Parseval's Identity for Fourier Transforms

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If $F(\omega)$ and $G(\omega)$ be the Fourier transforms of the functions $f(x)$ and $g(x)$ respectavily, then

1.
$$\int_{-\infty}^{\infty} F(\omega).\bar{G}(\omega)d\omega = \int_{-\infty}^{\infty} f(x).\bar{g}(x)dx$$

and

2.
$$\int_{-\infty}^{\infty} |F(\omega)|^2 d\omega = \int_{-\infty}^{\infty} |f(x)|^2 dx$$

where the bar indicates the complex congugate.

Parseval's Identity for sine transform: If $F_s(\omega)$ and $G_s(\omega)$ be the Fourier sine transforms of the functions $f(x)$ and $g(x)$ respectavily, then

1. $$\int_0^\infty F_s(\omega).G_s(\omega)d\omega = \int_0^\infty f(x).g(x)dx$$

   and

2. $$\int_0^\infty |F_s(\omega)|^2 d\omega = \int_0^\infty |f(x)|^2 dx$$

   .

Parseval's Identity for cosine transform: If $F_c(\omega)$ and $G_c(\omega)$ be the Fourier cosine transforms of the functions $f(x)$ and $g(x)$ respectavily, then

1. $$\int_0^\infty F_c(\omega).G_c(\omega)d\omega = \int_0^\infty f(x).g(x)dx$$

   and

2. $$\int_0^\infty |F_c(\omega)|^2 d\omega = \int_0^\infty |f(x)|^2 dx$$

   .

Find fourier sine transform of $f(x) = e^{-x}$ where , $x > 0$ and hence show that

$$\int_0^\infty \frac{x^2}{(1+x^2)^2} dx = \frac{\pi}{4}$$

Solution: The fourier sine transform of given fucntion is

$$\mathcal{F}_s(f) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(t) \sin \omega t dt$$

$$= \sqrt{\frac{2}{\pi}} \int\limits_0^\infty e^{-t} \sin \omega t dt$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{e^{-t}}{1^2 + \omega^2} \left(-1 \sin \omega t - \omega \cos \omega t \right) \right]_0^\infty$$

$$= \sqrt{\frac{2}{\pi}} \left[0 - \frac{e^0}{1+\omega^2} \left(0 - \omega.1 \right) \right]$$

$$\mathcal{F}_s(f) = \sqrt{\frac{2}{\pi}} \left(\frac{\omega}{1+\omega^2} \right)$$

which is required fourier sine transform of given function.

Next, using Parseval's identity for sine transform, we get

$$\int_0^\infty |f(x)|^2 dx = \int_0^\infty |F_s(\omega)|^2 d\omega$$

or,

$$\int_{0}^{\infty} \left(e^{-x}\right)^2 dx = \int_{0}^{\infty} \left\{ \sqrt{\frac{2}{\pi}} \left(\frac{\omega}{1+\omega^2} \right) \right\}^2 d\omega$$

or,

$$\int_{0}^{\infty} e^{-2x}\,dx = \frac{2}{\pi}\int_{0}^{\infty} \frac{\omega^2}{(1+\omega^2)^2}\,d\omega$$

or,

$$\frac{\pi}{2}\left[\frac{e^{-2x}}{-2}\right]_0^{\infty} = \int_0^{\infty} \frac{\omega^2}{(1+\omega^2)^2}\,d\omega$$

or,

$$\int_{0}^{\infty} \frac{\omega^2}{(1+\omega^2)^2} d\omega = \frac{\pi}{2}\left(0 - \frac{e^0}{(-2)}\right)$$

Replacing ω by x we get

$$\int_0^\infty \frac{x^2}{(1+x^2)^2}\,dx = \frac{\pi}{4}$$

which is required result.

Find fourier cosine transform of $f(x) = e^{-x}$ where , $x > 0$ and hence show that

$$\int_0^\infty \frac{1}{(1+x^2)^2}dx = \frac{\pi}{4}$$

Solution: The fourier cosine transform of given fucntion is

$$\mathcal{F}_c(f) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(t) \cos \omega t dt$$

$$= \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} e^{-t} \cos \omega t dt$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{e^{-t}}{1^2 + \omega^2} \left(-1 \cos \omega t + \omega \sin \omega t \right) \right]_0^\infty$$

$$= \sqrt{\frac{2}{\pi}} \left[0 - \frac{e^0}{1+\omega^2} \left(-1.1 + 0 \right) \right]$$

$$\mathcal{F}_c(f) = \sqrt{\frac{2}{\pi}}\left(\frac{1}{1+\omega^2}\right)$$

which is required Fourier cosine transform of given function.

Next, using Parseval's identity for cosine transform, we get

$$\int_{0}^{\infty} |f(x)|^2 dx = \int_{0}^{\infty} |F_c(\omega)|^2 d\omega$$

or,

$$\int_0^\infty \left(e^{-x}\right)^2 dx = \int_0^\infty \left\{ \sqrt{\frac{2}{\pi}} \left(\frac{1}{1+\omega^2} \right) \right\}^2 d\omega$$

or,

$$\int_0^\infty e^{-2x} dx = \frac{2}{\pi} \int_0^\infty \frac{1}{(1+\omega^2)^2} d\omega$$

or,

$$\frac{\pi}{2}\left[\frac{e^{-2x}}{-2}\right]_0^{\infty} = \int_0^{\infty} \frac{1}{(1+\omega^2)^2}\,d\omega$$

or,

$$\int_0^\infty \frac{1}{(1+\omega^2)^2} d\omega = \frac{\pi}{2}\left(0 - \frac{e^0}{(-2)}\right)$$

Replacing ω by x we get

$$\int_0^\infty \frac{1}{(1+x^2)^2}\,dx = \frac{\pi}{4}$$

which is required result.

Find Fourier sine and cosine transform of

$$f(x) = \begin{cases} 1 & \text{for} \quad 0 \leq x \leq 1 \\ 0 & \text{for} \quad x > 1 \end{cases}$$

And hence show that

1. $$\int_0^\infty \left(\frac{1 - \cos x}{x}\right)^2 = \frac{\pi}{2}$$

   and

2. $$\int_0^\infty \left(\frac{\sin x}{x}\right)^2 = \frac{\pi}{2}$$

Sol^n : The Fourier sine transform of given function $f(x)$ is

$$\mathcal{F}_s(f) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(t) \sin \omega t dt$$

$$= \sqrt{\frac{2}{\pi}} \int_{0}^{1} 1. \sin \omega t dt$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{-\cos \omega t}{\omega} \right]_0^1$$

$$\therefore \mathcal{F}_s(f) = \sqrt{\frac{2}{\pi}}\left(\frac{1-\cos\omega}{\omega}\right)$$

which is required Fourier sine transform of given function.

Next, using Parseval's identity for sine transform, we get

$$\int_{0}^{\infty} |F_s(\omega)|^2 d\omega = \int_{0}^{\infty} |f(x)|^2 dx$$

or,

$$\int_0^{\infty} \left\{ \sqrt{\frac{2}{\pi}} \left(\frac{1 - \cos \omega}{\omega} \right) \right\}^2 d\omega = \int_0^1 1 dx$$

or,

$$\int_0^\infty \left(\frac{1-\cos\omega}{\omega}\right)^2 d\omega = \frac{\pi}{2}[x]_0^1$$

Replacing ω by x we get

$$\int_{0}^{\infty} \left(\frac{1 - \cos x}{x} \right)^2 d\omega = \frac{\pi}{2}$$

which is required result.

Next,the Fourier cosine transform of given function $f(x)$ is

$$\mathcal{F}_c(f) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(t) \cos \omega t dt$$

$$= \sqrt{\frac{2}{\pi}} \int_{0}^{1} 1.\cos \omega t dt$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{\sin \omega t}{\omega} \right]_0^1$$

$$\therefore \mathcal{F}_c(f) = \sqrt{\frac{2}{\pi}} \left(\frac{\sin \omega}{\omega} \right)$$

which is required Fourier sine transform of given function.

Using Parseval's identity for cosine transform, we get

$$\int_0^\infty |F_c(\omega)|^2 d\omega = \int_0^\infty |f(x)|^2 dx$$

or,

$$\int_{0}^{\infty} \left\{ \sqrt{\frac{2}{\pi}} \left(\frac{\sin \omega}{\omega} \right) \right\}^2 d\omega = \int_{0}^{1} 1 dx$$

or,

$$\int_0^\infty \left(\frac{\sin\omega}{\omega}\right)^2 d\omega = \frac{\pi}{2}[x]_0^1$$

Replacing ω by x we get

$$\int_{0}^{\infty} \left(\frac{\sin x}{x} \right)^2 d\omega = \frac{\pi}{2}$$

which is required result.