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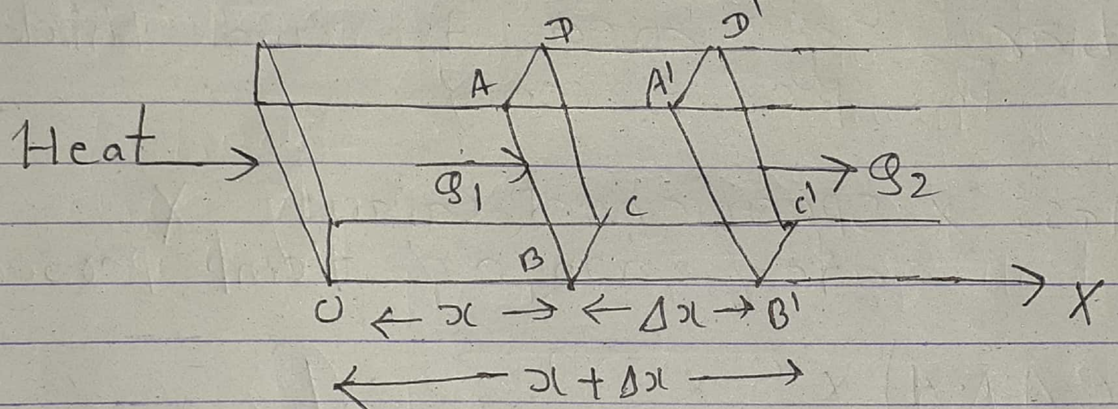
Heat equation or Diffusion eqn.

Derivation of one-dimensional heat eqn :-

Consider heat flows along a homogeneous bar of uniform cross-section. Suppose the sides of the bar are insulated and the loss of heat from the sides by conduction or radiation is negligible.



Taking one end of the bar at the origin and the direction of heat flows along the positive x-axis. If the heat flows along the positive x-axis then the temperature of the bar depends on the distance from the origin and time taken. Let u be the temperature, x be the distance from the origin where the heat flows at time t.

i.e $u = u(x, t)$

Taking cross-section area $= A$
density of bar $= \rho$
thermal conductivity $= K$
Temperature $= u$, specific heat $= S$

Consider the element of bar on the planes ABCD and A'B'C'D. plane ABCD is at a distance x from the origin and the plane A'B'C'D is at a distance $x+\Delta x$ from the origin. Then the rate of increase of heat in this element of bar of area A and thickness of Δx in per second

= Mass x specific heat capacity X rate of change of temp. per second

$$= (\rho \, \Delta x . A) \times S \times \frac{\partial u}{\partial t} \quad — (1)$$

Let q_1 and q_2 be the rate of heat inflow and outflow of heat in this bar.

Then In q_1

The rate of heat inflow at q_1 at a distance x from the origin is

$$= -KA \left(\frac{\partial u}{\partial x}\right)_{at\ x}$$

Similarly, the rate of heat outflow at q_2 at a distance $x+\Delta x$

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from the origin is

$$= -KA\left(\frac{\partial u}{\partial x}\right)_{at\ x+\Delta x}$$

$\left[\text{Rate of heat transfer} = -KA\dfrac{\partial T}{\partial x}\right]$

where K is thermal conductivity. It is the ability of substance to conduct heat. The negative sign appears at the heat flows from higher temperature to lower temperature.

Also the rate of increase of heat in this element of bar of area A and thickness Δx in one second

$$= \text{Heat inflow at } S_1 - \text{Heat out flow at } S_2$$

$$= -KA\left(\frac{\partial u}{\partial x}\right)_{at\ x} + KA\left(\frac{\partial u}{\partial x}\right)_{at\ x+\Delta x}$$

$$= KA\left(\frac{\partial u}{\partial x}\right)_{at\ x+\Delta x} - KA\left(\frac{\partial u}{\partial x}\right)_{at\ x} \quad \text{--- (2)}$$

From equation (1) and (2)

$$(\rho\,\Delta x\,A) \times s \times \frac{\partial u}{\partial t} = KA\left(\frac{\partial u}{\partial x}\right)_{x+\Delta x} - KA\left(\frac{\partial u}{\partial x}\right)_{x}$$

or $$\rho\,\Delta x \cdot A \times s \times \frac{\partial u}{\partial t} = KA\left[\left(\frac{\partial u}{\partial x}\right)_{x+\Delta x} - \left(\frac{\partial u}{\partial x}\right)_{x}\right]$$

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or $$\Delta x \cdot \frac{\partial u}{\partial t} = \frac{K}{s\rho}\left[\left(\frac{\partial u}{\partial x}\right)_{x+\Delta x} - \left(\frac{\partial u}{\partial x}\right)_x\right]$$

or $$\frac{\partial u}{\partial t} = \frac{K}{s\rho}\left[\frac{\left(\frac{\partial u}{\partial x}\right)_{x+\Delta x} - \left(\frac{\partial u}{\partial x}\right)_x}{\Delta x}\right]$$

Taking limit as $\Delta x \to 0$ on both sides, we get

$$\frac{\partial u}{\partial t} = \frac{K}{s\rho} \lim_{\Delta x \to 0}\left[\frac{\left(\frac{\partial u}{\partial x}\right)_{x+\Delta x} - \left(\frac{\partial u}{\partial x}\right)_x}{\Delta x}\right]$$

$$= \frac{K}{s\rho} \cdot \frac{\partial^2 u}{\partial x^2}$$

$$\boxed{\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}}$$ where

$$c^2 = \frac{K}{s\rho}$$

~~This~~ is one dimensional heat eqn.

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Solution of 1-D heat eqn :

We have one dimensional heat eqn is

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \quad \cdots (1)$$

under the boundary condition

$u(0,t) = 0$ and $u(L,t) = 0$

for all t $\cdots (2)$

With initial condition

$u(x,0) = f(x) \cdots (3)$ initial temperature

Let $u(x,t) = F(x) \cdot G(t) \cdots (4)$ be the solution of eqn (1) where F is function of x only and G be the function of t only.

By differentiating eqn (4)

$$\frac{\partial u}{\partial t} = F \cdot \dot{G} \quad \text{and} \quad \frac{\partial^2 u}{\partial x^2} = F''G.$$

Where dot denotes derivative w.r.to t and prime denotes derivative w.r.to x.

Then eqn (1) becomes

$$F \cdot \dot{G} = c^2 F'' G$$

or $$\frac{F''}{F} = \frac{\dot{G}}{c^2 G}$$

or $$\frac{F''}{F} = \frac{\dot{G}}{c^2 G} = K \text{ (Let)}$$

which implies that

$$F'' - FK = 0 \quad \cdots \quad (5)$$

$$\ddot{G} - c^2 K G = 0 \quad \cdots \quad (6)$$

For solution of (5)

we have from eqn (4)

$$u(x,t) = F(x) \cdot G(t)$$

i.e $u(0,t) = F(0) \cdot G(t) = 0$

$u(L,t) = F(L) \cdot G(t) = 0$

If $G = 0$ then $u = 0$ which is no meaning. So let $G \neq 0$. Then

$$F(0) = 0 = F(L) \quad \cdots \quad (7)$$

If K is zero then the solution of eqn (1) is $u = 0$ (being $F = 0$) which is no meaning.

Similarly if K is positive then F is also zero which implies $u = 0$. It is also no interest.

So let K is negative

Suppose $K = -p^2$. eqn (5) becomes

$$F'' + p^2 F = 0$$

Its auxillary equation is

$m^2 + p^2 = 0$

$\therefore m = \pm pi$ (purely imaginary)

Then general Solution

$F(x) = e^{px}(A\cos qx + B\sin qx)$

here real part $p = 0$ and imaginary part $q = p$

$\therefore F(x) = e^0 (A\cos px + B\sin px)$

$F(x) = A\cos px + B\sin px$ --- (8)

Where

A and B are constants

Applying boundary condition

$F(0) = A\cos 0 + B\sin 0$

or, $0 = A + 0$

$\therefore A = 0$

Again

$F(L) = A\cos pL + B\sin pL$

$0 = 0 \cdot \cos pL + B\sin pL$

or $B\sin pL = 0$

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Since $B \neq 0$, So

$$\sin pL = 0$$

or $\sin pL = \sin n\pi$

or $pL = n\pi$

or $p = \frac{n\pi}{L}$

Setting $B = 1$ eqn (8) becomes

$$F(x) = A\cos px + B\sin px$$

$$= 0 \cdot \cos px + 1 \cdot \sin \frac{n\pi}{L} x$$

$$F(x) = \sin \frac{n\pi}{L} x$$

i.e. $F_n(x) = \sin \frac{n\pi}{L} x \quad \cdots (9)$

for $n = 1, 2, 3, \cdots$

Again for solution of (6)

$$\dot{G} - c^2 k G = 0$$

or $\dot{G} - c^2(-p^2)G = 0 \quad [k = -p^2]$

or $\dot{G} + c^2 p^2 G = 0$

or $\dot{G} + c^2\left(\frac{n\pi}{L}\right)^2 G = 0$

or $\dot{G} + \left(\frac{cn\pi}{L}\right)^2 G = 0$

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or $\dot{G} + \lambda_n^2 G = 0 \quad \left[\lambda_n = \frac{cn\pi}{2}\right]$

or $\dot{G} = -\lambda_n^2 G$

or $\frac{\dot{G}}{G} = -\lambda_n^2$

integrating w.r.to t

or $\log G = -\lambda_n^2 t + C$

or $G = e^{-\lambda_n^2 t + C}$

or $G = e^{-\lambda_n^2 t} \cdot e^{C}$

or $G = e^{-\lambda_n^2 t} \cdot B_n$

The general Solution of ⑥ is

$$G(t) = B_n e^{-\lambda_n^2 t}$$

ie $G_n(t) = B_n e^{-\lambda_n^2 t} \quad \cdots (10)$

Therefore

$$U_n(x,t) = F_n(x) \cdot G_n(t)$$

$$= \sin\frac{n\pi}{2}x \cdot B_n e^{-\lambda_n^2 t}$$

$$U_n(x,t) = B_n \sin\frac{n\pi}{2}x \cdot e^{-\lambda_n^2 t}$$

By fundamental theorem

$$u(x,t) = \sum_{n=1}^{\infty} U_n(x,t)$$

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi}{l}x \cdot e^{-\lambda n^2 t}} \quad (11)$$

To get a solution also satisfying the initial condition.

$$u(x,0) = f(x)$$

eqn (11) becomes

$$\Rightarrow u(x,0) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi}{l}x$$

or $$f(x) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi}{l}x$$

Which is fourier sine series of $f(x)$ and coefficient B_n is

$$B_n = \frac{2}{l}\int_0^l f(x) \sin\frac{n\pi}{l}x\,dx$$

Therefore we get the solution of one dimensional heat eqn is

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$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi}{L} x \, e^{-\lambda_n^2 t}$$

where $\lambda_n = \frac{c n \pi}{L}$

$$B_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi}{L} x \, dx$$

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