Unit 3: Signals and System

Signal

- A signal is a function of one or more variables that conveys information about some (usually physical) phenomenon.
- A signal is a set of information of data
  - Any kind of physical variable subject to variations represents a signal
  - Both the independent variable and the physical variable can be either scalars or vectors
    - Independent variable: time (t), space (x, x=[x1,x2], x=[x1,x2,x3])
    - Signal: Electrocardiography signal (EEG) 1D, voice 1D, music 1D
    - Images (2D), video sequences (2D+time), volumetric data (3D)

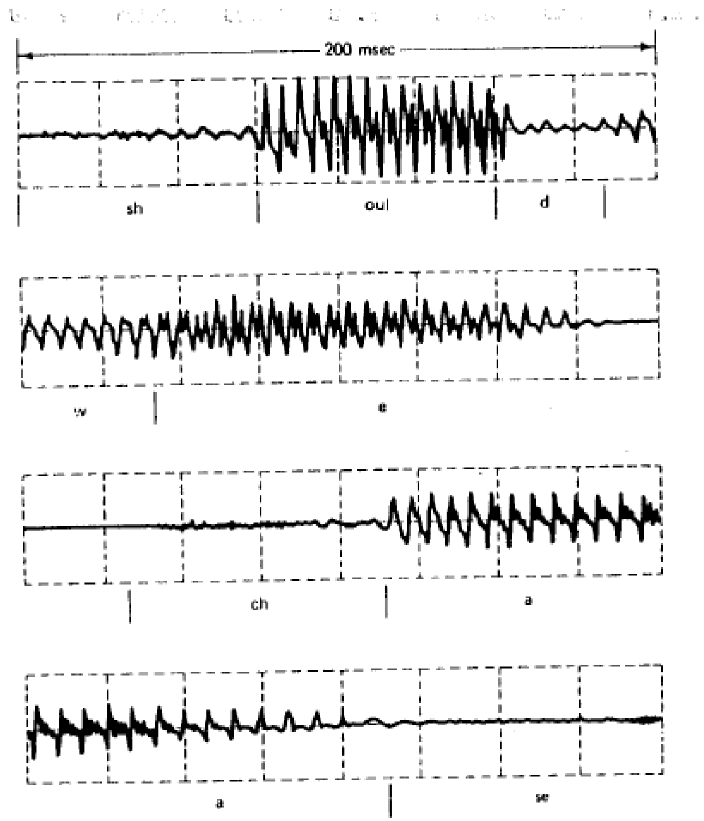


Figure 2.1 Example of a recording of speech. [Adapted from *Applications of Digital Signal Processing*, A. V. Oppenheim, ed. (Englewood Cliffs, N.J.: Prentice-Hall, Inc., 1978), p. 121.] The signal represents acoustic pressure variations as a function of time for the spoken words "should we chase." The top line of the figure corresponds to the word "should," the second line to the word "we," and the last two to the word "chase" (we have indicated the approximate beginnings and endings of each successive sound in each word).

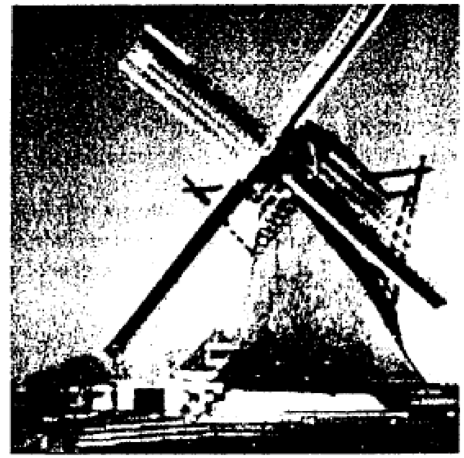


Figure 2.2 A monochromatic picture.

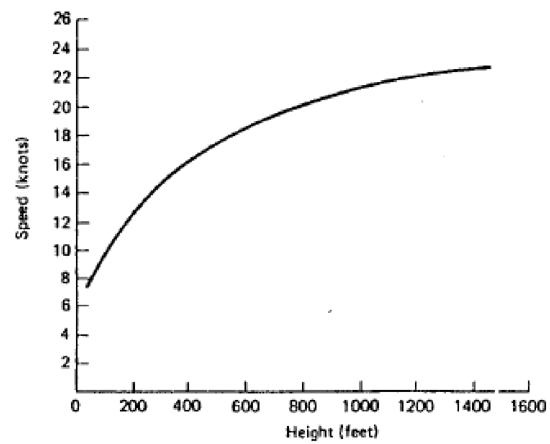


Figure 2.3 Typical annual average vertical wind profile. (Adapteu from Crawford and Hudson, National Severe Storms Laboratory Report', ESSA ERLTM-NSSL 48, August 1970.)

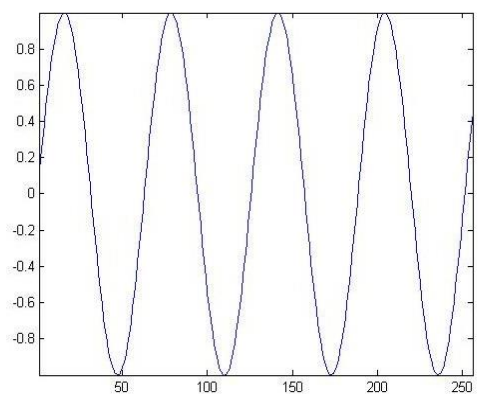
Classification of signals

- Continuous time – Discrete time
- Analog – Digital (numerical)
- Periodic – Aperiodic
- Energy – Power
- Deterministic – Random (probabilistic)
- Note
  - Such classes are not disjoint, so there are digital signals that are periodic of power type and others that are aperiodic of power type etc.
  - Any combination of single features from the different classes is possible

Continuous time – discrete time

- Continuous time signal: a signal that is specified for every real value of the independent variable
  - The independent variable is continuous, that is it takes any value on the real axis
  - The domain of the function representing the signal has the cardinality of real numbers
    - Signal ↔ f=f(t)
    - Independent variable ↔ time (t), position (x)
    - For continuous-time signals: $t \in \mathbb{R}$

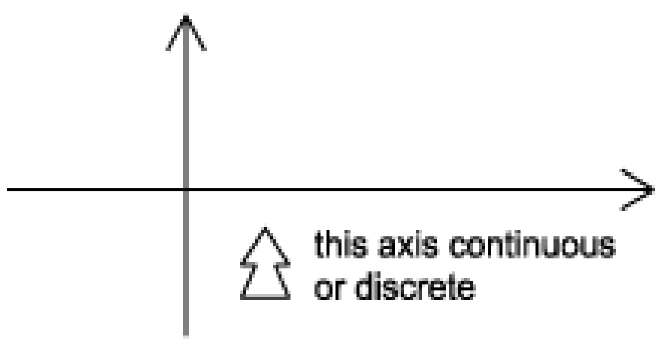
amplitude

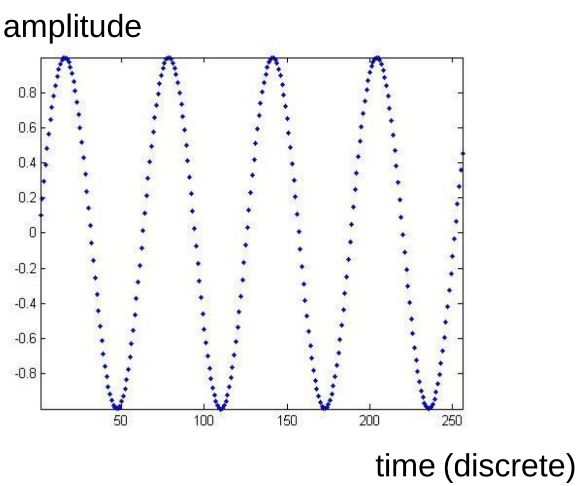


time

Continuous time – discrete time

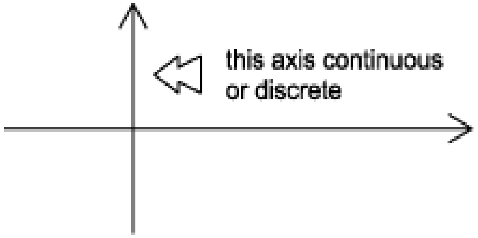
- Discrete time signal: a signal that is specified only for *discrete values* of the independent variable
  - It is usually generated by *sampling* so it will only have values at *equally spaced* intervals along the time axis
  - The domain of the function representing the signal has the cardinality of integer numbers
    - Signal ↔ f=f[n], also called "sequence"
    - Independent variable ↔ n
    - For discrete-time functions: $t \in \mathbf{Z}$





Analog - Digital

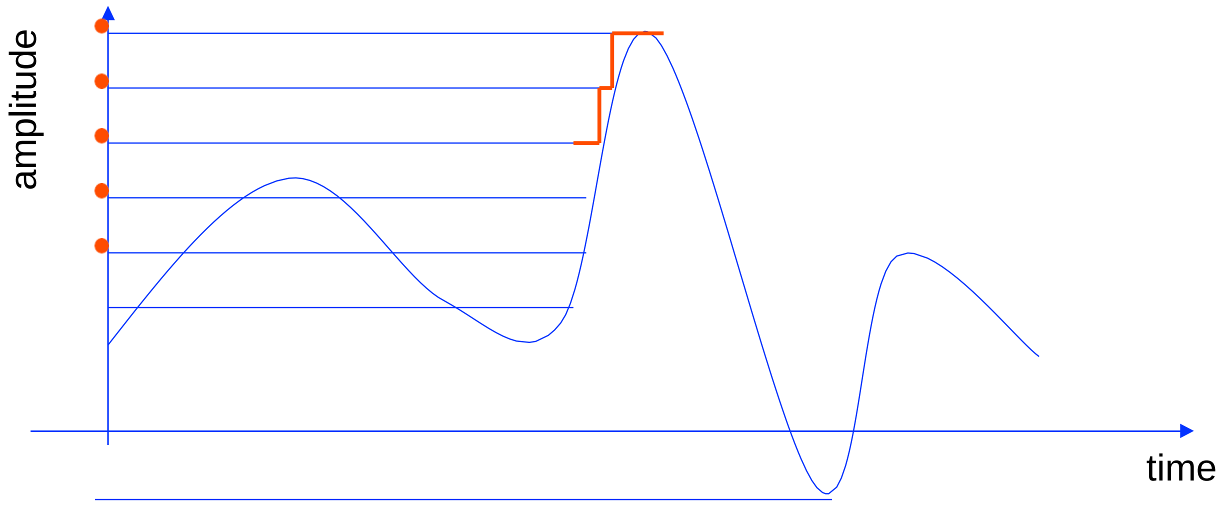
- **Analog signal**: signal whose amplitude can take on any value in a continuous range
  - The amplitude of the function f(t) (or f(x)) has the cardinality of real numbers
    - The difference between analog and digital is similar to the difference between continuous-time and discrete-time. In this case, however, the difference is with respect to the value of the function (y-axis)
  - Analog corresponds to a continuous y-axis, while digital corresponds to a discrete y-axis



- An analog signal can be both continuous time and discrete time

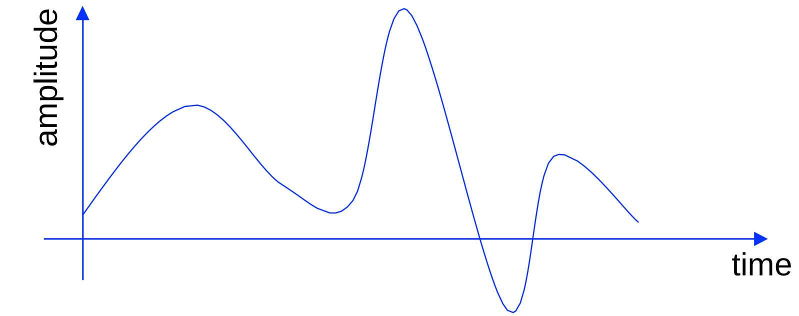
Analog - Digital

- **Digital signal**: a signal is one whose amplitude can take on only a finite number of values (thus it is quantized)
  - The amplitude of the function f() can take only a finite number of values
  - A digital signal whose amplitude can take only M different values is said to be M- ary
    - Binary signals are a special case for M=2



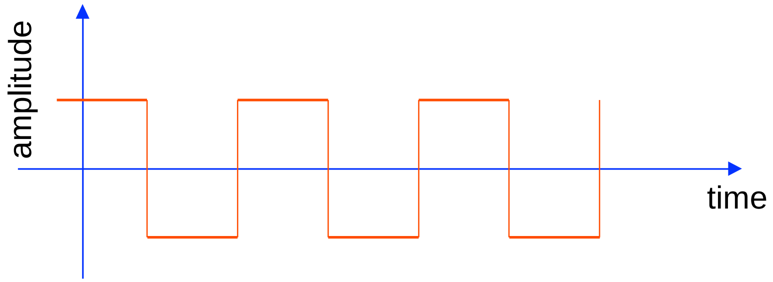
Example

– Continuous time
analog



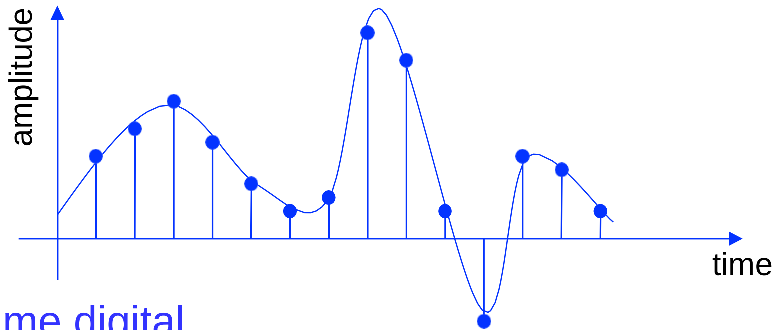
– Continuous time digital (or quantized)

- binary sequence, where the values of the function can only be one or zero.

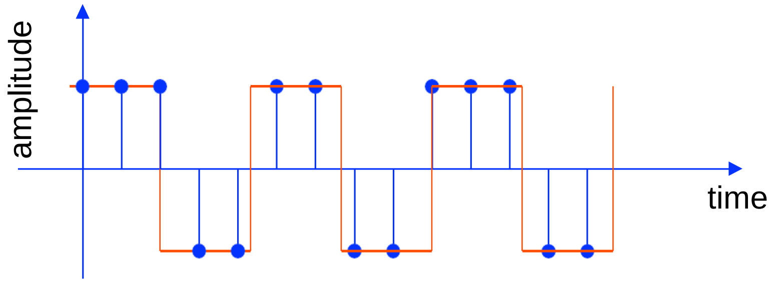


Example

- Discrete time analog



- Discrete time digital
  - binary sequence, where the values of the function can only be one or zero.



Periodic Signals

- A function x is said to be **periodic** with **period** T (or T**-periodic**) if, for some strictly-positive real constant T, the following condition holds:

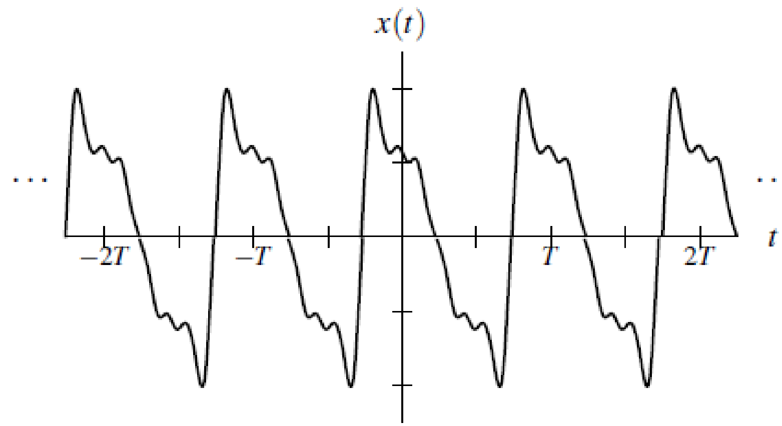
$$x(t) = x(t+T) \quad \text{for all } t.$$

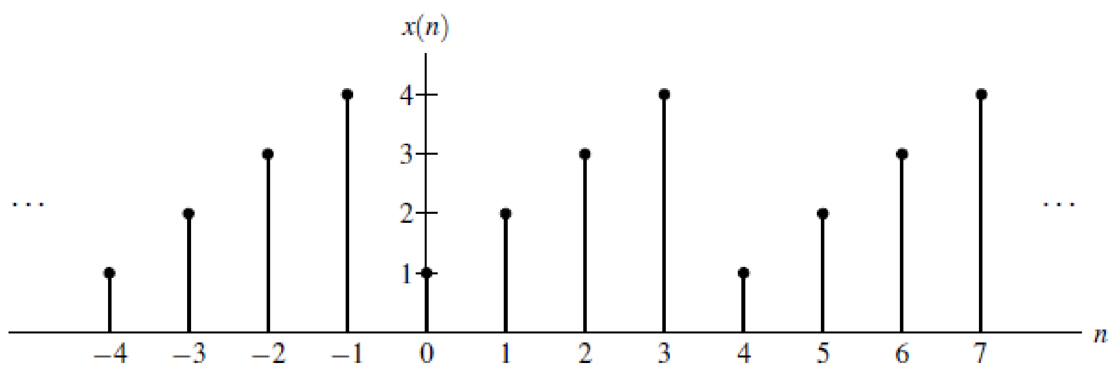
- A T-periodic function x is said to have **frequency** $\frac{1}{T}$ and **angular frequency** $\frac{2\pi}{T}$.
- A sequence x is said to be **periodic** with **period** N (or N**-periodic**) if, for some strictly-positive integer constant N, the following condition holds:

$$x(n) = x(n+N) \quad \text{for all } n.$$

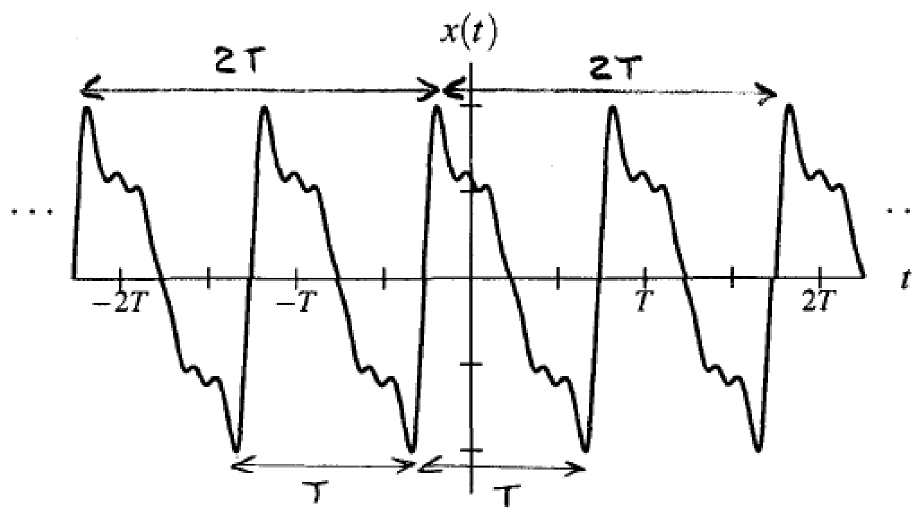
- An N-periodic sequence x is said to have **frequency** $\frac{1}{N}$ and **angular frequency** $\frac{2\pi}{N}$.
- A function/sequence that is not periodic is said to be **aperiodic**.

- Some examples of periodic signals are shown below.





- The period of a periodic signal is ***not unique***. That is, a signal that is periodic with period T is also periodic with period kT, for every (strictly) positive integer k.



- The smallest period with which a signal is periodic is called the **fundamental period** and its corresponding frequency is called the **fundamental frequency**.

Periodic - Aperiodic

- A signal f(t) is *periodic* if there exists a positive constant T_0 such that

$$f(t+T_0)=f(t) \qquad \forall t$$

  - The *smallest* value of T_0 which satisfies such relation is said the *period* of the function f(t)
  - A periodic signal remains unchanged when *time-shifted* of integer multiples of the period
  - Therefore, by definition, it starts at minus infinity and lasts forever

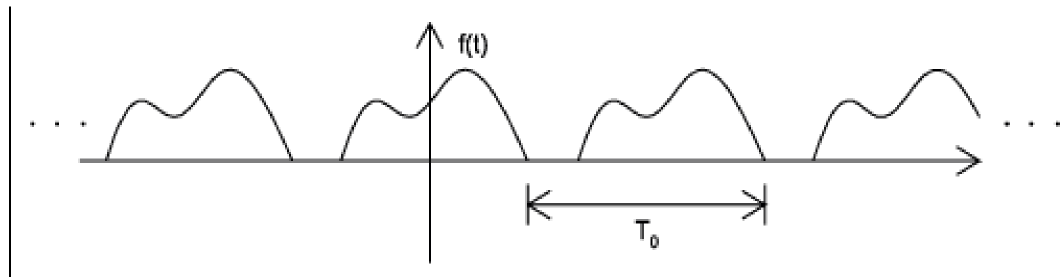
$$-\infty \leq t \leq +\infty \qquad t \in {}^\circ$$

$$-\infty \leq n \leq +\infty \qquad n \in \mathbf{Z}$$

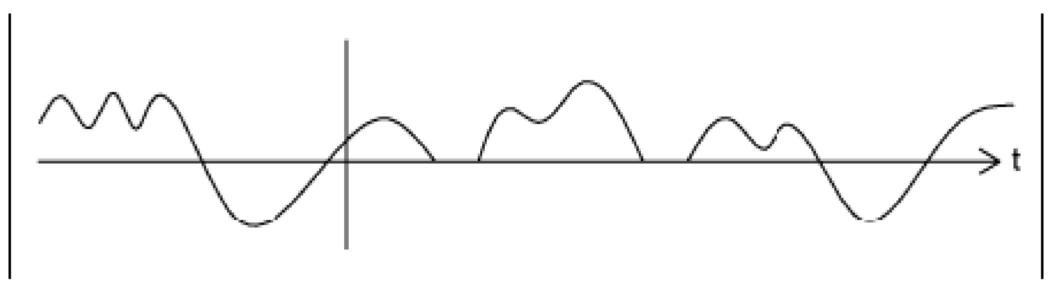
  - Periodic signals can be generated by *periodical extension*

Examples

- Periodic signal with period T_0

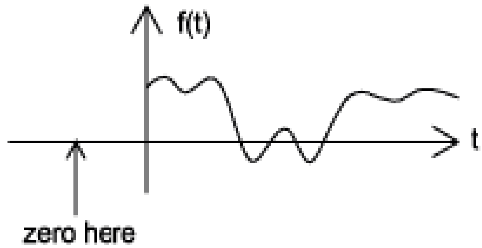


- Aperiodic signal



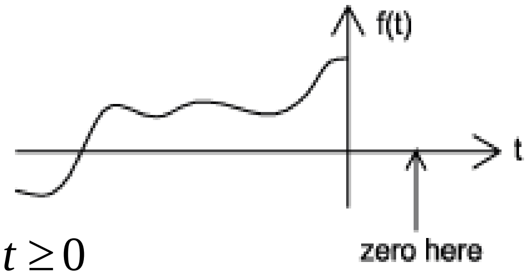
Causal and non-Causal signals

- *Causal* signals are signals that are *zero for all negative time (or spatial positions)*, while



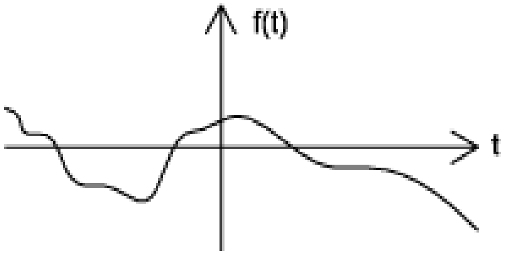
$$f(t) = 0 \qquad t < 0$$

- *Anticausal* are signals that are zero for all *positive time (or spatial positions)*.



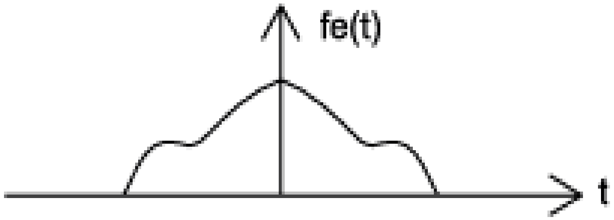
$$f(t) = 0 \qquad t \geq 0$$

- *Noncausal* signals are signals that have nonzero values in both positive and negative time

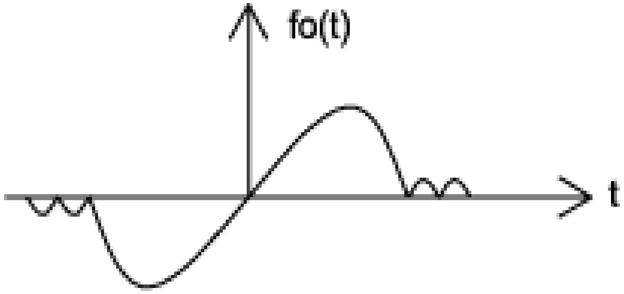


Even and Odd signals

- An even signal is any signal f such that f (t) = f (-t). Even signals can be easily spotted as they are symmetric around the vertical axis.



- An odd signal, on the other hand, is a signal f such that f (t)= - (f (-t))



Even Signals

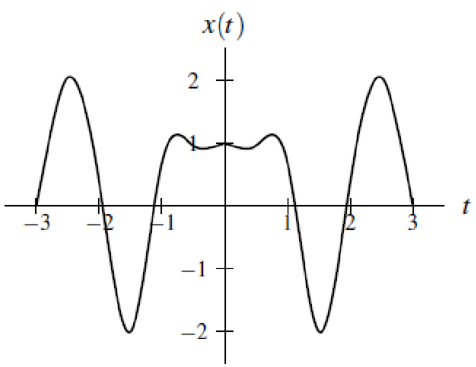
- A function x is said to be **even** if it satisfies

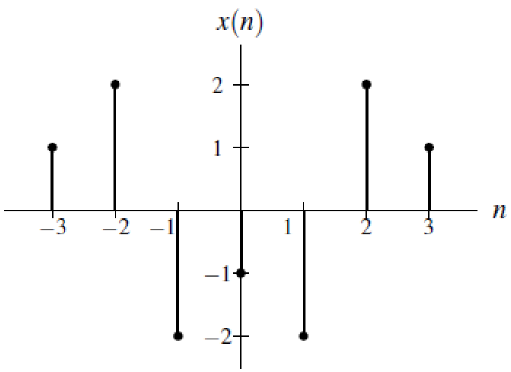
$$x(t) = x(-t) \quad \text{for all } t.$$

- A sequence x is said to be **even** if it satisfies

$$x(n) = x(-n) \quad \text{for all } n.$$

- Geometrically, the graph of an even signal is *symmetric* about the origin.
- Some examples of even signals are shown below.





Odd Signals

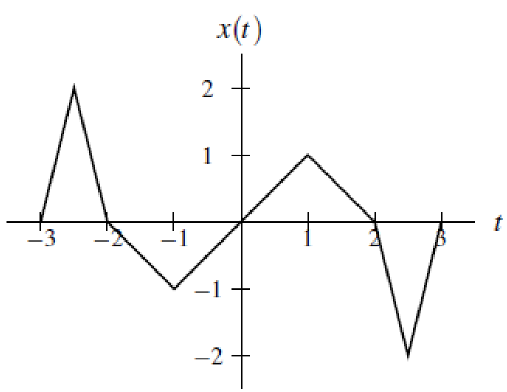
- A function x is said to be **odd** if it satisfies

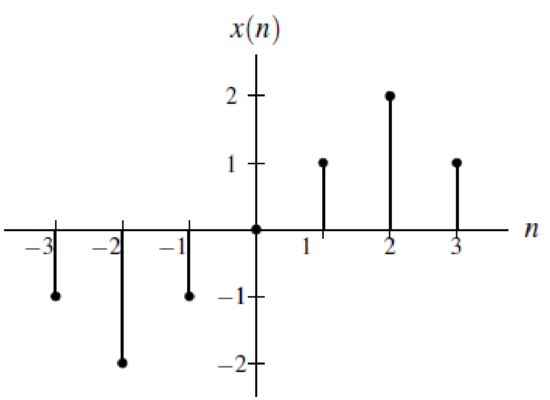
$$x(t) = -x(-t) \quad \text{for all } t.$$

- A sequence x is said to be **odd** if it satisfies

$$x(n) = -x(-n) \quad \text{for all } n.$$

- Geometrically, the graph of an odd signal is ***antisymmetric*** about the origin.
- An odd signal x must be such that $x(0) = 0$.
- Some examples of odd signals are shown below.





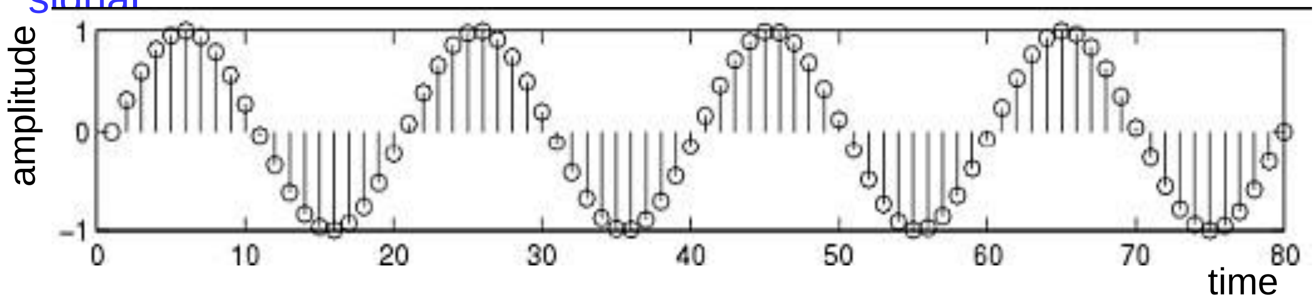
Deterministic - Probabilistic

- Deterministic signal: a signal whose physical description in known completely
- A deterministic signal is a signal in which each value of the signal is fixed and can be determined by a mathematical expression, rule, or table.
- Because of this the future values of the signal can be calculated from past values with complete confidence.
  - There is *no uncertainty* about its amplitude values
  - Examples: signals defined through a mathematical function or graph

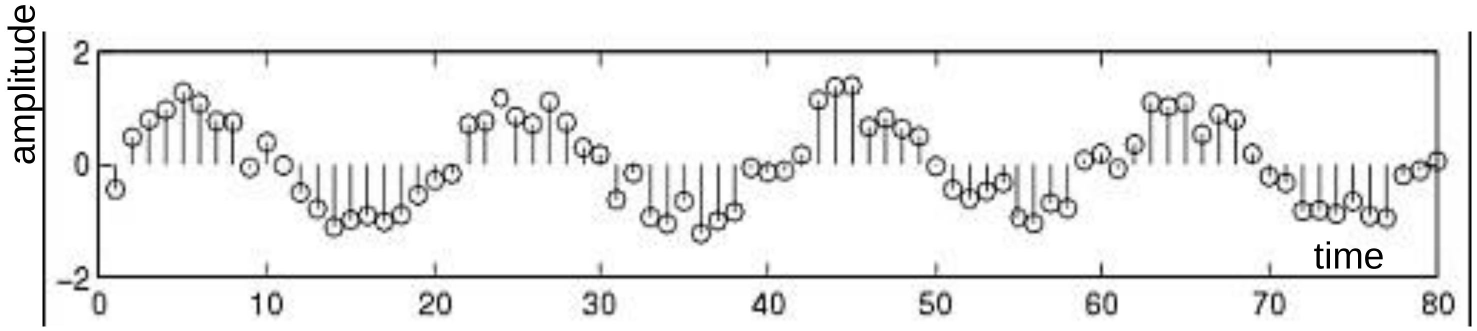
- Probabilistic (or random) signals: the amplitude values *cannot be predicted precisely* but are known only in terms of probabilistic descriptors
- The future values of a random signal cannot be accurately predicted and can usually only be guessed based on the averages of sets of signals
  - They are realization of a stochastic process for which a model could be available
  - Examples: EEG, evocated potentials, noise in CCD capture devices for digital cameras

Example

- Deterministic signal



- Random signal



Finite and Infinite length signals

- A finite length signal is non-zero over a finite set of values of the independent variable

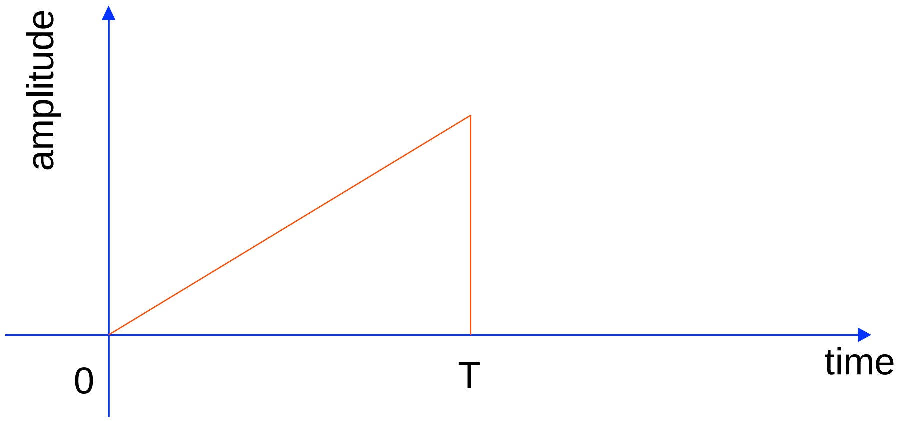
$$f = f(t), \ \forall t : t_1 \le t \le t_2$$

$$t_1 > -\infty, \ t_2 < +\infty$$

- An infinite length signal is non zero over an infinite set of values of the independent variable
  - For instance, a sinusoid $f(t)=\sin(\omega t)$ is an infinite length signal

Size of a signal: Norms

- "Size" indicates largeness or strength.
- The energy is represented by the area under the curve (of the squared signal)



Signal Energy and Power

- The **energy** E contained in the signal x is given by

$$E = \int_{-\infty}^{\infty} |x(t)|^2 \, dt.$$

- A signal with finite energy is said to be an **energy signal**.
- The **average power** P contained in the signal x is given by

$$P = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 \, dt.$$

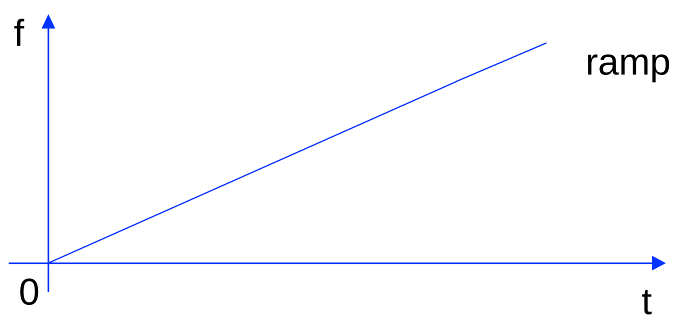
- A signal with (nonzero) finite average power is said to be a **power signal**.

Power - Energy

- The square root of the power is the root mean square (*rms*) value
  - This is a very important quantity as it is the most widespread measure of similarity/dissimilarity among signals
  - It is the basis for the definition of the Signal to Noise Ratio (SNR)

$$SNR = 20\log_{10}\left(\sqrt{\frac{P_{signal}}{P_{noise}}}\right)$$

  - It is such that a constant signal whose amplitude is =rms holds the same power content of the signal itself
- There exists signals for which neither the energy nor the power are finite



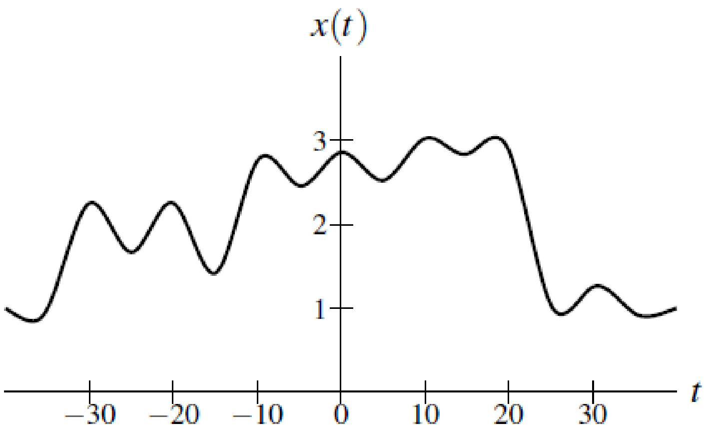
Energy and Power signals

- A signal with finite energy is an energy signal
  - Necessary condition for a signal to be of energy type is that the amplitude goes to zero as the independent variable tends to infinity
- A signal with finite and different from zero power is a power signal
  - The mean of an entity averaged over an infinite interval exists if either the entity is periodic or it has some statistical regularity
  - A power signal has infinite energy and an energy signal has zero power
  - There exist signals that are neither power nor energy, such as the ramp
- All practical signals have finite energy and thus are energy signals
  - It is impossible to generate a real power signal because this would have infinite duration and infinite energy, which is not doable.

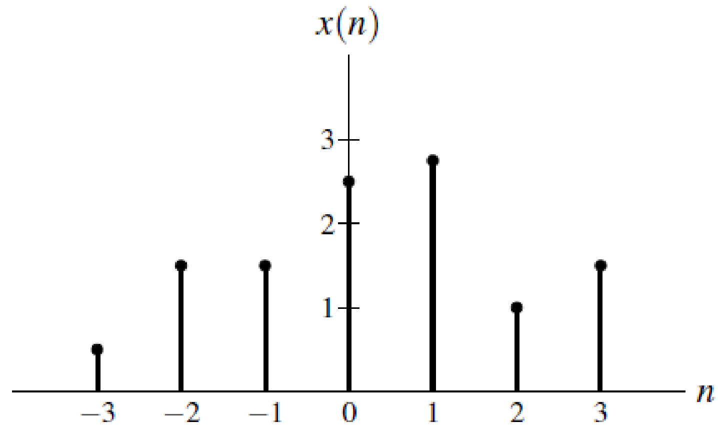
Classification of Signals: Summary

- Number of independent variables (i.e., dimensionality):
  - A signal with one independent variable is said to be one dimensional (e.g., audio).
  - A signal with more than one independent variable is said to be multi-dimensional (e.g., image).
- Continuous or discrete independent variables:
  - A signal with continuous independent variables is said to be continuous time (CT) (e.g., voltage waveform).
  - A signal with discrete independent variables is said to be discrete time (DT) (e.g., stock market index).
- Continuous or discrete dependent variable:
  - A signal with a continuous dependent variable is said to be continuous valued (e.g., voltage waveform).
  - A signal with a discrete dependent variable is said to be discrete valued (e.g., digital image).
- A continuous-valued CT signal is said to be analog (e.g., voltage waveform).
- A discrete-valued DT signal is said to be digital (e.g., digital audio).

Graphical Representation of Signals



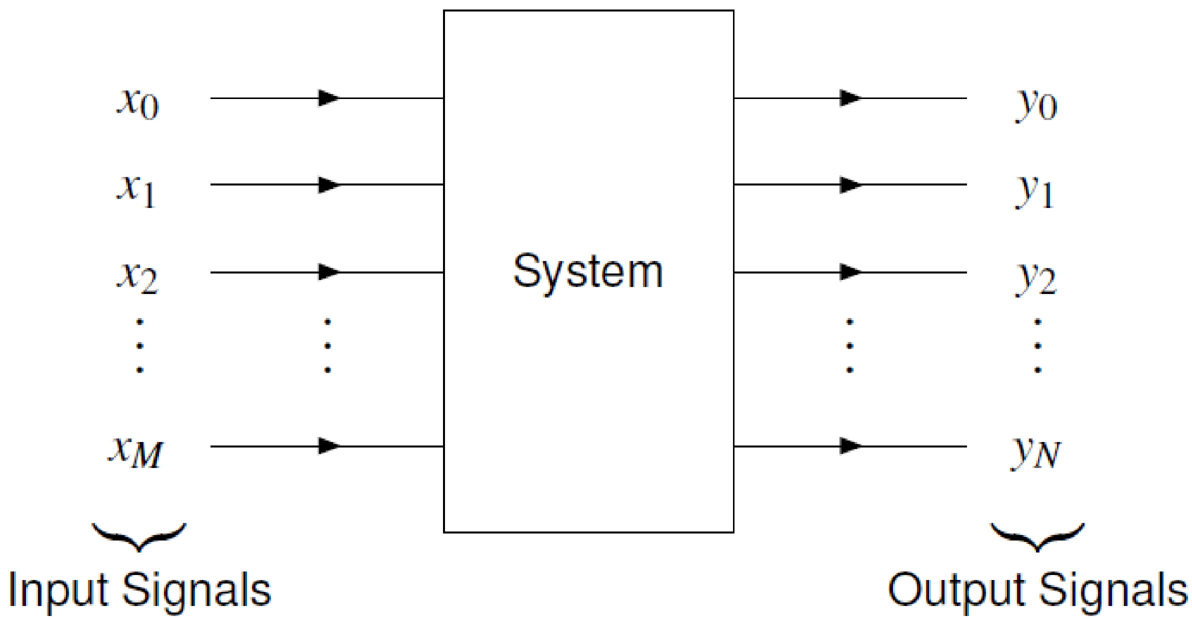
Continuous-Time (CT) Signal



Discrete-Time (DT) Signal

Systems

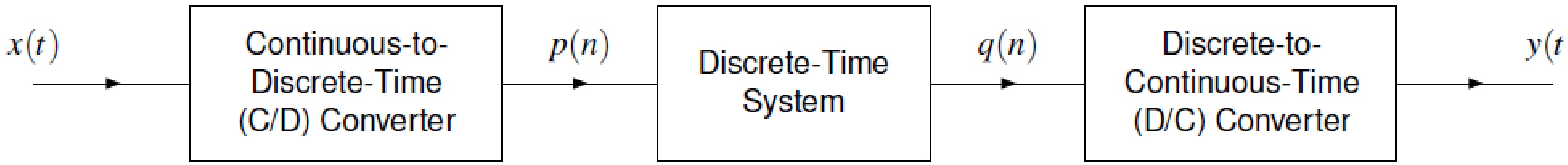
- A **system** is an entity that processes one or more input signals in order to produce one or more output signals.



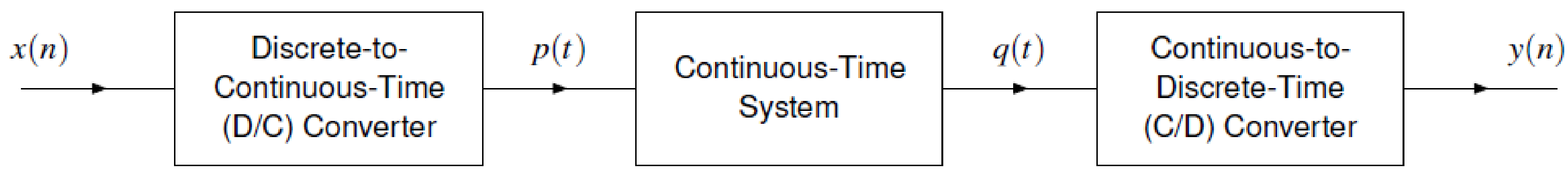
Classification of Systems

- Number of inputs:
  - A system with ***one*** input is said to be **single input (SI)**.
  - A system with ***more than one*** input is said to be **multiple input (MI)**.
- Number of outputs:
  - A system with ***one*** output is said to be **single output (SO)**.
  - A system with ***more than one*** output is said to be **multiple output (MO)**.
- Types of signals processed:
  - A system can be classified in terms of the ***types of signals*** that it processes.
  - Consequently, terms such as the following (which describe signals) can also be used to describe systems:
    - one-dimensional and multi-dimensional,
    - continuous-time (CT) and discrete-time (DT), and
    - analog and digital.
  - For example, a continuous-time (CT) system processes CT signals and a discrete-time (DT) system processes DT signals.

Signal Processing Systems

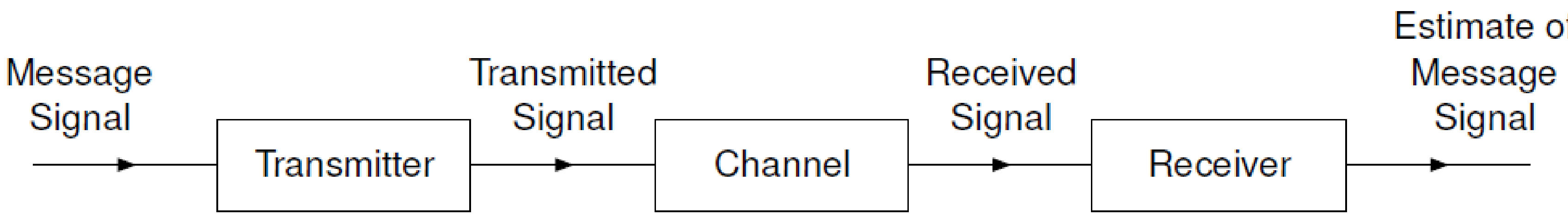


Processing a Continuous-Time Signal With a Discrete-Time System



Processing a Discrete-Time Signal With a Continuous-Time System

Communication Systems



General Structure of a Communication System

Continuous Time (CT) System

- A system with input x and output y can be described by the equation

$$y = \mathcal{H}\{x\},$$

  where $\mathcal{H}$ denotes an operator (i.e., transformation).
- Note that the operator $\mathcal{H}$ ***maps a function to a function*** (not a number to a number).
- Alternatively, we can express the above relationship using the notation

$$x \xrightarrow{\mathcal{H}} y.$$

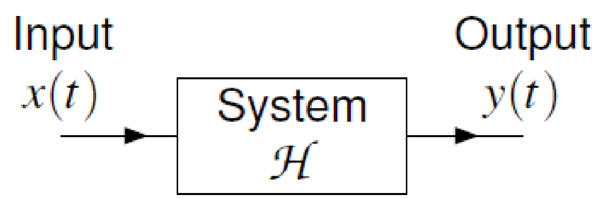
- If clear from the context, the operator $\mathcal{H}$ is often omitted, yielding the abbreviated notation

$$x \to y.$$

- Note that the symbols "$\to$" and "=" have ***very different*** meanings.
- The symbol "$\to$" should be read as "***produces***" (not as "equals").

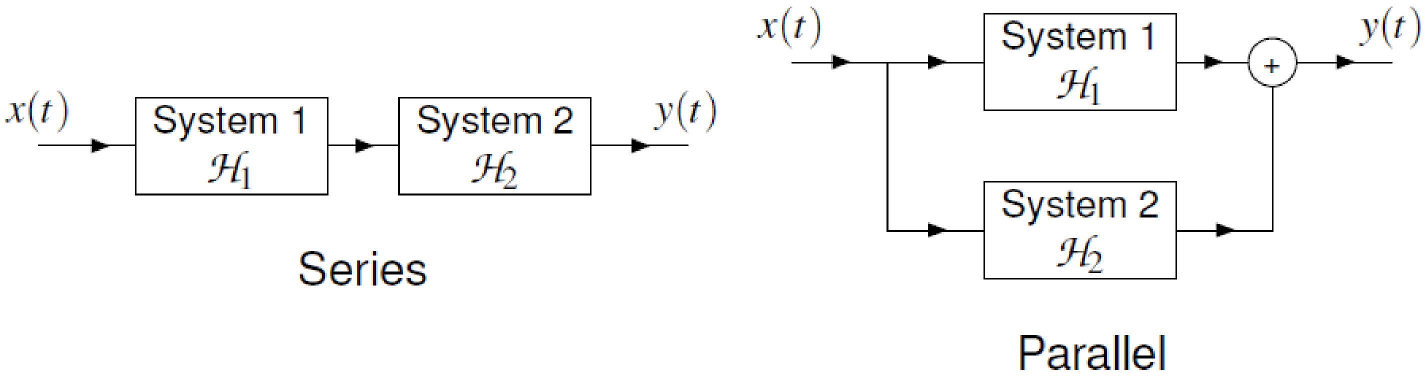
Block Diagram Representations

- Often, a system defined by the operator $\mathcal{H}$ and having the input x and output y is represented in the form of a ***block diagram*** as shown below.



Interconnection of Systems

- *Two basic ways* in which systems can be interconnected are shown below.



- A **series** (or **cascade**) connection ties the output of one system to the input of the other.
- The overall series-connected system is described by the equation

$$y = \mathcal{H}_2 \left\{ \mathcal{H}_1 \{x\} \right\}.$$

- A **parallel** connection ties the inputs of both systems together and sums their outputs.
- The overall parallel-connected system is described by the equation

$$y = \mathcal{H}_1\{x\} + \mathcal{H}_2\{x\}.$$

Properties of (CT) Systems: Memory and Causality

- A system with input x and output y is said to have **memory** if, for any real t_0, $y(t_0)$ depends on $x(t)$ for some $t \neq t_0$.
- A system that does not have memory is said to be **memoryless**.
- Although simple, a memoryless system is ***not very flexible***, since its current output value cannot rely on past or future values of the input.
- A system with input x and output y is said to be **causal** if, for every real t_0, $y(t_0)$ does not depend on $x(t)$ for some $t > t_0$.
- If the independent variable t represents time, a system must be causal in order to be ***physically realizable***.
- Noncausal systems can sometimes be useful in practice, however, since the independent variable ***need not always represent time***. For example, in some situations, the independent variable might represent position.

Inevitability

- The **inverse** of a system $\mathcal{H}$ is another system $\mathcal{H}^{-1}$ such that the combined effect of $\mathcal{H}$ cascaded with $\mathcal{H}^{-1}$ is a system where the input and output are equal.
- A system is said to be **invertible** if it has a corresponding inverse system (i.e., its inverse exists).
- Equivalently, a system is invertible if its input x can always be ***uniquely*** determined from its output y.
- Note that the invertibility of a system (which involves mappings between ***functions***) and the invertibility of a function (which involves mappings between ***numbers***) are ***fundamentally different*** things.
- An invertible system will always produce ***distinct outputs*** from any two ***distinct inputs***.
- To show that a system is ***invertible***, we simply find the ***inverse system***.
- To show that a system is ***not invertible***, we find ***two distinct inputs*** that result in ***identical outputs***.
- In practical terms, invertible systems are "nice" in the sense that their ***effects can be undone***.

Bounded-Input Bounded-Output (BIBO) Stability

- A system with input x and output y is **BIBO stable** if, for every bounded x, y is bounded (i.e., $|x(t)| < \infty$ for all t implies that $|y(t)| < \infty$ for all t).
- To show that a system is ***BIBO stable***, we must show that ***every bounded input*** leads to a ***bounded output***.
- To show that a system is ***not BIBO stable***, we only need to find a single ***bounded input*** that leads to an ***unbounded output***.
- In practical terms, a BIBO stable system is ***well behaved*** in the sense that, as long as the system input remains finite for all time, the output will also remain finite for all time.
- Usually, a system that is not BIBO stable will have ***serious safety issues***. For example, an iPod with a battery input of 3.7 volts and headset output of ∞ volts would result in one vaporized Apple customer and one big lawsuit.

Time Invariance (TI)

- A system $\mathcal{H}$ is said to be **time invariant (TI)** if, for every function x and every real number t_0, the following condition holds:

$$y(t-t_0) = \mathcal{H}x'(t) \quad \text{where} \quad y = \mathcal{H}x \quad \text{and} \quad x'(t) = x(t-t_0)$$

  (i.e., $\mathcal{H}$ ***commutes with time shifts***).
- In other words, a system is time invariant if a time shift (i.e., advance or delay) in the input always results only in an ***identical time shift*** in the output.
- A system that is not time invariant is said to be **time varying**.
- In simple terms, a time invariant system is a system whose behavior ***does not change*** with respect to time.
- Practically speaking, compared to time-varying systems, time-invariant systems are much ***easier to design and analyze***, since their behavior does not change with respect to time.

Additivity, Homogeneity, and Linearity

- A system $\mathcal{H}$ is said to be **additive** if, for all functions x_1 and x_2, the following condition holds:
$$\mathcal{H}(x_1 + x_2) = \mathcal{H}x_1 + \mathcal{H}x_2$$
(i.e., $\mathcal{H}$ *commutes with sums*).
- A system $\mathcal{H}$ is said to be **homogeneous** if, for every function x and every complex constant a, the following condition holds:
$$\mathcal{H}(ax) = a\mathcal{H}x$$
(i.e., $\mathcal{H}$ *commutes with multiplication by a constant*).
- A system that is both additive and homogeneous is said to be **linear**.
- In other words, a system $\mathcal{H}$ is *linear*, if for all functions x_1 and x_2 and all complex constants a_1 and a_2, the following condition holds:
$$\mathcal{H}(a_1x_1 + a_2x_2) = a_1\mathcal{H}x_1 + a_2\mathcal{H}x_2$$
(i.e., $\mathcal{H}$ *commutes with linear combinations*).
- The linearity property is also referred to as the **superposition** property.
- Practically speaking, linear systems are much *easier to design and analyze* than nonlinear systems.

Why Linear Time-Invariant (LTI) Systems?

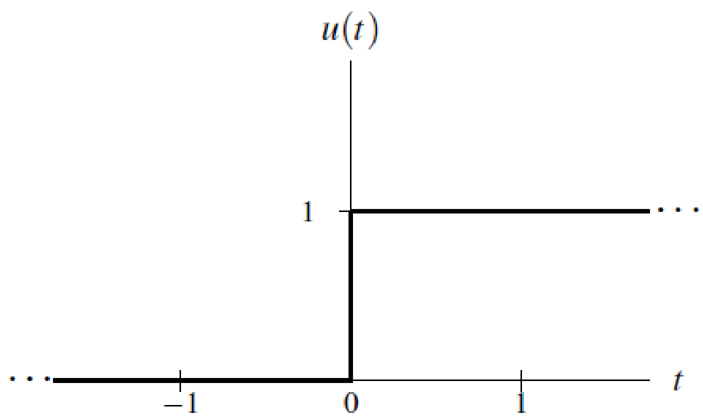
- In engineering, linear-time invariant (LTI) systems play a very important role.
- Very powerful mathematical tools have been developed for analyzing LTI systems.
- LTI systems are much easier to analyze than systems that are not LTI.
- In practice, systems that are not LTI can be well approximated using LTI models.
- So, even when dealing with systems that are not LTI, LTI systems still play an important role.

Unit-Step Function

- The **unit-step function** (also known as the **Heaviside function**), denoted u, is defined as

$$u(t) = \begin{cases} 1 & \text{if } t \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

- Due to the manner in which u is used in practice, the actual ***value of*** $u(0)$ is unimportant. Sometimes values of 0 and $\frac{1}{2}$ are also used for $u(0)$.
- A plot of this function is shown below.

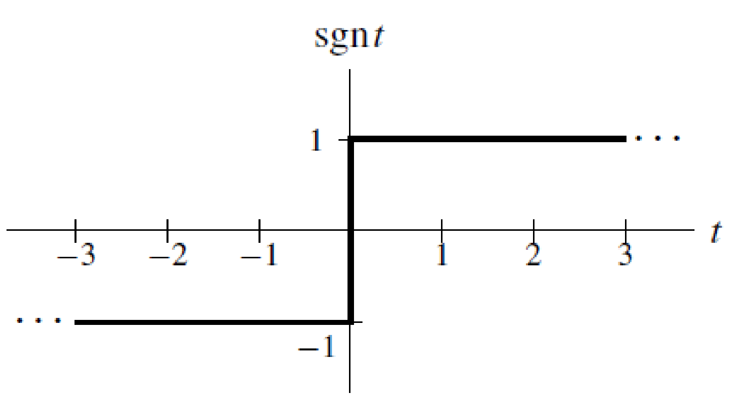


Signum Function

- The **signum function**, denoted sgn, is defined as

$$\operatorname{sgn} t = \begin{cases} 1 & \text{if } t > 0 \\ 0 & \text{if } t = 0 \\ -1 & \text{if } t < 0. \end{cases}$$

- From its definition, one can see that the signum function simply computes the ***sign*** of a number.
- A plot of this function is shown below.

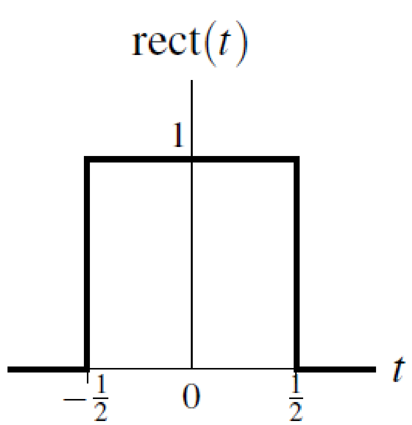


Rectangular Function

- The **rectangular function** (also called the unit-rectangular pulse function), denoted rect, is given by

$$\mathrm{rect}(t) = \begin{cases} 1 & \text{if } -\frac{1}{2} \leq t < \frac{1}{2} \\ 0 & \text{otherwise.} \end{cases}$$

- Due to the manner in which the rect function is used in practice, the actual ***value of*** $\mathrm{rect}(t)$ ***at*** $t = \pm\frac{1}{2}$ is unimportant. Sometimes different values are used from those specified above.
- A plot of this function is shown below.

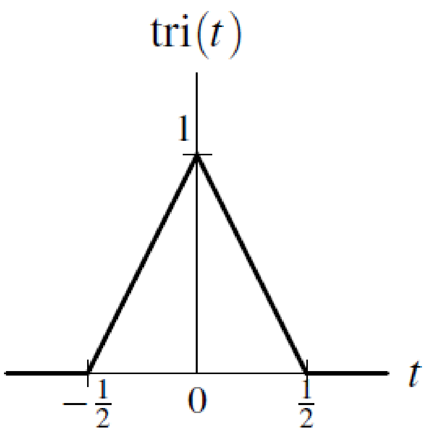


Triangular Function

- The **triangular function** (also called the unit-triangular pulse function), denoted tri, is defined as

$$\operatorname{tri}(t) = \begin{cases} 1 - 2|t| & |t| \le \frac{1}{2} \\ 0 & \text{otherwise.} \end{cases}$$

- A plot of this function is shown below.

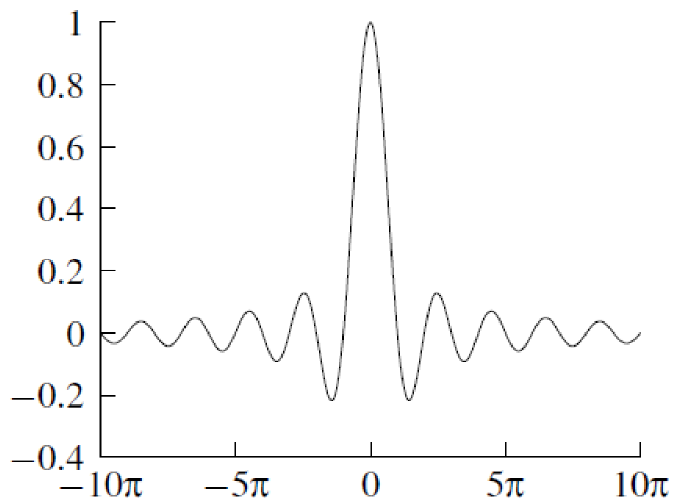


Cardinal Sine Function

- The **cardinal sine** function, denoted sinc, is given by

$$\operatorname{sinc}(t) = \frac{\sin t}{t}.$$

- By l'Hopital's rule, $\operatorname{sinc} 0 = 1$.
- A plot of this function for part of the real line is shown below. [Note that the oscillations in $\operatorname{sinc}(t)$ do not die out for finite t.]

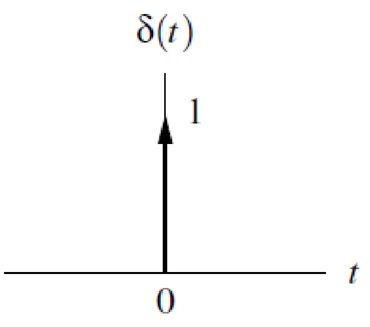


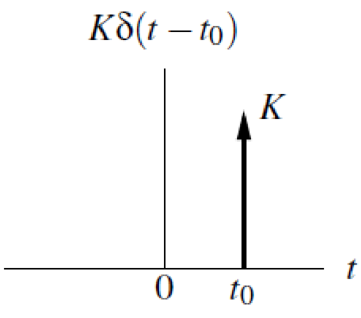
Unit-Impulse Function

- The **unit-impulse function** (also known as the **Dirac delta function** or **delta function**), denoted δ, is defined by the following two properties:

$$\delta(t) = 0 \quad \text{for } t \neq 0 \quad \text{and}$$
$$\int_{-\infty}^{\infty} \delta(t)\,dt = 1.$$

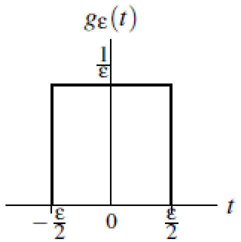
- Technically, δ is not a function in the ordinary sense. Rather, it is what is known as a ***generalized function***. Consequently, the δ function sometimes behaves in unusual ways.
- Graphically, the delta function is represented as shown below.





Unit-Impulse Function as a Limit

- Define
$$g_\varepsilon(t) = \begin{cases} 1/\varepsilon & \text{for } |t| < \varepsilon/2 \\ 0 & \text{otherwise.} \end{cases}$$
- The function g_ε has a plot of the form shown below.



- Clearly, for any choice of ε, $\int_{-\infty}^{\infty} g_\varepsilon(t)dt = 1$.
- The function δ can be obtained as the following limit:
$$\delta(t) = \lim_{\varepsilon \to 0} g_\varepsilon(t).$$
- That is, δ can be viewed as a *limiting case of a rectangular pulse* where the pulse width becomes infinitesimally small and the pulse height becomes infinitely large in such a way that the integral of the resulting function remains unity.

Properties of the Unit-Impulse Function

- **Equivalence property.** For any continuous function x and any real constant t_0,

$$x(t)\delta(t-t_0) = x(t_0)\delta(t-t_0).$$

- **Sifting property.** For any continuous function x and any real constant t_0,

$$\int_{-\infty}^{\infty} x(t)\delta(t-t_0)dt = x(t_0).$$

- The δ function also has the following properties:

$$\delta(t) = \delta(-t) \quad \text{and}$$
$$\delta(at) = \frac{1}{|a|}\delta(t),$$

where a is a nonzero real constant.

Representing a Rectangular Pulse Using Unit-Step Functions

- For real constants a and b where $a \leq b$, consider a function x of the form

$$x(t) = \begin{cases} 1 & \text{if } a \leq t < b \\ 0 & \text{otherwise} \end{cases}$$

  (i.e., $x(t)$ is a ***rectangular pulse*** of height one, with a ***rising edge at*** a and ***falling edge at*** b).
- The function x can be equivalently written as

$$x(t) = u(t-a) - u(t-b)$$

  (i.e., the difference of two time-shifted unit-step functions).
- Unlike the original expression for x, this latter expression for x ***does not involve multiple cases***.
- In effect, by using unit-step functions, we have collapsed a formula involving multiple cases into a single expression.

CT Convolution

- The (CT) **convolution** of the functions x and h, denoted $x * h$, is defined as the function

$$x * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau.$$

- The convolution result $x * h$ evaluated at the point t is simply a weighted average of the function x, where the weighting is given by h time reversed and shifted by t.
- Herein, the asterisk symbol (i.e., "$*$") will always be used to denote convolution, not multiplication.
- As we shall see, convolution is used extensively in systems theory.
- In particular, convolution has a special significance in the context of LTI systems.

Properties of Convolution

- The convolution operation is ***commutative***. That is, for any two functions x and h,

$$x * h = h * x.$$

- The convolution operation is ***associative***. That is, for any signals x, h_1, and h_2,

$$(x * h_1) * h_2 = x * (h_1 * h_2).$$

- The convolution operation is ***distributive*** with respect to addition. That is, for any signals x, h_1, and h_2,

$$x * (h_1 + h_2) = x * h_1 + x * h_2.$$

Representation of Signals Using Impulses

- For any function x,

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau = x * \delta(t).$$

- Thus, any function x can be written in terms of an expression involving δ.
- Moreover, δ is the ***convolutional identity***. That is, for any function x,

$$x * \delta = x.$$

Periodic Convolution

- The convolution of two periodic functions is usually not well defined.
- This motivates an alternative notion of convolution for periodic signals known as periodic convolution.
- The **periodic convolution** of the T-periodic functions x and h, denoted $x \circledast h$, is defined as

$$x \circledast h(t) = \int_T x(\tau)h(t-\tau)d\tau,$$

where $\int_T$ denotes integration over an interval of length T.

- The periodic convolution and (linear) convolution of the T-periodic functions x and h are related as follows:

$$x \circledast h(t) = x_0 * h(t) \quad \text{where} \quad x(t) = \sum_{k=-\infty}^{\infty} x_0(t-kT)$$

(i.e., $x_0(t)$ equals $x(t)$ over a single period of x and is zero elsewhere).

Impulse Response

- The response h of a system $\mathcal{H}$ to the input δ is called the **impulse response** of the system (i.e., $h = \mathcal{H}\{\delta\}$).
- For any LTI system with input x, output y, and impulse response h, the following relationship holds:

$$y = x * h.$$

- In other words, a LTI system simply ***computes a convolution***.
- Furthermore, a LTI system is ***completely characterized*** by its impulse response.
- That is, if the impulse response of a LTI system is known, we can determine the response of the system to any input.
- Since the impulse response of a LTI system is an extremely useful quantity, we often want to determine this quantity in a practical setting.
- Unfortunately, in practice, the impulse response of a system cannot be determined directly from the definition of the impulse response.

Step Response

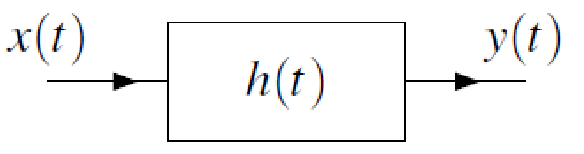
- The response s of a system $\mathcal{H}$ to the input u is called the **step response** of the system (i.e., $s = \mathcal{H}\{u\}$).
- The impulse response h and step response s of a system are related as

$$h(t) = \frac{ds(t)}{dt}.$$

- Therefore, the impulse response of a system can be determined from its step response by differentiation.
- The step response provides a practical means for determining the impulse response of a system.

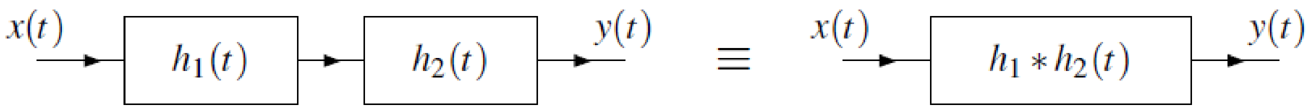
Block Diagram Representation of LTI Systems

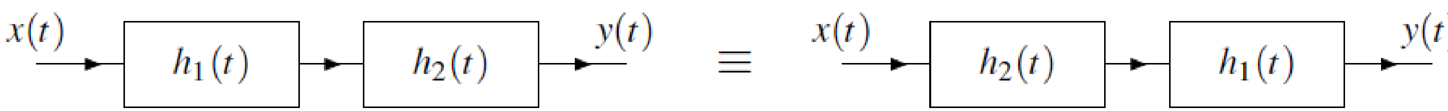
- Often, it is convenient to represent a (CT) LTI system in block diagram form.
- Since such systems are completely characterized by their impulse response, we often label a system with its impulse response.
- That is, we represent a system with input x, output y, and impulse response h, as shown below.



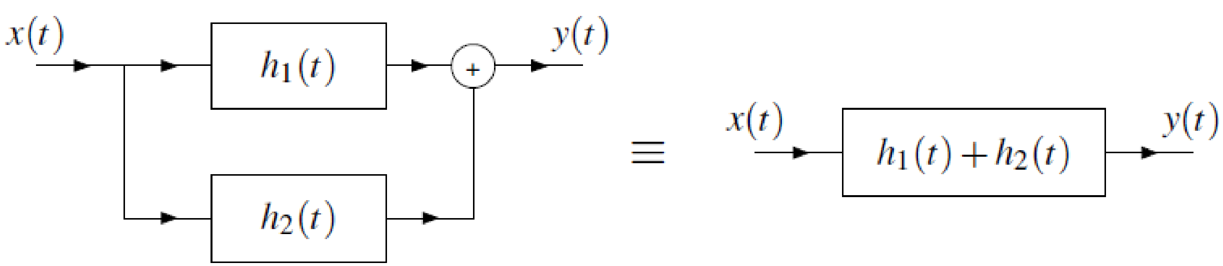
Interconnection of LTI Systems

- The *series* interconnection of the LTI systems with impulse responses h_1 and h_2 is the LTI system with impulse response $h = h_1 * h_2$. That is, we have the equivalences shown below.





- The *parallel* interconnection of the LTI systems with impulse responses h_1 and h_2 is a LTI system with the impulse response $h = h_1 + h_2$. That is, we have the equivalence shown below.



Memory

- A LTI system with impulse response h is memoryless if and only if

$$h(t) = 0 \quad \text{for all } t \neq 0.$$

- That is, a LTI system is memoryless if and only if its impulse response h is of the form

$$h(t) = K\delta(t),$$

  where K is a complex constant.

- Consequently, every memoryless LTI system with input x and output y is characterized by an equation of the form

$$y = x * (K\delta) = Kx$$

  (i.e., the system is an ideal amplifier).

- For a LTI system, the memoryless constraint is extremely restrictive (as every memoryless LTI system is an ideal amplifier).

Causality

- A LTI system with impulse response h is causal if and only if

$$h(t) = 0 \quad \text{for all } t < 0$$

  (i.e., h is a causal signal).
- It is due to the above relationship that we call a signal x, satisfying

$$x(t) = 0 \quad \text{for all } t < 0,$$

  a causal signal.

Invertibility

- The inverse of a LTI system, if such a system exists, is a LTI system.
- Let h and h_{inv} denote the impulse responses of a LTI system and its (LTI) inverse, respectively. Then,

$$h * h_{\text{inv}} = \delta.$$

- Consequently, a LTI system with impulse response h is invertible if and only if there exists a function h_{inv} such that

$$h * h_{\text{inv}} = \delta.$$

- Except in simple cases, the above condition is often quite difficult to test.

BIBO Stability

- A LTI system with impulse response h is BIBO stable if and only if

$$\int_{-\infty}^{\infty} |h(t)|\, dt < \infty$$

(i.e., h is ***absolutely integrable***).