Convolution Theorem

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Convolution between two functions $f(x)$ and $g(x)$ is defined by

$$f * g = \int_{-\infty}^{\infty} f(u)g(x-u)du$$

or

$$f * g = \int_{-\infty}^{\infty} g(u)f(x-u)du$$

Convolution Theorem: The Fourier transform of convolution between two functions $f(x)$ and $g(x)$ is $\sqrt{2\pi}$ times the product of their Fourier transforms.

$$i.e. \mathcal{F}(f * g) = \sqrt{2\pi}\mathcal{F}(f).\mathcal{F}(g)$$

Proof: The convolution between two functions $f(x)$ and $g(x)$ is given by

$$f * g = \int_{-\infty}^{\infty} f(u)g(x-u)du$$

taking Fourier transform on both sides, we get

$$\mathcal{F}(f * g) = \mathcal{F}\left(\int_{-\infty}^{\infty} f(u)g(x-u)du\right)$$

$$= \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} \left(\int\limits_{-\infty}^{\infty} f(u)g(t-u)du \right) e^{-i\omega t} dt$$

$$i.e.\mathcal{F}(f * g) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} \int\limits_{-\infty}^{\infty} f(u)g(t-u)e^{-i\omega t} dtdu.... \quad ... \quad (1)$$

Put

$$t - u = z$$

so that

$$t = u + z$$

and

$$dt = dz$$

Also when $t \to -\infty$ then $z \to -\infty$ and when $t \to \infty$, $z \to \infty$

So from equation (1) we get

$$\mathcal{F}(f * g) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(u)g(z).e^{-i\omega(u+z)} dzdu$$

$$= \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} \int\limits_{-\infty}^{\infty} f(u) g(z) . e^{-i\omega u} . e^{-i\omega z} dz du$$

$$= \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(u) e^{-i\omega u} du \right) . \left(\int_{-\infty}^{\infty} g(z) . e^{-i\omega z} dz \right)$$

$$= \mathcal{F}(f)\sqrt{2\pi}\left(\frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{\infty} g(z).e^{-i\omega z}dz\right)$$

$$= \mathcal{F}(f)\sqrt{2\pi}.\mathcal{F}(g)$$

Hence,

$$\mathcal{F}(f * g) = \sqrt{2\pi}\mathcal{F}(f).\mathcal{F}(g)$$

This completes the proof.

Find Fourier sine transform of

$$f(x) = \frac{e^{-ax}}{x}$$

for $x > 0$ and $a > 0$. Also, show that

$$\int_{0}^{\infty} tan^{-1}\left(\frac{x}{a}\right) sinxdx = \frac{\pi}{2} e^{-a}$$

The Fourier sine transform of given function is

$$\mathcal{F}_s(f) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(t) \sin \omega t dt = I(say)$$

$$i.e. I = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{e^{-at}}{t} \sin \omega t dt \ldots \quad ..(1)$$

Differentiating with respect to ω, we get

$$\frac{dI}{d\omega} = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \frac{e^{-at}}{t} \frac{d}{d\omega}(\sin \omega t) dt$$

$$or, \frac{dI}{d\omega} = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} \frac{e^{-at}}{t} t \cos \omega t dt$$

$$= \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} e^{-at} \cos \omega t dt$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{e^{-at}}{a^2 + \omega^2} \left(-a \cos \omega t + \omega \sin \omega t \right) \right]_0^\infty$$

$$= \sqrt{\frac{2}{\pi}} \left[0 - \frac{e^0}{a^2 + \omega^2} (-a.1 + 0) \right]$$

$$i.e. \frac{dI}{d\omega} = \sqrt{\frac{2}{\pi}} \left(\frac{a}{a^2 + \omega^2} \right)$$

Integrating w.r. to ω we get

$$I = \sqrt{\frac{2}{\pi}} \tan^{-1}\left(\frac{\omega}{a}\right) + C....(2)$$

where C is the constant of integration.

Initially, when $\omega = 0$, then from (1) $I = 0$ Also, from (2),

$$0 = 0 + C$$

$$i.e. C = 0$$

$$i.e. \mathcal{F}_s(f) = I = \sqrt{\frac{2}{\pi}} \tan^{-1}\left(\frac{\omega}{a}\right)$$

which is required Fourier sine transform of given function.

Next , using inversion formula for sine transform, we get

$$f(x) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} \{\mathcal{F}_s(f)\} \sin \omega x d\omega$$

$$\frac{e^{-ax}}{x} = \sqrt{\frac{2}{\pi}} \int_0^\infty \left\{ \sqrt{\frac{2}{\pi}} \tan^{-1}\left(\frac{\omega}{a}\right) \right\} \sin \omega x d\omega$$

Put $x = 1$, then we get

$$\frac{\pi}{2} e^{-a} = \int_0^{\infty} \tan^{-1}\left(\frac{\omega}{a}\right) \sin \omega d\omega$$

Replacing ω by x we get

$$\int_0^\infty \tan^{-1}\left(\frac{x}{a}\right)\sin x dx = \frac{\pi}{2}e^{-a}$$

which is required result.