Fourier Integral Theorem

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Fourier Integral Theorem:

It stats that, fourier integral of a function $f(x)$ is

$$f(x) = \int_0^\infty [A(\omega)\cos\omega x + B(\omega)\sin\omega x]\, d\omega$$

, where

$$A(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\cos\omega t dt$$

and

$$B(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\sin\omega t dt$$

Proof: We know, the fourier series of function $f(x)$ of period $2L$ defined on the interval $(-L, L)$ is a trigonometric series of the form

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right] ......(1)$$

where

$$a_0 = \frac{1}{L} \int_{-L}^{L} f(t) \quad dt,$$

$$a_n = \frac{1}{L} \int_{-L}^{L} f(t) \cos\left(\frac{n\pi t}{L}\right) dt$$

and

$$b_n = \frac{1}{L} \int_{-L}^{L} f(t) \sin\left(\frac{n\pi t}{L}\right) dt$$

Substituting the values of a_0, a_n and b_n in equation (1) we get,

$$\begin{aligned} f(x) &= \frac{1}{2L}\int_{-L}^{L} f(t) \quad dt \quad + \\ & \sum_{n=1}^{\infty}\left[\frac{1}{L}\int_{-L}^{L} f(t)\cos\left(\frac{n\pi t}{L}\right) dt.\cos\left(\frac{n\pi x}{L}\right)\right] \\ &+ \sum_{n=1}^{\infty}\left[\frac{1}{L}\int_{-L}^{L} f(t)\sin\left(\frac{n\pi t}{L}\right) dt.\sin\left(\frac{n\pi x}{L}\right)\right] \end{aligned}$$

$$or, f(x) = \frac{1}{2L}\int_{-L}^{L} f(t) \qquad \times$$

$$\left[1 + 2\sum_{n=1}^{\infty}\left\{\cos\left(\frac{n\pi t}{L}\right).\cos\left(\frac{n\pi x}{L}\right) + \sin\left(\frac{n\pi t}{L}\right).\sin\left(\frac{n\pi x}{L}\right)\right\}\right] dt$$

$$= \frac{1}{2L}\int_{-L}^{L} f(t)\left[1 + 2\sum_{n=1}^{\infty}\left\{\cos\left(\frac{n\pi t}{L} - \frac{n\pi x}{L}\right)\right\}\right] dt$$

$$= \frac{1}{2L} \int_{-L}^{L} f(t) \left[1 + 2 \sum_{n=1}^{\infty} \left\{ \cos \frac{n\pi}{L} (t - x) \right\} \right] dt$$

$$= \frac{1}{2L}\int_{-L}^{L} f(t)\left[\cos 0 + \sum_{n=1}^{\infty}\cos\frac{n\pi}{L}(t-x) + \sum_{n=1}^{\infty}\cos\frac{n\pi}{L}(t-x)\right] dt$$

$$\left[\because 1 = \cos 0 \quad and \quad 2\sum_{n=1}^{\infty} = \sum_{n=1}^{\infty} + \sum_{n=1}^{\infty}\right]$$

$$= \frac{1}{2L} \int_{-L}^{L} f(t) \left[\cos 0 + \sum_{n=1}^{\infty} \cos \frac{n\pi}{L} (t - x) + \sum_{n=1}^{\infty} \cos \frac{n\pi}{L} (-) (t - x) \right] dt$$

$$[\because \cos(-\theta) = \cos \theta]$$

$$= \frac{1}{2L} \int_{-L}^{L} f(t) \left[\cos 0 + \sum_{n=1}^{\infty} \cos \frac{n\pi}{L} (t - x) + \sum_{n=-\infty}^{-1} \cos \frac{n\pi}{L} (t - x) \right] dt$$

$$\therefore f(x) = \frac{1}{2L} \int_{-L}^{L} f(t) \left[\sum_{n=-\infty}^{\infty} \cos \frac{n\pi}{L} (t - x) \right] dt$$

$$\left[\because \sum_{n=-\infty}^{-1} + 0 + \sum_{n=1}^{\infty} = \sum_{n=-\infty}^{\infty} \right]$$

$$\therefore f(x) = \frac{1}{2\pi} \int_{-L}^{L} f(t) \left[\sum_{n=-\infty}^{\infty} (\frac{\pi}{L}) \cos \frac{n\pi}{L} (t - x) \right] dt.... \quad ...(2)$$

Let us assume that L increases indefinitely, so that we may write

$$\frac{n\pi}{L} = \omega$$

and

$$\frac{(n+1)\pi}{L} = \omega + d\omega$$

$$i.e. \quad d\omega = \frac{(n+1)\pi}{L} - \frac{n\pi}{L} = \frac{\pi}{L}$$

$i.e.$ in limiting case, when $L \to \infty$, we get

$$\lim_{L\to\infty}\left[\sum_{n=-\infty}^{\infty}(\frac{\pi}{L})\cos\frac{n\pi}{L}(t-x)\right] = \int_{-\infty}^{\infty}\cos\omega\,(t-x)\,d\omega$$

$$\because \left[\sum_{n=-\infty}^{\infty} = \int_{-\infty}^{\infty} and \lim_{L\to\infty}\frac{\pi}{L} = d\omega\right]$$

$$= 2\int_{0}^{\infty}\cos\omega\,(t-x)\,d\omega$$

So, from equation (2), we get

$$
\begin{aligned}
f(x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)\left[2\int_{0}^{\infty}\cos\omega\,(t-x)\,d\omega\right]dt \\
&= \int_{0}^{\infty}\left[\int_{-\infty}^{\infty}\frac{1}{\pi}f(t)\cos(\omega t-\omega x)dt\right]d\omega
\end{aligned}
$$

$$= \int_0^\infty \left[\frac{1}{\pi} \int_{-\infty}^\infty f(t) \left(\cos \omega t . \cos \omega x + \sin \omega t . \sin \omega x \right) dt \right] d\omega$$

$$= \int_0^\infty \left[\left\{ \frac{1}{\pi} \int_{-\infty}^{\infty} f(t) \cos \omega t dt \right\} \cos \omega x + \left\{ \frac{1}{\pi} \int_{-\infty}^{\infty} f(t) \sin \omega t dt \right\} \sin \omega x \right] d$$

$$\therefore f(x) = \int_0^{\infty} [A(\omega)\cos\omega x + B(\omega)\sin\omega x]\, d\omega$$

where

$$A(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\cos\omega t dt$$

and

$$B(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\sin\omega t dt$$

This completes the proof of the theorem.