Complex Fourier Integral and Fourier Transform

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From fourier integral theorem,

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(t) \left[2 \int_{0}^{\infty} \cos \omega \left(t - x \right) d\omega \right] dt$$

or,

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(t) \left[\int_{-\infty}^{\infty} \cos \omega \left(t - x \right) d\omega \right] dt$$

$$\left[\because 2 \int_{0}^{\infty} \cos \theta d\theta = \int_{-\infty}^{\infty} \cos \theta d\theta \right]$$

$$i.e. f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[f(t) \cos \omega \left(x - t \right) \right] dt d\omega ......(1)$$

$$\because \left[\cos \theta = \cos(-\theta) \right]$$

Since *sine* function is an odd function and

$$0 = \int_{-\infty}^{\infty} \sin \omega(x - t) d\omega$$

Multiplying both sides by

$$\frac{i}{2\pi}\int_{-\infty}^{\infty} f(t)dt$$

we get

$$0 = \frac{i}{2\pi}\int_{-\infty}^{\infty} f(t)dt.\int_{-\infty}^{\infty} \sin\omega(x-t)d\omega$$

$$i.e.0 = \frac{i}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} [f(t) \sin \omega (x - t)] \, dt d\omega ......(2)$$

Adding equations (1) and (2), we get

$$f(x) + 0 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t) \left[\cos \omega (x - t) + i \sin \omega (x - t)\right] dt d\omega$$

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t).e^{i\omega(x-t)} dt d\omega.....(3)$$

$$\left[\because \cos\theta + i\sin\theta = e^{i\theta}\right]$$

This equation is called complex fourier integral of given function.

It can be written as

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t).e^{i\omega x}.e^{-i\omega t} dt \quad d\omega$$

$$or, f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} f(t).e^{-iwt} dt \right\} e^{i\omega x} d\omega$$

$$i.e. f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left\{ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t).e^{-iwt} dt \right\} e^{i\omega x} d\omega....(4)$$

The expression in the bracket of equation (4) is called Fourier Transform of given function $f(x)$. It is denoted by symbol $\mathcal{F}$

$$i.e. \mathcal{F}[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t).e^{-iwt} dt.....(5)$$

And the function $f(x)$ itself is called inverse transform of $\mathcal{F}$.

$$i.e. \mathcal{F}^{-1} = f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \{\mathcal{F}[f(x)]\} e^{i\omega x} d\omega....(6)$$

Find the fourier transform of the function

$$f(x) = \begin{cases} e^{-kx} & \text{if} \quad x > 0, k > 0 \\ 0 & \text{if} \quad x < 0 \end{cases}$$

Sol^n : The fourier transform of given function is

$$\mathcal{F}[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t).e^{-iwt} dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} f(t).e^{-iwt} dt + \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} f(t).e^{-iwt} dt$$

$$= 0 + \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-kt}.e^{-iwt} dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-(k+i\omega)t} dt$$

$$= \frac{1}{\sqrt{2\pi}}(-)\left[\frac{e^{-(k+i\omega)t}}{(k+i\omega)}\right]_0^{\infty}$$

$$= -\frac{1}{\sqrt{2\pi}}\left[0 - \frac{1}{k + i\omega}\right]$$

$$= \frac{1}{\sqrt{2\pi}} \frac{k - i\omega}{(k + i\omega)(k - i\omega)}$$

$$\therefore \mathcal{F}(f) = \frac{1}{\sqrt{2\pi}} \cdot \frac{k - i\omega}{(k^2 + \omega^2)}$$

Find the fourier transform of

$$f(x) = e^{-x^2}$$

Sol^n : We have,

$$f(x) = e^{-x^2}$$

Its fourier transform is

$$\mathcal{F}(f) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t).e^{-i\omega t} dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-t^2}.e^{-i\omega t} dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(t^2 + i\omega t)} dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(t^2 + 2ti\omega/2 + (i\omega/2)^2 - (i\omega/2)^2)} dt$$

$$\left[\because (a+b)^2 = a^2 + 2ab + b^2\right]$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left\{t+\frac{i\omega}{2}\right\}^2 - \frac{\omega^2}{4}} dt$$

$$\therefore \mathcal{F}(f) = \frac{e^{-\frac{\omega^2}{4}}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left\{t+\frac{i\omega}{2}\right\}^2} dt..... \quad (1)$$

Put

$$t + i\omega/2 = y$$

so that

$$dt = dy$$

Also, when $t \to \infty$, then $y \to \infty$ and $t \to -\infty$, then $y \to -\infty$

So, from above,

$$\mathcal{F}(f) = \frac{e^{-\frac{\omega^2}{4}}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-y^2} dy$$

$$= \frac{e^{-\frac{\omega^2}{4}}}{\sqrt{2\pi}}.2\int_0^{\infty} e^{-y^2}dy$$

$$= \frac{e^{-\frac{\omega^2}{4}}}{\sqrt{2\pi}}.2\frac{\sqrt{\pi}}{2}$$

$$\left[\because \int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}\right]$$

Hence,

$$\mathcal{F}(f) = \frac{e^{-\frac{\omega^2}{4}}}{\sqrt{2}}$$

Find the fourier transform of

$$(1) f(x) = e^{\frac{-x^2}{2}}$$

and

$$(2) f(x) = e^{-ax^2}$$

Sol^n : We have,

$$f(x) = e^{-ax^2}$$

Its fourier transform is

$$\mathcal{F}(f) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t).e^{-i\omega t} dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-at^2}.e^{-i\omega t} dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(at^2 + i\omega t)} dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left\{(\sqrt{a}t)^2 + 2.(\sqrt{a}t).\frac{i\omega}{2.\sqrt{a}} + (\frac{i\omega}{2.\sqrt{a}})^2 - (\frac{i\omega}{2.\sqrt{a}})^2\right\}} dt$$

$$\left[\because (a+b)^2 = a^2 + 2ab + b^2\right]$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left(\sqrt{a}t + \frac{i\omega}{2.\sqrt{a}}\right)^2} . e^{-(\frac{\omega}{2.\sqrt{a}})^2} dt$$

$$\therefore \mathcal{F}(f) = \frac{e^{-\frac{\omega^2}{4a}}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left(\sqrt{a}t + \frac{i\omega}{2.\sqrt{a}}\right)^2} dt$$

Put,

$$\sqrt{a}t + \frac{i\omega}{2.\sqrt{a}} = y$$

so that

$$dt = \frac{dy}{\sqrt{a}}$$

Also, when $t \to \infty$, then $y \to \infty$ and $t \to -\infty$, then $y \to -\infty$

$$\mathcal{F}(f) = \frac{e^{-\frac{\omega^2}{4a}}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-y^2} . \frac{dy}{\sqrt{a}}$$

$$= \frac{e^{-\frac{\omega^2}{4a}}}{\sqrt{2\pi}} . 2\frac{\sqrt{\pi}}{2} . \frac{1}{\sqrt{a}}$$