Unit-10
Ring and Field:

⊛ Ring:

Definition → An algebraic structure $(R, +, \times)$ with the two binary operations addition $(+)$ and multiplication $(\times)$ that satisfies the following conditions is called a ring.

Abelian group
group

i) **Closure for addition**
$a+b \in R, \forall a, b \in R.$

ii) **Associativity**
$a+(b+c)=(a+b)+c, \forall a, b, c \in R.$

iii) **Existence of identity**
$\exists 0 \in R$ such that $0+a=a+0=a, \forall a \in R.$

iv) **Existence of inverse**
$\exists -a \in R$
$a+(-a)=(-a)+a=0, \forall a \in R.$

v) **Commutativity**
$a+b=b+a, \forall a, b \in R.$

vi) **Associativity for multiplication**
$a \cdot (b \cdot c)=(a \cdot b) \cdot c \quad \forall a, b, c \in R.$

viii) **Distributivity for multiplication over addition:**

(a) **Left distributive:** $a \cdot (b+c)=a \cdot b+a \cdot c \quad \forall a, b, c \in R.$

(b) **Right distributive:** $(a+b) \cdot c=a \cdot c+b \cdot c \quad \forall a, b, c \in R.$

Note: First four conditions show that R is a group under addition, & the first five conditions shows that R is an abelian group.

OR

Ring can also be defined as an algebraic structure $(R, +, \times)$ such that:

(a) R is an abelian group under $+$.

(b) Associativity holds for multiplication.

(c) Multiplication is distributive from left as well as right.

Commutative ring → A ring $(R, +, \times)$ is said to be commutative ring if multiplication operation is commutative.

Examples for commutative ring

① $(Z, +, \times)$ is a ring

Soln

For we have, Z is a non-empty set.

i) $a+b \in Z \quad \forall a, b \in Z$.

ii) $a+(b+c) = (a+b)+c \quad \forall a, b, c \in Z$.

iii) $\exists 0 \in Z : 0+a = a+0 = a \quad \forall a \in Z$.

iv) $\exists -a \in Z : a+(-a) = (-a)+a = 0 \quad \forall a \in Z$.

v) $a+b = b+a \quad \forall a, b \in Z$.

vi) $ab = ba \quad \forall a, b \in Z$.

vii) $a(bc) = (ab)c \quad \forall a, b \in Z$.

viii) (a). $a(b+c) = ab+bc$

(b) $(a+b) \cdot c = ab+bc$

It is commutative ring since $ab = ba \ \forall a, b \in Z$.

② The set of real numbers with the binary operations: $+, \times$ is a ring. i.e. $(R, +, \times)$ is a ring.

③ The set of rational numbers with the two binary operations addition, $+$ and multiplication $\times$, is a ring.

④ **Null (zero) ring** → The set $\{0\}$ with the two binary operation $+, \times$ constitutes a ring called null ring.

Related Questions:

① Show $Z_7 = \{0, 1, 2, 3, 4, 5, 6\}$ is a ring under the binary operation addition modulo $(+7)$ and multiplication modulo $(\times 7)$, 7.

Proof: The composition table for addition modulo and multiplication modulo.

$+_7$	0	1	2	3	4	5	6
0	0	1	2	3	4	5	6
1	1	2	3	4	5	6	0
2	2	3	4	5	6	0	1
3	3	4	5	6	0	1	2
4	4	5	6	0	1	2	3
5	5	6	0	1	2	3	4
6	6	0	1	2	3	4	5

max. num 6 हुन्छ त्यसैले 6 भन्दा बढि हुनु भएन भए 7 subtract गर्ने.
यसमा 1+6=7 भाको >6 त्यसैले 7-7=0 गरेको
similarly for others >6.

× 7	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6
2	0	2	4	6	1	3	5
3	0	3	6	2	5	1	4
4	0	4	1	5	2	6	3
5	0	5	3	1	6	4	2
6	0	6	5	4	3	2	1

हुनुपर्ने $3 \times 5 = 15$ which is > 2 times 6.
So, $7 \times 2 = 14$
subtracting from 15 we get 1 as remainder.
So, if > 6 divide by 7 and write the remainder.

Here the set is abelian group for addition modulo 7.

i) Closure → $\forall a, b \in Z_7$, $a +_7 b \in Z_7$.

ii) Associativity →

~~$2 +_7 (3 +_7 4) = (2 +_7 3) +_7 4$~~

$$(a +_7 b) +_7 c = a +_7 (b +_7 c) \ \forall a, b, c \in Z$$

iii) Additive identity → 0 is the additive identity.

iv) Existence of additive inverse → $\forall a \in Z_7$
$\exists a \in Z_7 : a + (-a) = 0$.

v) Commutativity holds → $a +_7 b = b +_7 a$.

vi) Closed for multiplication → $a \times_7 b \in Z_7$, $\forall a, b \in Z_7$.

vii) Associtivity for multiplication →
$a \times_7 (b \times_7 c) = (a \times_7 b) \times_7 c$, $\forall a, b, c \in Z_7$.

viii) Distributivity for multiplication over addition:
Left: $a \times_7 (b +_7 c) = a \times_7 b + a \times_7 c$.
Right: $(a +_7 b) \times_7 c = a \times_7 c + b \times_7 c$ $\forall a, b, c \in Z_7$.

Q. Evaluate: (12)(14) in Z_{21}.

Solution,

we have, $12 \times 14 = 168$
$= 8 \times 21 + 0$
$= 0$

Z_{21} दिइएकोले 21 ले multiply गरि remainder add गरेको र ans remainder लेखेको.

Q. Evaluate the sum $(1,2) + (3,5)$ in $Z_3 \times Z_7$.

Solution. $(1,2) + (3,5)$ in $Z_3 \times Z_7$
$= (1+3, 2+5)$
$= (4, 7)$
$= (1, 0)$

Z_3 र Z_7 दिएको छ > 3 र 7 आयो। यसैले 3 र 7 ले seperately divide गरि remainder लेखेको.

Q Compute the product in the given ring:

(a). $(12)(6) \in \mathbb{Z}_{25}$

(b) $(20)(-8) \in \mathbb{Z}_{26}$

(c) $(-3,5)(2,-4) \in \mathbb{Z}_4 \times \mathbb{Z}_{11}$

Solution:

(a). $(12)(6) \in \mathbb{Z}_{25}$.

we have, $12 \times 6 = 72$ — 25 ले expand गरेको

$= 25 \times 2 + 22$

$= 22$

(b) $(20)(-8) \in \mathbb{Z}_{26}$

we have, $(20).(-8) = -160$

$= -6 \times 26 + (-4)$

$= -4$

$= -4 + 26$ } — negative value भएकोले positive बनाउन 26 add गरेको

$= 22$

(c). $(-3,5)(2,-4) = (-6,-20)$

$= (-2,-9)$ → negative छ $\mathbb{Z}_4$ र $\mathbb{Z}_{11}$ दिएको छ त्यसैले 4 र 11 जोडेको

$= (2,2)$

* Properties of ring: (Not more imp).

Let $a, b \in (R,+,\times)$ 0 be an additive identity of the ring.

then,

(i) $a.0 = 0.a = 0$.

(ii) $a(-b) = (-a).b = -(a.b)$

(iii) $(-a).(-b) = -(a.b)$.

(i) Proof: we have, $a.0 = a.(0+0)$ $(\because 0 = 0+0)$

or, $a.0 = a.0 + a.0$ (distributivity property).

Since, $0, a \in R$

$a.0 \in R$

$0 + a.0 = a.0 + a.0$

$\Rightarrow 0 = a.0$

i.e, $a.0 = 0$ (right cancellation law)

$0.a = (0+0).a$

or, $0.a = 0.a + 0.a$ (right distributivity).

or, $0 + 0.a = 0.a + 0.a$

$\Rightarrow 0 = 0.a$ (right cancellation law).

(ii) Proof: we have,

$0 \cdot b = 0$

$[a + (-a)] \cdot b = 0$.

$\Rightarrow a \cdot b + (-a) \cdot b = 0$

$\Rightarrow (-a) \cdot b = -(a \cdot b)$ — (i).

Also,

$a \cdot 0 = 0$

$a \cdot [b + (-b)] = 0$

$\Rightarrow a \cdot b + a(-b) = 0$.

$\Rightarrow a \cdot (-b) = -(a \cdot b)$ — (ii).

From (i) and (ii) we get,

$a(-b) = (-a)b = -(a \cdot b)$

(iii). Proof:

we know that, $(+a) \cdot (-b) = -(a \cdot b)$

using $-a$ for a

$(-a) \cdot (-b) = -[(-a) \cdot b]$

$= -[-(a \cdot b)]$

$= ab$.

Zero divisor:

Let us consider a ring $(M_2(\mathbb{Z}, +, \times))$. In the ring $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is an identity element.

Let $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \in (M_2(\mathbb{Z}, +, \times))$ be non-zero elements such that their product is zero. The ring is called the ring with zero divisor.

Definition → A ring $(R, +, \times)$ is a ring with zero divisor.

Ring with no zero divisor → Let $(\mathbb{Z}, +, \times)$ be a ring with no zero divisor because, $ab = 0 \Rightarrow$ either $a = 0$ or $b = 0$ or both i.e, $ab = 0$ only when at least one is zero.

Integral domain → A Ring $(R, +, \times)$ is said to be integral domain if and only if (iff)

(i) R is commutative ring.

(ii) R has an identity element for multiplication.

(iii) R has no zero divisors.

THE END

Best of luck

Note:

⇒ Practice provided model questions and additional 4 sets also.


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Mathematics-II
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