One Dimensional Wave Equations

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One dimensional wave equation(Equation of vibrating string)

Let us consider a thin and homogeneous elastic string of length L and mass per unit length m, which is stretched and then fixed at its two ends along $x - axis$ in its equilibrium position. Let one end of the string is fixed at origin.

Now, we assume the following assumptions:
1. The string is perfectly elastic so that it can transmit tension only but not bending or shearing forces.

2. Each particle of the string moves in a direction perpendicular to the equilibrium position of the string.

3. The tension T, caused by stretching the string before fixing it at the end points is constant at all times at all points of the deflected string.

4.The tension is large enough compared to the weight of the string so that the gravitational force of attraction may be negligible.

5. The effect of frictional force is negligible.

6. The slope of deflection curve is small at all times and at all points.

When the string is in motion, let $u(x, t)$ be the displacement at any point of the string at time t.Consider the elementary portion PQ of the string of length Δx.
Since the string offers no resistance to bending, the tension is tangential to the curve of the string at each point.

Let α and β be the angles made by the tensions T_1 and T_2 along $x - axis$ at points P and Q respectively.Since the points of the string move vertically, there is no motion in horizontal direction.Hence the horizontal component of the tensions must be constant.

$$i.e. T_1 \cos\alpha = T_2 \cos\beta = T(say).... \quad ...(1)$$

The vertical component of tensions are $-T_1 \sin \alpha$ and $T_2 \sin \beta$, the minus sign indicates that the tension at P is directed downward. Therefore, the resultant force acting on the element PQ is

$$-T_1 \sin \alpha + T_2 \sin \beta$$

By Newton's second law of motion
resultant force $=$ mass $\times$ acceleration

$$i.e. - T_1 \sin\alpha + T_2 \sin\beta = m\Delta x.\frac{\partial^2 u}{\partial t^2}$$

Dividing by T we get

$$\frac{T_2 \sin \beta}{T} - \frac{T_1 \sin \alpha}{T} = \frac{m \Delta x}{T} \frac{\partial^2 u}{\partial t^2}$$

Using (1)

$$\frac{T_2 \sin\beta}{T_2 \cos\beta} - \frac{T_1 \sin\alpha}{T_1 \cos\alpha} = \frac{m\Delta x}{T}\frac{\partial^2 u}{\partial t^2}$$

$$or, \tan\beta - \tan\alpha = \frac{m\Delta x}{T}\frac{\partial^2 u}{\partial t^2} .....(2)$$

Since, $\tan\alpha$ and $\tan\beta$ are the slopes of the string at x and $x + \Delta x$, so

$$\tan\alpha = \frac{\partial u}{\partial x}|_x$$

and

$$\tan\beta = \frac{\partial u}{\partial x}|_{x+\Delta x}$$

So, from (2), we get

$$\left(\frac{\partial u}{\partial x}\right)_{x+\Delta x} - \left(\frac{\partial u}{\partial x}\right)_{x} = \frac{m\Delta x}{T}\frac{\partial^2 u}{\partial t^2}$$

Dividing by Δx we get

$$\frac{m}{T}\frac{\partial^2 u}{\partial t^2} = \left\{ \frac{\left(\frac{\partial u}{\partial x}\right)_{x+\Delta x} - \left(\frac{\partial u}{\partial x}\right)_{x}}{\Delta x} \right\}$$

Taking limit as $\Delta x \to 0$ we get

$$\frac{m}{T}\frac{\partial^2 u}{\partial t^2} = \lim_{\Delta x \to 0} \left\{ \frac{\left(\frac{\partial u}{\partial x}\right)_{x+\Delta x} - \left(\frac{\partial u}{\partial x}\right)_{x}}{\Delta x} \right\}$$

or,

$$\frac{m}{T}\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)$$

or,

$$\frac{\partial^2 u}{\partial t^2} = \frac{T}{m}\frac{\partial^2 u}{\partial x^2}$$

Hence,

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}$$

which is required one dimensional wave equation, where $c^2 = \frac{T}{m}$

Solution of one dimensional wave equation:

One dimensional wave equation is

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \ldots(1)$$

Since the string is fixed at its end points, so there is no deflection on the boundaries. Therefore, the boundary conditions are

$$u(0, t) = 0$$

and

$$u(L, t) = 0$$

The initial conditions are

$$u(x, 0) = f(x) \quad (initial \quad displacement)$$

and

$$u_t(x, 0) = g(x) \quad (initial \quad velocity)$$

Let

$$u(x,t) = X(x).T(t)$$

where, $X(x)$ is function of x alone and $T(t)$ is function of t alone, be the solution of (1).

Then, differentiating partially with respect to x we get

$$\frac{\partial u}{\partial x} = \frac{dX}{dx}.T$$

Also, differentiating partially with respect to t we get

$$\frac{\partial u}{\partial t} = X.\frac{dT}{dt}$$

Again, differentiating partially w.r to x , we get

$$\frac{\partial^2 u}{\partial x^2} = \frac{d^2 X}{dx^2}.T$$

Also,

$$\frac{\partial^2 u}{\partial t^2} = X.\frac{d^2 T}{dt^2}$$

Now, from equation (1),

$$X.\frac{d^2T}{dt^2} = c^2T.\frac{d^2X}{dx^2}$$

Dividing by c^2XT we get

$$\frac{1}{c^2T}\frac{d^2T}{dt^2} = \frac{1}{X}\frac{d^2X}{dx^2} = k(say)... \quad ...(2)$$

Now we have the following three cases.
Case I:
Let k is positive, say $k = \lambda^2$,then

$$\frac{1}{X}\frac{d^2X}{dx^2} = \lambda^2$$

and

$$\frac{1}{c^2T}\frac{d^2T}{dt^2} = \lambda^2$$

or,

$$\frac{d^2X}{dx^2} - \lambda^2 X = 0$$

and

$$\frac{d^2T}{dt^2} - c^2\lambda^2 T = 0$$

Solving we get

$$X = C_1 e^{\lambda x} + C_2 e^{-\lambda x}$$

and

$$T = C_3 e^{\lambda ct} + C_4 e^{-\lambda ct}$$

So,

$$u(x,t) = X.T$$

now becomes

$$u(x,t) = \left(C_1 e^{\lambda x} + C_2 e^{-\lambda x}\right)\left(C_3 e^{\lambda ct} + C_4 e^{-\lambda ct}\right) \text{....(3)}$$

Using the boundary condition $u(0,t) = 0$, we get

$$0 = \left(C_1 e^0 + C_2 e^0\right)\left(C_3 e^{\lambda ct} + C_4 e^{-\lambda ct}\right)$$

or

$$0 = (C_1 + C_2)\left(C_3 e^{\lambda ct} + C_4 e^{-\lambda ct}\right)$$

$$i.e. C_1 + C_2 = 0$$

since

$$C_3 e^{\lambda ct} + C_4 e^{-\lambda ct} \neq 0$$

Again, using the boundary condition $u(L, t) = L$, we get

$$0 = \left(C_1 e^{\lambda L} + C_2 e^{-\lambda L}\right)\left(C_3 e^{\lambda ct} + C_4 e^{-\lambda ct}\right)$$

or

$$0 = C_1 e^{\lambda L} - C_1 e^{-\lambda L}$$

$$\left[\because C_1 + C_2 = 0 \quad \text{and} \quad C_3 e^{\lambda ct} + C_4 e^{-\lambda ct} \neq 0\right]$$

$$or, e^{\lambda L} = e^{-\lambda L}$$

or,

$$e^{2\lambda L} = 1$$

or

$$e^{2\lambda L} = e^0$$

or,

$$2\lambda L = 0$$

or,

$$\lambda = 0, \quad \because 2L \neq 0$$

This is a contradiction to the fact that $k = \lambda^2$ is positive. So this solution is rejected.

Case II:
Let $k = 0$, then from (2)

$$\frac{1}{X}\frac{d^2X}{dx^2} = 0$$

and

$$\frac{1}{c^2T}\frac{d^2T}{dt^2} = 0$$

or

$$\frac{d^2X}{dx^2} = 0$$

and

$$\frac{d^2T}{dt^2} = 0$$

Integrating we get

$$\frac{dX}{dx} = A$$

and

$$\frac{dT}{dt} = C$$

where A and C are constants of integration.

Again, integrating we get

$$X = Ax + B$$

and

$$T = Ct + D$$

So,

$$u(x,t) = X.T$$

now becomes

$$u(x,t) = (Ax + B)(Ct + D)$$

Using the boundary condition $u(0, t) = 0$, we get

$$0 = (A.0 + B)(Ct + D)$$

i.e.

$$B = 0 \quad since \quad Ct + D \neq 0$$

Again, using the boundary condition $u(L,t) = 0$, we get

$$0 = (A.L + 0)(Ct + D)$$

i.e.

$$A = 0 \quad since \quad Ct + D \neq 0$$

$$\therefore u(x,t) = (Ax + B)(Ct + D) = 0 \quad \textit{since} \quad A = 0 \quad \textit{and} \quad B = 0$$

So only the zero solution is possible in this case. So, this solution is also discarded.

Case III:
Let k is negetive, say $k = -\lambda^2$,then

$$\frac{1}{X}\frac{d^2X}{dx^2} = -\lambda^2$$

and

$$\frac{1}{c^2T}\frac{d^2T}{dt^2} = -\lambda^2$$

or,

$$\frac{d^2X}{dx^2} + \lambda^2 X = 0$$

and

$$\frac{d^2T}{dt^2} + c^2\lambda^2 T = 0$$

Solving, we get

$$X = C_1 \cos \lambda x + C_2 \sin \lambda x$$

and

$$T = C_3 \cos \lambda ct + C_4 \sin \lambda ct$$

So,

$$u(x, t) = X.T$$

now becomes

$$u(x, t) = (C_1 \cos \lambda x + C_2 \sin \lambda x)(C_3 \cos \lambda ct + C_4 \sin \lambda ct) ... \quad ...(4)$$

Using the boundary condition $u(0, t) = 0$, we get

$$0 = (C_1.1 + C_2.0)(C_3 \cos \lambda ct + C_4 \sin \lambda ct)$$

i.e.

$$C_1 = 0 \quad \textit{since} \quad (C_3 \cos \lambda ct + C_4 \sin \lambda ct) \neq 0$$

Again, using the boundary condition $u(L, t) = 0$, we get

$$0 = (0 + C_2. \sin \lambda L)(C_3 \cos \lambda ct + C_4 \sin \lambda ct)$$

i.e.

$$\sin \lambda L = 0 \quad \textit{since} \quad C_2 \neq 0 \quad \textit{and} \quad (C_3 \cos \lambda ct + C_4 \sin \lambda ct) \neq 0$$

$$or, \sin \lambda L = 0 = \sin n\pi$$

or

$$\lambda L = n\pi$$

or,

$$\lambda = \frac{n\pi}{L} \quad n = 1, 2, 3, 4, ...$$

Now, from equation (4), we get

$$u(x,t) = C_2 \sin \frac{n\pi}{L} x (C_3 \cos \frac{n\pi}{L} ct + C_4 \sin \frac{n\pi}{L} ct)$$

i.e.

$$u_n(x,t) = \sin \frac{n\pi}{L} x (a_n \cos \frac{n\pi}{L} ct + b_n \sin \frac{n\pi}{L} ct)$$

where

$$a_n = C_2 C_3$$

and

$$b_n = C_2 C_4$$

Using principle of superposition, (i.e. adding all possible solutions) , we get

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t)$$

i.e.

$$u(x,t) = \sum_{n=1}^{\infty} \sin \frac{n\pi}{L} x \left[a_n \cos \frac{n\pi}{L} ct + b_n \sin \frac{n\pi}{L} ct \right] .....(5)$$

Differentiating with respect to time, we get

$$u_t(x,t) = \sum_{n=1}^{\infty} \sin\frac{n\pi}{L}x\left[a_n.(-)\frac{n\pi}{L}c.\sin\frac{n\pi}{L}ct + b_n.\frac{n\pi}{L}c\cos\frac{n\pi}{L}ct\right] .....(6)$$

Using the initial condition

$$u(x, 0) = f(x)$$

we get from (5)

$$f(x) = \sum_{n=1}^{\infty} \sin \frac{n\pi}{L} x \left[a_n \cos 0 + b_n \sin 0\right]$$

or

$$f(x) = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi}{L} x$$

which is a half range Fourier sine series, where

$$a_n = \frac{2}{L}\int\limits_{0}^{\infty} f(x)\sin\frac{n\pi}{L}x \quad dx \quad ....(7)$$

Again, using the initial condition

$$u_t(x,0) = g(x)$$

we get from (6)

$$g(x) = \sum_{n=1}^{\infty} \sin \frac{n\pi}{L} x \left[0 + b_n . \frac{n\pi}{L} c.1 \right]$$

or

$$g(x) = \sum_{n=1}^{\infty} b_n . \frac{n\pi}{L} c \sin \frac{n\pi}{L} x$$

which is also a half range Fourier sine series, where

$$b_n.\frac{n\pi}{L}c = \frac{2}{L}\int_0^{\infty} g(x)\sin\frac{n\pi}{L}x \quad dx$$

i.e.

$$b_n = \frac{2}{n\pi c}\int_0^{\infty} g(x)\sin\frac{n\pi}{L}x \quad dx \quad ....(8)$$

The equation (5), together with the equations (7) and (8) gives the required general solution of one dimensional wave equation.