(2, 3 topics may be earlier or later in chapter but contain all topics)

Unit → 2

A bit hard chapter to understand and solve questions others are easier

Transformation

⊛ Introduction to linear transformation:

Definition → A transformation (or function or mapping) T from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a rule that assigns to each vector x in $\mathbb{R}^n$ a vector $T(x)$ in $\mathbb{R}^m$. The set $\mathbb{R}^n$ is called the domain of T and $\mathbb{R}^m$ is called the codomain of T, and the set of all images $T(x)$ is called the range of T.

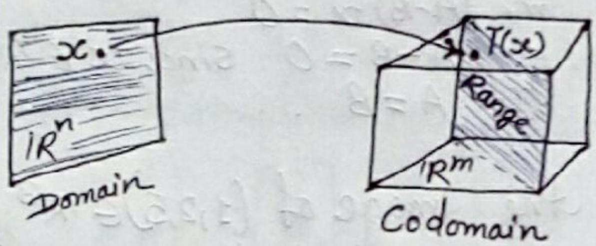

fig. Domain, codomain, and range of $T: R^n \to R^m$.

⊛ Linear transformation:

OR Def^n → Let $T: V \to W$ be a transformation (mapping or function) such that,

i) $T(cv) = c.T(v)$

ii) $T(u+v) = T(u) + T(v)$

$\forall c \in K$ and $u, v \in V$.

Example: Let $A = (a_{ij})_{m\times n}$ be an $m\times n$ matrix.

Let $T: \mathbb{R}^n \longrightarrow \mathbb{R}^m$

T: multiplication by the matrix A.

i.e, $T(X) = AX$ is a linear transformation.

⊛. Matrix transformations: (The matrix of linear transformation):

Contraction and Dilution transformation:

The transformation, $T: \mathbb{R}^2 \to \mathbb{R}^2$

$T(X) = rX$ is said to be contraction if $0 \le r \le 1$ (or $0 < r < 1$).

& The transformation is said to be dilation if $r > 1$.

If you are confused of what is R^2, R^3, R^n then have a look below:

$R^2 \to$ Matrix having 2 columns containg real numbers as it's elements

$R^3 \to$ " " 3 columns " " " " "

$R^n \to$ " " n " " " " "

Unique representation theorem: [Unit-5] → (unit 5 मा 2 मा Same)

Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation $x \to T(x) (= Ax)$, $\forall x \in \mathbb{R}^n$. Then there exists a unique matrix A of order $m \times n$, where $A = [T(e_1)\ T(e_2) \dots T(e_n)]$, e_i is the columns in identity matrix I_n.

Uniqueness → Let there exists another matrix $B_{m\times n}$ (say) (other than A) also,

$$T(x) = B.x \quad \forall\ x \in \mathbb{R}^n$$

Then, $Ax = B.x$.

or, $Ax - Bx = 0$

or, $(A-B)x = 0$

or, $A - B = 0$ since $x \in \mathbb{R}^n$ is non-zero also.

or, $A = B$.

Example: Find the image of $(1,2,5) \in \mathbb{R}^3$, under the transformation

$T: \mathbb{R}^3 \to \mathbb{R}^2$ such that $T(e_1) = \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix}$, $T(e_2) = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$

& $T(e_3) = \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}$

Soln

Here, $X = \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}$

$$A = [T(e_1)\ T(e_2)\ T(e_3)]$$

$$T(X) = AX$$

$$= \begin{bmatrix} 1 & 2 & -1 \\ 4 & -1 & -1 \\ 2 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}$$

$$= \begin{bmatrix} 1+4-5 \\ 4-2-5 \\ 2+2+10 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ -3 \\ 14 \end{bmatrix}$$

Co-ordinate vector →

Let, $v \in V$ be an arbitary element in V. Let, $B = \{b_1, b_2, \dots, b_n\}$ be a basis for V. Then there exists unique set of scalars $\{c_1, c_2, \dots, c_n\}$ such that,

$$v = c_1 b_1 + c_2 b_2 + \dots + c_n b_n.$$

The vector $\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$ is called the co-ordinate vector of v with respect to the basis denoted by $[v]_B$.

i.e, $$\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = [v]_B.$$

Example 1 → Find the standard matrix associated with the linear transformation $T: \mathbb{R}^3 \to \mathbb{R}^2$ such that,

$$T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 5 \end{bmatrix}, T\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -2 \\ 4 \end{bmatrix}.$$

Find the image of $\begin{bmatrix} 1 \\ 1 \\ 5 \end{bmatrix}$ under the transformation.

Solution:-

Here, $T: \mathbb{R}^3 \to \mathbb{R}^2$ such that $T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$,

$$T\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \;\&\; T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -2 \\ 4 \end{bmatrix}.$$

∴ Standard matrix, $A = [T(e_1) \; T(e_2) \cdot T(e_3)]$.

$$A = \begin{bmatrix} 2 & 1 & -2 \\ 5 & 3 & 4 \end{bmatrix}$$

Now, $T(X) = A(X)$

$$= \begin{bmatrix} 2 & 1 & -2 \\ 5 & 3 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 5 \end{bmatrix}$$

$$= 1\begin{bmatrix} 2 \\ 5 \end{bmatrix} + 1\begin{bmatrix} 1 \\ 3 \end{bmatrix} + 5\begin{bmatrix} -2 \\ 4 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 5 \end{bmatrix} + \begin{bmatrix} 1 \\ 3 \end{bmatrix} + \begin{bmatrix} -10 \\ 20 \end{bmatrix}$$

$$= \begin{bmatrix} -7 \\ 28 \end{bmatrix}$$

Note:- $T(X) = A(X)$, where $A = [T(e_1) . T(e_2) \dots T(e_n)]_{m \times n}$ called the standard matrix of transformation.

Q. Find the vector X in $\mathbb{R}^3$ where co-ordinate vector $[X]_B$ relative to the basis $B = \left\{ \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix} \right\}$ is $\begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}$ i.e. $[X]_B = \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}$.

Solution:-

Here, $B = \left\{ \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix} \right\}$

$$[X]_B = \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}$$

$X = ?$

we have, $X = 1 \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} + (-1) \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} + 4 \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}$

$$X = \begin{bmatrix} 2-1-4 \\ 1-1+4 \\ 1-3-4 \end{bmatrix}$$

$$X = \begin{bmatrix} -2 \\ 4 \\ -6 \end{bmatrix}$$

Q. Let. $B = \left\{ \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix} \right\}$ is a basis of $\mathbb{R}^3$. Find the co-ordinate vector of $\begin{bmatrix} 5 \\ 6 \\ 4 \end{bmatrix}$ with respect to the basis B.

Solution:-

Let $x_1 \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \\ 4 \end{bmatrix}$

Making the system of linear equations as

$$2x_1 + x_2 - x_3 = 5$$
$$x_1 + x_2 + x_3 = 6$$
$$x_1 + 3x_2 - x_3 = 4$$

and solving we get.

$$x_1 = 3$$
$$x_2 = 1$$
$$x_3 = 2$$

$\therefore$ Co-ordinate vector of $\begin{bmatrix} 5 \\ 6 \\ 4 \end{bmatrix}$ is $\begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$ with respect to the basis B.

⊛ Transformation related important questions and solutions.

Q1. Let $A=\begin{bmatrix}2&0\\0&2\end{bmatrix}$ be the given matrix and define $T:R^2\to R^2$ by $T(x)=Ax$. Find images under T of $u=\begin{bmatrix}1\\-3\end{bmatrix}$ and $v=\begin{bmatrix}a\\b\end{bmatrix}$.

Simple Transformation related

Solution:

Let $A=\begin{bmatrix}2&0\\0&2\end{bmatrix}$ and the transformation $T:R^2\to R^2$ defined by $T(x)=Ax$.

Also, let

$$u=\begin{bmatrix}1\\-3\end{bmatrix} \text{ and } v=\begin{bmatrix}a\\b\end{bmatrix}.$$

Then, $T(u)=Au=\begin{bmatrix}2&0\\0&2\end{bmatrix}\begin{bmatrix}1\\-3\end{bmatrix}=\begin{bmatrix}2\\-6\end{bmatrix}$.

and $T(v)=Av=\begin{bmatrix}2&0\\0&2\end{bmatrix}\begin{bmatrix}a\\b\end{bmatrix}=\begin{bmatrix}2a\\2b\end{bmatrix}$.

Thus the images of u and v under T are $\begin{bmatrix}2\\-6\end{bmatrix}$ and $\begin{bmatrix}2a\\2b\end{bmatrix}$.

Matrix transformation related

Q2. Let $A=\begin{bmatrix}1&-3\\3&5\\-1&7\end{bmatrix}$, $u=\begin{bmatrix}2\\-1\end{bmatrix}$, $b=\begin{bmatrix}3\\2\\-5\end{bmatrix}$, $c=\begin{bmatrix}3\\2\\5\end{bmatrix}$ and define a transformation $T:R^2\to R^3$ by $T(x)=Ax$ so that,

ⓐ. Find $T(u)$

ⓑ. Find x in R^2 whose image under T is b.

ⓒ. Is there more than one x whose image under T is b?

ⓓ. Determine if c is in the range of T.

Solution:

$$A=\begin{bmatrix}1&-3\\3&5\\-1&7\end{bmatrix},\ u=\begin{bmatrix}2\\-1\end{bmatrix},\ b=\begin{bmatrix}3\\2\\-5\end{bmatrix} \text{ and } c=\begin{bmatrix}3\\2\\5\end{bmatrix}.$$

Given that transformation $T:R^2\to R^3$ by $T(x)=Ax$.

Now,

ⓐ. $T(u)=Au=\begin{bmatrix}1&-3\\3&5\\-1&7\end{bmatrix}\begin{bmatrix}2\\-1\end{bmatrix}=\begin{bmatrix}2+3\\6-5\\-2-7\end{bmatrix}=\begin{bmatrix}5\\1\\-9\end{bmatrix}$.

(b). Let $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

Suppose x in $\mathbb{R}^2$ whose image under T is b. Then,

$T(x) = b$.

$\Rightarrow Ax = b$.

$\Rightarrow \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ -5 \end{bmatrix}$

The augmented matrix of $Ax = b$ is,

$$\begin{bmatrix} 1 & -3 & 3 \\ 3 & 5 & 2 \\ -1 & 7 & -5 \end{bmatrix}$$

$R_2 \to R_2 - 3R_1$
$R_3 \to R_3 + R_1$

$$\sim \begin{bmatrix} 1 & -3 & 3 \\ 0 & 14 & -7 \\ 0 & 4 & -2 \end{bmatrix}$$

$R_2 \to R_2/14$

$$\sim \begin{bmatrix} 1 & -3 & 3 \\ 0 & 1 & -0.5 \\ 0 & 4 & -2 \end{bmatrix}$$

$R_3 \to R_3 - 4R_2$

$$\sim \begin{bmatrix} 1 & -3 & 3 \\ 0 & 1 & -0.5 \\ 0 & 0 & 0 \end{bmatrix}$$

$R_1 \to R_1 - 3R_2$

$$\sim \begin{bmatrix} 1 & 0 & 1.5 \\ 0 & 1 & -0.5 \\ 0 & 0 & 0 \end{bmatrix}$$

This implies $x_1 = 1.5$ and $x_2 = -0.5$.

Thus, $x = \begin{bmatrix} 1.5 \\ -0.5 \end{bmatrix}$ in $\mathbb{R}^2$ whose image under T is b.

(c). In the above solution (b), x has no free variable, so the solution x is unique. This means there is exactly one x in $\mathbb{R}$ whose image under T is b.

(d). From (c), there is exactly one range b of T. So, c is not a range of T.

Note: For (d) we can proceed as in (b) with replacing value of b by c. Then we will get an inconsistent augmented matrix of $Ax = c$. This implies c is not a range of T.

Shear transformation → A transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(x) = Ax$ is called a shear transformation.

Q3. Prove that contradiction map is linear transformation.

Solⁿ

We know that map $T: R^2 \to R^2$ defined by $T(x) = rx$, where $0 \le r \le 1$ is called contradiction map.

Let $u, v \in R^2$ and c and d are scalar. Then

$$T(cu+dv) = r(cu+dv)$$
$$= rcu + rdv$$
$$= c(ru) + d(rv)$$
$$= cT(u) + dT(v)$$

$\therefore$ T is linear.

Q4. Show that the transformation T defined by $T(x_1, x_2) = (2x_1 - 3x_2, x_1+4, 5x_2)$ is not linear.

Solution: Let T is a transformation, defined by,

$$T(x_1, x_2) = (2x_1 - 3x_2, x_1+4, 5x_2).$$

Now,

$$T(u+v) = T(u_1+v_1, u_2+v_2)$$
$$= (2(u_1+v_1) - 3(u_2+v_2), (u_1+v_1)+4, 5(u_2+v_2))$$
$$= (2u_1+2v_1-3u_2-3v_2, u_1+v_1+4, 5u_2+5v_2)$$

and

$$T(u)+T(v) = T(u_1, u_2) + T(v_1, v_2)$$
$$= (2u_1 - 3u_2, u_1+4, 5u_2) + (2v_1 - 3v_2, v_1+4, 5v_2)$$
$$= (2u_1+2v_1-3u_2-3v_2, u_1+v_1+8, 5u_2+5v_2)$$
$$\ne T(u,v).$$

This implies that T is not a linear transformation or, for this transformation.

$$T(x_1, x_2) = (2x_1 - 3x_2, x_1+4, 5x_2).$$

⊛ Standard Matrix for linear Transformation T:

Defⁿ → Let $T: R^n \to R^m$ be a linear transformation defined by $T(x) = Ax$ for all x in R^n, where A is $m \times n$.

Clearly A is unique. Then $A = [T(e_1) \cdot T(e_2) \ldots T(e_n)]$ where e_j is the jth column of the identity matrix in $\mathbb{R}^n$.

Then the matrix A is called standard matrix for T.

Q5. Find the standard matrix A for linear transformation $T(x) = 2x$ for x in R^3.

Solution:

Let $T(x) = 2x$. In R^3.

$$T(e_1) = 2e_1 = 2\begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{bmatrix}2\\0\\0\end{bmatrix}$$

$$T(e_2) = 2e_2 = 2\begin{bmatrix}0\\1\\0\end{bmatrix} = \begin{bmatrix}0\\2\\0\end{bmatrix}$$

and $$T(e_3) = 2e_3 = 2\begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}0\\0\\2\end{bmatrix}$$

Now, the standard matrix A for $T(x) = 2x$ is,

$$A = [T(e_1)\ T(e_2)\ T(e_3)]$$

$$= \begin{bmatrix}2 & 0 & 0\\0 & 2 & 0\\0 & 0 & 2\end{bmatrix}$$

or, Given $T(x) = 2x$.

i.e, $T(x_1, x_2, x_3) = (2x_1, 2x_2, 2x_3)$.

or, $$T\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = \begin{pmatrix}2x_1\\2x_2\\2x_3\end{pmatrix}$$

$$= \begin{pmatrix}2x_1 + 0x_2 + 0x_3\\0x_1 + 2x_2 + 0x_3\\0x_1 + 0x_2 + x_3\end{pmatrix}$$

$$= \begin{pmatrix}2 & 0 & 0\\0 & 2 & 0\\0 & 0 & 2\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}$$

$\therefore T(x) = Ax$, when $A = \begin{pmatrix}2 & 0 & 0\\0 & 2 & 0\\0 & 0 & 2\end{pmatrix}$ is required matrix.

Onto:

A transformation $T: R^n \to R^m$ is said to be onto R^m if each b in R^m is the image of at least one x in R^n.

One-to-one:

A transformation $T: R^n \to R^m$ is said to be one-to-one if each b in R^m is the image of at most one x in R^n.

Theorem1: Let $T: R^n \to R^m$ be a linear transformation. Then T is one-to-one if and only if the equation $T(x)=0$ has only the trivial solution.

Proof: Let $T: R^n \to R^m$ is linear transformation.
Suppose that T is one-to-one. Then for any x in R^n.
$T(x) = 0 = T(0)$.
$\Rightarrow x = 0$ [$\because$ being T is one-to-one].
This means the equation $T(x)=0$ has only the trivial solution.

Theorem2: Let: $T: \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation and let A be the standard matrix for T. Then,

(a). T maps $\mathbb{R}^n$ onto $\mathbb{R}^m$ if and only if the columns of A span $\mathbb{R}^m$.

(b). T is one-to-one if and only if the columns of A are linearly independent.

Proof:

Let $T: R^n \to R^m$ be a linear transformation and let A be the standard matrix for T.

(a). Let T is onto $\Leftrightarrow$ for each $b \in \mathbb{R}^m$ $\exists x \in \mathbb{R}^n$ such that $T(x)=b$.
$\Leftrightarrow$ for each $b \in \mathbb{R}^m$ $Ax=b$ has solution, where A is $m \times n$ matrix.
$\Leftrightarrow$ column of A span $\mathbb{R}^m$.

(b). Let T is one to one $\Leftrightarrow$ equation $T(x)=0$ has only the trivial solution.
$\Leftrightarrow$ equation $Ax=0$ has only trivial solution.
$\Leftrightarrow$ column of A are linearly independent.