Ex 8.2

Q.N. (c) Find $u(x,t)$ of the string of length $L=\pi$ when $c^2=1$, the initial velocity is zero and initial deflection is $0.1x(\pi^2-x^2)$

Solution:- Here

Initial velocity $= g(x) = 0$

initial deflection $= f(x) = 0.1x(\pi^2-x^2)$

$L=\pi$, $c^2=1$.

We know that the solution of 1-D wave eqn is

$$u(x,t) = \sum_{n=1}^{\infty}\left(B_n \cos\lambda_n t + B_n^* \sin\lambda_n t\right)\sin\frac{n\pi}{L}.x \quad \ldots (1)$$

Where

$$B_n = \frac{2}{L}\int_0^L f(x).\sin\frac{n\pi}{L}x\,dx$$

$$B_n^* = \frac{2}{\lambda_n L}\int_0^L g(x).\sin\frac{n\pi}{L}.x\,dx$$

Since $g(x)=0$. So

$$B_n^* = \frac{2}{\lambda_n L}\int_0^L g(x)\sin\frac{n\pi}{L}x\,dx = 0$$

Then eqn (1) becomes

$$u(x,t) = \sum_{n=1}^{\infty} B_n \cos\lambda_n t.\sin\frac{n\pi}{L}x \quad \ldots (11)$$

Now

$$B_n = \frac{2}{L}\int_0^L f(x)\cdot \sin\frac{n\pi}{L}x\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} 0.1x(\pi^2-x^2)\,\sin\frac{n\pi}{\pi}x\,dx \quad [\text{put } L=\pi]$$

$$= \frac{0.2}{\pi}\int_0^{\pi}(\pi^2 x - x^3)\,\sin nx\,dx$$

$\pi^2 x - x^3$	$\sin nx$
$\pi^2 - 3x^2$	$-\frac{\cos nx}{n}$
$-6x$	$-\frac{\sin nx}{n^2}$
-6	$\frac{\cos nx}{n^3}$
0	$\frac{\sin nx}{n^4}$

$$= \frac{0.2}{\pi}\Bigg[(\pi^2 x - x^3)\cdot -\frac{\cos nx}{n} + (\pi^2 - 3x^2)\frac{\sin nx}{n^2} - 6x\frac{\cos nx}{n^3} - 6\frac{\sin nx}{n^4}\Bigg]_0^{\pi}$$

$$= \frac{0.2}{\pi}\Bigg[\cancel{(\pi^3-\pi^3)}\cdot -\frac{\cos n\pi}{n} + (\pi^2 - 3\pi^2)\cdot\frac{\cancel{\sin n\pi}}{n^2} - 6\pi\frac{\cos n\pi}{n^3} - 6\frac{\cancel{\sin n\pi}}{n^4}\Bigg]$$

$$= \frac{0.2}{\pi}\times -6\pi\frac{\cos n\pi}{n^3} = \frac{1.2\times -1\times(-1)^n}{n^3} = \frac{1.2(-1)^{n+1}}{n^3}$$

Eq^n (11) becomes

$$U(x,t) = \sum_{n=1}^{\infty} B_n \cos\lambda_n t \cdot \sin\frac{n\pi}{L}x$$

$$= \sum_{n=1}^{\infty} \frac{1.2}{n^3}(-1)^{n+1} \cdot \cos\frac{cn\pi}{L} \times t \cdot \sin\frac{n\pi}{\pi} \cdot x$$

$$\left[\lambda_n = \frac{cn\pi}{L}\right]$$

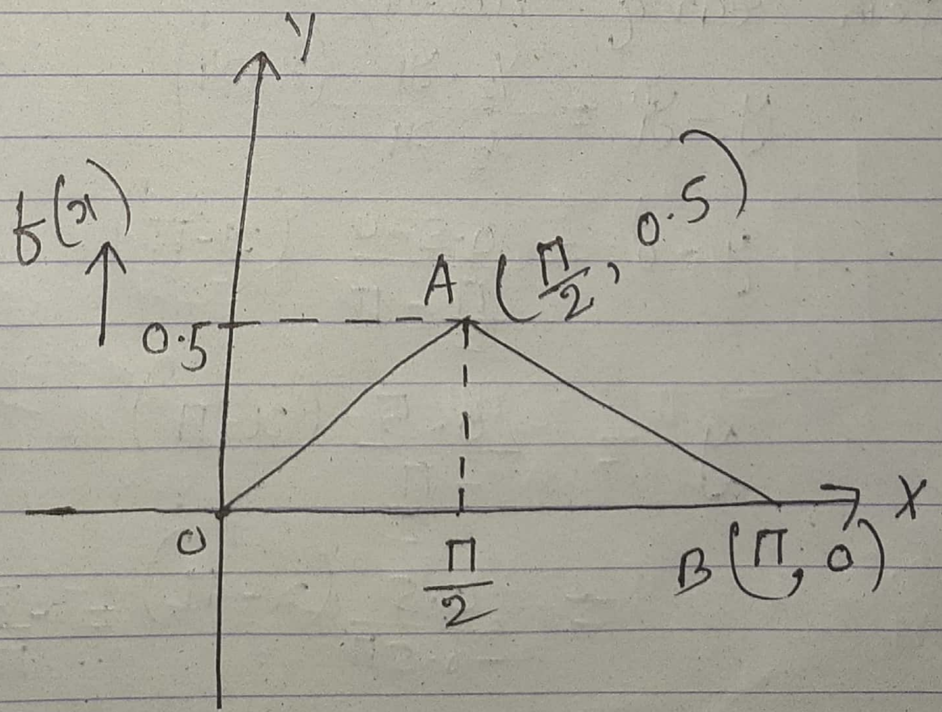
$$= 1.2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^3} \cdot \cos\frac{1\times n\pi}{\pi} \cdot t \cdot \sin nx \quad [L = \pi]$$

$$= 1.2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^3} \cos nt \cdot \sin nx$$

$$U(x,t) = 1.2\left(\cos t \sin x - \frac{1}{8}\cos 2t \sin 2x + \cdots\right)$$

~~is~~ the required ~~one~~ Solution.

(c)



(4)

Here, Length of string $= L = \pi$

$c^2 = 1$

Initial velocity $= g(x) = 0$

From the given figure Initial deflection is OA and AB

Now eqn of OA is given by

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

$$y - 0 = \frac{0.5 - 0}{\frac{\pi}{2} - 0}(x - 0)$$

$$y = \frac{0.5}{0.5\pi} \times x = \frac{x}{\pi}$$

ie $f(x) = \frac{x}{4}$

Again eqn of AB is

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

$$y - 0 = \frac{0.5 - 0}{\frac{\pi}{2} - \pi}(x - \pi)$$

$$y = \frac{0.5}{-\frac{\pi}{2}}(x - \pi)$$

$$= \frac{0.5}{-0.5\pi}(x - \pi) = -\frac{x}{\pi} + 1$$

Eqn of AB is

$$f(x) = 1 - \frac{x}{\pi}$$

Then Initial deflection is

$$f(x) = \begin{cases} \frac{x}{\pi} & \text{for } 0 < x < \frac{\pi}{2} \\ 1 - \frac{x}{\pi} & \text{for } \frac{\pi}{2} < x < \pi \end{cases}$$

We know that the solution of 1-D wave equation is

$$u(x,t) = \sum_{n=1}^{\infty} \left(B_n \cos \lambda_n t + B_n^* \sin \lambda_n t\right) \sin \frac{n\pi}{L} x \quad \text{--- (1)}$$

Where

$$B_n = \frac{2}{L} \int_0^L f(x) \cdot \sin \frac{n\pi}{L} x \, dx$$

$$B_n^* = \frac{2}{cn\pi} \int_0^L g(x) \sin \frac{n\pi}{L} x \, dx = 0$$

since $g(x) = 0$

Eqn (1) becomes

$$u(x,t) = \sum_{n=1}^{\infty} B_n \cos \lambda_n t \cdot \sin \frac{n\pi}{L} x \quad \text{--- (2)}$$

Now, We have to find B_n

$$B_n = \frac{2}{L}\int_0^L f(x)\cdot \sin\frac{n\pi}{L}x \cdot dx$$

$$= \frac{2}{\pi}\int_0^{\pi} f(x)\sin\frac{n\pi}{\pi}\cdot x\, dx$$

$$= \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\, dx$$

$$= \frac{2}{\pi}\left[\int_0^{\pi/2} f(x)\sin nx\, dx + \int_{\pi/2}^{\pi} f(x)\sin nx\, dx\right]$$

$$= \frac{2}{\pi}\left[\int_0^{\pi/2} \frac{x}{\pi}\sin nx\, dx + \int_{\frac{\pi}{2}}^{\pi}\left(1-\frac{x}{\pi}\right)\sin nx\, dx\right]$$

$$= \frac{2}{\pi^2}\int_0^{\pi/2} x\sin nx\, dx + \frac{2}{\pi^2}\int_{\pi/2}^{\pi}(\pi - x)\sin nx\, dx$$

$$\left[\begin{array}{llcll} x & \sin nx & \quad & \pi - x & \sin nx \\ 1 & -\frac{\cos nx}{n} & & -1 & -\frac{\cos nx}{n} \\ 0 & -\frac{\sin nx}{n^2} & & 0 & -\frac{\sin nx}{n^2} \end{array}\right.$$

(7)

$$= \frac{2}{\pi^2}\left[\left| -\frac{x\cos nx}{n} + \frac{\sin nx}{n^2} \right|_0^{\pi/2} + \left| (\pi-x)\cdot\frac{-\cos nx}{n} - \frac{\sin nx}{n^2} \right|_{\pi/2}^{\pi} \right]$$

$$= \frac{2}{\pi^2}\left[\cancel{-\frac{\frac{\pi}{2}\cdot\cos\frac{n\pi}{2}}{n}} + \frac{\sin\frac{n\pi}{2}}{n^2} - 0 + \cancel{(\pi-\pi)\cdot\frac{\cos n\pi}{n}} - \cancel{\frac{\sin n\pi}{n^2}} - \cancel{\left(\pi-\frac{\pi}{2}\right)\cdot\frac{-\cos\frac{n\pi}{2}}{n}} + \frac{\sin\frac{n\pi}{2}}{\cancel{2}} \right]$$

$$= \frac{4}{(n\pi)^2}\cdot\sin\frac{n\pi}{2}.$$

putting value of B_n in eqn (2)

$$u(x,t) = \sum_{n=1}^{\infty} \frac{4}{(n\pi)^2}\cdot\sin\frac{n\pi}{2}\cdot\cos\lambda_n t\cdot\sin\frac{n\pi}{L}\cdot x$$

$$= \frac{4}{\pi^2}\sum_{n=1}^{\infty} \frac{\sin\frac{n\pi}{2}}{n^2}\cdot\cos\frac{cn\pi}{L}t\,\sin\frac{n\pi}{\pi}\cdot x$$

$$= \frac{4}{\pi^2}\sum_{n=1}^{\infty} \frac{1}{n^2}\cdot\sin\frac{n\pi}{2}\,\cos\frac{1\cdot n\cdot\pi\cdot t}{\pi}\cdot\sin nx$$

$$= \frac{4}{\pi^2}\sum_{n=1}^{\infty} \frac{1}{n^2}\sin\frac{n\pi}{2}\cdot\cos nt\cdot\sin nx$$

$$u(x,t) = \frac{4}{\pi^2}\left(\cos t\cdot\sin x - \frac{1}{9}\cos 3t\cdot\sin 3x + \frac{1}{25}\cos 5t\cdot\sin 5x - \cdots\right)$$

(5) Find the deflection of the string of length $L=\pi$, $c^2=1$ and its initial deflection is zero and initial velocity is

$$g(x) = \begin{cases} 0.01x & \text{if } 0 < x < \frac{\pi}{2} \\ 0.01(\pi - x) & \text{if } \frac{\pi}{2} < x < \pi \end{cases}$$

Solution: Solution of 1-D is

$$u(x,t) = \sum_{n=1}^{\infty}\left(B_n \cos\lambda_n t + B_n^* \sin\lambda_n t\right)\sin\frac{n\pi}{L}x \quad \text{--- (1)}$$

where

$$B_n = \frac{2}{L}\int_0^L f(x)\cdot\sin\frac{n\pi}{L}x\,dx = 0$$

being initial deflection is zero

$$B_n^* = \frac{2}{cn\pi}\int_0^L g(x)\sin\frac{n\pi}{L}x\,dx$$

Eqn 1 becomes

$$U(x,t) = \sum_{n=1}^{\infty} B_n^* \sin \lambda_n t \cdot \sin \frac{n\pi}{L} x \quad (2)$$

Now

$$B_n^* = \frac{2}{cn\pi} \int_0^L g(x) \sin \frac{n\pi}{L} x \, dx$$

$$= \frac{2}{cn\pi} \int_0^\pi g(x) \sin \frac{n\pi}{\pi} \cdot x \, dx$$

$$= \frac{2}{cn\pi} \int_0^\pi g(x) \sin nx \, dx$$

$$= \frac{2}{cn\pi} \left[\int_0^{\pi/2} 0.01 x \sin nx \, dx + \int_{\pi/2}^{\pi} 0.01(\pi - x) \sin nx \, dx \right]$$

$$= \frac{0.02}{cn\pi} \left[\int_0^{\pi/2} x \sin nx \, dx + \int_{\pi/2}^{\pi} (\pi - x) \sin nx \, dx \right]$$

Same as Q.N. 1(c)

$$= \frac{0.02}{1 \times n\pi} \times \frac{2 \sin \frac{n\pi}{2}}{n^2}$$

$$= \frac{0.04}{n^3 \pi}$$

Putting value of B_n^* in eqn (2)

$$u(x,t) = \sum_{n=1}^{\infty} B_n^* \sin \lambda_n t \sin \frac{n\pi}{L} x$$

$$= \sum_{n=1}^{\infty} \frac{0.04}{n^3 \pi} \sin \frac{n\pi}{2} \cdot \sin \frac{(n\pi)}{\pi} \cdot t \cdot \sin \frac{nx \cdot \pi}{\pi}$$

$$= \sum_{n=1}^{\infty} \frac{0.04}{n^3 \pi} \cdot \sin \frac{n\pi}{2} \sin nt \cdot \sin nx$$

$$= \frac{0.04}{\pi} \sum_{n=1}^{\infty} \frac{1}{n^3} \sin \frac{n\pi}{2} \sin nt \sin nx$$

$$= \frac{0.04}{\pi} \left[1 \cdot \sin t \cdot \sin x - \frac{1}{27} \sin 3t \cdot \sin 3x + \cdots \right]$$

is the required Solution.

(6) A tightly stretched string of length L is drawn a side its mid point a distance $\frac{L}{2}$ perpendicular to the equilibrium position so that its initial position is given by

$$f(x) = \begin{cases} x & \text{if } 0 < x < \frac{L}{2} \\ (L-x) & \text{if } \frac{L}{2} < x < L \end{cases}$$

It is initially at rest and suddenly released. Find $u(x,t)$.

Scanned with CamScanner

Solution ÷ Here

Length of string $= L$

Initial velocity $= g(x) = 0$

Initial deflection is

$$f(x) = \begin{cases} x & \text{if } 0 < x < L/2 \\ L-x & \text{if } \frac{L}{2} < x < L \end{cases}$$

We know

$$u(x,t) = \sum_{n=1}^{\infty} \left(B_n \cos\lambda_n t + B_n^* \sin\lambda_n t\right) \sin\frac{n\pi}{L}x \quad -(1)$$

Since $g(x) = 0 \Rightarrow B_n^* = 0$

Eqn (1) becomes

$$u(x,t) = \sum_{n=1}^{\infty} B_n \cos\lambda_n t \cdot \sin\frac{n\pi}{L}x \quad -(2)$$

here $B_n = \frac{2}{L}\int_0^L f(x)\cdot \sin\frac{n\pi}{L}x\,dx$

$$= \frac{2}{L}\left[\int_0^{L/2} x \sin\frac{n\pi}{L}x\,dx + \int_{L/2}^{L} (L-x)\sin\frac{n\pi}{L}x\,dx\right]$$

$$\left[\begin{array}{ll} x & \sin\frac{n\pi}{L}x \\ 1 & \frac{-\cos\frac{n\pi}{L}x}{\frac{n\pi}{L}} \\ 0 & \frac{-\sin\frac{n\pi}{L}x}{\left(\frac{n\pi}{L}\right)^2} \end{array}\right.
\qquad
\begin{array}{ll} L-x & \sin\frac{n\pi}{L}x \\ -1 & \frac{-\cos\frac{n\pi}{L}x}{\frac{n\pi}{L}} \\ 0 & \frac{-\sin\frac{n\pi}{L}x}{\left(\frac{n\pi}{L}\right)^2} \end{array}$$

$$= \frac{2}{L}\left[\left| \frac{-x\cos\frac{n\pi}{L}\cdot x}{\frac{n\pi}{L}} + \frac{\sin\frac{n\pi}{L}\cdot x}{\left(\frac{n\pi}{L}\right)^2} \right|_0^{L/2} \right.$$

$$\left. + \left| (L-x)\cdot - \frac{\cos\frac{n\pi}{L}\cdot x}{\frac{n\pi}{L}} - \frac{\sin\frac{n\pi}{L}\cdot x}{\left(\frac{n\pi}{L}\right)^2} \right|_{L/2}^{L} \right]$$

$$= \frac{2}{L}\left[-\frac{L/2\cdot L}{n\pi}\cdot\cos\frac{n\pi}{\cancel{L}}\times \cancel{L}/2 + \sin\frac{n\pi}{\cancel{L}}\times\frac{\cancel{L}}{2}\times\frac{L^2}{(n\pi)^2} \right.$$

$$- 0 + \cancel{(L-L)}^{0}\cdot - \frac{\cos\frac{n\pi}{L}\cdot L}{\frac{n\pi}{L}}\cdot$$

$$+ \frac{L^2}{(n\pi)^2}\cdot\sin\frac{n\pi}{\cancel{L}}\cdot\cancel{L} + \left(L - \frac{L}{2}\right)\cdot\cos\frac{n\pi}{L}\cdot\frac{L}{2}\times\frac{L}{n\pi}$$

$$\left. + \frac{\sin\frac{n\pi}{\cancel{L}}\times\frac{\cancel{L}}{2}}{\left(\frac{n\pi}{L}\right)^2} \right]$$

$$= \frac{2}{L}\left[\cancel{-\frac{L^2}{2n\pi}\cos\frac{n\pi}{2}} + \sin\frac{n\pi}{2}\cdot\frac{L^2}{(n\pi)^2} \right.$$

$$\left. + \frac{L^2}{(n\pi)^2}\cdot\cancel{\sin n\pi}^{0} + \cancel{\frac{L^2}{2n\pi}\cdot\cos\frac{n\pi}{2}} + \frac{L}{(n\pi)^2}\cdot\sin\frac{n\pi}{2} \right]$$

$$B_n = \frac{4L}{(n\pi)^2}\cdot\sin\frac{n\pi}{2} \Rightarrow$$ Putting value in eqn (2) which ~~is~~ required soln.