Cyclic Histogram Thresholding and Multithresholding [(*)]

Dimo Dimov, and Lasko Laskov

***Abstract**: The paper concerns the problem of thresholding of an integer domain of 1D cyclic histogram (periodic function) resulting in two or more consecutive regions (classes). An optimal solution is searched for in the terms of the statistical criterion well known in the pattern recognition area as Fisher's LDA (Linear Discriminant Analysis) and also successfully applied for image binarization by Otsu (1979). An effective (quadratic complexity) extension of the Otsu's method is also known, which segments the image by respective thresholding of the image intensity histogram into arbitrary number of classes. We propose one more extension of this approach for the case of the cyclic histograms. Similar problem can be brought by the optimal segmentation of color images based on their HSV histogram, and more general in all problems which try to approximate a given periodic function with a predefined number of Gaussians. The paper describes the theoretical basis and the experimental evaluation of the proposed approach.*

***Key words**: Cyclic histogram thresholding and multithresholding, Periodic function approximation by Gaussians, Image processing.*

1. INTRODUCTION

The examined problem usually is a result of so called *histogram* approaches to image binarization [1, 2, 3]. The histogram that is a statistical function of the image intensity, is being divided according to an optimality criterion into two compact parts, and during the binarization, one of the parts is labeled as background (e.g. white), the other – as object of interest (e.g. black). Analogously, one can binarize also color images, for example using the corresponding Hue-histogram of the HSV color scheme of the image, but in this case the applied histogram is a cyclic (periodic) one, [4].

More generally, if we exclude the physical meaning of the term histogram, the considered problem can be brought by attempts to approximate a periodic function (e.g. statistical) by a given number of simple functions, e.g. statistical distributions like Gaussians.

Our approach to the examined problem is an extension of the classical approaches for thresholding (and multi-level thresholding) which divide a histogram in two or more compact sequential parts, but in the case of a cyclic histogram. As a base approach to this extension we adopt the Otsu's method [2].

2. BACKGROUND

For thresholding of a given histogram $H(i)$, $i = 0,1,...,(T-1)$ into two Gaussian components, Otsu [2] applies an approach frequently associated with the name of Fisher in the Linear Discriminant Analysis (LDA), see [5]. More precisely, Otsu searches a discriminant point (a threshold) t, $0 \le t < T$, via the criterion $\lambda = \sigma^2_{\mathrm{Btw}} / \sigma^2_{\mathrm{Wth}} = \max$, where σ^2_{Wth} is the so called *within-class* variance, and σ^2_{Btw} the *between-class* variance $\sigma^2_{\mathrm{Wth}} = \sigma^2_0 + \sigma^2_1$, $\sigma^2_{\mathrm{Btw}} = \omega_0(\mu_0 - \mu_{\mathrm{img}})^2 + \omega_1(\mu_1 - \mu_{\mathrm{img}})^2$. Here (μ_0, σ^2_0), (μ_1, σ^2_1) and $(\mu_{\mathrm{img}}, \sigma^2_{\mathrm{img}})$ are the parameters (mean and variance) of the corresponding Gaussian models for the object class, for the background class and for the whole image, and ω_0, ω_1, $\omega_0 + \omega_1 = 1$ play the role of normalizing coefficients.

Since the two classes are initially unknown the preliminary statistics accumulation using Fisher's LDA would lead to an inefficient procedure. That is why, Otsu proposes an

[(*)] This work was supported by following grants of the Institute of Information Technologies (IIT) at Bulgarian Academy of Sciences (BAS): Grant # DO-02-275/2008 and Grant # VU-MI-204/2006 of the National Science Fund at Bulgarian Ministry of Education & Science, and Grant # 010088/2007 of BAS.

equivalent but more efficient in terms of calculations criterion $\eta = \sigma^2_{\mathrm{Btw}} / \sigma^2_{\mathrm{img}} \sim \sigma^2_{\mathrm{Btw}}$, $\sigma^2_{\mathrm{img}} = \mathrm{cte}$, which is maximized by an item-by-item examination:

$$t_{\mathrm{opti}} = \arg\max_{0 \le t < T} \eta(t) = \arg\max_{0 \le t < T} \left(\omega_0(t)\omega_1(t)\left(\mu_0(t) - \mu_1(t)\right)^2 \right), \tag{1}$$

$$\omega_0(t) = \frac{1}{\Omega}\sum_{i=0}^{t} H(i),\ \omega_1(t) = \frac{1}{\Omega}\sum_{i=t+1}^{T-1} H(i),\ \mu_0(t) = \frac{1}{\Omega\omega_0(t)}\sum_{i=0}^{t} iH(i),\ \mu_1(t) = \frac{1}{\Omega\omega_1(t)}\sum_{i=t+1}^{T-1} iH(i),\ \Omega = \sum_{i=0}^{T-1} H(i).$$

This method is of complexity $\sim T$, where in the case of gray-level images usually $T = 256$. The iterative extension of the method for M classes $(M > 2)$ leads to exponential complexity $\sim T^{M-1}$, [2], see also [6]. But, few years before the efforts in [6], an effective algorithm was already proposed, based on dynamic programming and having a complexity of $\sim T^2 M$, see Kurita, Otsu, and Abdelmalek [3].

3. CYCLIC THRESHOLDING IN TWO CLASSES

The approach towards the problem solution can be iterative like the proposed solution in [2] for three $(M = 3)$ classes.

3.1. Intuitive approach towards the solution. Algorithm $\mathcal{A}0$.

We can take as a solution the couple (t_0, t_1), which maximizes the criterion $\eta = \eta(t \mid t_0)$, introduced for the cyclic histogram $H(t) = H(t+T)$, $t = 0,1,\ldots,(T-1)$ by analogy with (1). Thus, for all the possible starting points $t_0,\ t_0 = -1,0,1,\ldots,(T-2)$, we can define the threshold t_1 as: $t_1 = \arg\max_{t_0 < t < T + t_0,\ -1 \le t_0 \le T-2} (\eta(t \mid t_0))$.

The following are the considerations on which the intuitive approach ($\mathcal{A}0$) has been designed in [4], on the example of the HSV scheme interpretation for a given image:

- There are two thresholds, t_0 and t_1, to separate two classes (continuous areas) in the HS-histogram (Fig.1) resulting in a periodic H-histogram (Fig.2).
- Let us suppose that the histogram start-point coincides with the threshold t_0. Then we have to calculate the threshold t_1 to maximize the criterion $\eta(t \mid t_0)$.
- As t_0 is a priori unknown we have to repeat the above procedure for each t_0, $t_0 = -1,0,1,\ldots,(T-2)$, and to get as result this couple (t_0, t_1) which maximizes $\eta(t \mid t_0)$.

The intuitive approach $\mathcal{A}0$ is implausible, for example considering the results of binarization of sequential frames of a video-clip [7], because:

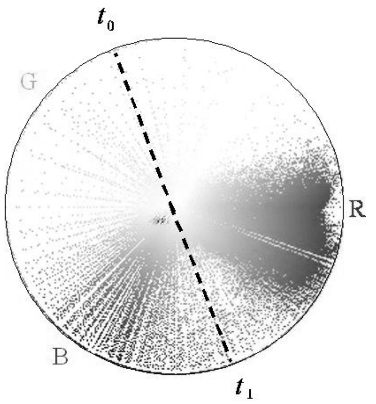


Fig.1. The HS color histogram of an image; both H-thresholds, t_0 = 110°, and t_1 = 291°, are outlined.

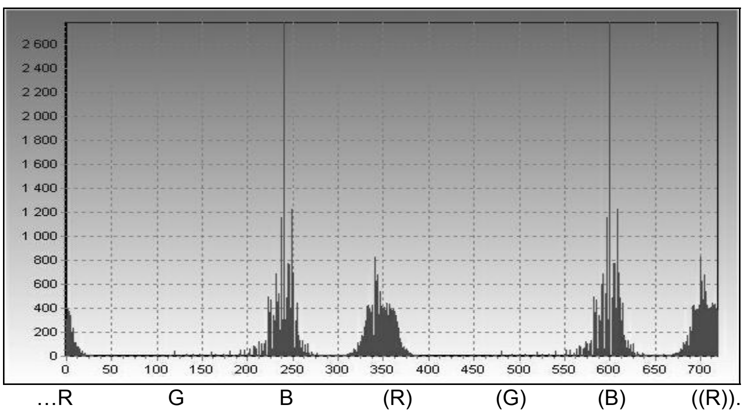


Fig.2. An optimal couple of thresholds (t_0, t_1), t_0=110, t_1=291= $\mathcal{A}$(110) that is equivalent to t_0=470 = $\mathcal{A}$(291)=110 mod(T), t_1=651= $\mathcal{A}$(470)=291 mod(T), because of periodicity of the histogram $H(t)$=$H(t+T)$, T=360.

♦ In the case of smooth changes in two sequential frames, we expect that the corresponding smooth changes will occur in the averaged intensities μ_0, μ_1 as well as in the other statistical characteristics σ_0, σ_1, ω_0 ,ω_1 of the two classes (object and background). The intuitive $\boldsymbol{A0}$ is not proved to fulfill this requirement. In contrary:

♦ The experiment with binarization of a large number of video-clips from the type "face on blue background shot by moving camera" [7], shows obvious leaps of the averaged intensities μ_0 and μ_1 of the two classes.

♦ Interpretation of the corresponding histograms: The criterion function $\eta = \eta(t \mid t_0)$ is not guaranteed to be one-modal and frequently it has two (or more) well distinguishable local maxima. For example, as the video clip advances, a given local maximum can grow up exceeding the current global maximum. Then the position of the global maximum catastrophically changes, reflecting in the choice of μ_1 depending on μ_0, see also Fig 3.

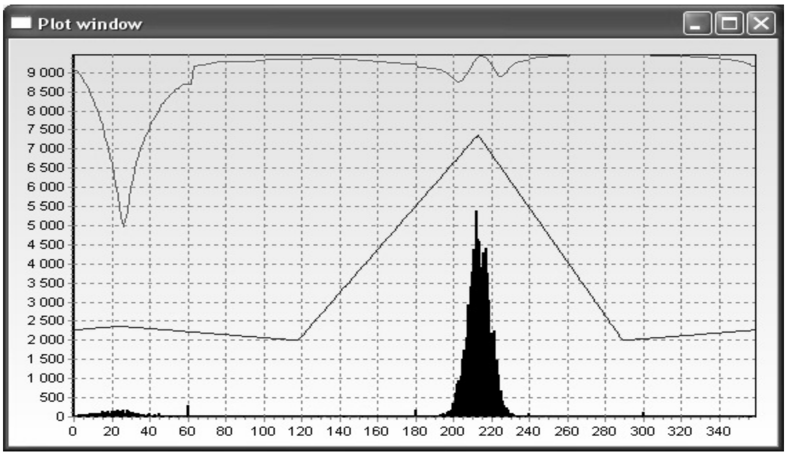
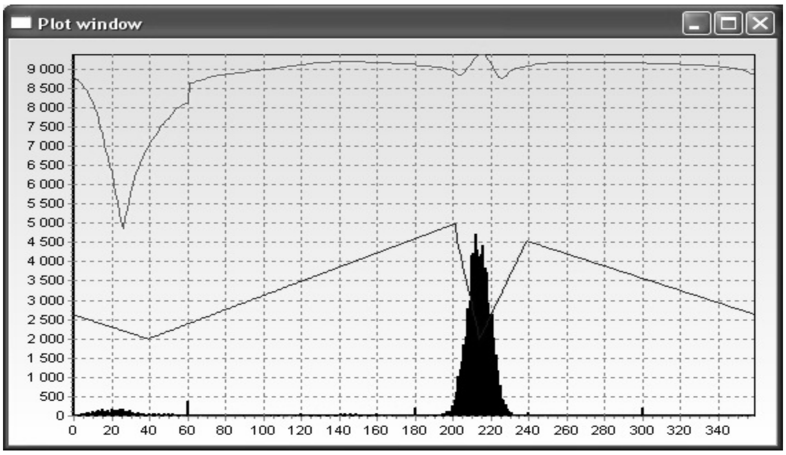


Fig.3. A catastrophic jump of the optimal decision (t_0,t_1) is shown when the "maximum of $\eta(t|t_0)$" criterion is used. On the right of two consecutive images from a video the respective graphics are shown – the H-histogram bars (each below), both the classes (marked artificially in the middles) and the $\eta(t|t_0)$ function (each above). The means μ_0 and μ_1 corresponding to the found thresholds t_0 and t_1 are also shown.



3.2. New idea for the problem solution

Let us denote by $\boldsymbol{A}$ the algorithm for optimal solution t_1, $t_1 \in \{-1,0,1,...,(T-2)\}$ of the descried problem following (1). Let us denote by $\boldsymbol{A}(t_0)$ the extension of $\boldsymbol{A}$ for some initial value t_0: $t_0 \in \{-1,0,1,...,(T-2)\}$, $t_1 = \boldsymbol{A}(t_0)$. Then, by considerations of symmetry it must be also fulfilled:

$$t_0 \cong \boldsymbol{A}(t_1) \equiv \boldsymbol{A}(\boldsymbol{A}(t_0)) \bmod(T), \quad t_0 \in \{-1,0,1,...,(T-2)\}. \tag{2}$$

Moreover, the above must be true for each of the threshold values t_0 and t_1 of the optimal pair (t_0, t_1). If we consider that t_0 and t_1 are integer values, and that $t_1 \neq t_0$, then we can propose the following sequence of actions to determine the optimal pair (t_0, t_1):

- For a given $t_0, t_0 \in \{-1,0,1,...,(T-2)\}$ we have: $t_1 = \mathcal{A}(t_0)$, $t_1 \in \{t_0 + t \mid t = 1,2,...,(T-1)\}$.
- From the threshold t_1 we calculate a new start position $\tilde{t}_0$, $\tilde{t}_0 \equiv (t_1 + t_0 + 1) \bmod(T)$.
- For $\tilde{t}_0$, $\tilde{t}_0 \in \{-1,0,1,...,(T-2)\}$ we get the new threshold $\tilde{t}_1 = \mathcal{A}(\tilde{t}_0)$, $\tilde{t}_1 \in \{\tilde{t}_0 + t \mid t = 1,2,...(T-1)\}$.
- From $\tilde{t}_1$ we calculate the next start position $\approx{t}_0$, $\approx{t}_0 \equiv (\tilde{t}_1 + \tilde{t}_0 + 1) \bmod(T)$.
- The new start position $\approx{t}_0$ must coincide with the initial one t_0 in the frames of periodicity T, i.e.: $\approx{t}_0 \equiv t_0 \bmod(T)$.

Or putted together we obtain:

$$\mathcal{A}\big((\mathcal{A}(t_0) + t_0 + 1) \bmod(T)\big) + \mathcal{A}(t_0) + t_0 + 2 \equiv t_0 \bmod(T), \tag{3}$$

or equivalently:

$$\mathcal{A}\big((t_1 + t_0 + 1) \bmod(T)\big) + t_1 + 2 \equiv 0 \bmod(T), \quad t_1 = \mathcal{A}(t_0). \tag{3a}$$

Unlike (2), the equation (3a) requires the minimal (only single) extension (see also Fig.2) of the base histogram $H(t)$ to $\tilde{H}(t)$:

$$\tilde{H}(t) = \begin{Bmatrix} H(t) & , \; t \in \{0,1,...,(T-1)\} \\ H(t-T), & t \in \{T,(T+1),...,(2T-1)\} \end{Bmatrix}, \tag{4}$$

that has been implemented in the next algorithm $\mathcal{A}1$ for the case of two classes:

3.3. *Algorithm $\mathcal{A}1$:*

$\mathcal{A}1$.*step 1*: For all t_0, $t_0 = -1,0,1,...,(T-2)$, find the corresponding $t_1 = \mathcal{A}(t_0)$.

$\mathcal{A}1$.*step 2*: Separate all pairs (t_0, t_1), for which the equation (3a) is fulfilled. The number of the found couples is even, i.e. there exists at least two couples (t_0, t_1) and (t_1, t_0), which are symmetric.

$\mathcal{A}1$.*step 3*: If the number of the found couples is greater than two, choose the couple that maximizes the criterion function $\eta(t \mid t_0)$ from (1). (*End of* $\mathcal{A}1$).

The complexity of the algorithm $\mathcal{A}1$ is evaluated to ~T^2.

4. CYCLIC MULTI-LEVEL THRESHOLDING

Suppose we have an algorithm $\mathcal{B}$ for optimal segmentation of $(M+1)$ successive regions (classes) of a conventional histogram H, by M, $M \geq 1$ thresholds $(t_1, t_2, ...t_M)$, $0 \leq t_1 < t_2 < ... < t_M < T$, where T defines the domain of the histogram $H(t)$, $t \in \{0,1,...,(T-1)\}$.

Let us denote by $\mathcal{B}(t_0)$ the extension of the algorithm $\mathcal{B}$ for a starting value t_0, namely: $(t_1, t_2, ...t_M) = \mathcal{B}(t_0)$, $t_0 \in \{-1,0,1,...,(T-2)\}$. In this way, the base algorithm $\mathcal{B}$ is represented as $\mathcal{B}(-1)$. The complexity of $\mathcal{B}$, is evaluated to ~$(T-M)^2 M$, see also [3].

The complexity of $\mathcal{B}(.)$ is similar, since it can be implemented from $\mathcal{B}$ by simple readdressing of the extended $\tilde{H}$ from (4), because of the H periodicity, see also Fig.2.

For the sake of concreteness of $\mathcal{B}(.)$, we will examine in analogy with (1) the following extended criterion function:

$$\eta = \eta(t_M, t_{M-1}, ...t_1 \mid t_0) = \sigma^2_{\mathrm{Btw}}(t_0) \big/ \sigma^2_{\mathrm{img}}(t_0), \tag{5}$$

where

$$\sigma^2_{\mathrm{Btw}}(t_0) = \sum_{i=0}^{M} \omega_i(t_0)\big(\mu_i(t_0) - \mu_{\mathrm{img}}(t_0)\big)^2 ,$$

$$\omega_i(t_0) = \frac{1}{\Omega}\sum_{j=t_i}^{t_{i+1}} \tilde{H}(j),\ \mu_i(t_0) = \frac{1}{\Omega\omega_i(t_0)}\sum_{j=t_i}^{t_{i+1}} j\tilde{H}(j),\ i = 0,1,2,...,M ; \tag{5a}$$

$$\sigma^2_{\mathrm{img}}(t_0) = \frac{1}{\Omega}\sum_{j=t_0}^{T-1+t_0} \big(j - \mu_{\mathrm{img}}(t_0)\big)^2 \tilde{H}(j),\ \mu_{\mathrm{img}}(t_0) = \frac{1}{\Omega}\sum_{j=t_0}^{T-1+t_0} j\tilde{H}(j),\ t_0 = -1,0,1,2,...,(T-2);$$

$$\Omega = \sum_{j=0}^{T-1} H(j),\ t_{M+1} = T-1 .$$

These considerations allow the extension of the algorithm $\mathcal{A}1$ to the following algorithm $\mathcal{A}2$ for periodic histogram thresholding in $M+1$ classes, $M \geq 1$:

4.1. Algorithm $\mathcal{A}2$:

$\mathcal{A}2$*.step 1*: For each starting value t_0, $t_0 = -1,0,1,...,(T-2)$, calculate M optimal thresholds $(t_1, t_2, ..., t_M)$ using the algorithm $\mathcal{B}(.)$, applied on the extended histogram $\tilde{H}(t)$ from (4). Place the results in an array $\mathcal{Mot}$ (Matrix of **o**ptimal **t**hresholds) of size $T \times (M+1)$ and description as follows:

$$\begin{array}{lll}
\text{row } 0: & (t_{0,0}, t_{0,1}, t_{0,2}, t_{0,3}, ..., t_{0,M}) & , \quad -1 = t_{0,0} < t_{0,1} < t_{0,2} < ... < t_{0,M} < (2T-1) \\
\text{row } 1: & (t_{1,0}, t_{1,1}, t_{1,2}, t_{1,3}, ..., t_{1,M}) & , \quad 0 = t_{1,0} < t_{1,1} < t_{1,2} < ... < t_{1,M} < (2T-1) \\
... & & \\
\text{row } k: & (t_{k,0}, t_{k,1}, t_{k,2}, t_{k,3}, ..., t_{k,M}) & , \quad (k-1) = t_{k,0} < t_{k,1} < t_{k,2} < ... < t_{k,M} < (2T-1) \\
... & & \\
\text{row}(T-1): & (t_{T-1,0}, t_{T-1,1}, t_{T-1,2}, ..., t_{T-1,M}) & , \quad (T-2) = t_{T-1,0} < t_{T-1,1} < ... < ... < t_{T-1,M} < (2T-1),
\end{array}$$

where each of the sequences $(t_{k,0}, t_{k,1}, t_{k,2}, t_{k,3}, ..., t_{k,M})$, $k = 0,1,...,(T-1)$ corresponds to some concrete solution $t_0 < t_1 < t_2 < ... < t_M < (2T-1)$, i.e. $(t_1, t_2, ... t_M) = \mathcal{B}(t_0)$, for $t_0 = k-1$.

$\mathcal{A}2$*.step 2*: Extend each row (k), $k = 0,1,...,(T-1)$ of $\mathcal{Mot}$ to the matrix $\mathcal{Mrot}(k)$ (Matrix of **r**otated **o**ptimal **t**hresholds) with dimension $(M+1)\times(M+1)$, where its rows (m), $m = 1,2,...,M$, are the corresponding rows $(t_{m,0})$ of $\mathcal{Mot}$, chosen by the rule $t_{m,0} = t_{k,m}$, $t_{k,m} \in \mathcal{Mrot}(k)$, and after that respectively "shifted cyclically to right" with m positions:

$$\begin{array}{llllllll}
m=0: & (t_{k,0}) & , t_{k,1} & , t_{k,2} & , t_{k,3}, ..., & t_{k,(M-1)} & , t_{k,M} & \\
\hline
m=1: \Rightarrow & t_{S,M} & , (t_{S,0}) & , t_{S,1} & , t_{S,2}, ..., & t_{S,(M-2)} & , t_{S,(M-1)} & , S = t_{k,1} \\
m=2: \Rightarrow & t_{S,(M-1)} & , t_{S,M} & , (t_{S,0}) & , t_{S,1}, ..., & t_{S,(M-3)} & , t_{S,(M-2)} & , S = t_{k,2} \\
... & ... & ... & ... & ... & ... & ... & \\
m=M: \Rightarrow & t_{S,1} & , t_{s,2} & , t_{S,3} & , t_{S,4}, ..., & t_{S,M} & , (t_{S,0}) & , S = t_{k,M}
\end{array}$$

Extend $\mathcal{Mot}$ to $\mathcal{Marot}$ (Matrix of **a**ll **r**otated **o**ptimal **t**hresholds), using vertical concatenation of the corresponding matrices $\mathcal{Mrot}(k)$, $k = 0,1,...,(T-1)$. The resulting $\mathcal{Marot}$:

$$\mathcal{Marot} \equiv \mathcal{Mrot}(0) \wedge \mathcal{Mrot}(1) \wedge ... \wedge \mathcal{Mrot}(k) \wedge ... \wedge \mathcal{Mrot}(T-1),$$

is a three-dimensional matrix with size $T \times (M+1) \times (M+1)$.

$\mathcal{A}2$*.step 3*: Recalculate the elements of $\mathcal{Marot}$ according to the beginning $(t=0)$ of the original histogram H, considering also its cyclic recurrence $H(t) = H(t+T)$:

$$\tau_{k,m,i} \equiv t_{k,m,i} \bmod(T),\ 0 \leq \tau_{k,m,i} < T,\ i \in \{0,1,...,M\},\ m \in \{0,1,...,M\},\ k \in \{0,1,...,(T-1)\}.$$

The row content (m), $m=1,2,...,M$ of a given $\mathcal{Mrot}(k)$, represents all possible cyclic sequences of solutions $(t_{S,1},t_{S,2},...t_{S,M})=\mathbf{B}(t_{k,m})$, $S=t_{k,m}$, for a given solution $(t_{k,1},t_{k,2},...,t_{k,m},...,t_{k,M})=\mathbf{B}(t_{k,0})$, recorded in the row (k), $k=0,1,...,(T-1)$ of $\mathcal{Mot}$.

The column content (i), $i=0,1,2,...,M$ of a given $\mathcal{Mrot}(k)$, represents all possible values of the i-th threshold, which can occur in the solutions $(t_{S,1},t_{S,2},...,t_{S,i},...,t_{S,M})=\mathbf{B}(t_{k,m})$, $S=t_{k,m}$, $m=1,2,...,M$, recorded in $\mathcal{Mrot}(k)$, $k=0,1,...,(T-1)$. Because of symmetry reasons, for the optimal solution k_{opti} (i.e. one of the $\mathcal{Mot}$ rows), we expect that the deviations in the columns between the thresholds (i), $i=0,1,2,...,M$ of the corresponding $\mathcal{Mrot}(k_{\text{opti}})$ to be minimal.

$\mathcal{A2}$.*step 4*: For each one $\mathcal{Mrot}(k)$, $k=0,1,...,(T-1)$ calculate the average values $E_{k,i}$ of the thresholds in the rows (i), $i=0,1,2,...,M$ and recalculate (center) the elements of $\mathcal{Mrot}(k)$ according to these average values:

$$\Delta_{k,m,i}=\tau_{k,m,i}-E_{k,i}\ ,\ m\in\{0,1,...,M\}\ ,\quad E_{k,i}=\frac{1}{M+1}\sum_{m=0}^{M}\tau_{k,m,i}\ ,\ i\in\{0,1,...,M\}\ ,$$

where the new values $\Delta_{k,m,i}$ of $\mathcal{Mrot}(k)$ represent the relative deviations of the old $\tau_{k,m,i}$ from the corresponding centers $E_{k,i}$.

$\mathcal{A2}$.*step 5*: For each one $\mathcal{Mrot}(k)$, $k=0,1,...,(T-1)$ calculate the averaged absolute deviations $\varepsilon_{k,m}$ in its rows (m):

$$\varepsilon_{k,m}=\frac{1}{M+1}\sum_{i=0}^{M}|\,\Delta_{k,m,i}\,|\ ,\ m=0,1,...,M\ ,$$

as well as the value of the possible minimum $\varepsilon_{\min}(k)=\min\limits_{0\le m\le M}\varepsilon_{k,m}$. Obviously, $\varepsilon_{\min}(k)\ge 0$.

The numbers $\varepsilon_{\min}(k)$, $k=0,1,...,(T-1)$ are regarded as a measure of the closeness of the corresponding solutions $(t_{k,1},t_{k,2},...,t_{k,m},...,t_{k,M})=\mathbf{B}(t_{k,0})$, recorded in $\mathcal{Mot}$, to the optimal solution (k_{opti}), which corresponds to the minimum $\varepsilon_{\min}$, $\varepsilon_{\min}=\min\limits_{0\le k<T}(\varepsilon_{\min}(k))$, $k_{\text{opti}}=\arg\min\limits_{0\le k<T}(\varepsilon_{\min}(k))$.

$\mathcal{A2}$.*step 6*: Calculate the set $\mathbf{K}$ from the rows numbers (k), for which the corresponding $\varepsilon_{\min}(k)$ reaches the minimum $\varepsilon_{\min}$. Thus, each row (k), $k\in\mathbf{K}$ of the initial matrix $\mathcal{Mot}$ (step *S1*), which takes part by definition in the recalculated $\mathcal{Mrot}(k)$ (steps *S2÷S4*), can be an optimum solution candidate.

Apparently, because of considerations for symmetry, for the size of $\mathbf{K}$ we have that $|\mathbf{K}|\ge M$. Moreover: $|\mathbf{K}|=nM$, $n\ge 1$, n is integer, and $\mathbf{K}$ is divided in n classes of equivalence $\mathbf{K}_j$, $j=1,2...,n$, where each of the rows (k), $k\in\mathbf{K}_j$, is obtained from one of the others (r), $r\in\mathbf{K}\backslash\mathbf{K}_j$ by a cyclic shift to the left by $(k-r)$ positions.

$\mathcal{A2}$.*step 7*: If $|\mathbf{K}|=M$, for definiteness, for *optimal decision* we choose only one of the rows (k), $k\in\mathbf{K}$, for example the one for which $\tau_{k,0,0}=\min$ (see Step 3).

Otherwise, if $|\mathbf{K}|=nM$, $n>1$, we define additionally the class $\mathbf{K}_j$, $j\in\{1,2...,n\}$ of optimal solutions, using the maximum of the extended base criterion $\eta=\eta(t_M,t_{M-1},...t_1\,|\,t_0)$ from (5). Again, for definiteness, for *optimal decision* we choose only one of the rows (k), $k\in\mathbf{K}_j$, for example the one for which $\tau_{k,0,0}=\min$. (*End of* $\mathcal{A2}$).

4.2. Additional explanations for the algorithm $\mathbf{A2}$:

• General optimizations of the program structures for implementation of matrices $\mathbf{Mot}$, $\mathbf{Mrot}(.)$ and $\mathbf{Marot}$. We will note that the matrices $\mathbf{Mrot}(k)$, $k = 0,1,...,(T-1)$ can be reduced to a single working matrix $\mathbf{Mrot}$ with dimensions $(M+1)\times(M+1)$, which reduces the "big" matrix $\mathbf{Marot}$ to the base matrix $\mathbf{Mot}$.

• At the same time, in Step 5 the classical "least square method" can be used, which will be a slight drawback in the terms of processing speed.

• The complexity of $\mathbf{A2}$ is determined mainly by the complexity of its Step 1 and is calculated to ~$T(T-M)^2M$, M the number of classes, i.e. the number of cyclic thresholds.

• Apparently, for the case of two classes $(M=2)$, it is more efficiently to use the algorithm $\mathbf{A1}$ (complexity ~T^2) instead of $\mathbf{A2}$ (complexity ~T^3).

4. EXPERIMENTS AND RESULTS

We carried out an experimental analysis of the proposed approach through an arbitrary picture of outdoor view (Fig.4) to assure a larger spectrum of colors most of all for the tests of algorithm $\mathbf{A2}$. The experimental software is a C++ written Windows-XP application operating on an IBM compatible PC: Intel Pentium 4 CPU 2.8GHz, MM 2,0GB.

The results of the experiments (execution times) of the proposed algorithms are represented in Table 1, where the discrete Hue-histogram is considered a priori calculated for $T = 360$ (angular degrees).

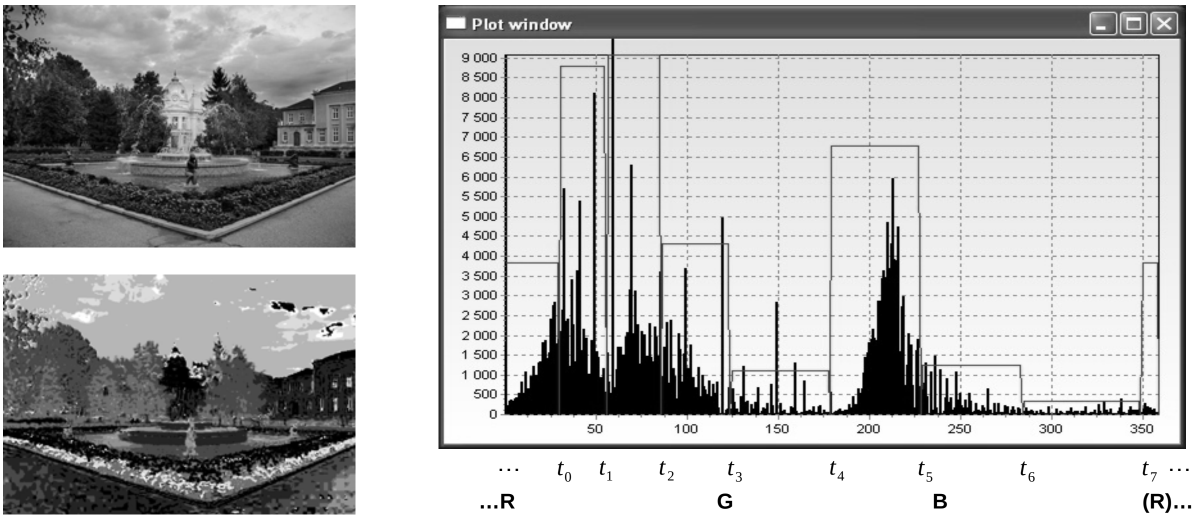


Fig.4. An arbitrary picture of outdoor view (left above), its Hue-histogram optimal thresholding in 8 levels (on the right), and its segmentation in corresponding 8 color means (left down) are shown. Regularly increasing gray intensities are used to represent the respective Hue-means, see also row ($\mathbf{A2}$,*M*=8) in Table 1.

Table 1. Experiment results' comparison among the algorithms $\mathbf{A0}$, $\mathbf{A1}$, and $\mathbf{A2}$, on the picture example of Fig.4.

Algorithm & number (***M***) of classes	Processing speed (***S***) [s]	Processing speed per class (***S/M***) [s]	Threshold series found (by ***T*=360**)
$\mathbf{A0}$, *M*=2	0.007	0.003	126, 313
$\mathbf{A1}$, *M*=2	0.006	0.003	135, 315
$\mathbf{A2}$, *M*=2	0.656	0.328	135, 315
$\mathbf{A2}$, *M*=3	1.062	0.354	63, 154, 305
$\mathbf{A2}$, *M*=4	1.422	0.356	46, 93, 167, 300
$\mathbf{A2}$, *M*=8	2.641	0.330	29, 55, 85, 123, 178, 228, 284, 349
$\mathbf{A2}$, *M*=16	4.562	0.285	12, 27, 39, 52, 65, 78, 93, 111, 137, 176, 205, 219, 237, 262, 301, 344

The measured times S correspond to the theoretical valuations of the algorithms:
$S(\boldsymbol{A0}) \Leftrightarrow O(\boldsymbol{A0}) \sim T^2$, $S(\boldsymbol{A1}) \Leftrightarrow O(\boldsymbol{A1}) \sim T^2$ and $S(\boldsymbol{A2}, M) \Leftrightarrow O(\boldsymbol{A2}, M) \sim T(T-M)^2 M$, demonstrated by the ratio S/M (column 3 of the table) that slightly diminish with M at $\boldsymbol{A2}$. $\boldsymbol{A2}$ should be preferred in all cases of $M > 3$, while $\boldsymbol{A1}$: for $M = 2$. Only for $M = 3$ classes, experiments show that the inner $\boldsymbol{B}(.)$ of $\boldsymbol{A2}$ is better to perform in a classical way, e.g. [6].

5. CONCLUSION

In this paper, following the Otsu's criterion [2], a new method is proposed for optimal thresholding and multithresholding of cyclic histograms. The method preserves the symmetry of the solution for the expense of insignificant (in most cases) deviations from the optimum of an intuitive iterative extension of the Otsu's criterion, cf. [4]. The method can be modified for other approaches for thresholding and multithresholding of histograms, e.g. entropy approach mentioned in [1, 3].

Two algorithms are proposed, $\boldsymbol{A1}$ and $\boldsymbol{A2}$. $\boldsymbol{A1}$ concerns thresholding in 2 classes and has quadratic complexity, while $\boldsymbol{A2}$ concerns multithresholding in more than 2 classes and has cubic complexity (per class). This speed up is provided by dynamic programming for the base algorithms (i.e. for the non-cyclic case) in analogy with the idea from [3].

The motivation for the proposed approach development is connected with discovering of effective and statistically optimal method for color images segmentation based on the HSV or HLS color schemes, which histograms become cyclic. Unlike the intuitive approach proposed in [4], marked here as $\boldsymbol{A0}$, the current method $(\boldsymbol{A1}, \boldsymbol{A2})$ is optimized for color segmentation of image sequences (video-clips). The method provides a smooth change of the parameters of approximating Gaussians with the smooth change of the position of the camera, which is usually connected with unwanted change of illumination, see [7].

The method is applicable in the multiple cases of discrete approximation of a function by Gaussians and particularly when a stress is put on the symmetry (periodicity) of the approximation. In this prospective as future work the authors intend to investigate a similar approach based on the wavelet and Fourier analysis.

REFERENCES

[1] Sahoo, P.K., S. Soltani, A.K.C. Wong and Y.C. Chen, A survey of Thresholding Techniques, *Comp. Vision, Graphics, and Image Proc.*, Vol.41, pp.233-260, 1988.

[2] Otsu N.: A Threshold Selection Method from Gray-Level Historgams, *IEEE Trans. on Systems, Man, and Cybernetics*, Vol.9, No.1, pp.62-66, 1979.

[3] Kurita, T., N. Otsu, and N. Abdelmalek, Maximum Likelihood Thresholding Based on Population Mixture Models, *Pattern Recognition*, Vol.25, No.10, pp.1231-1240, 1992.

[4] Laskov L., and D. Dimov: Color Image Segmentation for Neume Note Recognition, In: Proceedings of the Int.Conf. A&I'07, 03-06.10.07, Sofia, 2007, pp. III.37-41.

[5] Duda, R.O., P.E. Hart, and D.G. Stock, Pattern Classification (2d ed.), John Wiley & Sons, Inc., 2001.

[6] Ping-Sung Liao, Tse-Sheng Chen, and Pau-Choo Chung, A Fast Algorithm for Multilevel Thresholding, *Journal of Information Science and Engineering*, Vol.17, pp.713-727, 2001.

[7] Dimov D.T., N. Zlateva, and A. Marinov: CBIR Approach to Face Recognition, In Proc. of A&I'08 Conf., 01-04.10.2008, Sofia, Workshop on Multisensor Signal, Image and Data Processing, 02.10.2008, pp. IV.21-26.

ABOUT THE AUTHORS

Dimo Dimov, Assoc. Prof., Ph.D., Institute of Information Technologies (IIT) at Bulgarian Academy of Sciences (BAS), Tel: (+359 2) 870 6493, E-mail: dtdim@iinf.bas.bg

Lasko Laskov, Ph.D. Student, IIT-BAS, and Institute of Mathematics and Informatics (IMI) at BAS, Tel: (+359 2) 979 2843, E-mail: llaskov@iinf.bas.bg