Partial Differential Equation

A partial differential equation is a relation between a dependent variable, one or more independent variable and partial derivatives of the dependent variable with respect to the independent variable. If z is taken as a dependent variable which is function of x and y. Then it is written as

$$z = f(x, y)$$

The partial differential co-efficients of z with respect to x and y are $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ respectively.

Similarly the second order partial derivatives are $\frac{\partial^2 z}{\partial x^2}$, $\frac{\partial^2 z}{\partial y^2}$, $\frac{\partial^2 z}{\partial x \partial y}$

Where z is dependent variable and x, y are Independent variable.

The simplest physical problem can be modeled by ordinary differential equation where as most problems in fluid mechanics, elasticity, heat transfer and quantum mechanics lead to partial differential equations.

Important formula of partial differential equation of second order

① One dimensional Wave equation :-

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

② One dimensional heat Equation :-

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$$

③ Two dimensional Laplace equation :-

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

④ Two dimensional poisson equation :-

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y)$$

⑤ Two dimensional Wave equation :-

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)$$

⑥ Two dimensional heat equation :-

$$\frac{\partial u}{\partial t} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)$$

Where c is constant, t is time period and x, y are cartesian co-ordinates.

Example

Verify $u = x^2 + t^2$ is the solution of one dimensional wave equation.

Solution

We have one dimensional wave equation is

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad \text{.........(1)}$$

But we have

$$u = x^2 + t^2$$

$\Rightarrow$ Differentiating w.r.to x and t partially we get,

$$\frac{\partial u}{\partial t} = 2t \text{ and } \frac{\partial^2 u}{\partial t^2} = 2$$

and

$$\frac{\partial u}{\partial x} = 2x, \quad \frac{\partial^2 u}{\partial x^2} = 2.$$

From equation (1), we get

$$2 = 2c^2$$

$$c^2 = 1$$

$$\Rightarrow \quad c = \pm 1$$

$\therefore$ Thus we get, the given function $u = x^2 + t^2$ satisfy one dimensional wave equation only when $c = \pm 1$,

Example

Show that $u = e^{-w^2c^2t}$ sinwx is the solution of one dimensional heat equation.

Solution

We have, one dimensional heat equation is

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad \text{...........(1)}$$

Also, we have

$$u = e^{-w^2c^2t} \sin wx$$

Differentiating partially we get,

$$\frac{\partial u}{\partial t} = -w^2c^2\, e^{-w^2c^2t} \sin wx$$

$$\frac{\partial u}{\partial x} = e^{-w^2c^2t}\, w\cos wx$$

and

$$\frac{\partial^2 u}{\partial x^2} = -w^2\, e^{-w^2c^2t} \sin wx$$

From equation (1) we get,

$$-w^2c^2\, e^{-w^2c^2t} \sin wx = c^2\left[-w^2\, e^{-w^2c^2t} \sin wx\right]$$

Thus we get the given function is the solution of one dimensional heat equation,

Example

Show that $u = \cos x \sinh y$, satisfy two dimensional laplace equation.

Solution

We know two dimensional laplace equation is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad \text{............(1)}$$

We have u = cosx sinhy

Differentiating partially with respect to x and y, we get

$$\frac{\partial u}{\partial x} = -\sin x \sinh y$$

$$\frac{\partial^2 u}{\partial x^2} = -\cos x \sinh y$$

Also,
$$\frac{\partial u}{\partial y} = \cos x \cosh y$$

$$\frac{\partial^2 u}{\partial y^2} = \cos x \sinh y$$

There fore we get

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -\cos x \sinh y + \cos x \sinh y$$

$$= 0.$$

Hence the given function satisfy laplace equation.

Example

Solve $u_{xx} + 9u = 0$

Solution

We have given differential equation is

$$u_{xx} + 9u = 0 \qquad \text{.........(1)}$$

Its auxiliary equation is

$$m^2+9 = 0, m = \pm i3$$

Thus required solution of the given differential equation is,

$$u = A \cos 3x + B \sin 3x$$

where A and B are constants.

Example

Solve $u_y + 2yu = 0$

Solution

The given differential equation is

$$u_y + 2yu = 0$$

$$\Rightarrow \quad \frac{\partial u}{\partial y} + 2yu = 0$$

$$\Rightarrow \quad \frac{\partial u}{u} + 2y\,\partial y = 0$$

Integrating both sides, we get

$$\log u + y^2 = c, \qquad \text{where c is constant}$$

$$\Rightarrow \quad u = e^{c-y^2}$$

This is the required solution of the given differential equation.

Example

Solve the partial differential equation $u_{yy} = u$.

Solution

We have

$$u_{yy} = u \quad \Rightarrow \quad \frac{\partial^2 u}{\partial y^2} - u = 0$$

Its auxiliary equation is

$$m^2 - 1 = 0 \quad \Rightarrow \quad m = \pm 1.$$

Then its solution is

$$u = Ae^y + Be^{-y}$$

where A and B are function of x or constants.

Example

Solve $u_{yy} = u_y$

Solution

We have given differential equation is,

$$u_{yy} = u_y \quad \Rightarrow \quad u_{yy} - u_y = 0$$

Its quxiliary equation is

$$m^2 - m = 0 \quad \Rightarrow \quad m = 0, 1$$

Therefore required solution is

$$u = A + Be^y$$

where A and B are constant or function of x.

EX. 8.1

(1) Verify the given function to satisfy one dimensional wave equation

(a) $U = \sin 9t . \sin \frac{x}{4}$

Solution: We have

$U = \sin 9t . \sin \frac{x}{4}$

We know one dimensional wave eqn is

$$\frac{\partial^2 U}{\partial t^2} = c^2 \frac{\partial^2 U}{\partial x^2} \quad \ldots \text{(1)}$$

Now,

$$\frac{\partial U}{\partial t} = 9 \cos 9t . \sin \frac{x}{4}$$

$$\frac{\partial^2 U}{\partial t^2} = -81 \sin 9t . \sin \frac{x}{4}$$

Again

$$\frac{\partial U}{\partial x} = \frac{1}{4} \sin 9t . \cos \frac{x}{4}$$

$$\frac{\partial^2 U}{\partial x^2} = -\frac{1}{16} \sin 9t . \sin \frac{x}{4}$$

Putting these values in eqn (1), we get

$$-81 \sin 9t . \sin \frac{x}{4} = -\frac{1}{16} \sin 9t . \sin \frac{x}{4}$$

$$c^2 = 1296$$

$$c = \pm 36$$

Thus U is satisfy eqn (1) with $c = \pm 36$

2. Verify one dimensional heat eqn

(b) $U = e^{-4t} \cos 3x$

Solution :- We have

$$U = e^{-4t} \cos 3x$$

We know one dimensional heat eqn is

$$\frac{\partial U}{\partial t} = c^2 \frac{\partial^2 U}{\partial x^2} \quad \text{--- (1)}$$

Now,

$$\frac{\partial U}{\partial t} = -4 e^{-4t} \cos 3x$$

Again

$$\frac{\partial U}{\partial x} = -3 e^{-4t} \sin 3x$$

$$\frac{\partial^2 U}{\partial x^2} = -9 e^{-4t} \cos 3x$$

putting these values in eqn (1) we get

$$-4 e^{-4t} \cos 3x = c^2 \left(-9 e^{-4t} \cos 3x\right)$$

or $4 = c^2 9$

or $c^2 = \frac{4}{9}$

$\therefore c = \pm \frac{2}{3}$

Thus U satisfy (1) with $c = \pm \frac{2}{3}$

③ Veriby the function to satisby two dimensional Laplace equation

ⓒ $U = \tan^{-1}\left(\frac{y}{x}\right)$

Solution: We know two dimensional Laplace equation is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{---} (1)$$

here

$$U = \tan^{-1}\left(\frac{y}{x}\right)$$

then $$\frac{\partial U}{\partial x} = \frac{\partial \tan^{-1}(y/x)}{\partial (y/x)} \times \frac{\partial (y/x)}{\partial x}$$

$$= \frac{1}{1+\frac{y^2}{x^2}} \times -\frac{y}{x^2}$$

$$= \frac{-y}{x^2+y^2}$$

$$\therefore \frac{\partial^2 U}{\partial x^2} = \frac{\partial\left(-\frac{y}{x^2+y^2}\right)}{\partial x}$$

$$= \frac{2xy}{(x^2+y^2)^2}$$

Also $$\frac{\partial U}{\partial y} = \frac{x}{x^2+y^2} \Rightarrow \frac{\partial^2 U}{\partial y^2} = -\frac{2xy}{(x^2+y^2)^2}$$

Putting these values in eqn (1), we get

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = \frac{2xy}{(x^2+y^2)^2} - \frac{2xy}{(x^2+y^2)^2}$$

$$= 0$$

Thus U Satisfy Laplace equation.

(4) Solve the following partial differential eqn

(a) $U_y = U$

or $\frac{\partial U}{\partial y} = U \Rightarrow \frac{\partial U}{U} = \partial y$

integrating both sides

$\log U = y + C$ where c is constant

$U = e^{y+C}$

$U = e^{y} . e^{C}$

$\therefore U = K . e^{y}$ where $e^{C} = K$

(e) $U_{xx} - U = 0$

Solution:- The auxillary eqn is

$m^2 - 1 = 0$

or $m^2 = 1$

or $m = \pm 1$

Then the general solution is

$U = C_1 . e^{x} + C_2 . e^{-x}$ //

Exercise 8.1

1. **Verify the given function to satisfy one dimensional wave equation.**

a) $u = \sin 9t \sin\frac{x}{4}$ b) $u = \cos 4t \sin 2x$

c) $u = \sin ct \sin x$

2. **Verify the given function to satisfy one dimensional heat equation.**

a) $u = e^{-t} \sin x$ b) $u = e^{-4t} \cos 3x$

c) $u = e^{-9t} \cos wx$

3. **Verify the given function to satisfy two dimensional laplace equation.**

a) $u = 2xy$ b) $u = e^x \sin y$

c) $u = \tan^{-1}\left(\frac{y}{x}\right)$

4. **Solve the following partial differential equations:**

a) $u_y = u$ b) $u_{yy} = 0$

c) $u_{xy} = u_x$ d) $u_y = 2xy_u$

e) $u_{xx} - u = 0$

Answers

4. (a) $u = c(x)\, e^y$ (b) $u = h(x)\, y + k(x)$

(c) $u = c(x)\, e^y + h(y)$ (d) $u = c(x)\, e^{xy^2}$

(e) $u = A(y)\, e^x + B(y)\, e^{-x}$.