One Dimensional Heat Equations ++

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Q. Find the temperature in a laterally insulated bar of length L, whose ends are kept at zero temperatures, assuming that the initial temperature is

$$f(x) = \begin{cases} x & \text{for} \quad 0 \leq x \leq \frac{L}{2} \\ L - x & \text{for} \quad \frac{L}{2} \leq x \leq L \end{cases}$$

Sol^n : It is the case of one dimensional heat equation with its equation

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} ...(1)$$

The boundary conditions are

$$u(0, t) = 0$$

and

$$u(L, t) = 0$$

The initial temperature on the rod is

$$u(x, 0) = \begin{cases} x & \text{for} \quad 0 \leq x \leq \frac{L}{2} \\ L - x & \text{for} \quad \frac{L}{2} \leq x \leq L \end{cases}$$

Its general solution is

$$u(x,t) = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi}{L} x . e^{-\left(\frac{n\pi}{L}\right)^2 c^2 t} ......(2)$$

Using the initial condition,

$$u(x,0) = f(x) = \begin{cases} x & \text{for} \quad 0 \leq x \leq \frac{L}{2} \\ L - x & \text{for} \quad \frac{L}{2} \leq x \leq L \end{cases}$$

we get

$$f(x) = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi}{L} x.e^0$$

or,

$$f(x) = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi}{L} x$$

Which is a half range Fourier sine series, where

$$a_n = \frac{2}{L} \int_{0}^{L} f(x) \sin \frac{n\pi}{L} x \quad dx$$

$$
\begin{aligned}
&= \frac{2}{L}\int_{0}^{L/2} f(x)\sin\frac{n\pi}{L}x\,dx + \frac{2}{L}\int_{L/2}^{L} f(x)\sin\frac{n\pi}{L}x\,dx \\
&= \frac{2}{L}\int_{0}^{L/2} x\sin\frac{n\pi}{L}x\,dx + \frac{2}{L}\int_{L/2}^{L} (L-x)\sin\frac{n\pi}{L}x\,dx
\end{aligned}
$$

$$
\begin{aligned}
= \quad & \frac{2}{L}\left[x.\frac{L}{n\pi}(-)\cos\frac{n\pi}{L}x - 1.\left(\frac{L}{n\pi}\right)^2(-)\sin\frac{n\pi}{L}x\right]_0^{L/2} \\
+ \quad & \frac{2}{L}\left[(L-x).\frac{L}{n\pi}(-)\cos\frac{n\pi}{L}x - (-1).\left(\frac{L}{n\pi}\right)^2(-)\sin\frac{n\pi}{L}x\right]_{L/2}^{L}
\end{aligned}
$$

$$
\begin{aligned}
= \quad & \frac{2}{L}\left[(-)\frac{L}{2}.\frac{L}{n\pi}\cos\frac{n\pi}{2}+\left(\frac{L}{n\pi}\right)^2\sin\frac{n\pi}{2}-0+0\right] \\
+ \quad & \frac{2}{L}\left[0-0-\left\{(-)\frac{L}{2}.\frac{L}{n\pi}\cos\frac{n\pi}{2}-\left(\frac{L}{n\pi}\right)^2\sin\frac{n\pi}{2}\right\}\right]
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{2}{L}.2.\left(\frac{L}{n\pi}\right)^2 \sin\frac{n\pi}{2} \\
&= \frac{4L}{n^2\pi^2}\sin\frac{n\pi}{2}
\end{aligned}
$$

$$\therefore a_n = \begin{cases} \frac{4L}{n^2\pi^2} \sin \frac{n\pi}{2} & \text{for} \quad n = odd \\ 0 & \text{for} \quad n = even \end{cases}$$

Hence, from equation (2), we get

$$u(x,t) = \frac{4L}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \sin \frac{n\pi}{L} x . e^{-\left(\frac{n\pi}{L}\right)^2 c^2 t}$$

where $n = odd$

This is required temperature distribution.

Determine the solution of one dimensional heat equation under the boundary condition $u(0, t) = 0, u(L, t) = 0$ and the initial temperature is $u(x, 0) = x$, where L is the length of the bar.

Sol^n : It is the case of one dimensional heat equation with its equation

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} ...(1)$$

The boundary conditions are

$$u(0, t) = 0$$

and

$$u(L, t) = 0$$

The initial temperature on the rod is

$$u(x, 0) = f(x) = x$$

Its general solution is

$$u(x,t) = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi}{L} x . e^{-\left(\frac{n\pi}{L}\right)^2 c^2 t} ......(2)$$

Using the initial condition,

$$u(x, 0) = f(x)$$

we get

$$f(x) = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi}{L} x . e^{0}$$

or,

$$f(x) = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi}{L} x$$

Which is a half range Fourier sine series, where

$$a_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi}{L} x \quad dx$$

$$= \frac{2}{L} \int_{0}^{L} x . \sin \frac{n\pi}{L} x dx$$

$$= \frac{2}{L}\left[x.\frac{L}{n\pi}(-)\cos\frac{n\pi}{L}x - 1.\left(\frac{L}{n\pi}\right)^2(-)\sin\frac{n\pi}{L}x\right]_0^L$$

$$= \frac{2}{L}\left[L.\frac{L}{n\pi}(-)\cos n\pi - 0 - 1.\,(0-0)\right]$$

$$\therefore a_n = -\frac{2L}{n\pi}\cos n\pi$$

Hence, from (1), we get

$$u(x,t) = -\frac{2L}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\cos n\pi . \sin\frac{n\pi}{L}x . e^{-\left(\frac{n\pi}{L}\right)^2 c^2 t}$$

This is required temperature distribution.

Steady state condition:

The condition at whcih temperature is independent of time is called steady state condition.
In steady state ,

$$\frac{\partial u}{\partial t} = 0$$

The one dimensional heat equation

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} ...(1)$$

now becomes

$$0 = c^2 \frac{\partial^2 u}{\partial x^2}$$

or,

$$\frac{\partial^2 u}{\partial x^2} = 0$$

Integrating we get

$$\frac{\partial u}{\partial x} = A$$

Again, integrating, we get

$$u = Ax + B$$

which is temperature distribution on the bar in steady state condition.

Q. An insulated rod of length L has its ends P and Q maintained at temperatures $0^o C$ and $100^o C$ respectively, until the steady state condition prevails.If the end Q is suddenly reduced to $0^o C$ and maintained at $0^o C$, find the temperature distribution on the rod.

Sol^n: In steady state,

$$\frac{\partial^2 u}{\partial x^2} = 0$$

Solving,

$$u(x) = Ax + B$$

Using the boundary condition, $u = 0$ at $x = 0$, we get

$$0 = A.0 + B$$

i.e.

$$B = 0$$

Again, using the boundary condition, $u = 100$ at $x = L$, we get

$$100 = A.L + 0$$

i.e.

$$A = \frac{100}{L}$$

So,

$$u = Ax + B = \frac{100}{L}x + 0$$

which is initial temperature on the rod.

Now, we have to solve the one dimensional heat equation with its equation

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} ...(1)$$

The boundary conditions are

$$u(0, t) = 0$$

and

$$u(L, t) = 0$$

And the initial condition is,

$$u(x, 0) = f(x) = \frac{100}{L} x$$

Its general solution is

$$u(x,t) = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi}{L} x . e^{-\left(\frac{n\pi}{L}\right)^2 c^2 t} ......(2)$$

Using the initial condition,

$$u(x,0) = f(x) = \frac{100}{L}x$$

we get

$$f(x) = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi}{L}x.e^0$$

or,

$$f(x) = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi}{L} x$$

Which is a half range Fourier sine series, where

$$a_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi}{L} x \quad dx$$

$$= \frac{2}{L}\int_{0}^{L}\frac{100x}{L}.\sin\frac{n\pi}{L}xdx$$

$$= \frac{200}{L^2}\left[x.\frac{L}{n\pi}(-)\cos\frac{n\pi}{L}x - 1.\left(\frac{L}{n\pi}\right)^2(-)\sin\frac{n\pi}{L}x\right]_0^L$$

$$= \frac{200}{L^2}\left[L.\frac{L}{n\pi}(-)\cos n\pi - 0 - 1.\left(0-0\right)\right]$$

$$\therefore a_n = -\frac{200}{n\pi}\cos n\pi$$

Hence, from (1), we get

$$u(x,t) = -\frac{200}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\cos n\pi.\sin\frac{n\pi}{L}x.e^{-\left(\frac{n\pi}{L}\right)^2 c^2 t}$$

This is required temperature distribution.

Solution of non homogeneous boundary conditions

If temperatures at end points of the rod are different from zero, the boundary conditions are called non-homogeneous.
In such case, the solution of one dimensional heat equation is given by

$$u(x,t) = u_s(x) + u_1(x,t)$$

where,

$$u_s(x)$$

is solution of steady state condition i.e.

$$u_s(x) = Ax + B$$

and

$$u_1(x,t)$$

satisfies the heat equation with homogeneous boundary conditions

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial t^2}$$

i.e.

$$\frac{\partial u_1}{\partial t} = c^2 \frac{\partial^2 u_1}{\partial t^2}$$

$$\frac{\partial u_1}{\partial t} = c^2 \frac{\partial^2 u_1}{\partial t^2}$$

with the initial condition

$$u_1(x, 0) = u(x, 0) - u_s$$

The ends A and B of the rod of length 20 cm have the temperatures at $30^o C$ and at $80^o C$ respectively,until the steady state condition prevails.The temperatures of the ends are changed to $40^o C$ and $60^o C$ respectively. Find the temperature distribution in the rod at any time t.

Sol^n: First, we find the initial temperature on the rod. For, in steady state,

$$\frac{\partial^2 u}{\partial x^2} = 0$$

Solving,

$$u(x) = Ax + B$$

Using the boundary condition, $u = 30$ at $x = 0$, we get

$$30 = A.0 + B$$

i.e.

$$B = 30$$

Again, using the boundary condition, $u = 80$ at $x = 20$, we get

$$80 = A.20 + 30$$

i.e.

$$A = \frac{5}{2}$$

So,

$$u = Ax + B = \frac{5}{2}x + 30$$

which is initial temperature on the rod.

Now, we have to solve the one dimensional heat equation

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} ...(1)$$

The boundary conditions are

$$u(0, t) = 40$$

and

$$u(L, t) = 60$$

And the initial condition is,

$$u(x, 0) = f(x) = \frac{5}{2}x + 30$$

Its solution is

$$u(x,t) = u_s(x) + u_1(x,t).......(2)$$

where

$$u_s = Cx + D$$

using $u = 40 \quad at \quad x = 0$, we get $D = 40$ and using $u = 60 \quad at \quad x = 20$, we get $C = 1$.

$$\therefore u_s = x + 40$$

And

$$u_1(x,t)$$

satisfies the heat equation

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$$

with the homogeneous boundary conditions. So, its solution is,

$$u_1(x,t) = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi}{L} x . e^{-\left(\frac{n\pi}{L}\right)^2 c^2 t} .....(3)$$

with the initial condition

$$u_1(x,0) = u(x) - u_s$$

So,using the initial condition ,

$$u_1(x,0) = f(x) = \frac{5}{2}x + 30 - (x+40) = \frac{3}{2}x - 10$$

we get

$$f(x) = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi}{L} x$$

which is a half range Fourier sine series, where

$$a_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi}{L} x \quad dx$$

$$= \frac{2}{20} \int_{0}^{20} \left(\frac{3}{2}x - 10 \right) \sin \frac{n\pi}{20}x \quad dx$$

$$= \frac{1}{10}\left[\left(\frac{3}{2}x - 10\right).\frac{20}{n\pi}(-)\cos\frac{n\pi}{20}x - \frac{3}{2}.\left(\frac{20}{n\pi}\right)^2(-)\sin\frac{n\pi}{20}x\right]_0^{20}$$

$$= \frac{1}{10}\left[\left\{20.\frac{20}{n\pi}(-)\cos n\pi - 0\right\} - \left\{(-10).\frac{20}{n\pi}(-)\right\}\right]$$

$$\therefore a_n = -\frac{20}{n\pi}\left[1 + 2\cos n\pi\right]$$

Hence, from (2),

$$u(x,t) = u_s(x) + u_1(x,t)$$

we get

$$u(x,t) = x + 40 - \frac{20}{\pi} \sum_{n=1}^{\infty} \frac{(1 + \cos n\pi)}{n} \sin \frac{n\pi}{20} x . e^{-\left(\frac{n\pi}{20}\right)^2 c^2 t}$$

which is required solution to the given problem.

A rod of length L has its ends A and B kept at temperatures at $0^o C$ and at $100^o C$ respectively,until the steady state condition prevails.If the changes consists of raising the temperature of A to $25^o C$ and reducing that of B to $75^o C$ respectively. Find the temperature distribution in the rod at any time t.

Sol^n: First, we find the initial temperature on the rod. For, in steady state,

$$\frac{\partial^2 u}{\partial x^2} = 0$$

Solving,

$$u(x) = Ax + B$$

Using the boundary condition, $u = 30$ at $x = 0$, we get

$$0 = A.0 + B$$

i.e.

$$B = 0$$

Again, using the boundary condition, $u = 100$ at $x = L$, we get

$$100 = A.L + 0$$

i.e.

$$A = \frac{100}{L}$$

So,

$$u = Ax + B = \frac{100}{L}x$$

which is initial temperature on the rod.

Now, we have to solve the one dimensional heat equation

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \dots (1)$$

With the boundary conditions

$$u(0, t) = 25$$

and

$$u(L, t) = 75$$

And the initial condition is,

$$u(x, 0) = f(x) = \frac{100}{L} x$$

Its solution is

$$u(x,t) = u_s(x) + u_1(x,t) ....(2)$$

where

$$u_s = Cx + D$$

using $u = 25 \quad at \quad x = 0$, we get $D = 25$ and using $u = 75 \quad at \quad x = L$, we get $C = \frac{50}{L}$.

$$\therefore u_s = \frac{50}{L}x + 25$$

And

$$u_1(x,t)$$

satisfies the heat equation

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$$

with the homogeneous boundary conditions. So, its solution is,

$$u_1(x,t) = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi}{L} x . e^{-\left(\frac{n\pi}{L}\right)^2 c^2 t} .....(3)$$

with the initial condition

$$u_1(x,0) = u(x) - u_s$$

So,using the initial condition ,

$$u_1(x,0) = f(x) = \frac{100}{L}x - (\frac{50}{L}x + 25) = \frac{50}{L}x - 25$$

we get

$$f(x) = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi}{L}x$$

which is a half range Fourier sine series, where

$$a_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi}{L}x \quad dx$$

$$= \frac{2}{L} \int_{0}^{L} \left(\frac{50}{L} x - 25 \right) \sin \frac{n\pi}{L} x \quad dx$$

$$= \frac{2}{L}\left[\left(\frac{50}{L}x - 25\right).\frac{L}{n\pi}(-)\cos\frac{n\pi}{L}x - \frac{50}{L}.\left(\frac{L}{n\pi}\right)^2(-)\sin\frac{n\pi}{L}x\right]_0^L$$

$$= \frac{2}{L}\left[\left\{25.\frac{L}{n\pi}(-)\cos n\pi - 0\right\} - \left\{(-25).\frac{L}{n\pi}(-1) - 0\right\}\right]$$

$$\therefore a_n = -\frac{50}{n\pi}\left[1 + \cos n\pi\right]$$

Hence, from (2),

$$u(x, t) = u_s(x) + u_1(x, t)$$

we get

$$u(x,t) = \frac{50}{L}x + 25 - \frac{50}{\pi}\sum_{n=1}^{\infty}\frac{(1+\cos n\pi)}{n}\sin\frac{n\pi}{L}x.e^{-\left(\frac{n\pi}{L}\right)^2 c^2 t}$$

which is required solution to the given problem.

A rod of length L has its ends A and B kept at temperatures at $0^o C$ and at $100^o C$ respectively,until the steady state condition prevails.If the changes consists of raising the temperature of A to $20^o C$ and reducing that of B to $80^o C$ respectively. Find the temperature distribution in the rod at any time t.

Sol^n: First, we find the initial temperature on the rod. For, in steady state,

$$\frac{\partial^2 u}{\partial x^2} = 0$$

Solving,

$$u(x) = Ax + B$$

Using the boundary condition, $u = 30$ at $x = 0$, we get

$$0 = A.0 + B$$

i.e.

$$B = 0$$

Again, using the boundary condition, $u = 100$ at $x = L$, we get

$$100 = A.L + 0$$

i.e.

$$A = \frac{100}{L}$$

So,

$$u = Ax + B = \frac{100}{L}x$$

which is initial temperature on the rod.

Now, we have to solve the one dimensional heat equation

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} ...(1)$$

With the boundary conditions

$$u(0, t) = 20$$

and

$$u(L, t) = 80$$

And the initial condition is,

$$u(x, 0) = f(x) = \frac{100}{L} x$$

Its solution is

$$u(x,t) = u_s(x) + u_1(x,t).......(2)$$

where

$$u_s = Cx + D$$

using $u = 20 \quad at \quad x = 0$, we get $D = 20$ and using $u = 80 \quad at \quad x = L$, we get $C = \frac{60}{L}$.

$$\therefore u_s = \frac{60}{L}x + 20$$

And

$$u_1(x,t)$$

satisfies the heat equation

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$$

with the homogeneous boundary conditions. So, its solution is,

$$u_1(x,t) = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi}{L} x . e^{-\left(\frac{n\pi}{L}\right)^2 c^2 t} \text{.....(3)}$$

with the initial condition

$$u_1(x,0) = u(x) - u_s$$

So,using the initial condition ,

$$u_1(x,0) = f(x) = \frac{100}{L}x - (\frac{60}{L}x + 20) = \frac{40}{L}x - 20$$

we get

$$f(x) = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi}{L}x$$

which is a half range Fourier sine series, where

$$a_n = \frac{2}{L}\int_0^L f(x) \sin \frac{n\pi}{L}x \quad dx$$

$$= \frac{2}{L} \int_{0}^{L} \left(\frac{40}{L} x - 20 \right) \sin \frac{n\pi}{L} x \quad dx$$

$$= \frac{2}{L}\left[\left(\frac{40}{L}x - 20\right).\frac{L}{n\pi}(-)\cos\frac{n\pi}{L}x - \frac{40}{L}.\left(\frac{L}{n\pi}\right)^2(-)\sin\frac{n\pi}{L}x\right]_0^L$$

$$= \frac{2}{L}\left[\left\{20.\frac{L}{n\pi}(-)\cos n\pi - 0\right\} - \left\{(-20).\frac{L}{n\pi}(-1) - 0\right\}\right]$$

$$\therefore a_n = -\frac{40}{n\pi}\left[1 + \cos n\pi\right]$$

Hence, from (2),

$$u(x,t) = u_s(x) + u_1(x,t)$$

we get

$$u(x,t) = \frac{60}{L}x + 20 - \frac{40}{\pi}\sum_{n=1}^{\infty}\frac{(1+\cos n\pi)}{n}\sin\frac{n\pi}{L}x.e^{-\left(\frac{n\pi}{L}\right)^2 c^2 t}$$

which is required solution to the given problem.