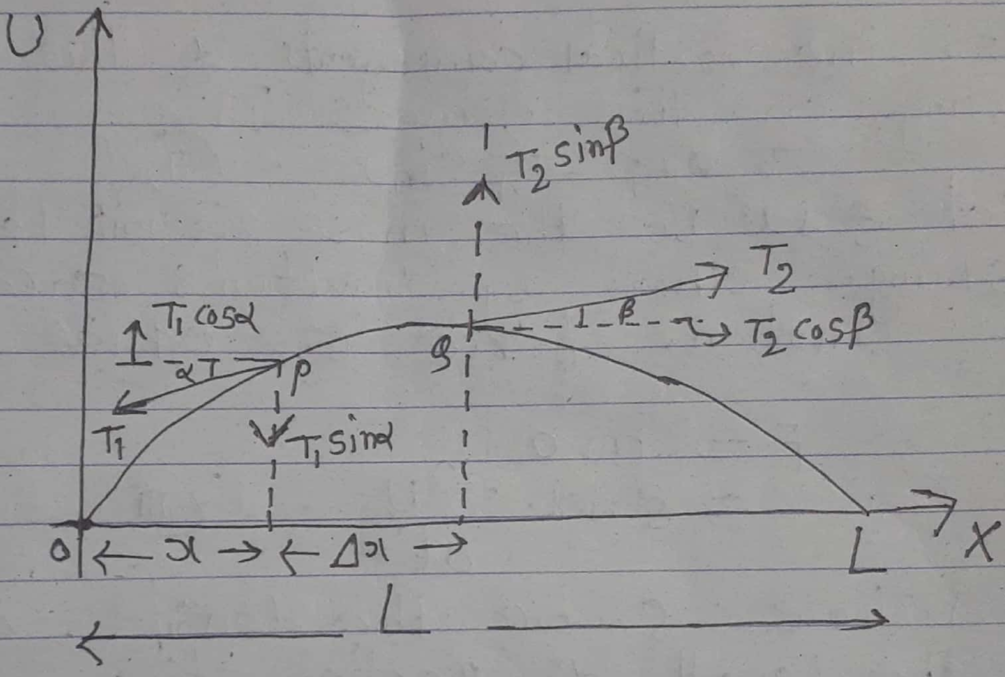
Wave equation (vibration of string)

Derivation of one dimensional wave equation



Consider a tightly stretched elastic string of length $OL = L$ and fix it at the ends O and L. When the string vibrates, each point of the string makes small vibration on U-axis which is shown in figure.

Let $P(x, u)$ and $Q(x+\Delta x, u+\Delta u)$ be two neighbouring points on the string. Let T_1 and T_2 be the tension at the points P and Q where the tensions makes angles α and β with X-axis. Since the point of the string moves vertically, there is no motion in the horizontal direction. Thus the horizontal components of the tension must be equal i.e constant

$$\therefore \quad T_1 \cos\alpha = T_2 \cos\beta = T \text{ (say)} \cdots (1)$$

Also the vertical components of the force acting on this elements is

$$T_2 \sin\beta - T_1 \sin\alpha \cdots (II)$$

Let $\rho \Delta x$ be the mass per unit length of string. Then by Newton's second law of motion, force is equal to

$$F = ma$$
$$= \rho \Delta x . \frac{\partial^2 u}{\partial t^2} \cdots (III)$$

Where ρ be the density, Δx is the length of portion string and $\frac{\partial^2 u}{\partial t^2}$ is acceleration

From eqn (II) and (III)

$$T_2 \sin\beta - T_1 \sin\alpha = \rho \Delta x . \frac{\partial^2 u}{\partial t^2} \cdots (IV)$$

Now dividing eqn (IV) by (II), we get

$$\frac{T_2 \sin\beta}{T_2 \cos\beta} - \frac{T_1 \sin\alpha}{T_1 \cos\alpha} = \frac{\rho \Delta x . \frac{\partial^2 u}{\partial t^2}}{T}$$

$$\tan\beta - \tan\alpha = \frac{\rho \Delta x}{T} . \frac{\partial^2 u}{\partial t^2} \cdots (V)$$

We know that $\tan\alpha$ and $\tan\beta$ are the slopes of the string at x and $x+\Delta x$ respectively

Where $\tan\alpha = \frac{\partial U}{\partial x}\Big|_{at\ x}$

$\tan\beta = \frac{\partial U}{\partial x}\Big|_{at\ x+\Delta x}$

Eqn ⑤ becomes, we get

$$\left[\frac{\partial U}{\partial x}\Big|_{at\ x+\Delta x} - \frac{\partial U}{\partial x}\Big|_{at\ x}\right] = \frac{\rho\,\Delta x}{T}\cdot\frac{\partial^2 U}{\partial t^2}$$

or, $$\frac{1}{\Delta x}\left[\frac{\partial U}{\partial x}\Big|_{at\ x+\Delta x} - \frac{\partial U}{\partial x}\Big|_{at\ x}\right] = \frac{\rho}{T}\cdot\frac{\partial^2 U}{\partial t^2}$$

Taking limit $\Delta x \to 0$ on both sides, we get

or $$\lim_{\Delta x\to 0}\frac{1}{\Delta x}\left[\frac{\partial U}{\partial x}\Big|_{x+\Delta x} - \frac{\partial U}{\partial x}\Big|_{x}\right] = \lim_{\Delta x\to 0}\frac{\rho}{T}\frac{\partial^2 U}{\partial t^2}$$

or $$\lim_{\Delta x\to 0}\left[\frac{\frac{\partial U}{\partial x}\Big|_{x+\Delta x} - \frac{\partial U}{\partial x}\Big|_{x}}{\Delta x}\right] = \frac{\rho}{T}\cdot\frac{\partial^2 U}{\partial t^2}$$

or $$\frac{\partial^2 u}{\partial x^2} = \frac{\rho}{T} \cdot \frac{\partial^2 u}{\partial t^2}$$

or, $$\frac{\rho}{T} \cdot \frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$$

or $$\frac{\partial^2 u}{\partial t^2} = \frac{T}{\rho} \frac{\partial^2 u}{\partial x^2}$$

or, $$\boxed{\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}}$$

Where $c^2 = \frac{T}{\rho}$

This is the required one dimensional wave equation.

Solution of one dimensional wave equation under certain initial and Boundary conditions

Solution:- We have one dimensional wave eqn is

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \quad \cdots (1)$$

Where $u(x, t)$ is the deflection of the string with boundary condition

$u(0, t) = 0$ and $u(L, t) = 0$ for all t .... (2)

then its initial condition is

$u(x, 0) = f(x)$ ---(3) initial deflection

$\left.\frac{\partial u}{\partial t}\right|_{t=0} = g(x)$ ---(4) initial velocity

Suppose $u(x, t) = F(x) \cdot G(t)$ ---(5)

be the Solution of eqn (1)

Differentiating eqn (5) w.r. to x and t

$$\frac{\partial^2 u}{\partial t^2} = F \cdot \ddot{G} \quad \text{and} \quad \frac{\partial^2 u}{\partial x^2} = F'' G$$

Where

dots denotes derivative w.r. to t

prime denotes derivative w.r. to x

putting these values in eqn (1) we get

$$F\ddot{G} = c^2 F'' G$$

$$\Rightarrow \frac{F''}{F} = \frac{\ddot{G}}{G \cdot c^2}$$

$$\Rightarrow \frac{F''}{F} = \frac{\ddot{G}}{G \cdot c^2} = k \text{ (say)}$$

which gives

$$F'' - Fk = 0 \quad \cdots (6)$$

$$\ddot{G} - c^2 k G = 0 \quad \cdots (7)$$

These are second order ordinary differential equation. We have to determine F and G from eqn (6) and (7) under boundary conditions.

Also we have

$$u(x, t) = F(x) \cdot G(t)$$

i.e $u(0, t) = F(0) \cdot G(t) = 0$

$$u(L, t) = F(L) \cdot G(t) = 0$$

for all t

For solving eqn (6) we have to generate boundary condition (b.c.)

If $G = 0$, then $u = 0$ which is no meaning or interest. So $G \neq 0$

Then we get

$$F(0) = 0 = F(L) \quad \cdots (8)$$

case (i) Let $K = 0$ then eqn (6) becomes

$$F'' = 0$$

integrating

$$F' = a$$

again integrating

$$F = ax + b \quad \ldots (9)$$

Where a and b are constants. We have to find a and b under boundary condition

We have

$$F(0) = 0 \text{ and } F(L) = 0$$

$\therefore$ $F(x) = ax + b$ Again $F(L) = aL + b$

or $F(0) = a \cdot 0 + b$ $0 = aL + 0$

$0 = b$ $\therefore a = 0$

$\therefore$ $b = 0$

We get $a = b = 0$

putting these values of a and b in eqn (9)

$$F = 0 \cdot x + 0 \Rightarrow F = 0$$

Which is no interest.

case (ii) Suppose that $K > 0$
ie K is positive

Let $K = \lambda^2$

Then eqn (6) becomes

$$F'' - \lambda^2 F = 0$$

its auxillary eqn is

$$m^2 - \lambda^2 = 0$$

$$m = \pm \lambda$$

its general solution is

$$F(x) = A.e^{\lambda x} + B.e^{-\lambda x} \quad \cdots (10)$$

Where A and B are constants to be determine under boundary condition.

i.e $F(0) = A.e^{0} + B.e^{0}$

$$0 = A + B$$

$$A + B = 0 \quad \cdots (i)$$

Again $F(L) = A.e^{\lambda L} + B.e^{-\lambda L}$

$$0 = A.e^{\lambda L} + B.e^{-\lambda L}$$

$$A.e^{\lambda L} + B.e^{-\lambda L} = 0 \quad \cdots (ii)$$

Solving (i) and (ii) we get

$$A = B = 0.$$

$\therefore$ Eqn (10) becomes

$$F = 0 \Rightarrow U = 0$$

This is also no interest.

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case (iii) Let $K < 0$

i.e K is negative

Suppose $K = -p^2$

Eqn (6) becomes

$$F'' + p^2 F = 0$$

A. E. is

$$m^2 + p^2 = 0$$

$$m = \pm pi$$

its general solution is

$$F(x) = e^{px}(A\cos qx + B\sin qx)$$

where

p = real part

q = imaginary part

$$F(x) = e^{0 \cdot x}(A\cos px + B\sin px)$$

$$F(x) = A\cos px + B\sin px \quad \text{--- (11)}$$

using boundary condition

$$F(0) = A\cos 0 + B\sin 0$$

$$0 = A \cdot 1 + 0$$

$$\therefore A = 0$$

Also

$$F(L) = A\cos pL + B\sin pL$$

$$0 = 0 \cdot \cos pL + B\sin pL$$

$$\therefore B\sin pL = 0$$

Let $B \neq 0$. So

$$\sin pL = 0$$

$$\sin pL = \sin n\pi$$

$$\therefore pL = n\pi$$

$$p = \frac{n\pi}{L}$$ where n is an integer

We have $B \neq 0$, setting $B = 1$

Eqn (11) becomes

$$F(x) = 0 \cdot \cos px + 1 \cdot \sin \frac{n\pi}{L} \cdot x$$

$$F(x) = \sin \frac{n\pi}{L} \cdot x$$

We obtain infinitely many solⁿ

$$F(x) = F_n(x)$$

i.e $F_n(x) = \sin \frac{n\pi}{L} \cdot x \quad \cdots (12)$

for $n = 1, 2, 3, \ldots$

Similarly for Solution of eqn (7) i.e

$$\ddot{G} - c^2 K G = 0$$

We have $K = -p^2 = -\left(\frac{n\pi}{L}\right)^2$

Then Eqn (7) becomes

$$\ddot{G} + c^2\left(\frac{n\pi}{L}\right)^2 G = 0$$

or $$\ddot{G} + \left(\frac{cn\pi}{L}\right)^2 G = 0$$

$$\ddot{G} + \lambda_n^2 G = 0 \quad \text{where } \lambda_n = \frac{cn\pi}{L}$$

its auxillary eqn is

$$m^2 + \lambda_n^2 = 0$$

$$\therefore m = \pm \lambda_n i \quad \text{here } p = 0,\ q = \lambda_n$$

its general solution

$$G(t) = e^{pt}\left(B_n \cos\lambda_n t + B_n^* \sin\lambda_n t\right)$$

$$G(t) = e^{0}\left(B_n \cos\lambda_n t + B_n^* \sin\lambda_n t\right)$$

$$G(t) = B_n \cos\lambda_n t + B_n^* \sin\lambda_n t$$

i.e $$G_n(t) = B_n \cos\lambda_n t + B_n^* \sin\lambda_n t$$

Therefore required solution of eqn (1) is

$$U_n(x,t) = F_n(x) \cdot G_n(t)$$

$$= \sin\frac{n\pi}{L}x\left(B_n \cos\lambda_n t + B_n^* \sin\lambda_n t\right)$$

By fundamental theorem the solution of given wave eqn is the infinitely many u_n. Thus

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t)$$

$$\Rightarrow u(x,t) = \sum_{n=1}^{\infty} \left(B_n \cos\lambda_n t + B_n^* \sin\lambda_n t\right) \sin\frac{n\pi}{L}\cdot x \quad \text{---(13)}$$

From eqn (3) [initial condition]

$$u(x,0) = f(x)$$

Then eqn (13) becomes

$$u(x,0) = \sum_{n=1}^{\infty} \left(B_n \cos 0 + B_n^* \sin 0\right) \cdot \sin\frac{n\pi}{L} x$$

i.e $$f(x) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi}{L} x$$

Which is fourier sine series and its co-efficient B_n is given by

$$B_n = \frac{2}{L}\int_0^L f(x) \sin\frac{n\pi}{L} x \, dx$$

Also, we have

$$\left.\frac{\partial u}{\partial t}\right|_{t=0} = g(x)$$

Eqn (13) becomes

(13)

$$\Rightarrow \left[\sum_{n=1}^{\infty} \left(-B_n \lambda_n \sin \lambda_n t + B_n^* \lambda_n \cos \lambda_n t \right) \sin \frac{n\pi}{L} x \right]_{t=0} = g(x)$$

put $t = 0$

$$\Rightarrow \sum_{n=1}^{\infty} B_n^* \lambda_n \sin \frac{n\pi}{L} x = g(x)$$

$$\text{or} \quad g(x) = \sum_{n=1}^{\infty} B_n^* \lambda_n \sin \frac{n\pi}{L} x$$

Which is also fourier sine series of $g(x)$. its co-efficient is given by

$$B_n^* \lambda_n = \frac{2}{L} \int_0^L g(x) \sin \frac{n\pi}{L} x \, dx$$

$$B_n^* = \frac{2}{\lambda_n L} \int_0^L g(x) \sin \frac{n\pi}{L} x \, dx$$

$$= \frac{2}{\frac{cn\pi}{L} \times L} \int_0^L g(x) \sin \frac{n\pi}{L} x \, dx$$

where $\lambda_n = \frac{cn\pi}{L}$

$$= \frac{2}{cn\pi} \int_0^L g(x) \sin \frac{n\pi}{L} x \, dx$$

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Therefore we get required Solution of one dimensional wave eqn is

$$u(x,t) = \sum_{n=1}^{\infty} \left(B_n \cos \lambda_n t + B_n^* \sin \lambda_n t\right) \sin \frac{n\pi}{L} x$$

Where

$$B_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi}{L} x \, dx$$

$$B_n^* = \frac{2}{cn\pi} \int_0^L g(x) \sin \frac{n\pi}{L} x \, dx$$

Where $f(x)$ and $g(x)$ are initial deflection and initial velocity of the string respectively.