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Mathematics-II
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# UNIT-1

# LINEAR EQUATIONS IN LINEAR ALGEBRA

⊛ Linear equation(s) → An eq$^n$ of the form $a_1x_1+a_2x_2+\ldots+a_nx_n=b$, where $a_i(i=1,\ldots,n)$ and b are real (or complex) numbers is a linear equation between the variables $x_1,\ldots,x_n$ (unknowns).

For example

$$3x_1-2x_1+x_3=5 \text{ is a linear equation}$$

but $2\sqrt{x_1}+3x_2-x_3=1$ is not a linear equation.

⊛. System of linear equations:-

A collection of one or more linear equations between the same variables constitutes a system.

For example:

$$3x_1+x_2-x_3=1$$
$$x_1+x_3=4$$
$$x_1+3x_2+4x_3=-1$$

is a system of linear equations consisting of 3 variables.

⊛ Solution of linear equations:-

Def$^n$ → It is the set of the values of the variables satisfying the given system of linear equations. A system of linear equations may be consistent or inconsistent.

Consistent → Having unique solution or infinitely many solutions.

Inconsistent → If it has no solution.

⊛. Echelon & Row reduced echelon form of a matrix:- ( V.Imp)

Echelon form → A matrix is said to be in echelon form if it satisfies the following three conditions.

i) Non-zero rows are above the zero rows.
i.e, All the zero rows if exists are to be at the bottom.

ii) The leading entity (non-zero element) of any row (except the first) should be in the column to the right of the leading entity of the row above.

iii) All the members in the column below the leading entity should be zero.

## Row reduced echelon form of a matrix (RREF):-

In addition to previous three conditions if a matrix satisfies other two following conditions also then it is said to be in RREF.

i) Leading entity in a row should be unity only.

ii) Unity is only the non-zero entity in the column in which it belongs.

# Below are two structures that demonstrate echelon and row reduced echelon forms.

$$\begin{bmatrix} 0 & \boxed{*} & * & * & * & * \\ 0 & 0 & \boxed{*} & * & * & * \\ 0 & 0 & 0 & 0 & \boxed{*} & * \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & \boxed{1} & 0 & * & 0 & * \\ 0 & 0 & \boxed{1} & * & 0 & * \\ 0 & 0 & 0 & 0 & \boxed{1} & * \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

It is in echelon form — It is row reduced echelon form (RRE

Representation in above diagrams or structures

□ → represents leading entity or called pivot element and it's column is pivot column.

* → represents any non-zero number.

## ⊛. Reduction of a matrix into echelon or row reduced echelon form

A matrix can be reduced into the echelon or row reduced echelon form by performing following three operations.

i) Interchange → Any two rows can be interchanged symbol '⟷' represents interchange.

ii) Scaling → Any row can be multiplied by a scalar.

iii) Replacement → Any row can be replaced by sum of it and scalar multiple of the other.

'→' sign represents replacement or scaling.

'∼' represents equivalent.

* **Example** to demonstrate echelon form and row reduced echelon form.

# Reduce the matrix say $A=\begin{bmatrix} 2 & 1 & 5 & 3 & 4 \\ 0 & 7 & 9 & 1 & 8 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 3 & 4 & 2 & -1 \end{bmatrix}$

Solution

Here, matrix $A=\begin{bmatrix} 2 & 1 & 5 & 3 & 4 \\ 0 & 7 & 9 & 1 & 8 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 3 & 4 & 2 & -1 \end{bmatrix}$

not necessary to write for understanding only

Now using the conditions for echelon form that we wrote before, using row operations.

$R_3 \leftrightarrow R_4$

$$A=\begin{bmatrix} 2 & 1 & 5 & 3 & 4 \\ 0 & 7 & 9 & 1 & 8 \\ 1 & 3 & 4 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

non-zero rows should be above zero rows

$R_1 \leftrightarrow R_3$

$$A=\begin{bmatrix} 1 & 3 & 4 & 2 & -1 \\ 0 & 7 & 9 & 1 & 8 \\ 2 & 1 & 5 & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_3 \rightarrow R_3 + (-2)R_1$

$$A=\begin{bmatrix} 1 & 3 & 4 & 2 & -1 \\ 0 & 7 & 9 & 1 & 8 \\ 0 & -5 & -3 & -1 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_3 \rightarrow R_3 + \frac{5}{7}R_2$

$$A=\begin{bmatrix} \boxed{1} & 3 & 4 & 2 & -1 \\ 0 & \boxed{7} & 9 & 1 & 8 \\ 0 & 0 & \boxed{24/7} & -2/7 & 82/7 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Now it is in echelon form.

understanding only

In addition to make it RREF we make pivot elements (□) 1 (unity) and making the pivot column elements all zero except unity pivot element.

Again

$R_3 \to \frac{7}{24} R_3$

$$A = \begin{bmatrix} 1 & 3 & 4 & 2 & -1 \\ 0 & 7 & 9 & 1 & 8 \\ 0 & 0 & 1 & -1/12 & 41/12 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_2 \to \frac{1}{7} R_2$

$$A = \begin{bmatrix} 1 & 3 & 4 & 2 & -1 \\ 0 & 1 & 9/7 & 1/7 & 8/7 \\ 0 & 0 & 1 & -1/12 & 41/12 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_1 \to R_1 + (-3) R_2$

$$A = \begin{bmatrix} 1 & 0 & 1/7 & 11/7 & -31/7 \\ 0 & 1 & 9/7 & 1/7 & 8/7 \\ 0 & 0 & 1 & -1/12 & 41/12 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_1 \to R_1 + \left(-\frac{1}{7}\right) . R_3$

$$A = \begin{bmatrix} 1 & 0 & 0 & 133/84 & -413/84 \\ 0 & 1 & 9/7 & 1/7 & 8/7 \\ 0 & 0 & 1 & -1/12 & 41/12 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_2 \to R_2 + \left(-9/7\right) . R_3$

box not compulsory only for understanding pivot element

$$A = \begin{bmatrix} \boxed{1} & 0 & 0 & 133/84 & -413/84 \\ 0 & \boxed{1} & 0 & 21/84 & -273/84 \\ 0 & 0 & \boxed{1} & -1/12 & 41/12 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Now it is in row reduced echelon form (RREF).

# Solution of system of linear equations:-

Procedure:

i) Construct the augmented matrix of the system as in below example.

ii) Reduce the augmented matrix into echelon form by the row operations interchange, scaling, replacement.

iii) If the right most column in echelon form be the pivot column, then no solution exists otherwise solution of system exists.

iv) Reduce the echelon form of the matrix into the row reduced echelon form.

v) Find general solution as in example below.

Example 1

Solve the system of linear equations:-

$$3x_1 + 2x_2 - x_3 = 4$$
$$x_1 + 3x_2 + x_3 = 1$$
$$4x_1 + x_2 - 2x_3 = 0.$$

Now, consider the augmented matrix of the system.

$$\begin{bmatrix} 3 & 2 & -1 & 4 \\ 1 & 3 & 1 & 1 \\ 4 & 1 & -2 & 0 \end{bmatrix}$$

$R_1 \longleftrightarrow R_2$

$$\begin{bmatrix} 1 & 3 & 1 & 1 \\ 3 & 2 & -1 & 4 \\ 4 & 1 & -2 & 0 \end{bmatrix}$$

$R_2 \rightarrow R_2 + (-3)R_1$

& $R_3 \rightarrow R_3 + (-4)R_1$

$$\begin{bmatrix} 1 & 3 & 1 & 1 \\ 0 & -7 & -4 & 1 \\ 0 & -11 & -6 & -4 \end{bmatrix}$$

$R_2 \rightarrow \left(-\frac{1}{7}\right)R_2$

$$\begin{bmatrix} 1 & 3 & 1 & 1 \\ 0 & 1 & 4/4 & -1/7 \\ 0 & -11 & -6 & -4 \end{bmatrix}$$

$R_3 \to R_3 + 11R_2$

$$\begin{bmatrix} 1 & 3 & 1 & 1 \\ 0 & 1 & 4/7 & -1/7 \\ 0 & 0 & 2/7 & -39/7 \end{bmatrix}$$

$R_3 \to \frac{7}{2}R_3$

$$\begin{bmatrix} 1 & 3 & 1 & 1 \\ 0 & 1 & 4/7 & -1/7 \\ 0 & 0 & 1 & -39/2 \end{bmatrix}$$

$R_1 \to R_1 + (-1)R_3$

$R_2 \to R_2 + \left(-\frac{4}{7}\right) \cdot R_3$

$$\begin{bmatrix} 1 & 3 & 0 & 41/2 \\ 0 & 1 & 0 & 11 \\ 0 & 0 & 1 & -39/2 \end{bmatrix}$$

$R_1 \to R_1 + (-3)R_2$

$$\begin{array}{cccc} x_1 & x_2 & x_3 & \\ \end{array}$$
$$\begin{bmatrix} 1 & 0 & 0 & -25/2 \\ 0 & 1 & 0 & 11 \\ 0 & 0 & 1 & -39/2 \end{bmatrix}$$

$\therefore$ The system has unique solution.

$x_1 = -25/2$

$x_2 = 11$

& $x_3 = -39/2$

Example 2:- Solve the system of linear equations

$$x_1+2x_2-x_3=4$$
$$-2x_1+x_2+5x_3=-1$$

Here,

Augmented matrix of the given system is

$$\begin{bmatrix} 1 & 2 & -1 & 4 \\ -2 & 1 & 5 & -1 \end{bmatrix}$$

$R_2 \to R_2+2R_1$

$$\begin{bmatrix} 1 & 2 & -1 & 4 \\ 0 & 5 & 3 & 7 \end{bmatrix}$$

$R_2 \to \frac{1}{5}R_2$

$$\begin{bmatrix} 1 & 2 & -1 & 4 \\ 0 & 1 & 3/5 & 7/5 \end{bmatrix}$$

$R_1 \to R_1+(-2)R_2$

$$\begin{bmatrix} 1 & 0 & -11/5 & 6/5 \\ 0 & 1 & 3/5 & 7/5 \end{bmatrix}$$

The augmented matrix corresponds to the system

$$x_1-11/5\,x_3=6/5$$
$$x_2+\tfrac{3}{5}x_3=7/5$$
$$x_3=x_3$$

Reason

$x_3$ छैन त्यसमा free छ but 4 columns छन् so, 3 eqn हुँदैन (column भन्दा एक कम) त्यसैले यस्तो लेख्न मिल्छ free variable भाको बेला

The general solution is,

$$x_1=11/5\,x_3+6/5$$
$$x_2=-3/5\,x_3+7/5$$
$$x_3=1x_3+0$$

$\therefore$ The solution is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 11/5 \\ -3/5 \\ 1 \end{bmatrix} x_3 + \begin{bmatrix} 6/5 \\ 7/5 \\ 0 \end{bmatrix}.$$

Example 3:- Determine the solution of the system.

$$2x_1 + x_2 + 4x_3 - x_4 = 0$$
$$4x_1 + 3x_2 + x_3 + 2x_4 = 0$$
$$x_1 + 2x_2 + 5x_3 + 3x_4 = 0.$$

Soln

Augmented matrix of the system is.

$$\begin{bmatrix} 2 & 1 & 4 & -1 & 0 \\ 4 & 3 & 1 & 2 & 0 \\ 1 & 2 & 5 & 3 & 0 \end{bmatrix}$$

$R_1 \leftrightarrow R_3$

$$\begin{bmatrix} 1 & 2 & 5 & 3 & 0 \\ 4 & 3 & 1 & 2 & 0 \\ 2 & 1 & 4 & -1 & 0 \end{bmatrix}$$

$R_2 \rightarrow R_2 + (-4)R_1$

$R_3 \rightarrow R_3 + (-2)R_1$

$$\begin{bmatrix} 1 & 2 & 5 & 3 & 0 \\ 0 & -5 & -19 & -10 & 0 \\ 0 & -3 & -6 & -7 & 0 \end{bmatrix}$$

$R_2 \rightarrow -\frac{1}{5}R_2$

$$\begin{bmatrix} 1 & 2 & 5 & 3 & 0 \\ 0 & 1 & 19/5 & 2 & 0 \\ 0 & -3 & -6 & -7 & 0 \end{bmatrix}$$

$R_3 \rightarrow R_3 + 3R_2$

$$\begin{bmatrix} 1 & 2 & 5 & 3 & 0 \\ 0 & 1 & 15/5 & 2 & 0 \\ 0 & 0 & 27/5 & -1 & 0 \end{bmatrix}$$

$R_3 \rightarrow \frac{5}{27} R_3$

$$\begin{bmatrix} 1 & 2 & 5 & 3 & 0 \\ 0 & 1 & 19/5 & 2 & 0 \\ 0 & 0 & 1 & -5/27 & 0 \end{bmatrix}$$

$R_1 \rightarrow R_1 + (-5)R_3$

$R_2 \rightarrow R_2 + (-19/5)R_3$

$$\begin{bmatrix} 1 & 2 & 0 & \frac{106}{27} & 0 \\ 0 & 1 & 0 & 73/27 & 0 \\ 0 & 0 & 1 & -5/27 & 0 \end{bmatrix}$$

$R_1 \to R_1 + (-2)R_1$

$$\begin{bmatrix} 1 & 0 & 0 & -40/27 & 0 \\ 0 & 1 & 0 & 73/27 & 0 \\ 0 & 0 & 1 & -5/27 & 0 \end{bmatrix}$$

There are total 5 columns so it contain 4 variables in total. The variables in above $x_1, x_2, x_3$ ~~are~~ corrosponding to pivot columns are basic variable and $x_4$ is absent so, $x_4$ is free variable. The RREF of augmented matrix equivalent to this system is;

$$x_1 - \frac{40}{27}x_4 = 0.$$

$$x_2 + \frac{73}{27}x_4 = 0$$

$$x_3 - \frac{5}{27}x_4 = 0$$

$x_4$ is free.

$\therefore$ The general solution is $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} \frac{40}{27} \\ -73/27 \\ 5/27 \\ 1 \end{bmatrix} x_4.$

Here, $b = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$.

Hence, According to the value of $x_4$ of any choice (since it is free) the system has infinitely many solutions.

Note: 1) The system of linear equations having matrix equation $Ax=b$ is homogenous if $b=0$. as we saw in example 3 & the system is non-homogenous if $b \neq 0$ as in example 1 and 2.

2) The homogeneous equation $Ax=0$ is satisfied $x=0$ (obviously). The solution is called the trivial solution & The homogenous eqn $Ax=0$ may also be satisfied for $x \neq 0$. Then the solution is called the non-trivial solution.

## Applications of system of linear equations:-

### Balencing Chemical Reactions:

**Example 1:** $x_1 H_2 + x_2 O_2 \longrightarrow x_3 H_2 O$

**Sol$^n$**

$x_1$ मा 2H छन्  H $x_2$ मा छैन

$$x_1 \begin{bmatrix} 2 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 2 \end{bmatrix} = x_3 \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$x_1$ मा O छैन  दुईवटा O

Now,

$$x_1 \begin{bmatrix} 2 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 2 \end{bmatrix} + x_3 \begin{bmatrix} -2 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

i.e, $2x_1 + 0.x_2 - 2x_3 = 0$

& $0.x_1 + 2x_2 - 1.x_3 = 0$

Augmented matrix of system is,

$$\begin{bmatrix} 2 & 0 & -2 & 0 \\ 0 & 2 & -1 & 0 \end{bmatrix}$$

$R_1 \to \frac{1}{2} R_1$ , $R_2 \to \frac{1}{2} R_1$

$$\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -1/2 & 0 \end{bmatrix}$$

Now $x_1 - x_3 = 0$ or, $x_1 = x_3$ / $x_2 - \frac{1}{2}x_3 = 0$ or, $x_2 = \frac{1}{2}x_3$.

$$\therefore X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1/2 \\ 1 \end{bmatrix} x_3$$

Since molecule can not be in fraction let we take $x_3 = 2$ then,

$x_1 = 2, x_2 = 1$ and $x_3 = 2$.

Now, chemical reaction becomes.

$\therefore 2H_2 + O_2 \to 2H_2O$

which is balanced.

Example 2: Balance the following chemical reaction.

Soln $Al + O_2 \rightarrow Al_2O_3$

Let, $x_1 Al + x_2 O_2 \rightarrow x_3 Al_2O_3$.

$$x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 2 \end{bmatrix} = x_3 \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 2 \end{bmatrix} - x_3 \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 2 \end{bmatrix} + x_3 \begin{bmatrix} -2 \\ -3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

i.e, $1x_1 + 0.x_2 - 2x_3 = 0$.

$0.x_1 + 2.x_2 - 3.x_3 = 0$.

Augmented matrix of the system is,

$$\begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 2 & -3 & 0 \end{bmatrix}$$

$R_2 \rightarrow \frac{1}{2} R_2$

$$\begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & -3/2 & 0 \end{bmatrix}$$

Now, $x_1 - 2x_3 = 0$ or, $x_1 = 2x_3$

& $x_2 - \frac{3}{2} x_3 = 0$ or, $x_2 = \frac{3}{2} x_3$

$$\therefore X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 3/2 \\ 1 \end{bmatrix} x_3.$$

Since molecule can not be in fraction let we take $x_3 = 2$ than,

$x_1 = 4$, $x_2 = 3$ and $x_3 = 2$.

Now, the chemical reaction becomes.

$4Al + 3O_2 \rightarrow 2Al_2O_3$ which is balanced.

## Linearly dependent and independent vectors:

A set of vectors $\{v_1, v_2, \ldots, v_n\}$ in V is said to be linearly dependent if their linear combination is zero for at least one scalar is non-zero.

i.e, $\{v_1, v_2, \ldots v_n\}$ is linearly dependent $\Leftrightarrow$ In the linear combination $c_1v_1 + c_2v_2 + \ldots + c_nv_n = 0$, for at least one $c_i\,(i=1,\ldots,n) \neq 0$.

Linearly independent → A set of vectors $\{v_1, v_2, \ldots v_n\}$ is a vector space V is said to be linearly independent if their linear combination $c_1v_1 + c_2v_2 + \ldots + c_nv_n$ is zero only when each scalar is zero.

i.e, $\{v_1, v_2, \ldots v_n\}$ is linearly independent $\Leftrightarrow c_1 = c_2 = \ldots = c_n = 0$, for $c_1v_1 + c_2v_2 + \ldots + c_nv_n = 0$.

**Note:**

i) A single zero vector is obviously linearly dependent.

ii) A set of vectors with two or more element is linearly dependent if one of the vector among the set can be expressed as the linear combination of the remaning.

For e.g.
$$\text{at least}\begin{cases} \text{i) } 2(1,3) + 2(-1,-3) = (0,0) \\ \text{ii) } 2(0,0) + 0(1,3) = (0,0) \end{cases} \Bigg\} \text{ linearly dependent or (non-trivial solution)}$$

iii) A linearly independent set of vectors can be made linearly dependent by introducing (including) a zero vector in the set.

for e.g. $0(5,-3) + 0(2,3) = (0,0)$ → linearly independent (or trivial solution)

only for each