Unit-7
Eigen values and Eigen vectors

Eigen value:

Definition: If A is $n \times n$ matrix, then a scalar λ is called an eigen value of matrix A if equation $Ax = \lambda x$ has a non-trivial solution. Such an x is called eigen vector corresponding to eigen value λ.

Eigen vector:

Definition: If A is $n \times n$ matrix, then a non-zero vector $x \in R^n$ is called an eigen-vector of matrix A if $Ax = \lambda x$, where λ is scalar.

Example 1: Is $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ an eigen vector of $\begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}$?

Solution:

Since $A = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}$ and $x = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

So, $Ax = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \lambda x$

Hence, $x = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is eigen vector of $\begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}$

Note: If $Ax \neq \lambda x$ then x is not eigen vector of A.

Example 2: Show that -2 is eigen value of $\begin{bmatrix} 7 & 3 \\ 3 & -1 \end{bmatrix}$

Solution:

Given, $\lambda = -2$ and $A = \begin{bmatrix} 7 & 3 \\ 3 & -1 \end{bmatrix}$

If $Ax = \lambda x$

or, $Ax = -2x$

or, $(A+2I)x = 0$ — (i), where I = identity matrix

has non-trivial solution, then $\lambda = -2$ is eigen value of $\begin{bmatrix} 7 & 3 \\ 3 & -1 \end{bmatrix}$

Since, $A+2I = \begin{bmatrix} 7 & 3 \\ 3 & -1 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 9 & 3 \\ 3 & 1 \end{bmatrix}$.

So, row reduced augmented matrix is;

$[A+2I \quad 0]$

$\sim \begin{bmatrix} 9 & 3 & 0 \\ 3 & 1 & 0 \end{bmatrix}$

$R_1 \to \frac{1}{3}R_1 \quad \sim \begin{bmatrix} 3 & 1 & 0 \\ 3 & 1 & 0 \end{bmatrix}$

$R_2 \to R_2 - R_1 \quad \sim \begin{bmatrix} 3 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Thus homogenous system has free variable (here x_2 is free variable), so eqn (i) has non-trivial soln. Thus $\lambda = -2$ is eigen value of given matrix A.

Moreover

For finding corresponding eigenvectors;

The general solution form $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -x_2/3 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} -1/3 \\ 1 \end{bmatrix}$

($\because$ from row reduced augmented matrix $3x_1 + x_2 = 0$
or, $3x_1 = -x_2$
or, $x_1 = -x_2/3$.)

So, $x_2 \begin{bmatrix} -1/3 \\ 1 \end{bmatrix}$, where $x_2 \neq 0$ are the eigen vectors corresponding to eigen value $\lambda = -2$.

Note:- If the homogenous system has no free variable after constructing row reduced augmented matrix, then the equation has trivial solution and λ is not eigen value of A.

Example 3: Find the basis for the eigen space corresponding to listed eigenvalue, where $A = \begin{bmatrix} 4 & 2 & 3 \\ -1 & 1 & -3 \\ 2 & 4 & 9 \end{bmatrix}$ and $\lambda = 3$.

Solution:

Since $\lambda = 3$ is eigen value for given matrix A, so, $Ax = 3x$ has non-trivial solution. i.e, $(A - 3I)x = 0$ — (i).

Here,

$$A - 3I = \begin{bmatrix} 4 & 2 & 3 \\ -1 & 1 & -3 \\ 2 & 4 & 9 \end{bmatrix} - \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 & 3 \\ -1 & -2 & -3 \\ 2 & 4 & 6 \end{bmatrix}$$

So, reduced augmented matrix $[A-3I \quad 0]$

$$= \begin{bmatrix} 1 & 2 & 3 & 0 \\ -1 & -2 & -3 & 0 \\ 2 & 4 & 6 & 0 \end{bmatrix}$$

$R_2 \to R_2 + R_1$ & $R_3 \to R_3 - 2R_1$.

$$\sim \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

remember

fte ← free variable भए trivial हुन्छ र λ eigen value हुन्छ A को लागि.

$\Rightarrow$ reverse for no free variable, no trivial solution, λ not eigen value of A.

Thus the homogenous system has free variable so the system has non-trivial solution.

$$x_1 + 2x_2 + 3x_3 = 0$$
x_2 is free
x_3 is free

This implies

$$x_1 = -2x_2 - 3x_3$$
$$x_2 = x_2$$
$$x_3 = x_3$$

Hence, $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2x_2 - 3x_3 \\ x_2 \\ x_3 \end{bmatrix} = \left\{ x_2 \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix} \right\}$ is eigen space

and basis for eigenspace is $\left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix} \right\}$.

⊛. The characteristic Equation:

Definition (Characteric Polynomial, Characteristic Equation):

If λ be an eigen value of a square matrix A, then $\det(A-\lambda I)$ is called characteristic polynomial and $\det(A-\lambda I)=0$ is called characteristic equation of the matrix A.

Example1: Find the characteristic polynomial of matrix $\begin{bmatrix} 2 & -1 \\ 1 & 4 \end{bmatrix}$ and find it's eigen value.

Solution:

Characteristic polynomial is $|A-\lambda I|$,

where,
$$A-\lambda I = \begin{bmatrix} 2 & -1 \\ 1 & 4 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$$
$$= \begin{bmatrix} 2-\lambda & -1 \\ 1 & 4-\lambda \end{bmatrix}$$

Therefore characteristic polynomial is, $\begin{vmatrix} 2-\lambda & -1 \\ 1 & 4-\lambda \end{vmatrix}$

$$= (2-\lambda)(4-\lambda)+1$$
$$= \lambda^2-6\lambda+9.$$

So, characteristic equation is $|A-\lambda I|=0$

or, $\lambda^2-6\lambda+9=0$

or, $\lambda^2-\lambda(3+3)+9=0$

or, $\lambda^2-3\lambda-3\lambda+9=0$

or, $\lambda(\lambda-3)-3(\lambda-3)=0$

or, $(\lambda-3)(\lambda-3)=0$

or, $\lambda=3$.

$\therefore \lambda=3$ is eigenvalue of matrix $\begin{bmatrix} 2 & 1 \\ -1 & 4 \end{bmatrix}$

Example2: Find the characteristic equation and eigen value of A where $A=\begin{bmatrix} 1 & -4 \\ 4 & 2 \end{bmatrix}$.

Solution: Given, $A=\begin{bmatrix} 1 & -4 \\ 4 & 2 \end{bmatrix}$

So, the characteristic eqn of A is $|A-\lambda I|=0$.

or, $A-\lambda I = \begin{bmatrix} 1 & -4 \\ 4 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 1-\lambda & -4 \\ 4 & 2-\lambda \end{bmatrix}$

Thus characteristic eqn of A is $|A-\lambda I|=0$

or $\begin{vmatrix} 1-\lambda & -4 \\ 4 & 2-\lambda \end{vmatrix}=0$

or, $(1-\lambda)(2-\lambda)+16=0$

or, $\lambda^2-3\lambda+18=0$.

This gives $\lambda = \dfrac{3\pm\sqrt{(-3)^2-4\times1\times18}}{2} = \dfrac{3\pm\sqrt{-63}}{2}$

This gives the imiganiary value of λ. Therefore the matrix A has no real eigen value.

Diagonalization:

Definition → A square matrix A is called diagonalizable if there exist an invertible matrix P and diagonal matrix D such that $A = PDP^{-1}$ [Equivalently $AP = PD$].

Procedure for Diagonalizing a matrix:

Step1: Find n linearly independent eigen vectors of A. say $v_1, v_2, \dots, v_n$.

step2: For matrix P having $v_1, v_2, \dots, v_n$ as its column vectors.

step3: The matrix D will be the diagonal matrix with $\lambda_1, \lambda_2, \dots, \lambda_n$ as its successive diagonal entries, where λ_i is the eigenvalue corresponding to v_i for $i = 1, 2, \dots, n$.

Here, $A = PDP^{-1}$ or $AP = PD$, if so our P and D really work as $AP = PD$ then the matrix A is diagonalizible.

Example 1: Diagonalize the matrix $\begin{bmatrix} -1 & 4 & -2 \\ -3 & 4 & 0 \\ -3 & 1 & 3 \end{bmatrix}$, if exist.

Solution: Let $A = \begin{bmatrix} -1 & 4 & -2 \\ -3 & 4 & 0 \\ -3 & 1 & 3 \end{bmatrix}$.

The characteristic polynomial of A is $A - \lambda I = \begin{bmatrix} -1-\lambda & 4 & -2 \\ -3 & 4-\lambda & 0 \\ -3 & 1 & 3-\lambda \end{bmatrix}$

Therefore, the characteristic equation of A is $A - \lambda I = 0$.

$$\Rightarrow \begin{vmatrix} -1-\lambda & 4 & -2 \\ -3 & 4-\lambda & 0 \\ -3 & 1 & 3-\lambda \end{vmatrix} = 0$$

$R_3 \to R_3 - R_2$

$$\Rightarrow \begin{vmatrix} -1-\lambda & 4 & -2 \\ -3 & 4-\lambda & 0 \\ 0 & -3+\lambda & 3-\lambda \end{vmatrix} = 0.$$

$$\Rightarrow (3-\lambda) \begin{vmatrix} -1-\lambda & 4 & -2 \\ -3 & 4-\lambda & 0 \\ 0 & -1 & 1 \end{vmatrix} = 0.$$

Either $(3-\lambda) = 0$ or $\begin{vmatrix} -1-\lambda & 4 & -2 \\ -3 & 4-\lambda & 0 \\ 0 & -1 & 1 \end{vmatrix} = 0.$

$$\Rightarrow \begin{vmatrix} -1-\lambda & 2 & -2 \\ -3 & 4-\lambda & 0 \\ 0 & 0 & 1 \end{vmatrix} = 0.$$

$$\Rightarrow \begin{vmatrix} -1-\lambda & 2 \\ -3 & 4-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda (-1-\lambda)(4-\lambda) + 6 = 0$$

$$\Rightarrow \lambda^2 - 3\lambda + 2 = 0$$
$$\Rightarrow \lambda^2 - 2\lambda - \lambda + 2 = 0$$
$$\Rightarrow \lambda(\lambda-2) - 1(\lambda-2) = 0$$
$$\Rightarrow (\lambda-2)(\lambda-1) = 0$$

$\lambda = 1$ र 2 यो बाट
3 चाहिं either $(3-\lambda)=0$ बाट

Therefore $\lambda = 1, 2, 3$.

For $\lambda = 1$

Since $Ax = \lambda x$.
So $(A - \lambda I)x = 0$.

And it's augmented matrix is $[A - \lambda I \quad 0]$

$$= [A - I \quad 0], \text{ being } \lambda = 1.$$

$$= \begin{bmatrix} -2 & 4 & -2 & 0 \\ -3 & 3 & 0 & 0 \\ -3 & 1 & 2 & 0 \end{bmatrix}$$

$R_1 \to -\frac{1}{2} R_1$

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ -3 & 3 & 0 & 0 \\ -3 & 1 & 2 & 0 \end{bmatrix}$$

$R_2 \to R_2 + 3R_1, \; R_3 \to R_3 + 3R_1$

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & -3 & 3 & 0 \\ 0 & -5 & 5 & 0 \end{bmatrix}$$

$R_2 \to -\frac{1}{3} R_2$

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -5 & 5 & 0 \end{bmatrix}$$

$R_3 \to R_3 + 5R_2$

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_1 \to R_1 + 2R_2$

$$\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Now,
$$x_1 - x_3 = 0 \Rightarrow x_1 = x_3$$
$$x_2 - x_3 = 0 \Rightarrow x_2 = x_3$$
$$x_3 \text{ is free} \Rightarrow x_3 = x_3$$

$$\therefore X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} x_3 . \quad \text{Let } x_3 = 1 \quad \therefore X = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

For $\lambda = 2$

Augmented matrix is $[A-2I \quad 0]$

$$= \begin{bmatrix} -3 & 4 & -2 & 0 \\ -3 & 2 & 0 & 0 \\ -3 & 1 & 1 & 0 \end{bmatrix}$$

$R_2 \to R_2 - R_1, R_3 \to R_3 - R_1$.

$$\begin{bmatrix} -3 & 4 & -2 & 0 \\ 0 & -2 & 2 & 0 \\ 0 & -3 & 3 & 0 \end{bmatrix}$$

$R_2 \to R_2 - R_3$

$$\begin{bmatrix} -3 & 4 & -2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -3 & 3 & 0 \end{bmatrix}$$

$R_3 \to R_3 + 3R_2$

$$\begin{bmatrix} -3 & 4 & -2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_1 \to R_1 + (-4)R_2$

$$\begin{bmatrix} -3 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_1 \to -1/3 R_1$

$$\begin{bmatrix} 1 & 0 & -2/3 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Now,

$x_1 - 2/3 x_3 = 0 \quad \Rightarrow x_1 = 2/3 x_3$

$x_2 - x_3 = 0 \quad \Rightarrow x_2 = x_3$

x_3 is free $\quad \Rightarrow x_3 = x_3$.

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2/3 \\ 1 \\ 1 \end{bmatrix} x_3.$$

Let $x_3 = 3 \quad \therefore X = \begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix}$.

For $\lambda = 3$

Augmented matrix is $[A-3I \quad 0]$

$$= \begin{bmatrix} -4 & 4 & -2 & 0 \\ -3 & 1 & 0 & 0 \\ -3 & 1 & 0 & 0 \end{bmatrix}$$

$R_1 \to \frac{-1}{4} R_1$

$$\begin{bmatrix} 1 & -1 & 1/2 & 0 \\ -3 & 1 & 0 & 0 \\ -3 & 1 & 0 & 0 \end{bmatrix}$$

$R_2 \to R_2 + 3R_1$ & $R_3 \to R_3 + 3R_1$

$$\begin{bmatrix} 1 & -1 & 1/2 & 0 \\ 0 & -2 & 3/2 & 0 \\ 0 & -2 & 3/2 & 0 \end{bmatrix}$$

$R_2 \to -\frac{1}{2} R_2$ $$\begin{bmatrix} 1 & -1 & 1/2 & 0 \\ 0 & 1 & -3/4 & 0 \\ 0 & -2 & 3/2 & 0 \end{bmatrix}$$

$R_3 \to R_3 + 2R_2$ $$\begin{bmatrix} 1 & -1 & 1/2 & 0 \\ 0 & 1 & -3/4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_1 \to R_1 + R_2$ $$\begin{bmatrix} 1 & 0 & -1/4 & 0 \\ 0 & 1 & -3/4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Now, $x_1 - 1/4\, x_3 = 0 \Rightarrow x_1 = 1/4\, x_3$

$x_2 - 3/4\, x_3 = 0 \Rightarrow x_2 = 3/4\, x_3$

x_3 is free $\Rightarrow x_3 = x_3$.

$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1/4 \\ 3/4 \\ 1 \end{bmatrix} x_3$. let $x_3 = 4$ $\therefore X = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$

There are 3 base vectors in total which are L.I. (linearly independent)

So, $P = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 3 \\ 1 & 3 & 4 \end{bmatrix}$ & $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

collection of value of X

values of λ in diagonal others zero

Also,

$$PD = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 3 \\ 1 & 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 4 & 3 \\ 1 & 6 & 9 \\ 1 & 6 & 12 \end{bmatrix}.$$

and $$AP = \begin{bmatrix} -1 & 4 & -2 \\ -3 & 4 & 0 \\ -3 & 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 3 \\ 1 & 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 4 & 3 \\ 1 & 6 & 9 \\ 1 & 6 & 12 \end{bmatrix}.$$

Thus, $AP = PD$ or equivalently, $A = PDP^{-1}$

Therefore, A is diagonalizable

//

Example 2: Diagonalizable the matrix $A = \begin{bmatrix} 4 & 0 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 5 \end{bmatrix}$, if possible.

Solution:

Given, $A = \begin{bmatrix} 4 & 0 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

So, the characteristic polynomial of A is $A - \lambda I = \begin{bmatrix} 4-\lambda & 0 & 0 \\ 1 & 4-\lambda & 0 \\ 0 & 0 & 5-\lambda \end{bmatrix}$

Therefore, the characteristic equation of A is $|A - \lambda I| = 0$.

$$\Rightarrow \begin{vmatrix} 4-\lambda & 0 & 0 \\ 1 & 4-\lambda & 0 \\ 0 & 0 & 5-\lambda \end{vmatrix} = 0.$$

This determinant is an lower triangular. So we get,

$\lambda = 4, 5.$

For $\lambda = 4$

Since $Ax = \lambda x$. So $(A - \lambda I)x = 0$.

And, its augmented matrix is $[A - \lambda I \;\; 0]$

$= [A - 4I \;\; 0] \quad (\because \lambda = 4)$

$$\sim \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

This is reduced echelon form.

From this last matrix, x_2 is free variable.

and $x_1 = 0$

$x_3 = 0.$

Therefore, $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$

There the basis for eigen space (for $\lambda = 4$) is $= \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = v_1$

for $\lambda = 5$ आफैं solve गर्नें

Similarly the eigen space (for $\lambda = 5$) is $= \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = v_2$.

There are only two vector (v_1 & v_2) in basis and is linearly independent. But we need three independent eigen vectors to form P. So, P doesn't exist. Hence, A is not diagonalizable.

Theorem: An $n*n$ matrix with n distinct eigenvalues is diagonalizable.

Example 1: Is matrix $A=\begin{bmatrix}5 & 0 & 0\\ 1 & 2 & 0\\ -3 & 2 & 3\end{bmatrix}$ is diagonalizable?

Solution:

Since matrix is triangular and there are three distinct eigenvalues (i.e, $\lambda=2,3$ and 5) and matrix is 3×3. So it is diagonalizable.

Example 2: Let $A=PDP^{-1}$, compute A^4; if $P=\begin{bmatrix}5 & 7\\ 2 & 3\end{bmatrix}$ and $D=\begin{bmatrix}2 & 0\\ 0 & 1\end{bmatrix}$

Solution: We know that,

$$A^4=PD^4P^{-1}=\begin{bmatrix}5 & 7\\ 2 & 3\end{bmatrix}\begin{bmatrix}2 & 0\\ 0 & 1\end{bmatrix}^4\begin{bmatrix}5 & 7\\ 2 & 3\end{bmatrix}^{-1}$$

$$=\begin{bmatrix}5 & 7\\ 2 & 3\end{bmatrix}\begin{bmatrix}2^4 & 0\\ 0 & 1^4\end{bmatrix}\left(\frac{1}{15-14}\right)\begin{bmatrix}3 & -7\\ -2 & 5\end{bmatrix}$$

$$=\begin{bmatrix}5 & 7\\ 2 & 3\end{bmatrix}\begin{bmatrix}16 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}3 & -7\\ -2 & 5\end{bmatrix}$$

$$=\begin{bmatrix}80 & 7\\ 32 & 3\end{bmatrix}\begin{bmatrix}3 & -7\\ -2 & 5\end{bmatrix}$$

$$=\begin{bmatrix}226 & -525\\ 90 & -209\end{bmatrix}.$$

Note: If only A is given in question first we find P and D same as we used in ~~diagoliz~~ diagonalization of matrix then we follow same process as in example 2.

Example 3: Let $A=\begin{bmatrix}-3 & 12\\ -2 & 7\end{bmatrix}$, $v_1=\begin{bmatrix}3\\ 1\end{bmatrix}$ and $v_2=\begin{bmatrix}2\\ 1\end{bmatrix}$. Suppose you are told that v_1 and v_2 are eigenvectors of A. Use the information to diagonalize A.

Soln:

To diagonalize A, we must find the value of P and D.

For these, we need the eigenvalue λ of A.

For the eigen value λ corresponding to eigen vector $v_1=\begin{bmatrix}3\\ 1\end{bmatrix}$.

Let, $Av_1=\begin{bmatrix}-3 & 12\\ -2 & 7\end{bmatrix}\begin{bmatrix}3\\ 1\end{bmatrix}=\begin{bmatrix}-9+12\\ -6+7\end{bmatrix}=\begin{bmatrix}3\\ 1\end{bmatrix}=1\begin{bmatrix}3\\ 1\end{bmatrix}=1.v_1.$

This shows that $\lambda=1$.

For $\lambda=-1$,
$$(A-\lambda I)x=0$$

For the eigenvalue λ corresponding to eigen vector $v_2=\begin{bmatrix}2\\1\end{bmatrix}$.

Let, $Av_2=\begin{bmatrix}-3 & 12\\-2 & 7\end{bmatrix}\begin{bmatrix}2\\1\end{bmatrix}=\begin{bmatrix}-6+12\\-4+7\end{bmatrix}=\begin{bmatrix}6\\3\end{bmatrix}=3\begin{bmatrix}2\\1\end{bmatrix}=3v_2$.

This shows that $\lambda=3$.

So, $P=\begin{bmatrix}3 & 2\\1 & 1\end{bmatrix}$ and $D=\begin{bmatrix}1 & 0\\0 & 3\end{bmatrix}$.

And, $AP=\begin{bmatrix}-3 & 12\\-2 & 7\end{bmatrix}\begin{bmatrix}3 & 2\\1 & 1\end{bmatrix}=\begin{bmatrix}-9+12 & -6+12\\-6+7 & -4+7\end{bmatrix}=\begin{bmatrix}3 & 6\\1 & 3\end{bmatrix}$.

$PD=\begin{bmatrix}3 & 2\\1 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\0 & 3\end{bmatrix}=\begin{bmatrix}3 & 6\\1 & 3\end{bmatrix}$.

This shows $AP=PD$ or equivalently $A=PDP^{-1}$

So, A is diagonalizable.

⊛ Complex Eigen values:

[TU 2067 B.S.C. CSIT exam asked question]

Example 1: If $A=\begin{bmatrix}0 & -1\\1 & 0\end{bmatrix}$. Find eigenvalue and corresponding eigen vector.

Solution:

Given $A=\begin{bmatrix}0 & -1\\1 & 0\end{bmatrix}$

The characteristic equation is, $|A-\lambda I|=0$.

$$\Rightarrow \begin{vmatrix}-\lambda & -1\\1 & -\lambda\end{vmatrix}=0$$

$$\Rightarrow \lambda^2+1=0$$

$$\Rightarrow \lambda=\pm i \text{ (complex eigen values).}$$

For $\lambda=i$,

$Ax=\lambda x$, $x\neq 0$

i.e, $(A-\lambda I)x=0$.

having non-trivial solution, then x is eigenvector of Eigenvalue λ.

or, $\begin{pmatrix}-i & -1\\1 & -i\end{pmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}=0$.

$\Rightarrow -ix_1-x_2=0 \dots (i)$

Here, both eq^n are identical $\Rightarrow x_1-ix_2=0 \dots (ii)$

Take eq^n (ii)

$x_1=ix_2$

Put $x_2=1$ then $x_1=ix_2$

Put $x_2=1$ then $x_1=i$.

Hence, eigen vector is $x=\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{pmatrix}i\\1\end{pmatrix}$ corresponding $\lambda=i$.

Hence, eigen vector $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2+4i \\ 5 \end{bmatrix}$ corresponding to eigen value $\lambda = 0.8 + (0.6)i$.

And, the basis for the corresponding to $\lambda = 0.8 + (0.6)i$ is,

$$v_1 = \begin{bmatrix} -2+4i \\ 5 \end{bmatrix}.$$

For $\lambda = 0.8 - (0.6)i$,

$$(A - \lambda I)x = 0.$$

$$\Rightarrow \begin{pmatrix} -0.3+(0.6)i & -0.6 \\ 0.75 & 0.3+0.6i \end{pmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\Rightarrow (-0.3 + (0.6)i)x_1 - (0.6)x_2 = 0 \quad \text{--- (iii)}$$

And $0.75x_1 + (0.3 + (0.6)i)x_2 = 0$ --- (iv).

Here both (iii) and (iv) are identical so it has non-trivial solution.

Taking (iv)

$$0.75x_1 + (0.3 + (0.6)i)x_2 = 0$$

$$\Rightarrow 0.75x_1 = -(0.3 + (0.6)i)x_2$$

$$\Rightarrow x_1 = \frac{-1}{0.75}(0.3 + (0.6)i)x_2$$

$$\Rightarrow x_1 = \left(-\frac{2}{5} - \frac{4}{5}i\right)x_2$$

Put $x_2 = 5$, then $x_1 = -2 - 4i$

Hence, eigen vector $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2-4i \\ 5 \end{bmatrix}$ corresponding to eigenvector.

$\lambda = 0.8 - (0.6)i$. A basis for the corresponding to $\lambda = 0.8 - (0.6)i$ is $v_2 = \begin{bmatrix} -2-4i \\ 5 \end{bmatrix}$.