INFORMAL NOTES ON CATEGORY THEORY
PART 2

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Project 5. PRODUCTS. The paradigm example of products in the Category of Sets is the ***cartesian product*** (= the set of ordered pairs from two sets). This also works in the Category of Groups and Homomorphisms and many similar categories. The problem is not to ***construct*** a product but to ***characterize*** it in general categorical-theoretic terms.

Here is the formal definition:

$$\begin{aligned}\mathrm{Product}(a,b,c) \Longleftrightarrow \exists\, p_1, p_2\, [\; & p_1 : c \longrightarrow a \wedge p_2 : c \longrightarrow b \;\wedge \\ & \forall x,f,g\,[[f : x \longrightarrow a \wedge g : x \longrightarrow b\,] \Longrightarrow \\ & \exists!\, h\,[h : x \longrightarrow c \wedge p_1 \circ h = f \wedge p_2 \circ h = g]]]\end{aligned}$$

Exercise 5.1. Explain why this formal definition is correct in the Category of Sets.

Exercise 5.2. Find some easy to understand categories where products ***never*** exist.

Exercise 5.3. Show that Product(a, b, c) and Product(a, b, d) always imply the isomorphism $c \cong d$.

Exercise 5.4. Find some easy to understand categories where some Product(a, b, c) holds but maps p_1 and p_2 are ***not*** unique.

Exercise 5.5. Show that Product(a, b, c) implies Product(b, a, c).

Exercise 5.6. Show that Product(a, b, u), Product(u, c, v), Product(a, s, t), and Product(b, c, s), always imply the isomorphism $v \cong t$.

Exercise 5.7. Show that if Initial(z), then for any type a we have Product(z, a, z).

Exercise 5.8. Show that if Final(u), then for any type a we have Product(u, a, a).

Note: This standard definition of product is be found at:
https://en.wikipedia.org/wiki/Product_(category_theory)