INFORMAL NOTES ON CATEGORY THEORY
PART 1

Dana S. Scott
November 24 2019

Now that we have the basic axioms for Category Theory fixed in Free Logic, we have to do more work in the way of expanding the ***development*** of the theory using ATP.

The work on Modeloids was excellent, but they are of course very special categories. The idea I discussed at our Berlin Meeting is that **Type Theory** is better when done in **Category Theory.** My position is that

Category Theory is the ***algebra*** of Type Theory.

Just as Algebra generalizes the study of Integer Arithmetic, so Category Theory generalizes Type Theory.

Church-style types are based first on ***type symbols,*** and this is just a way of formulating the ***free algebra of types.*** I claim it is better to make type constructs ***relative to*** given types instead of having to associate every type with a special type symbol. And this is what
Category Theory does – when developed beyond just the basic axioms and then heading to **Topos Theory.**

In other words we have to find the additional axioms for ***kinds*** of types and for ***constructs*** of new types from given types. I say "we" because category people already know how to do this in their language. What is needed is to give translations into our kind of logical language.

And then we will see that **Topos Theory** becomes the category-theoretic version of **Higher Type Theory.** What is "higher" about it is the constructs (functors) can be iterated over and over to give more and more

types,
plus there is an analogue to the formation of ***powersets*** corresponding to using the Comprehension Axiom in Set Theory. Standard texts say a ***topos*** is a category with (a) finite limits, (b) exponents, and (c) a subobject classifier. This is the brief explanation. It is set out in

https://en.wikipedia.org/wiki/Topos

— but beginners might not be able to understand everything there. So John Baez has given a more elementary presentation in his essay ***Topos Theory in a Nutshell*** to be found at

http://math.ucr.edu/home/baez/topos.html

I recommend looking at it. As he says, a well known text is "advanced enough to make any beginner run away screaming! I love it now, but that took years." Our project should be to reduce the time needed to get to a basic understanding!

I will now begin the outlining of (I hope) easy exercises aimed at this goal. It will take some time to go through the necessary definitions, so this message will no doubt have to be continued. One terminological convention: The standard books on Category Theory speak of ***maps*** and ***objects***. I will here try to replace the second word by ***types***. Every

category is a system of types, but special categories get
to a general system of higher types.

Project 1. Initial Types. The paradigm example of an initial
type is ***the empty set*** in the Category of Sets and Functions.
In our formulation of the general theory, the types ***are*** the
identity functions. They are elements t such that

$$t = \mathrm{dom}(t) \quad (\text{equivalently, } t = \mathrm{cod}(t)).$$

Using the arrow notation this should be equivalent to

$$t : t \longrightarrow t. \quad (\text{Yes?})$$

(In general $f : x \longrightarrow y$ means $\mathrm{dom}(f) = x \wedge \mathrm{cod}(f) = y$.)

The reason an initial type z is called "initial" is that
we can always find trivially a function $f: t \longrightarrow z$ for any
arbitrary type t. And in fact the f is unique (in, say,
the category of sets when z is the empty set). Here is
the formal definition:

$$\mathrm{Initial}(z) \iff \forall t\, \exists!\, f\, [\mathrm{dom}(f) = z \wedge \mathrm{cod}(f) = t]$$

Exercise 1.1. Show that any two initial types are isomorphic.
Exercise 1.2. Show that if Initial(z) and $f : z \longrightarrow z$, then
$f = z$.
Exercise 1.3. Find some easy to understand categories which
have ***no*** initial types. Need the initial type be unique
as it is in the Category of Sets?

Project 2. Final Types. The paradigm example of a final type is ***the unit set*** {0} in the Category of Sets. Here is the formal definition:

$$\mathrm{Final}(u) \iff \forall t\, \exists! f\, [\mathrm{dom}(f) = t \wedge \mathrm{cod}(f) = u]$$

This definition is ***dual*** to the definition of **Project 1.**

Exercise 2.1. What is the meaning of ***duality*** in Category Theory? (You may have to look this up.)

Exercise 2.2. Formulate and prove the dual versions of Exercises 1.1, 1.2, and 1.3.

Exercise 2.3. Find some easy to understand categories which have no initial types but do have final types.

Exercise 2.4. Find some easy to understand categories which have initial types which are at the same time final types.

Project 3. Monic Maps. The paradigm example of a monic map in the Category of Sets is a mapping which is ***one-one and into***. Here is the formal definition:

$$\mathrm{Monic}(m) \iff \forall x,y,f,g\, [[f : x \longrightarrow y \wedge g : x \longrightarrow y \wedge m \circ f = m \circ g] \Longrightarrow f = g]$$

Exercise 3.1. Explain why this formal definition is correct in the Category of Sets.

Exercise 3.2. If Monic(m) and Monic(n) and dom(m) = dom(n) and cod(m) = cod(n), need it be the case that n = m?

Exercise 3.3. Find some easy to understand categories which have no monic maps.

Project 4. Epic Maps. The paradigm example of an epi map
in the Category of Sets is a mapping which is ***onto.***

Exercise 4.1. Formulate the definition of Epi(e) dual to
the definition of Project 3. And explain why this formal
definition is correct in the Category of Sets.

Exercise 4.2. What are maps which are both monic and epi?
What is the relation to a map having an ***inverse***?

Exercise 4.3. Are there categories where all monics are epis?

TO BE CONTINUED