Partial Differential Equations

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Solve the following by using method of separating the variables:

1. $$u_{xx} + 9u = 0$$
2. $$u_{xy} - u = 0$$
3. $$xu_{xy} + 2yu = 0$$
4. $$u_x + u_y = (x + y)u$$
5. $$u_x = yu_y$$

1.Sol^n : Given equation is

$$\frac{\partial^2 u}{\partial x^2} + 9u = 0.... \quad ...(1), \qquad u = u(x, y)$$

Let

$$u(x, y) = X(x).Y(y)$$

be the solution of (1).
Then, differentiating with respect to x we get

$$\frac{\partial u}{\partial x} = \frac{dX}{dx}.Y$$

Also,

$$\frac{\partial^2 u}{\partial x^2} = \frac{d^2 X}{dx^2}.Y$$

So, from equation (1), we get

$$\frac{d^2X}{dx^2}.Y + 9X.Y = 0$$

$$or, \frac{d^2X}{dx^2}.Y = -9X.Y$$

dividing by XY, we get

$$\frac{1}{X}\frac{d^2X}{dx^2} = -9$$

or,

$$\frac{d^2X}{dx^2} - 9X = 0$$

It's auxiliary equation is

$$m^2 + 9 = 0$$

$$or, m^2 = -9$$

$$i.e. m = \pm 3i$$

which are two imaginary values. So,

$$X = C_1 \cos 3x + C_2 \sin 3x$$

Hence, from (1),

$$u = X.Y$$

becomes

$$u = (C_1 \cos 3x + C_2 \sin 3x)\, Y$$

which is required solution.

2.Sol^n : Given equation is

$$\frac{\partial^2 u}{\partial x \partial y} - u = 0.... \quad ...(1), \qquad u = u(x,y)$$

Let

$$u(x, y) = X(x).Y(y)$$

be the solution of (1).
Then, differentiating partially with respect to x we get

$$\frac{\partial u}{\partial x} = \frac{dX}{dx}.Y$$

Again, differentiating partially with respect to y we get

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{dX}{dx}.\frac{dY}{dy}$$

So, from equation (1), we get

$$\frac{dX}{dx}.\frac{dY}{dy} - X.Y = 0$$

$$or, \frac{dX}{dx}.\frac{dY}{dy} = X.Y$$

$$\frac{1}{X}\frac{dX}{dx} = \frac{Y}{\frac{dY}{dy}} = k(say)$$

then,

$$\frac{1}{X}\frac{dX}{dx} = k... \quad ...(2)$$

and

$$\frac{Y}{\frac{dY}{dy}} = k... \quad ...(3)$$

From (2), separating the variables we get

$$\frac{dX}{X} = kdx$$

Integrating,

$$\log X = kx + \log C_1$$
$$or, \log \frac{X}{C_1} = kx$$
$$i.e. X = C_1 e^{kx}$$

From (3),

$$Y = k\frac{dY}{dy}$$

$$or, \frac{dY}{Y} = \frac{dy}{k}$$

Integrating,

$$\log Y = \frac{1}{k}.y + \log C_2$$

$$or, \log \frac{Y}{C_2} = \frac{1}{k}.y$$

$$i.e. Y = C_2 e^{y/k}$$

Hence,

$$u = X.Y$$

now becomes

$$u = C_1 e^{kx}.C_2 e^{y/k}$$

$$i.e. u = Ce^{kx+y/k}$$

which is required solution, where $C = C_1.C_2$

3.Sol^n : Given equation is

$$x \frac{\partial^2 u}{\partial x \partial y} + 2yu = 0.... \quad ...(1), \qquad u = u(x, y)$$

Let

$$u(x, y) = X(x).Y(y)$$

be the solution of (1).
Then, differentiating partially with respect to x we get

$$\frac{\partial u}{\partial x} = \frac{dX}{dx}.Y$$

Again, differentiating partially with respect to y we get

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{dX}{dx}.\frac{dY}{dy}$$

So, from (1)

$$x\frac{dX}{dx}.\frac{dY}{dy} + 2yX.Y = 0$$

$$or, x\frac{dX}{dx}.\frac{dY}{dy} = -2yX.Y$$

separating the variables, we get

$$\frac{x}{X}\cdot\frac{dX}{dx} = \frac{-2yY}{\frac{dY}{dy}} = k(say)$$

Then,

$$\frac{x}{X} \cdot \frac{dX}{dx} = k \ldots \quad (2)$$

and

$$\frac{-2yY}{\frac{dY}{dy}} = k \ldots. \quad (3)$$

From (2), we get

$$\frac{dX}{X} = k\frac{dx}{x}$$

Integrating,

$$\log X = k \log x + \log C_1$$

$$or, \log X = \log x^k + \log C_1 = \log(C_1 x^k)$$

$$\therefore X = C_1 x^k$$

From, equation (3), we get

$$-2yY = k.\frac{dY}{dy}$$

or,

$$\frac{dY}{Y} = -\frac{2}{k}ydy$$

Integrating,

$$\log Y = -\frac{2}{k}.\frac{y^2}{2} + \log C_2$$

i.e.

$$Y = C_2 e^{-\frac{y^2}{k}}$$

Hence

$$u = X.Y$$

becomes

$$u = C_1 x^k . C_2 e^{-\frac{y^2}{k}}$$

$$\therefore u = C.x^k.e^{-\frac{y^2}{k}}$$

which is required solution..

4.Sol^n : Given equation is

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = (x + y)u.... \quad ...(1), \qquad u = u(x, y)$$

Let

$$u(x, y) = X(x).Y(y)$$

be the solution of (1).
Then, differentiating partially with respect to x we get

$$\frac{\partial u}{\partial x} = \frac{dX}{dx}.Y$$

Also, differentiating partially with respect to y we get

$$\frac{\partial u}{\partial y} = X.\frac{dY}{dy}$$

So, from (1)

$$\frac{dX}{dx}.Y + X.\frac{dY}{dy} = (x+y)XY$$

Dividing by XY

$$\frac{1}{X}\frac{dX}{dx} + \frac{1}{Y}\frac{dY}{dy} = x + y$$

or

$$\frac{1}{X}\frac{dX}{dx} - x = y - \frac{1}{Y}\frac{dY}{dy} = k(say)$$

Then,

$$\frac{1}{X}\frac{dX}{dx} - x = k... \quad (2)$$

and

$$y - \frac{1}{Y}\frac{dY}{dy} = k... \quad (3)$$

From (2)

$$\frac{1}{X}\frac{dX}{dx} = k + x$$

or

$$\frac{dX}{X} = (k + x)dx$$

Integrating,

$$\int \frac{dX}{X} = \int (k + x)dx + constant$$

or,

$$\log X = kx + \frac{x^2}{2} + \log C_1$$

$$i.e. X = c_1 e^{\left(kx+\frac{x^2}{2}\right)}$$

From (3)

$$\frac{1}{Y}\frac{dY}{dy} = y - k$$

or

$$\frac{dY}{Y} = (y - k)dy$$

Integrating,

$$\int \frac{dY}{Y} = \int (y - k)dy + constant$$

or,

$$\log Y = \frac{y^2}{2} - ky + \log C_2$$

$$i.e. Y = c_2 e^{(\frac{y^2}{2} - ky)}$$

Hence

$$u = X.Y$$

becomes

$$u = c_1 e^{(kx+\frac{x^2}{2})}.c_2 e^{(\frac{y^2}{2}-ky)}$$

$$\therefore u = Ce^{\left\{k(x-y)+\frac{1}{2}(x^2+y^2)\right\}}$$

which is required solution.