Solutions of Questions Related to Fourier Sine and Cosine Transforms

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Find Fourier sine and cosine transform of

$$f(x) = 2e^{-5x} + 5e^{-2x}$$

The fourier sine transform of given function is

$$\mathcal{F}_s(f) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(t) \sin \omega t dt$$

$$= \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} (2e^{-5t} + 5e^{-2t}) \sin \omega t dt$$

$$= 2\sqrt{\frac{2}{\pi}} \int_0^\infty e^{-5t} \sin \omega t dt$$

$$+5\sqrt{\frac{2}{\pi}} \int_0^\infty e^{-2t} \sin \omega t dt$$

$$= 2\sqrt{\frac{2}{\pi}}\left[\frac{e^{-5t}}{5^2+\omega^2}\left(-5\sin\omega t - \omega\cos\omega t\right)\right]_0^{\infty}$$

$$+5\sqrt{\frac{2}{\pi}}\left[\frac{e^{-2t}}{2^2+\omega^2}\left(-2\sin\omega t - \omega\cos\omega t\right)\right]_0^{\infty}$$

$$= 2\sqrt{\frac{2}{\pi}}\left[0 - \frac{e^0}{25+\omega^2}(0-\omega.1)\right]$$

$$+5\sqrt{\frac{2}{\pi}}\left[0 - \frac{e^0}{4+\omega^2}(0-\omega.1)\right]$$

$$\mathcal{F}_s(f) = \sqrt{\frac{2}{\pi}}\left(\frac{2\omega}{25+\omega^2} + \frac{5\omega}{4+\omega^2}\right)$$

which is required fourier sine transform of given function.

Also, the fourier cosine transform of given function is

$$\mathcal{F}_c(f) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(t) \cos \omega t dt$$

$$= \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} (2e^{-5t} + 5e^{-2t}) \cos \omega t dt$$

$$= 2\sqrt{\frac{2}{\pi}} \int_0^\infty e^{-5t} \cos \omega t dt$$

$$+5\sqrt{\frac{2}{\pi}} \int_0^\infty e^{-2t} \cos \omega t dt$$

$$
= 2\sqrt{\frac{2}{\pi}}\left[\frac{e^{-5t}}{5^2+\omega^2}\left(-5\cos\omega t + \omega\sin\omega t\right)\right]_0^\infty
$$

$$
+5\sqrt{\frac{2}{\pi}}\left[\frac{e^{-2t}}{2^2+\omega^2}\left(-2\cos\omega t + \omega\sin\omega t\right)\right]_0^\infty
$$

$$= 2\sqrt{\frac{2}{\pi}}\left[0 - \frac{e^0}{25+\omega^2}(-5.1+0)\right]$$

$$+5\sqrt{\frac{2}{\pi}}\left[0 - \frac{e^0}{4+\omega^2}(-2.1+0)\right]$$

$$\mathcal{F}_s(f) = 10\sqrt{\frac{2}{\pi}}\left(\frac{1}{25+\omega^2} + \frac{1}{4+\omega^2}\right)$$

which is required fourier cosine transform of given function.

Find fourier cosine transform of $f(x) = e^{-mx}$ where , $m > 0$ and hence show that

$$\int_0^\infty \frac{\cos kx}{1+x^2} dx = \frac{\pi}{2} e^{-k}$$

Solution: The fourier cosine transform of given fucntion is

$$\mathcal{F}_c(f) = \sqrt{\frac{2}{\pi}} \int\limits_0^\infty f(t) \cos \omega t dt$$

$$= \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} e^{-mt} \cos \omega t dt$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{e^{-mt}}{m^2 + \omega^2} \left(-m \cos \omega t + \omega \sin \omega t \right) \right]_0^\infty$$

$$= \sqrt{\frac{2}{\pi}} \left[0 - \frac{e^0}{m^2 + \omega^2} \left(-m.1 + 0 \right) \right]$$

$$\mathcal{F}_c(f) = \sqrt{\frac{2}{\pi}} \left(\frac{m}{m^2 + \omega^2} \right)$$

which is required fourier cosine transform of given function.

Next part, using inversion formula for cosine transform, we get

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \{\mathcal{F}_c(f)\} \cos \omega x d\omega$$

$$or, e^{-mx} = \sqrt{\frac{2}{\pi}} \int_0^\infty \left\{ \sqrt{\frac{2}{\pi}} \left(\frac{m}{m^2 + \omega^2} \right) \right\} \cos \omega x d\omega$$

$$or, e^{-mx} = \frac{2}{\pi} \int_{0}^{\infty} \left(\frac{m}{m^2 + \omega^2} \right) \cos \omega x d\omega$$

Put $m = 1$, then

$$\frac{\pi}{2} e^{-x} = \int_{0}^{\infty} \frac{\cos \omega x}{1 + \omega^2} d\omega$$

Replacing x by k we get

$$\frac{\pi}{2}e^{-k} = \int_{0}^{\infty} \frac{\cos k\omega}{1+\omega^2}\,d\omega$$

Replacing ω by x we get

$$\int_0^\infty \frac{\cos kx}{1+x^2}\,dx = \frac{\pi}{2}e^{-k}$$

which is required result.

Find fourier sine transform of $f(x) = e^{-x}$ where , $x > 0$ and hence show that

$$\int_{0}^{\infty} \frac{x \sin mx}{1+x^2} dx = \frac{\pi}{2} e^{-m}$$

Solution: The fourier sine transform of given fucntion is

$$\mathcal{F}_s(f) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(t) \sin \omega t dt$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-t} \sin \omega t dt$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{e^{-t}}{1^2 + \omega^2} \left(-1 \sin \omega t - \omega \cos \omega t \right) \right]_0^\infty$$

$$= \sqrt{\frac{2}{\pi}} \left[0 - \frac{e^0}{1+\omega^2} \left(0 - \omega .1 \right) \right]$$

$$\mathcal{F}_s(f) = \sqrt{\frac{2}{\pi}}\left(\frac{\omega}{1+\omega^2}\right)$$

which is required fourier sine transform of given function.

Next , using inversion formula for sine transform, we get

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \{\mathcal{F}_s(f)\} \sin \omega x d\omega$$

$$or, e^{-x} = \sqrt{\frac{2}{\pi}} \int_0^\infty \left\{ \sqrt{\frac{2}{\pi}} \left(\frac{\omega}{1+\omega^2} \right) \right\} \sin \omega x d\omega$$

$$or,\, e^{-x} = \frac{2}{\pi} \int\limits_{0}^{\infty} \left(\frac{\omega \sin \omega x}{1+\omega^2} \right) d\omega$$

Replacing x by m we get

$$\frac{\pi}{2}e^{-m} = \int_{0}^{\infty} \frac{\omega \sin m\omega}{1+\omega^2}\, d\omega$$

Replacing ω by x we get

$$\int_0^\infty \frac{x \sin mx}{1+x^2}\,dx = \frac{\pi}{2}e^{-m}$$

which is required result.