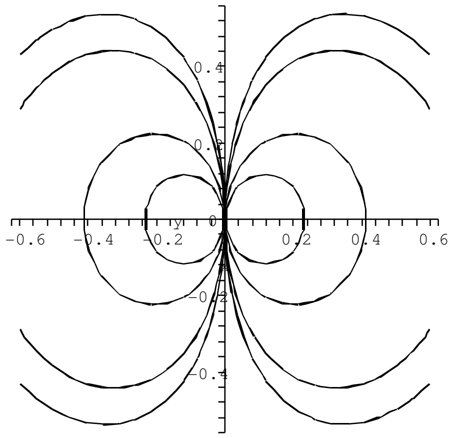

Figure 23.15: Streamlines in Problem 5.

7. From the solution to Problem 6, we have

$$(f'(z))^2 = \frac{9a^4K^2}{(z - ia\sqrt{3}/2)^2(z + ia\sqrt{3}/2)^2}.$$

By Blasius's theorem, the thrust of the fluid outside the barrier $4x^2 + 4(y-a)^2 = a^2$ is the vector $A\mathbf{i} + B\mathbf{j}$, where

$$\begin{aligned}
A - Bi &= \frac{1}{2} i\rho \oint_\gamma (f'(z))^2\, dz \\
&= \frac{i\rho}{2} \oint_\gamma \frac{9a^4K^2}{(z - ia\sqrt{3})/2^2(z + ia\sqrt{3}/2)^2}\, dz \\
&= \pi\rho \mathrm{Res}\left((f'(z))^2, ia\sqrt{3}/2\right) \\
&= -\pi\rho(9a^4K^2)\frac{d}{dz}\left[\left(z + \frac{ia\sqrt{3}}{2}\right)^{-2}\right]_{z=ia\sqrt{3}/2} \\
&= -9\pi a^4K^2\rho(-2(ia\sqrt{3})^{-3}) \\
&= -\frac{18\pi a^4K^2\rho}{3\sqrt{3}a^3} i.
\end{aligned}$$

The vertical component of the thrust is

$$B = 2\sqrt{3}\pi a\rho K^2.$$

8. Write

$$\begin{aligned} f(z) &= K\mathrm{Log}\left(\frac{z-a}{z-b}\right) \\ &= \frac{K}{2}\ln\left(\frac{|z|^2+|a|^2-2\mathrm{Re}(a\overline{z})}{|z|^2+|b|^2-2\mathrm{Re}(b\overline{z})}\right) + iK\arg\left(\frac{z-a}{z-b}\right) \\ &= \varphi(x,y)+i\psi(x,y). \end{aligned}$$

To analyze the equipotential curves, let $a = a_1 + ia_2$, $b = b_1 + ib_2$ and $z = x + iy$. An equipotential curve $\varphi(x, y) = c$ is the graph of

$$(x^2+y^2)(1-c) - 2(a_1x + a_2y - c(b_1x + b_2y)) + a_1^2 + a_2^2 - c(b_1^2 + b_2^2) = 0.$$

If $c = 1$, this is the line

$$(a_1 - b_1)x + (a_2 - b_2)y = \frac{1}{2}[(a_1^2 + a_2^2) - (b_1^2 + b_2^2)].$$

If $c \neq 1$, we get an equation

$$\left(x - \frac{a_1 - cb_1}{1-c}\right)^2 + \left(y - \frac{a_2 - cb_2}{1-c}\right)^2 = r^2,$$

where

$$r^2 = \frac{c}{(1-c)^2}[(a_1 - b_1)^2 + (a_2 - b_2)^2].$$

These are circles if $c > 0$. Notice that the centers of these circles all lie on the line

$$(a_2 - b_2)x - (a_1 - b_1)y + a_1b_2 - a_2b_1 = 0.$$

This line connects $a = a_1 + a_2i$ and $b = b_1 + b_2i$ in the complex plane. This line containing the centers of the equipotential curves (for $c \neq 1$) is orthogonal to the equipotential curve obtained when $c = 1$, and these two lines intersect at $((a_1 + b_1)/2, (a_2 + b_2)/2)$, which is the midpoint of the segment connecting a and b in the complex plane.

For the streamlines, write

$$\begin{aligned} \frac{z-a}{z-b} &= \frac{(z-a)(\overline{z}-\overline{b})}{|z-b|^2} \\ &= \frac{|z|^2 - (a\overline{z} + \overline{b}z) + a\overline{b}}{|z-b|^2} \\ &= \frac{x^2+y^2-[(a_1+b_1)x + (a_2+b_2)y] + a_1b_1 + a_2b_2}{(x-b_1)^2 + (y-b_2)^2} \\ &\quad + i\frac{a_2b_1 - a_1b_2 - x(a_2-b_2) + y(a_1-b_1)}{(x-b_1)^2+(y-b_2)^2}. \end{aligned}$$

A streamline $\arg((z - a)/(z - b)) = k$ has the form

$$\frac{a_2b_1 - a_1b_2 - (a_2 - b_2)x + y(a_1 - b_1)}{x^2 + y^2 - [(a_1 + b_1)x + (a_2 + b_2)y] + a_1b_1 + a_2b_2} = k.$$

If $k = 0$ we obtain the line

$$(a_2 - b_2)x - (a_1 - b_1)y + a_1b_2 - a_2b_1 = 0,$$

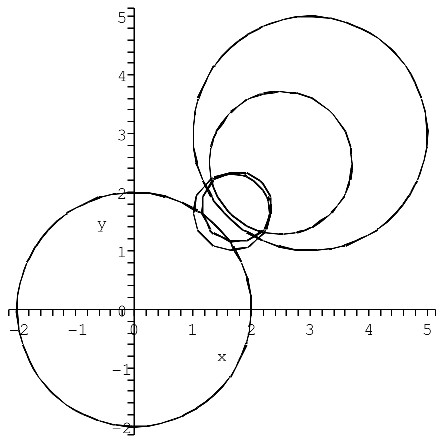

Figure 23.16: Equipotential curves in Problem 8.

which connects a to b in the complex plane. If $k \neq 0$, we obtain

$$\left(x + \frac{a_2 - b_2 - k(a_1 + b_1)}{2k}\right)^2 + \left(y - \frac{a_1 - b_1 + k(a_2 + b_2)}{2k}\right)^2 = r^2,$$

where

$$r^2 = \frac{1 + k^2}{4k^2}\left[(a_1 - b_1)^2 + (a_2 - b_2)^2\right].$$

This is the equation of a circle of radius r. The centers of these circles lie on the line

$$(a_1 - b_1)x + (a_2 - b_2)y - \frac{1}{2}[(a_1 - b_1)^2 + (a_2 - b_2)^2] = 0.$$

This is the perpendicular bisector of the segment between a and b, and each circle passes through both a and b.

Figure 23.16 shows some equipotential curves for the case $a = 1 + i$ and $b = 2 + 2i$. Now these curves are are circles with centers on the line $y = x$. Figure 23.17 shows some streamlines for this case. Centers of the streamlines lie on the perpendicular bisector of the line $y = x$ between $1 + i$ and $2 + 2i$.

9. Write

$$f(z) = K \log(z - z_0) = K \ln|z - z_0| + iK \arg(z - z_0),$$

so equipotential curves are graphs of

$$\varphi(x, y) = K \ln|z - z_0| = c,$$

which are concentric circles about z_0, and streamlines are graphs of

$$\psi(x, y) = iK \arg(z - z_0) = k.$$

These are half-lines emanating from z_0.

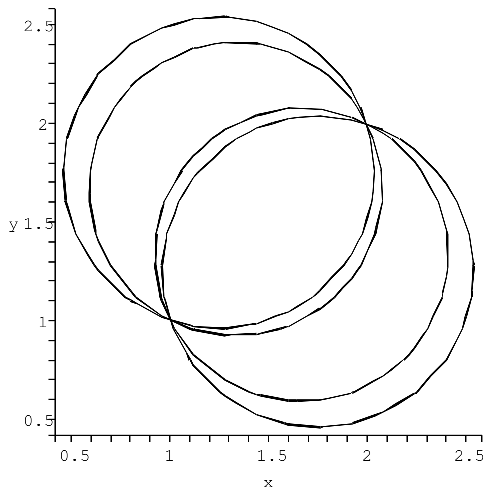

Figure 23.17: Streamlines in Problem 8.

This flow has no stagnation points. The velocity is

$$\overline{f'(z)} = \frac{\overline{K}}{|z - z_0|^2}\,(x - x_0 + i(y - y_0)) = u(x, y) + iv(x, y).$$

For any circle $\gamma : |z - z_0| = r$, compute

$$\begin{aligned}
&\oint_\gamma -v\,dx + u\,dy \\
&= \int_0^{2\pi} \left[-\frac{K}{r^2}(r\sin(t))(-r\sin(t)) + \frac{K}{r^2}(r\cos(t))(r\cos(t)) \right] dt \\
&= 2\pi K.
\end{aligned}$$

Therefore z_0 is a source if $K > 0$ and a sink if $K < 0$.

10. **Hint** Using part of the solution of Problem 8, write

$$\begin{aligned}
f(z) &= \frac{m - ik}{2\pi} \left[\frac{|z|^2 + |a|^2 - 2\mathrm{Re}(a\overline{z})}{|z|^2 + |b|^2 - 2\mathrm{Re}(b\overline{z})} + i\arg\left(\frac{z - a}{z - b}\right) \right] \\
&= \varphi(x, y) + i\psi(x, y).
\end{aligned}$$

Compute

$$f'(z) = \frac{m - ik}{2\pi} \left(\frac{a - b}{(z - a)(z - b)} \right).$$

Since the velocity is

$$V(x, y) = \overline{f'(z)} = u(x, y) + iv(x, y),$$

then

$$f'(z) = u(x, y) - iv(x, y)$$

so

$$f'(z)\,dz = (u - iv)(dx + i\,dy) = (u\,dx + v\,dy) + i(-v\,dx + u\,dy).$$

Therefore, for any closed path γ,

$$\oint_\gamma f'(z)\,dz = \oint_\gamma (u\,dx + v\,dy) + i\oint_\gamma(-v\,dx + u\,dy).$$

Now write

$$\oint_\gamma f'(z)\,dz = 2\pi i\left(\frac{m - ik}{2\pi}\right)\sum \text{Res}\left(\frac{a-b}{(z-a)(z-b)}\right),$$

and take cases on whether a or b is enclosed by γ to evaluate the integral.

11. Let $z = x + iy$ to obtain

$$\begin{aligned} f(z) &= K\left(x + iy + \frac{1}{x+iy}\right) + \frac{ib}{2\pi}\text{Log}(x+iy) \\ &= \frac{Kx(x^2+y^2+1)}{x^2+y^2} - \frac{b}{2\pi}\arg(x+iy) \\ &\quad + i\left[\frac{Ky(x^2+y^2-1)}{x^2+y^2} + \frac{b}{4\pi}\ln(x^2+y^2)\right]. \end{aligned}$$

Equipotential curves are graphs of

$$\varphi(x,y) = \frac{Kx(x^2+y^2+1)}{x^2+y^2} - \frac{b}{2\pi}\arg(x+iy) = c_1.$$

Streamlines are graphs of

$$\psi(x,y) = \frac{Ky(x^2+y^2-1)}{x^2+y^2} + \frac{b}{4\pi}\ln(x^2+y^2) = c_2.$$

Some equipotential lines are shown in Figure 23.18 for $K = 1$ and $b = 2\pi$. Streamlines are shown in Figure 23.19 for $K = 1$ and $b = 4\pi$.

Compute

$$f'(z) = K\left(1 - \frac{1}{z^2}\right) + \frac{ib}{2\pi z} = \frac{1}{z^2}\left[kz^2 + \frac{ib}{2\pi}z - k\right].$$

Stagnation points occur where $f'(z) = 0$. These points are

$$z = -\frac{ib}{4\pi k} \pm \sqrt{1 - \frac{b^2}{16\pi^2 K^2}}.$$

These points lie on the unit circle symmetrically across the imaginary axis.

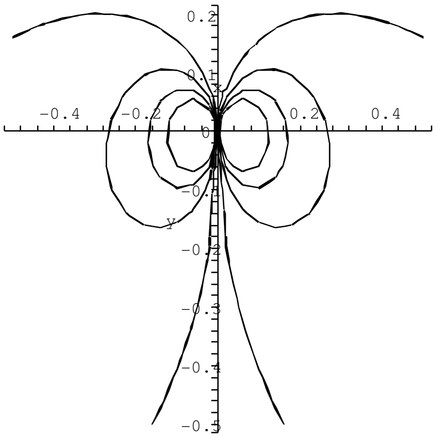

Figure 23.18: Equipotential curves in Problem 11.

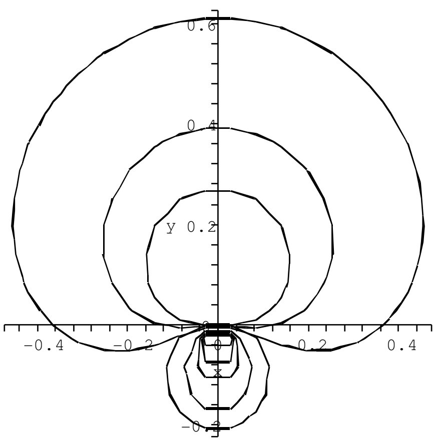

Figure 23.19: Streamlines in Problem 11.