One Dimensional Wave Equations

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Find the deflection

$$u(x, t)$$

of a vibrating string of length π and $c^2 = 4$ for zero initial velocity and initial deflection is $\sin 5x$

Sol^n : Given problem is a one dimensional wave equation with its PDE:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \ldots \quad \ldots(1)$$

with the BC's:

$$u(0, t) = 0$$

and

$$u(L, t) = 0$$

The initial conditions are

$$u(x, 0) = f(x) = \sin 5x \quad (initial \quad displacement)$$

and

$$u_t(x, 0) = g(x) = 0 \quad (zero \quad initial \quad velocity)$$

Its general solution is

$$u(x,t) = (C_1 \cos \lambda x + C_2 \sin \lambda x)(C_3 \cos \lambda ct + C_4 \sin \lambda ct) \dots \quad \dots(2)$$

Using the BC:

$$u(0, t) = 0$$

we get,

$$C_1 = 0$$

And using the BC:

$$u(L, t) = 0$$

we get,

$$\lambda = \frac{n\pi}{L} = n, \quad since \quad L = \pi$$

So, from (2), we get

$$u(x, t) = C_2 \sin \frac{n\pi}{L} x (C_3 \cos \frac{n\pi}{L} ct + C_4 \sin \frac{n\pi}{L} ct)$$

i.e.

$$u_n(x, t) = \sin nx (a_n \cos nct + b_n \sin nct)$$

where

$$a_n = C_2 C_3$$

and

$$b_n = C_2 C_4$$

Using principle of superposition, (i.e. adding all possible solutions) , we get

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t)$$

i.e.

$$u(x,t) = \sum_{n=1}^{\infty} \sin nx \left[a_n \cos nct + b_n \sin nct\right] .....(3)$$

Differentiating with respect to time, we get

$$u_t(x, t) = \sum_{n=1}^{\infty} \sin nx \left[a_n.(-)nc. \sin nct + b_n.nc \cos nct\right] .....(4)$$

using the initial condition

$$u_t(x,0) = g(x) = 0$$

we get

$$b_n = 0$$

Using the initial conditions

$$u(x,0) = f(x) = \sin 5x$$

we get

$$\sin 5x = \sum_{n=1}^{\infty} a_n \sin nx$$

or

$$\sin 5x = a_1 \sin 1.x + a_2 \sin 2x + ... + a_5 \sin 5x + ...$$

Comparing, we get

$$a_5 = 1$$

and rest are zeros. Hence, from (3), required solution is

$$u(x,t) = a_5 \sin 5x \cos 5ct$$

i.e.

$$u(x,t) = a_5 \sin 5x \cos 10t$$

since $c^2 = 4$

If the string be fixed at both ends, find the displacement with the following initial conditions.
The initial displacement

$$u(x,0) = u_0 \sin \frac{\pi}{L} x$$

and the initial velocity is zero.

Sol^n : Given problem is a one dimensional wave equation with its PDE:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \dots \quad \dots(1)$$

with the BC's:

$$u(0, t) = 0$$

and

$$u(L, t) = 0$$

The initial conditions are

$$u(x, 0) = f(x) = u_0 \sin \frac{\pi}{L} x \quad (initial \quad displacement)$$

and

$$u_t(x, 0) = g(x) = 0 \quad (zero \quad initial \quad velocity)$$

Its general solution is

$$u(x,t) = \sum_{n=1}^{\infty} \sin \frac{n\pi}{L} x \left[a_n \cos \frac{n\pi}{L} ct + b_n \sin \frac{n\pi}{L} ct \right] .....(2)$$

using the initial condition

$$u_t(x,0) = g(x) = 0$$

we get

$$b_n = 0$$

Again, using the IC:

$$u(x,0) = f(x) = u_0 \sin \frac{\pi}{L} x$$

we get

$$u_0 \sin \frac{\pi}{L} x = \sum_{n=1}^{\infty} a_n . \sin \frac{n\pi}{L} x$$

or,

$$u_0 \sin \frac{\pi}{L} x = a_1 . \sin \frac{1\pi}{L} x + a_2 . \sin \frac{2\pi}{L} x + a_3 . \sin \frac{3\pi}{L} x + ...$$

Comparing the similar coefficients, we get

$$a_1 = u_0$$

and the others are zeros.
Hence, from (2), we get

$$u(x,t) = a_1 \sin \frac{\pi}{L} x \cos \frac{\pi}{L} ct$$

i.e.

$$u(x,t) = u_0 \sin \frac{\pi}{L} x \cos \frac{\pi}{L} ct$$

which is required solution.

A tightly stretched string with fixed end points $x = 0$ and $x = L$ is initially in a position given by

$$u(x, 0) = u_0 \sin^3\left(\frac{\pi x}{L}\right)$$

If it is released from rest from this position find the displacement.

Sol^n : Given problem is a one dimensional wave equation with its PDE:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \ldots \quad \ldots(1)$$

with the BC's:

$$u(0, t) = 0$$

and

$$u(L, t) = 0$$

The initial conditions are

$$u(x, 0) = f(x) = u_0 \sin^3\left(\frac{\pi x}{L}\right) \quad (initial \quad displacement)$$

and

$$u_t(x, 0) = g(x) = 0 \quad (zero \quad initial \quad velocity)$$

Its general solution is

$$u(x,t) = \sum_{n=1}^{\infty} \sin \frac{n\pi}{L} x \left[a_n \cos \frac{n\pi}{L} ct + b_n \sin \frac{n\pi}{L} ct \right] \text{.....(2)}$$

using the initial condition

$$u_t(x, 0) = g(x) = 0$$

we get

$$b_n = 0$$

Again, using the IC:

$$u(x,0) = f(x) = u_0 \sin^3\left(\frac{\pi x}{L}\right)$$

we get

$$u_0 \sin^3\left(\frac{\pi x}{L}\right) = \sum_{n=1}^{\infty} a_n . \sin\frac{n\pi}{L}x$$

or,

$$u_0.\frac{1}{4}\left(3\sin\frac{\pi x}{L}-\sin\frac{3\pi x}{L}\right)=a_1.\sin\frac{1\pi}{L}x+a_2.\sin\frac{2\pi}{L}x+a_3.\sin\frac{3\pi}{L}x+....$$

$$[\because \sin 3A = 3\sin A - 4\sin^3 A]$$

Comparing similar coefficients, we get $a_1=\frac{3u_0}{4}$, $a_2=0$, $a_3=-\frac{u_0}{4}$ and rest are zeros.

Hence, from (2), we get

$$u(x,t) = a_1 \sin\frac{\pi x}{L}\cos\frac{\pi ct}{L} + 0 + a_3 \sin\frac{3\pi x}{L}\cos\frac{3\pi ct}{L} + 0 + 0 + ...$$

i.e.

$$u(x,t) = \frac{3u_0}{4}\sin\frac{\pi x}{L}\cos\frac{\pi ct}{L} - \frac{u_0}{4}\sin\frac{3\pi x}{L}\cos\frac{3\pi ct}{L}$$

which is required solution.

The vibration of an elastic string is governed by the PDE

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

and

$$c^2 = 1$$

. The length of the string is π and the ends are fixed. The initial velocity is zero and the initial deflection is $f(x) = 2(\sin x + \sin 3x)$. Find the deflection of the vibrating string.

Find the deflection $u(x, t)$ of a vibrating string of length $L = \pi$ and $c^2 = 1$,if its initial velocity is zero and initial deflection is

$$0.0x \quad for \quad 0 < x < \frac{\pi}{2}$$

and

$$0.01(\pi - x) \quad for \quad \frac{\pi}{2} < x < \pi$$

A tightly stretched string with fixed ends at $x = 0$ and $x = L$ is initially at rest in its equilibrium position. If it is set vibrating by giving to each of its points a displacement $3(Lx - x^2)$, find $u(x, t)$.

Sol^n : Given problem is a one dimensional wave equation with its PDE:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \dots \quad \dots(1)$$

with the BC's:

$$u(0, t) = 0$$

and

$$u(L, t) = 0$$

The initial conditions are

$$u(x, 0) = f(x) = 3(Lx - x^2)$$

and

$$u_t(x, 0) = g(x) = 0$$

Its general solution is

$$u(x,t) = \sum_{n=1}^{\infty} \sin \frac{n\pi}{L} x \left[a_n \cos \frac{n\pi}{L} ct + b_n \sin \frac{n\pi}{L} ct \right] .....(2)$$

using the initial condition

$$u_t(x,0) = g(x) = 0$$

we get

$$b_n = 0$$

Again, using the IC:

$$u(x,0) = f(x) = 3(Lx - x^2)$$

we get

$$3(Lx - x^2) = \sum_{n=1}^{\infty} a_n . \sin \frac{n\pi}{L} x$$

which is a Fourier sine series, where

$$a_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi}{L}x \quad dx$$

$$= \frac{2}{L} \int_0^L 3(Lx - x^2) \sin \frac{n\pi}{L} x \quad dx$$

$$= \frac{6}{L}\left[(Lx - x^2).\left\{\frac{(-)\cos\frac{n\pi x}{L}}{\frac{n\pi}{L}}\right\} + (L-2x).\left\{\frac{\sin\frac{n\pi x}{L}}{\left(\frac{n\pi}{L}\right)^2}\right\} - 2\left\{\frac{\cos\frac{n\pi x}{L}}{\left(\frac{n\pi}{L}\right)^3}\right\}\right]_0^L$$

$$= \frac{6}{L}\left[\left\{0+0-2\frac{\cos n\pi}{\left(\frac{n\pi}{L}\right)^3}\right\}-\left\{0+0-\frac{2}{\left(\frac{n\pi}{L}\right)^3}\right\}\right]$$

$$= \frac{12L^2}{n^3\pi^3}[1 - \cos n\pi]$$

$$= \frac{12L^2}{n^3\pi^3}[1-(-1)^n]$$

$$[\because \cos n\pi = (-1)^n]$$

$$= \frac{24L^2}{n^3\pi^3} \quad if \quad n = odd \quad zero \quad if \quad n = even$$

Hence, from (2), we get

$$u(x,t) = \frac{24L^2}{\pi^3} \sum_{n=1}^{\infty} \frac{1}{n^3} \sin \frac{n\pi}{L} x \cos \frac{n\pi}{L} ct$$

where n= odd integer.

A string is stretched and then fastened to two points L apart. Motion is started by displacing the string from

$$u = k(Lx - x^2)$$

from which it is released at time $t = 0$. Find the displacement.