Analytical Function:

A Function $F(x)$ in a domain D is said to be analytic if it is differentiable everywhere in D.

Theorem: Necessary condⁿ for a function to be analytic

let $F(z) = u(x,y) + iv(x,y)$ is a complex Function, which is differentiable in the neighbourhood of z itself (i.e analytic)

$$u_x = v_y \text{ and } u_y = -v_x$$

These equation is called C-R eqⁿ.

proof:

Let $F(z) = u(x,y) + i.v(x,y)$ be a analytic function, then this function is differentiable in the neigh. of z itself.

so, $$F'(z) = \lim_{\Delta z \to 0} \frac{F(z+\Delta z) - F(z)}{\Delta z}$$

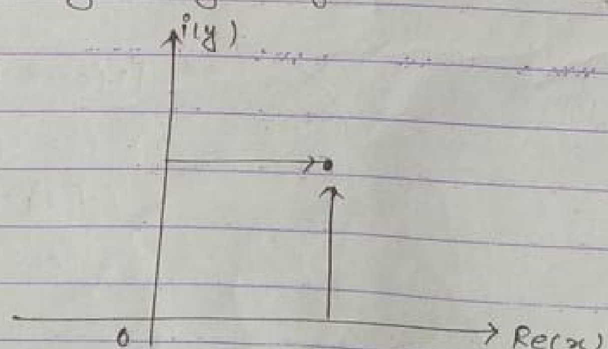
since, $z = x + iy$

$$\Delta z = \Delta x + i\Delta y$$

$$F(z+\Delta z) = u(x+\Delta x, y+\Delta y) + iv(x+\Delta x, y+\Delta y)$$

$$F'(z) = \lim_{\Delta z \to 0} \frac{u(x+\Delta x, y+\Delta y) + i.v(x+\Delta x, y+\Delta y) - u(x,y) - iv(x,y)}{\Delta x + i\Delta y} \quad (1)$$

There are infinitely many ways to become Δz zero.



case I: consider $\Delta x = 0$, $i\Delta y = 0$

Then eqⁿ (1) becomes,

$\Delta z = \Delta x + i\Delta y$

$$F'(z) = \lim_{i\Delta y \to 0} \frac{u(x, y+\Delta y) + iv(x, y+\Delta y) - u(x,y) - iv(x,y)}{i\Delta y}$$

$$= \frac{u_y + iv_y}{i}$$

$$= -iu_y + v_y \quad \text{---(II)}$$

case II : consider $\Delta y = 0$, $\Delta x \to 0$ then.

eqⁿ ① becomes,

$$F'(z) = \lim_{\Delta x \to 0} \frac{u(x+\Delta x, y) + i \cdot v(x+\Delta x, y) - u(x,y) - iv(x,y)}{\Delta x}$$

$$F'(z) = u_x + iv_x \quad \text{---(III)}$$

since, the function is analytic so eqⁿ (II) & (III)

$$\therefore u_x = v_y$$

$$\& \; u_y = -v_x$$

C-R equations in polar form:

we have C-R equation:

$$u_x = v_y \quad \& \quad u_y = -v_x$$

set $x = r\cos\theta \qquad y = r\sin\theta$

so that, $r^2 = x^2 + y^2$ and $\theta = \tan^{-1}(y/x)$

$$\frac{\partial y}{\partial x} = \frac{x}{r}, \quad \frac{\partial r}{\partial y} = y/r$$

$$\frac{\partial \theta}{\partial y} = \frac{1}{1 + y^2/x^2} \cdot 1/x$$

$$= \frac{x}{x^2 + y^2} = \frac{x}{r^2}$$

$$= \frac{1}{r} \cdot \frac{x}{r}$$

$$= \frac{1}{r} \cdot \frac{\partial r}{\partial x}$$

now,

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r} \cdot \frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta} \cdot \frac{\partial \theta}{\partial x}$$

$$= \frac{\partial u}{\partial r} \cdot \frac{x}{r} + \frac{\partial u}{\partial \theta}\left(-\frac{1}{r} \cdot \frac{\partial r}{\partial y}\right)$$

$$= \frac{\partial u}{\partial r} \cos\theta - \frac{1}{r} \cdot \frac{\partial u}{\partial \theta} \cdot \frac{\partial r}{\partial y}$$

$$= \frac{\partial u}{\partial r} \cos\theta - \frac{1}{r} \cdot \frac{\partial u}{\partial \theta} \cdot y/r$$

$$u_x = \frac{\partial u}{\partial r} \cos\theta - \frac{1}{r} \frac{\partial u}{\partial \theta} \cdot \sin\theta$$

$$u_y = \frac{\partial u}{\partial y} = \frac{\partial u}{\partial r} \cdot \frac{\partial r}{\partial y} + \frac{\partial u}{\partial \theta} \cdot \frac{\partial \theta}{\partial y}$$

$$= \frac{\partial u}{\partial r}\sin\theta + \frac{\partial u}{\partial \theta}\cdot\frac{\cos\theta}{r}$$

Similarly,

$$v_x = \frac{\partial v}{\partial r}\cos\theta - \frac{\partial v}{\partial \theta}\frac{\sin\theta}{r}$$

$$v_y = \frac{\partial v}{\partial r}\sin\theta + \frac{\partial v}{\partial \theta}\cdot\frac{\cos\theta}{r}$$

Since,

$$u_x = v_y$$

$$\frac{\partial u}{\partial r}\cos\theta - \frac{1}{r}\frac{\partial u}{\partial \theta}\cdot\sin\theta = \frac{\partial v}{\partial r}\sin\theta + \frac{\partial v}{\partial \theta}\frac{\cos\theta}{r}$$

and $u_y = -v_x$

$$\frac{\partial u}{\partial r}\cdot\sin\theta + \frac{\partial u}{\partial \theta}\cdot\frac{\cos\theta}{r} = \frac{\partial v}{\partial \theta}\frac{\sin\theta}{r} - \frac{\partial v}{\partial r}\cos\theta$$

solving ⓐ and ⓑ

$$\left.\begin{aligned} \frac{\partial u}{\partial r} &= \frac{1}{r}\cdot\frac{\partial v}{\partial \theta} \\ \frac{\partial u}{\partial \theta} &= -r\cdot\frac{\partial u}{\partial r} \end{aligned}\right\} \text{ - C-R equation in polar form}$$

Check the analytical or not:

$f(x) = z^n$

$= (re^{i\theta})^n$

$= r^n e^{i\theta n}$

$= r^n(\cos n\theta + i\sin n\theta)$

$u = r^n\cos n\theta$

$v = r^n\sin n\theta$

$$\frac{\partial u}{\partial r} = n r^{n-1} \cos n\theta$$

$$\frac{\partial v}{\partial r} = n r^{n-1} \sin n\theta$$

$$\frac{\partial u}{\partial \theta} = - r^{n} \cdot n \sin n\theta$$

$$\frac{\partial v}{\partial \theta} = r^{n} \cdot n \cos n\theta$$

$$\frac{\partial u}{\partial r} = n r^{n-1}$$

$$= \int_c x\,dx + i\int_c x\,dy$$

$$= \int_1^3 x\,dx + i\int_1^3 x\,\frac{dx}{2}$$

$$= \left[\frac{x^2}{2}\right]_1^3 + \frac{i}{2}\left[\frac{x^2}{2}\right]_1^3$$

$$= \left(\frac{9}{2}-\frac{1}{2}\right) + \frac{i}{2}\left(\frac{9}{2}-\frac{1}{2}\right)$$

$$= 4+2i$$

*. Simply connected domain:

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*. Cauchy's integral theorem/ Cauchy's fundamental theorem:

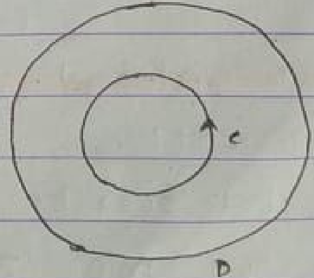
Statement:

if $f(z)$ is analytic in a simply connected domain D, for every simple closed path c in D, then.

$$\oint_c f(z)\,dz = 0$$

P.f:

Let $f(z)$ be analytic function in a simply connected domain D.



we have,

$$f(z) = u+iv$$
$$z = x+iy$$
$$dz = dx+i\,dy$$

now,

$$\oint_c f(z)\,dz = \oint_c (u+iv)(dx+i\,dy) = \oint (u\,dx - v\,dy + i\oint_c (v\,dx + u\,dy) \quad \text{---} (1)$$

Since $f(z)$ is analytic so it is continuous. Then by Greens theor in plane.

$$\int_c u\,dx - v\,dy = \iint_R \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy$$

$$\int_c \phi\,dx + \psi\,dy = \iint_R \left(\frac{\partial \psi}{\partial x}\right.$$

dx

& $$\int_c v\,dx + u\,dy = \iint_R \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dx\,dy$$

As $f(z)$ is analytic function Then u & v statisfy CR eq^n.

$\therefore u_x = v_y$ & $u_y = -v_x$

So, $\oint_c u\,dx - v\,dy = 0$ & $\oint_c v\,dx + u\,dy = 0$

$\therefore$ Equation ① becomes

$$\oint_c f(z)\,dz = 0$$

proved

Q. $\oint_c f(z)\,dz$, where $f(z) = \tan z/2$ and c is unit circle in counterclockwise direction.

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Cauchy's Integral formula:

statement:

Let $f(z)$ be analytic in a simply connected domain D. Then for any point z_0 in D and any simple closed path c in D that encloses z_0 such that,

$$\oint_c \frac{f(z)}{z - z_0}\,dz = 2\pi i f(z_0)$$

proof:

let $f(z)$ is analytic on c

also, $\frac{f(z)}{z - z_0}$ is also analytic on c except at z_0

Then form a circle C_1 centre at z_0 with small radius r.

Then $\frac{f(z)}{z-z_0}$ is analytic in the region bounded by C & C_1 by cauchy's theorem in multiple connected domain

$$\oint_C \frac{f(z)}{z-z_0} dz = \oint_{C_1} \frac{f(z)}{z-z_0} dz$$

set. $z = z_0 + re^{i\theta}$ on C_1

on differentiating,

$$dz = r.ie^{i\theta} d\theta$$

$$\therefore \oint \frac{f(z)\,dz}{z-z_0} = \int_0^{2\pi} \frac{f(z_0 + re^{i\theta})}{\cancel{re^{i\theta}}} . \cancel{r}i\cancel{e^{i\theta}} d\theta$$

$$= i \int_0^{2\pi} f(z_0 + re^{i\theta}) d\theta$$

As $r \longrightarrow 0$. we get,

$$\oint_C \frac{f(z)}{z-z_0} dz = i \int_0^{2\pi} f(z_0) d\theta$$

$$= if(z_0) \int_0^{2\pi} \cancel{f(z_0)} d\theta$$

$$= if(z_0) \left[\theta\right]_0^{2\pi}$$

$$= 2\pi if(z_0) \left[\theta\right]_0^{2\pi}$$

$$= 2\pi if(z_0)$$

$$\therefore \oint_C f$$

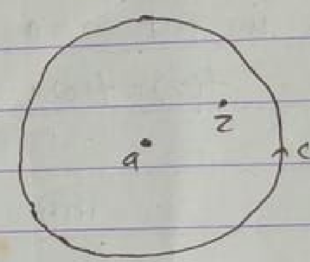
Sequence & Series:

Taylor series expansion:-

Taylor's Theorem: let $f(z)$ be an analytic function in the circle c, of radius R and centre 'a'. Then for any point z inside c, $f(z)$ can be expressed as,

$$f(z) = f(a) + (z-a) f'(a) + \frac{(z-a)^2}{2!} f''(a) + \cdots + \frac{(z-a)^n}{n!} f^n(a) + \cdots$$

Pf:



Let $f(z)$ is analytic function in circle C of centre a and radius R. Take any point z inside C. Draw a circle C_1 of centre a ~~the~~ that encloses z.

Take a point w on C_1.

Now, $$\frac{1}{w-z} = \frac{1}{w-a+a-z} = \frac{1}{w-a-(z-a)} = \frac{1}{(w-a)\left(1-\frac{z-a}{w-a}\right)}$$

$(1-x)^{-1} = 1 + x + x^2 + \cdots$

$$\therefore \frac{1}{w-z} = \frac{1}{(w-a)}\left[1 - \frac{z-a}{w-a}\right]^{-1}$$

$$= \frac{1}{w-a}\left[1 + \frac{z-a}{w-a} + \left(\frac{z-a}{w-a}\right)^2 + \cdots + \left(\frac{z-a}{w-a}\right)^n + \cdots\right]$$

$$= \frac{1}{w-a} + \frac{z-a}{(w-a)^2} + \frac{(z-a)^2}{(w-a)^3} + \cdots + \frac{(z-a)^n}{(w-a)^{n+1}} + \cdots \quad \text{—(1)}$$

Since. so series (1) is absolutely convergent and hence each term is integrable. Multiplying (1) by ~~the~~ $f(w)$ and integrating with respect to w over the circle C_1.

$$\oint_{C_1} \frac{f(w)}{w-z}\,dw = \oint_{C_1} \frac{f(w)}{w-a}\,dw + (z-a)\oint_{C_1} \frac{f(w)}{(w-a)^2}\,dw + \cdots (z-a)^n \oint_C \frac{f(w)}{(w-a)^{n+1}}\,dw.$$

By using cauchy integral formula.

$$2\pi i f(z) = 2\pi i f(a) + \frac{2\pi i}{1!}(z-a)f'(a) + \cdots + \frac{(z-a)^n 2\pi i f^n(a)}{n!}$$

$$\therefore f(z) = f(a) + (z-a)f'(a) + \frac{(z-a)^2}{2!}f''(a) + \cdots + \frac{(z-a)^n f^n(a)}{n!} \quad \text{--- (2)}$$

$$f(z) = \sum_{n=0}^{\infty} \frac{(z-a)^n f^n(a)}{n!}$$

if we put $a=0$ on (2) we get

$$f(z) = f(0) + zf'(0) + \frac{z^2}{2!}f''(0) + \cdots + \frac{z^n}{n!}f^n(0) + \cdots$$

This series is called Maclaurin's series.

S. I. Formulae.

1. $\frac{1}{1-x} = 1 + x + x^2 + \cdots$

2. $\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n$

3. $\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$

4. $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$

5. $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

$$e^{-z} = \sum_{n=0}^{\infty} \frac{(-1)^n z^n}{n!}$$

Z- transform:

Z-transform of a function f(t) of variable t on a discrete time period T is denoted by $z\{f(t)\}$ or $F(z)$ and is defined by $z\{f(t)\} = \sum_{n=0}^{\infty} f(t) z^{-n} = \sum_{n=0}^{\infty} f(nT) z^{-n}$

$f(t)$ → variable; nT → Time period (non -ve)

example:

Find $z\{1\}$.

Soln: $z\{1\} = \sum_{n=0}^{\infty} 1 \cdot z^{-n} = 1 + \frac{1}{z} + \frac{1}{z^2} + \cdots = \frac{1}{1-\frac{1}{z}} = \frac{z}{z-1}$

$$z\{a^n\} = \sum_{n=0}^{\infty} a^n \cdot z^{-n} = 1 + \frac{a}{z} + \left(\frac{a}{z}\right)^2 + \cdots = \frac{1}{1-\frac{a}{z}} = \frac{z}{z-a}$$

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First shifting Theorem:

statement: if $z\{f(t)\} = F(z)$, then

$$z\{e^{-at} f(t)\} = F(ze^{aT}) = \{F(z)\}_{z \to ze^{aT}}$$

proof: we have,

$$z\{f(t)\} = F(z) = \sum_{n=0}^{\infty} f(t) z^{-n} = \sum_{n=0}^{\infty} f(nT) z^{-n}$$

Now,

$$z\{e^{-at} f(t)\} = \sum_{n=0}^{\infty} e^{-at} f(t) z^{-n}$$

Replacing t by nT

$$= \sum_{n=0}^{\infty} e^{-anT} f(nT) z^{-n}$$

$$= \sum_{n=0}^{\infty} f(nT) \left(e^{aT} z\right)^{-n}$$

$$= F(e^{aT} z) = \{F(z)\}_{z \to ze^{aT}}$$

proved

skew lines - parallel नि हुँदैन

12 intersect " "

Second Shifting Theorem:

statement: if $Z\{f(t)\} = F(z)$, then

$$Z\{f(t+T)\} = z[F(z) - f(0)]$$

<u>proof:</u> we have,

$$Z\{f(t)\} = F(z) = \sum_{n=0}^{\infty} f(t) z^{-n} = \sum_{n=0}^{\infty} f(nT) z^{-n}$$

now,

$$z\{f(t+T)\} = \sum_{n=0}^{\infty} f(t+T) z^{-n}$$

Replacing t by nT,

$$= \sum_{n=0}^{\infty} f(nT+T) z^{-n}$$

$$= \sum_{n=0}^{\infty} f[(n+1)]T z^{-n}$$

put $n+1 = k$ $\left(\begin{array}{l} n=\infty, k=\infty \\ n=0, k=1 \end{array}\right)$

$$= \sum_{k=1}^{\infty} f(kT) z^{-(k-1)} \rightarrow z^{-k} z^{1}$$

$$= z \sum_{k=1}^{\infty} f(kT) z^{-k}$$

$$= z\left\{ \sum_{k=1}^{\infty} f(kT) z^{-k} + f(0) - f(0) \cancel{z^{-k}} \right\}$$

$$= z\left\{ \sum_{k=0}^{\infty} f(kT) z^{-k} - f(0) \right\}$$

$$= z[F(z) - f(0)]$$ proved ✓.

Show that

$$Z\{t^k\} = -zT\frac{d}{dz}\left[Z\{t^{k-1}\}\right]$$

<u>proof:</u> we have,

$$Z\{t^k\} = \sum_{n=0}^{\infty} t^k z^{-n} = \sum_{n=0}^{\infty} (nT)^k z^{-n}$$

$$Z\{t^{k-1}\} = \sum_{n=0}^{\infty} t^{k-1} z^{-n} = \sum_{n=0}^{\infty} (nT)^{k-1} z^{-n}$$

Now,

$$\frac{d}{dz}\left[Z\{t^{k-1}\}\right] = \frac{d}{dz}\left[\sum_{n=0}^{\infty} (nT)^{k-1} z^{-n}\right]$$

$$= \sum_{n=0}^{\infty} (nT)^{k-1} (-n).\, z^{-n-1}$$

$$= -(zT)^{-1} \sum_{n=0}^{\infty} (nT)^k z^{-n}$$

$$\frac{d}{dz}\left[Z\{t^{k-1}\}\right] = -(zT)^{-1} Z\{t^k\}$$

$$\therefore -zT\frac{d}{dz}\left[Z\{t^{k-1}\}\right] = Z\{t^k\}$$

Q. Find $Z\{t^2\}$

$$Z\{t^2\} = \sum_{n=0}^{\infty} t^2 z^{-n} = \sum_{n=0}^{\infty} (nT)^k z^{-n}$$

$$\frac{d}{dz}\left[Z\{t^{2-1}\}\right] = \sum_{n=0}^{\infty} t^{2-1} z^{-n} = \sum_{n=0}^{\infty} (nT)^{2-1} z^{-n}$$

$$= \sum_{n=0}^{\infty} (nT)^1 z^{-n}$$

Initial value theorem:

If $Z\{f(t)\} = F(z)$, then

$$\lim_{t\to 0} f(t) = \lim_{z\to\infty} F(z)$$

proof:

$$F(z) = Z\{f(t)\} = \sum_{n=0}^{\infty} f(t) z^{-n} = \sum_{n=0}^{\infty} F(nT) z^{-n}$$

$$= f(0) + f(T)z^{-1} + f(2T)z^{-2} + \cdots$$

Now,

$$\lim_{z\to\infty} F(z) = \lim_{z\to\infty}\left[f(0) + \frac{f(T)}{z} + \frac{f(2T)}{z^2} + \cdots\right] = f(0)$$

$$\lim_{z\to\infty} F(z) = \lim_{t\to 0} f(t) \quad \text{proved}$$

Final value Theorem:

If $Z\{f(t)\} = F(z)$, then

$$\lim_{t\to\infty} f(t) = \lim_{z\to 1}(z-1)F(z)$$

proof: we have,

$$Z\{f(t)\} = F(z) = \sum_{n=0}^{\infty} f(t) z^{-n} = \sum_{n=0}^{\infty} f(nT) z^{-n}$$

$$\& \; Z\{f(t+T)\} = z[F(z) - F(0)] \quad [\because \text{2nd shifting theorem}]$$

Now,

$$zF(z) - zf(0) - f(z) = Z\{f(t+T)\} - Z\{f(t)\}$$

$$= \sum_{n=0}^{\infty} f(t+T) z^{-n} - \sum_{n=0}^{\infty} f(t) z^{-n}$$

$$= \sum_{n=0}^{\infty} f(nT+T) z^{-n} - \sum_{n=0}^{\infty} f(nT) z^{-n}$$

$$= \sum_{n=0}^{\infty} [f(nT+T) - f(nT)] z^{-n}$$

$$= \lim_{n\to\infty} [f(T) + f(2T) + \dots + f[(n+1)T] - f(0) - f(T) - \dots - f(nT)]$$

$$= \lim_{n\to\infty} [f(n+1)T - F(0)]$$

$$= \lim_{n\to\infty} [f(n+1)T - f(0)]$$

i.e $\lim_{z\to 1} [zF(z) - F(z)] - \cancel{f(0)} = \lim_{n\to\infty} f(n+1)T - \cancel{f(0)}$

i.e. $\lim_{z\to 1} [(z-1)F(z)] = f(\infty)$

$\lim_{z\to 1} [(z-1)F(z)] = \lim_{t\to\infty} F(t)$ proved

Convolution of two functions:

let $f(t)$ and $g(t)$ of two functions of variable. Then convolution of f and g is denoted by $f*g$ and is given by

$$f*g = \sum_{K=0}^{\infty} f(KT)\, g[(n-K)T]$$

Convolution Theorem:

If $F(z)$ and $G(z)$ is a z- transform of a fx^n. $f(t)$ & $g(t)$ then, $z\{f*g\} = F(z)G(z)$

proof:- we have,

$$F(z) = Z\{f(t)\} = \sum_{n=0}^{\infty} f(nT) z^{-n}$$

$$G(z) = Z\{g(t)\} = \sum_{n=0}^{\infty} g(nT) z^{-n}$$

Now,

$$F(z)\,G(z) = \left[\sum_{n=0}^{\infty} f(nT) z^{-n}\right]\left[\sum_{n=0}^{\infty} g(nT) z^{-n}\right]$$

$$= \left[f(0) + f(T) z^{-1} + f(2T) z^{-2} + \cdots\right]\left[g(0) + g(T) z^{-1} + g(2T) z^{-2} + \cdots\right]$$

$$= f(0) g(0) + \left[f(T) g(0) + f(0) g(T)\right] z^{-1} + \left[f(0) g(2T) + f(T) g(T) + f(2T) g(0)\right] z^{-2} + \cdots$$

$$f * g = \sum_{k=0}^{\infty} f(kT)\, g[(n-k)T]$$

$$= \sum_{n=0}^{\infty}\left[\sum_{k=0}^{\infty} f(kT)\, g(n-k)T\right] z^{-n}$$

$$= \sum_{n=0}^{\infty} (f*g) z^{-n}$$

$$= Z\{f*g\}$$ proved.

Fourier Transform:

Fourier Series:

The Fourier series of function f(a) with period $2l$ is $f(x) =$

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{l}\right)x + b_n \sin\left(\frac{n\pi}{l}\right)x\,dx$$

~~where, a~~

Fourier series in complex form:

we have fourier series of function f(x) in period 2π is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx \quad \text{—— (1)}$$

we know,

$$\cos nx = \frac{e^{inx} + e^{-inx}}{2} \quad \text{and} \quad \sin nx = \frac{e^{inx} - e^{-inx}}{2i}$$

Then,

$$a_n \cos nx + b_n \sin nx = a_n \cdot \frac{1}{2}(e^{inx} + e^{-inx}) + b_n \cdot \frac{1}{2}(e^{inx} - e^{-inx})$$

$$= \frac{1}{2}(a_n - ib_n)e^{inx} + \frac{1}{2}(a_n + ib_n)e^{-inx}$$

put, $c_0 = \frac{a_0}{2}$, $c_n = \frac{1}{2}(a_n - ib_n)$, $c_{-n} = \frac{1}{2}(a_n + ib_n)$

Then eq^n. (1) becomes,

$$f(x) = c_0 + \sum_{n=1}^{\infty} (c_n e^{inx} + c_{-n} e^{-inx})$$

$$\therefore f x = \sum_{n=-\infty}^{\infty} c_n e^{inx}$$

which

Fourier Integral:

we have fourier series of function $f(x)$ with period $2l$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{l}\right)x + b_n \sin\left(\frac{n\pi}{l}\right)x \quad \text{—— (I)}$$

where, $a_0 = \frac{1}{l}\int_{-l}^{l} f(t)\,dt,$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(t)\cos\left(\frac{n\pi}{l}\right)t\,dt$$

$$\& \; b_n = \frac{1}{l}\int_{-l}^{l} f(t)\sin\left(\frac{n\pi}{2}\right)t\,dt$$

Replace, $\frac{n\pi}{l}$ by w_n. (i.e. $w_n = \frac{n\pi}{2}$). Then

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos w_n x + b_n \sin w_n x) \quad \text{—— (II)}$$

where,

$$a_n = \frac{1}{l}\int_{-l}^{l} f(t)\cos w_n t\,dt$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(t)\sin w_n t\,dt$$

Substituting a_0, a_n & b_n on (II),

$$f(x) = \frac{1}{2l}\int_{-l}^{l} f(t)\,dt + \frac{1}{l}\sum_{n=1}^{\infty} \cos w_n x \int_{-l}^{l} f(t)\cos w_n t\,dt + \sin w_n x \int_{-l}^{l} f(t)\sin w_n t\,dt. \quad \text{—— (III)}$$

Setting $\Delta w = w_{n+1} - w_n = \frac{(n+1)\pi}{2} - \frac{n\pi}{l} = \frac{\pi}{l}$

$$\Rightarrow \frac{1}{l} = \frac{\Delta w}{\pi}$$

Then eq^n 3 reduces to,

$$f(x) = \frac{1}{2l}\int_{-l}^{l} f(t)\,dt + \frac{1}{\pi}\sum_{n=1}^{\infty}\left[\cos w_n x\,\Delta w \int_{-l}^{l} f(t)\cos w_n t\,dt + \sin w_n x\,\Delta w \int_{-l}^{l} f(t)\sin w_n t\,dt\right] \quad \text{— (iv)}$$

This representation is valid for all l. So $\frac{1}{l} \to 0$ as $l \to \infty$. Suppose that $\lim_{l\to\infty} f(x)$ is integrable on x-axis. So replace Δw by dw and $\sum_{n=1}^{\infty}$ by $\int_0^{\infty}$

Now Eq^n 4 becomes,

$$f(x) = \frac{1}{\pi}\int_0^{\infty}\left[\cos wx \int_{-\infty}^{\infty} f(t)\cos wt\,dt + \sin wx \int_{-\infty}^{\infty} f(t)\sin wt\,dt\right] dw$$

$$= \int_0^{\infty}\left[A(w)\cos wx + B(w)\sin wx\right] dw$$

where,

$$A(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\cos wt\,dt$$

& $$B(w) = \frac{1}{\pi}\int_{-\infty}^{\pi} f(t)\sin wt\,dt$$

which is Fourier integral of $f(x)$.

Fourier cosine and sine integral:

we have, Fourier integral is,

$$F(x) = \int_0^{\infty}\left[A(w)\cos wx + B(w)\sin wx\right] dw \quad \text{— (1)}$$

where,

$$A(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\cos wt\,dt$$

$$B(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\sin wt\, dt$$

case I: If $f(x)$ is even function, then $B(w)=0$ Then ① becomes,

$f(x) = \int_0^{\infty} A(w)\cos wx\, dw$ is Fourier cosine integral.

where, $A(w) = 2\int_0^{\alpha} f(t)\cos wt\,dt$

case II: if $f(x)$ is odd fxn. then $A(w)=0$. Then eqn ① becomes

$f(x) = \int_0^{\infty} B(w)\sin wx\, dw$ is F.S.I of $f(x)$

where, $B(w) = 2\int_0^{\infty} f(t)\sin wt\, dt$.

Fourier integral in complex form:

we have, fourier integral of $f(x)$ is

$$F(x) = \int_0^{\infty} (A(w)\cos wx + B(w)\sin wx)\,dw \quad \text{——} ①$$

where, $A(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\cos wt\,dt$

$B(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\sin wt\, dt$

Substituting $A(w)$ and $B(w)$ on ①, we get.

$$f(x) = \frac{1}{\pi}\int_0^{\infty} \int_{-\infty}^{\infty} f(t)\,(\cos wt\cos wx + \sin wt \sin wx)\,dt$$

$$= \frac{1}{\pi}\int_0^{\infty}\int_{-\infty}^{\infty} f(t)\cos(wx - wt)\,dt\, dw$$