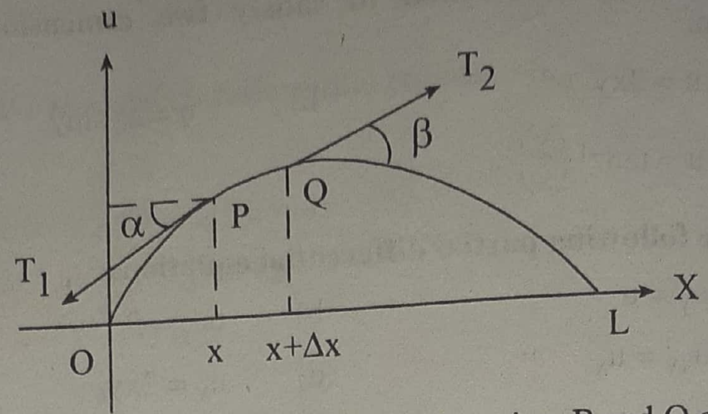
One Dimensional Wave Equation

An elastic string is tightly stretched to length L and two ends are fixed such that

1. The mass of the string per unit length is constant. The string is perfectly elastic and doesnot offer any resistance to bending.
2. The tension caused by stretching the string before fixing it at the ends is so large that the action of the gravitational force on the string can be neglected.
3. The string performs small transverse motions in a vertical plane; that is, every particle of the string moves strictly vertically and so that the deflection and the slope at every point of the string always remain small in absolute value.

Derivation of one Dimensional Wave Equation

Let us suppose that the force acting on small portion of the string. Since the string does not offer resistance to bending, the tension is tangential to the curve of the string at each point.



Let T_1 and T_2 are tensions at the ends points P and Q of that portion. Since the points of the string move vertically, there is no motion in the horizontal direction. Therefore the horizontal components of the tension must be constant.

Therefore from figure, we get

$T_1 \cos\alpha = T_2 \cos\beta = T$ (say) = constant ....(1)

In the vertical direction, forces in vertical components are $-T_1 \sin\alpha$ and $T_2 \sin\beta$ of T_1 and T_2. Here negative sign appears because the component at P is directed downward.

Then, by Newtons second law, we get,

Resultant force is equals to mass of the portion times the acceleration. That is,

$$T_2 \sin\beta - T_1 \sin\alpha = \rho\Delta x \frac{\partial^2 u}{\partial t^2} \quad \text{........(2)}$$

where ρ be the mass of undetected string per unit length, Δx is the length of the portion of the undeflected string.

Dividing equation (2) by equation (1) we get,

$$\frac{T_2 \sin\beta}{T_2 \cos\beta} - \frac{T_1 \sin\alpha}{T_1 \cos\alpha} = \frac{\rho\Delta x}{T} \cdot \frac{\partial^2 u}{\partial t^2}$$

$$\Rightarrow \quad \tan\beta - \tan\alpha = \frac{\rho\Delta x}{T} \cdot \frac{\partial^2 u}{\partial t^2} \quad \text{...........(3)}$$

We know, tanα and tan β are the slopes of the string at x and x+Δx respectively.

Where, $\tan\alpha = \left.\frac{\partial u}{\partial x}\right|_{\text{at } x}$

and,
$$\tan\beta = \left.\frac{\partial u}{\partial x}\right|_{\text{at } x+\Delta x}$$

From equation (3), we get

$$\left[\left.\frac{\partial u}{\partial x}\right|_{x+\Delta x} - \left.\frac{\partial u}{\partial x}\right|_{x}\right] = \frac{\rho\Delta x}{T}\frac{\partial^2 u}{\partial t^2}$$

$$\Rightarrow \quad \frac{1}{\Delta x}\left[\left.\frac{\partial u}{\partial x}\right|_{x+\Delta x} - \left.\frac{\partial u}{\partial x}\right|_{x}\right] = \frac{\rho}{T}\frac{\partial^2 u}{\partial t^2}$$

Taking limit Δx tends to zero on both side, we get,

$$\lim_{\Delta x \to 0}\frac{1}{\Delta x}\left[\left.\frac{\partial u}{\partial x}\right|_{x+\Delta x} - \left.\frac{\partial u}{\partial x}\right|_{x}\right] = \frac{\rho}{T}\frac{\partial^2 u}{\partial t^2}$$

$$\Rightarrow \quad \frac{\partial^2 u}{\partial x^2} = \frac{\rho}{T}\frac{\partial^2 u}{\partial t^2}$$

$$\Rightarrow \quad \frac{T}{\rho}\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$$

$$\therefore \quad \frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}, \text{ where } c^2 = \frac{T}{\rho}.$$

This is the required one dimensional wave equation.

Solution of Dimensional Wave Equation under Certain Initial and Boundary Conditions

We have one dimensional wave equation is

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2} \quad \text{.........(1)}$$

where $u(x,t)$ is the deflection of the string, with boundary condition

$$u(0,t) = 0 \text{ and } u(L,t) = 0 \text{ for all t. ....(2)}$$

And initial condition

$$u(x,0) = f(x) \ \text{(initial deflection)} \quad \text{.........3)}$$

$$\left.\frac{\partial u}{\partial t}\right|_{t=0} = g(x) \ \text{(initial velocity)} \quad \text{..........(4)}$$

$$\text{Let } u(x,t) = F(x)\,G(t) \quad \text{.........5)}$$

be the solution of (1). Then by differentiating we get

$$\frac{\partial^2 u}{\partial t^2} = F\ddot{G} \text{ and } \frac{\partial^2 u}{\partial x^2} = F''G$$

where dot denotes derivative with respect to t and prime denotes derivative with respect to x.

Putting these values in equation (1) we get

$$F\ddot{G} = c^2 F''G$$

$$\Rightarrow \quad \frac{\ddot{G}}{c^2 G} = \frac{F''}{F} = k \text{ (say)}$$

This gives

$$F'' - kF = 0 \qquad \text{.........(6)}$$

and

$$\ddot{G} - c^2 kG = 0 \qquad \text{.........(7)}$$

which are ordinary differential equation.

We have to find F and G from equation (6) and (7) under the boundary condition (2).

$0 = u(0,t) = F(0)\,G(t)$

and $\quad 0 = u(L, t) = F(L)\,G(t) \quad$ for all t.

Solving Equation (6):

If $G = 0$, then $u = 0$, which is of no interest. So $G \neq 0$ and we get

$$F(0) = 0 = F(L) \qquad \text{.........(8)}$$

For $k = 0$, then the general solution of equation (6) is

$F = ax + b$

and from (8), we get $a = 0 = b$. Then $F = 0$, which is of no interest because $u = 0$.

Again if k is positive and put $k = \lambda^2$, then the general solution of (6) is

$F = Ae^{\lambda x} + Be^{-\lambda x}$

and from equation (8), we get $F = 0$ which is of no interest. Thus we left with all possibilities except k is negative.

let $k = -p^2$. Then equation (6) can be written as

$F'' + P^2F = 0$

Its general solution is

$F(x) = A\cos px + B\sin px$

From equation (8), we get

$F(0) = A = 0$

$F(L) = B \sin pL = 0$

let $B \neq 0$, Since otherwise $F = 0$. So, we get $\sin pL = 0$

$PL = n\pi$

$\Rightarrow$

$P = \frac{n\pi}{L}$ for n is an integer.

we have $B \neq 0$, setting $B = 1$, we get

$F(x) = \sin \frac{n\pi}{L} x$

We obtain infinitely many solutions

$F(x) = F_n(x)$, thus

$$F_n(x) = \sin\frac{n\pi}{L} x \text{ for } n = 1, 2, 3, \qquad \text{.........(9)}$$

Solving equation (7):

We have $k = -p^2 = -\left(\frac{n\pi}{L}\right)^2$. Thus equation (7) is

$$\ddot{G} - c^2 k G = 0$$

$$\Rightarrow \quad \ddot{G} + \lambda_n^2 G = 0 \qquad \text{where } \lambda_n = \frac{cn\pi}{L}$$

Its solution is $\quad G_n(t) = B_n \cos\lambda_n t + B_n^* \sin\lambda_n t$

Therefore required solution of equation (1) is

$$u_n(x, t) = (B_n \cos\lambda_n t + B_n^* \sin\lambda_n t) \sin\frac{n\pi}{L} x$$

for $n = 1, 2, 3, \ldots$

These functions are called the eigen functions or characteristics functions and the values $\lambda_n = \frac{cn\pi}{L}$ are called eigen values or characteristics values of the vibrating string. The set $\{\lambda_1, \lambda_2, \lambda_3, \ldots\ldots\}$ is called the spectrum.

Again, a single solution $u_n(x, t)$ will not a general solution of given equation. Then by fundamental theorem the solution of given wave is the sum of finitely many u_n.

Thus,

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t)$$

$$u(x,t) = \sum_{n=1}^{\infty} (\cos\lambda_n t + B_n \sin\lambda_n t) \sin\frac{n\pi}{L} x \qquad \text{.........(10)}$$

From equation (3) and (4)

$$u(x,0) = f(x) \quad \text{and} \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = g(x).$$

From equation (10) we get,

$$u(x, 0) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi}{L} x = f(x)$$

$$\Rightarrow \qquad f(x) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi}{L} x$$

which is fourier sine series, the coefficient

$$B_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi}{L} x\, dx$$

Also, we have

$$\left.\frac{\partial u}{\partial t}\right|_{t=0} = g(x).$$

$$\Rightarrow \qquad \left[\sum_{n=1}^{\infty} (-B_n\lambda_n \sin\lambda_n t + B_n^*\lambda_n \cos\lambda_n t)\sin\frac{n\pi}{L}x\right]_{t=0} = g(x)$$

$$\Rightarrow \qquad \sum_{n=1}^{\infty} B_n^*\lambda_n \sin\frac{n\pi}{L}x = g(x)$$

Which is the fourier sine series of g(x) with period 2L. Then the coefficient

$$B_n^*\lambda_n = \frac{2}{L}\int_0^L g(x)\sin\frac{n\pi}{L}x\, dx\ .$$

Therefore we get required solution of one dimensional wave equation is

$$u = \sum_{n=1}^{\infty} (B_n \cos\lambda_n t + B_n^* \sin\lambda_n t)\sin\frac{n\pi}{L} x$$

$$\text{where } \lambda_n = \frac{cn\pi}{L}, \qquad B_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi}{L}x\, dx$$

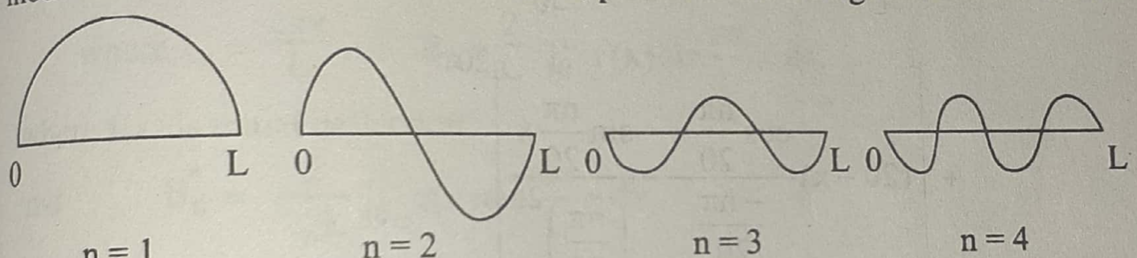
$$B_n^* = \frac{2}{L\lambda_n}\int_0^L g(x)\sin\frac{n\pi}{L}x\,dx$$

where f(x) and g(x) are initial deflection of string and initial velocity of the string respectively.

Note:

We get each u_n represents a harmonic motion having the frequency $\frac{\lambda_n}{2\pi} = \frac{cn}{2L}$ cycles per unit time. This motion is called the n^{th} normal mode of the string. The first normal mode is known as the fundamental mode (n = 1), and the others are known as overtones; musically they give the octave plus fifth, etc.

We have, $\sin\frac{n\pi}{L}x = 0$ at $x = \frac{L}{n}, \frac{2L}{n}, \ldots, \frac{(n-1)L}{2}$, the n^{th} normal mode has n – 1 so called nodes. That is, points of the string that denote move.



Normal modes of the vibrating string

Tuning is done by changing the tension T. We have the frequency $\frac{\lambda_n}{2\pi} = \frac{cn}{2L}$ of u_n with $c = T/\rho$ confirms that effect because it shows that the frequency is proportional to the tension T.

Example

A tight string of length 20 cm fastened at both ends is displaced from its position of equilibrium, by imparting to each of its points at initial velocity $g(x) = x$, for $0 \le x \le 10$

$$= 20 - x, \text{ for } 10 \le x \le 20.$$

where x being distance from one end. Find the displacement at any time t.

Solution

We have, solution of one dimensional wave equation

$$u(x, t) = \sum_{n=1}^{\infty}(B_n \cos\lambda_n t + B^*_n \sin\lambda_n t)\sin\frac{n\pi}{L}x$$

where $B_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi}{L}x\,dx = 0$, since $f(x) = 0$

and $B^*_n = \frac{2}{cn\pi}\int_0^L g(x)\sin\frac{n\pi}{L}x\,dx$

$$= \frac{2}{cn\pi}\int_0^{20} g(x)\sin\frac{n\pi}{20}x\,dx$$

$$= \frac{2}{cn\pi}\left[\int_0^{10} x\sin\frac{n\pi}{20}x\,dx + \int_{10}^{20}(20-x)\sin\frac{n\pi}{20}x\,dx\right]$$

$$= \frac{2}{cn\pi}\left[x\frac{\cos\frac{n\pi}{20}x}{\frac{-n\pi}{20}} + \frac{\sin\frac{n\pi}{20}x}{\left(\frac{n\pi}{20}\right)^2}\right]_0^{10}$$

$$+\left[(20-x)\frac{\cos\frac{n\pi}{20}x}{\frac{-n\pi}{20}} - \frac{\sin\frac{n\pi}{20}x}{\left(\frac{n\pi}{20}\right)^2}\right]_{10}^{20}$$

$$= \frac{2}{cn\pi}\left(\frac{-200}{n\pi}\cos\frac{n\pi}{2} + \frac{400}{n^2\pi^2}\sin\frac{n\pi}{2} + \frac{200}{n\pi}\cos\frac{n\pi}{2} + \frac{400}{n^2\pi^2}\sin\frac{n\pi}{2}\right)$$

$$= \frac{2}{cn\pi}\frac{800}{n^2\pi^2}\sin n\pi/2$$

$$B^*_n = \frac{1600}{cn^3\pi^3}\sin n\pi/2$$

Substituting these values in equation (1), we get

$$u(x,t) = \sum_{n=1}^{\infty}\frac{1600}{cn^3\pi^3}\sin\frac{n\pi}{2}\sin\lambda_n t\sin\frac{n\pi}{L}x$$

where, $\lambda_n = \frac{cn\pi}{L} = \frac{cn\pi}{20}$

$$\therefore\ u(x,t) = \frac{1600}{c\pi^3}\sum_{n=1}^{\infty}\sin\frac{n\pi}{2}\ \sin\frac{cn\pi}{20}t\ \sin\frac{n\pi}{20}x$$

This is the required solution of the given vibrating string.

Example

Find the solution of one dimensional wave equation corresponding to the triangular initial deflection

$$f(x) = \begin{cases} \frac{2k}{L}x & \text{if } 0 < x < \frac{L}{2} \\ \frac{2k}{L}(L-x) & \text{if } \frac{L}{2} < x < L \end{cases}$$ **and its initial velocity is zero.**

Solution

We know solution of one dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \text{ is}$$

$$u(x, t) = \sum_{n=1}^{\infty} (B_n \cos\lambda_n t + B_n^* \sin\lambda_n t) \sin\frac{n\pi}{L}x \quad \ldots(1)$$

where $\lambda_n = \frac{cn\pi}{L}$, $\quad B_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi}{L}x\,dx$

where f(x) is initial deflection

and $\quad B_n^* = \frac{2}{\lambda_n L}\int_0^L g(x)\sin\frac{n\pi}{L}x\,dx$

where g(x) is initial velocity.

But we have, $f(x) = \begin{cases} \frac{2k}{L}x & \text{if } 0 < x < \frac{L}{2} \\ \frac{2k}{L}(L-x) & \text{if } \frac{L}{2} < x < L \end{cases}$

and $\quad g(x) = 0.$

Therefore we get,

$$B_n = \frac{2}{L}\left[\int_0^{L/2} \frac{2k}{L}\sin\frac{n\pi}{L}x\,dx + \int_{L/2}^{L} \frac{2k}{L}(L-x)\sin\frac{n\pi}{L}x\,dx\right]$$

$$B_n = \frac{4k}{L^2}\left[\int_0^{L/2} x\sin\frac{n\pi}{L}x\,dx + \int_{L/2}^{L}(L-x)\sin\frac{n\pi}{L}x\,dx\right] \quad \ldots\ldots(2)$$

Here $\quad \int_0^{L/2} x\sin\frac{n\pi}{L}x\,dx = \left[\frac{x\cos\frac{n\pi}{L}x}{-\frac{n\pi}{L}} + \frac{\sin\frac{n\pi}{L}x}{\left(\frac{n\pi}{L}\right)^2}\right]_0^{\frac{L}{2}}$

$$= \left[\frac{\frac{L}{2}\cos\frac{n\pi}{2}x}{-\frac{n\pi}{L}} + \frac{\sin\frac{n\pi}{2}x}{\left(\frac{n\pi}{L}\right)^2} \right] = \frac{\sin n\pi/2}{\left(\frac{n\pi}{L}\right)^2}$$

Also,
$$\int_{L/2}^{L}(L-x)\sin\frac{n\pi}{L}x\,dx = \left[\frac{(L-x)\cos\frac{n\pi}{L}x}{-\frac{n\pi}{L}} - \frac{\sin\frac{n\pi}{L}x}{\left(\frac{n\pi}{L}\right)^2} \right]_{L/2}^{L}$$

$$= \left[0 - \frac{\frac{L}{2}\cos\frac{n\pi}{2}}{-\frac{n\pi}{L}} + \frac{\sin\frac{n\pi}{2}}{\left(\frac{n\pi}{L}\right)^2} \right] = \frac{\sin\frac{n\pi}{2}}{\left(\frac{n\pi}{L}\right)^2}$$

Therefore from equation (2) we get,

$$B_n = \frac{4k}{L^2}\,\frac{2\sin\frac{n\pi}{2}}{\left(\frac{n\pi}{L}\right)^2},\ B_n = \frac{8k}{(n\pi)^2}\sin\frac{n\pi}{2}$$

Also, $B_n^* = 0$, since $g(x) = 0$.

Therefore from equation (1),

$$u(x, t) = \sum_{n=1}^{\infty}\left(\frac{8k}{(n\pi)^2}\sin\frac{n\pi}{2}\cos\lambda_n t\right)\sin\frac{n\pi}{L}x$$

$$= \frac{8k}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\frac{n\pi}{2}\cos\frac{cn\pi}{L}t$$

$$u(x,t) = \frac{8k}{\pi^2}\left[\sin\frac{\pi}{L}x\cos\frac{\pi c}{L}t - \frac{1}{3^2}\sin\frac{3\pi}{L}\cos\frac{3\pi c}{L}t + \ldots\right]$$

This is the required solution of one dimensional wave equation with given initial velocity and initial deflection.

Example

Solve one dimensional wave equation with initial deflection is 0.01 sin3x and initial velocity is zero and $L = \pi$, $c^2 = 1$.

Scanned with CamScanner

Solution

We have one dimensional wave equation is

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

And its solution is

$$u(x,t) = \sum_{n=1}^{\infty} (B_n \cos\lambda_n t + B_n^* \sin\lambda_n t) \sin\frac{n\pi}{L} x \qquad \text{......(1)}$$

where $B_n = \frac{2}{L} \int_0^L f(x) \sin\frac{n\pi}{L} x \, dx$, λ_n; $B_n^* = \frac{2}{L} \int_0^L g(x) \sin\frac{n\pi}{L} x \, dx$

where f(x) is initial definition of the string and g(x) is initial velocity of the string.

Also, we have

$$f(x) = 0.01 \sin 3x$$

and $g(x) = 0$. Then we get, $B_n^* = 0$ and

We have

$$u(x, 0) = 0.01 \sin 3x$$

$$\text{and } u(x, 0) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi}{L} x,$$

$$\Rightarrow 0.01 \sin 3x = B_1 \sin x + B_2 \sin 2x + B_3 \sin 3x + \ldots$$

Equating coefficients, we get

$$B_1 = 0, B_2 = 0, B_3 = 0.01, B_4 = 0 = B_5 = \ldots$$

Thus, we get, required solution is

$$u(x, t) = 0.01 \cos\lambda_3 t \sin x$$

Where

$$\lambda n = \frac{cn\pi}{L}$$

$$\lambda n = \pm n$$

$$\therefore \quad u(x, t) = 0.01 \cos 3t \sin 3x$$

This is the required solution of one dimensional wave equation with given initial velocity and initial deflection.

Example

Find u(x, t) of the string of length $L = \pi$ when $c^2 = 1$, the initial velocity is zero and the initial deification is $(0.1)\, x\, (\pi - x)$.

Solution

We know solution of one dimensional equation is

$$u(x, t) = \sum_{n=1}^{\infty} (B_n \cos\lambda_n t + B_n^* \sin\lambda_n t) \sin\frac{n\pi}{L}x \qquad \ldots\ldots(1)$$

where $\quad Bn = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi}{L}x\,dx$, $\lambda_n B_n^* = \frac{2}{L}\int_0^L g(x)\sin\frac{n\pi}{L}x\,dx$

where $f(x) = (0.1)\,x\,(\pi - x)$, is initial defection and $g(x) = 0$, is initial velocity of the string.

clearly $B_n^* = 0$.

And $\quad B_n = \frac{2}{\pi}\int_0^{\pi}(0.1)x(\pi - x)\sin\frac{n\pi}{\pi}x\,dx$

$$= \frac{0.2}{\pi}\int_0^{\pi} x(\pi - x)\sin nx\,dx$$

$$= \frac{0.2}{\pi}\int_0^{\pi}(\pi x - x^2)\sin nx\,dx$$

$$= \frac{0.2}{\pi}\left[\left\{(\pi x - x^2)\frac{\cos nx}{-n}\right\}_0^{\pi} + (\pi - 2x)\frac{\cos nx}{n}dx\right]$$

$$= \frac{0.2}{\pi}\frac{1}{n}\left[(\pi - 2x)\frac{\sin nx}{n} - 2\frac{\cos nx}{n^2}\right]_0^{\pi}$$

$$= \frac{0.2}{n\pi}\left(\frac{-\pi\sin n\pi}{n} - \frac{2\cos n\pi}{n^2} + \frac{2}{n^2}\right)$$

$$B_n = \frac{0.2}{n\pi}\frac{2}{n^2}[1 - \cos n\pi]$$

Therefore from equation (1), we get,

$$u(x,t) = \sum_{n=1}^{\infty}\frac{0.4}{\pi n^3}(1 - \cos n\pi)\cos\lambda_n t\sin\frac{n\pi}{L}x$$

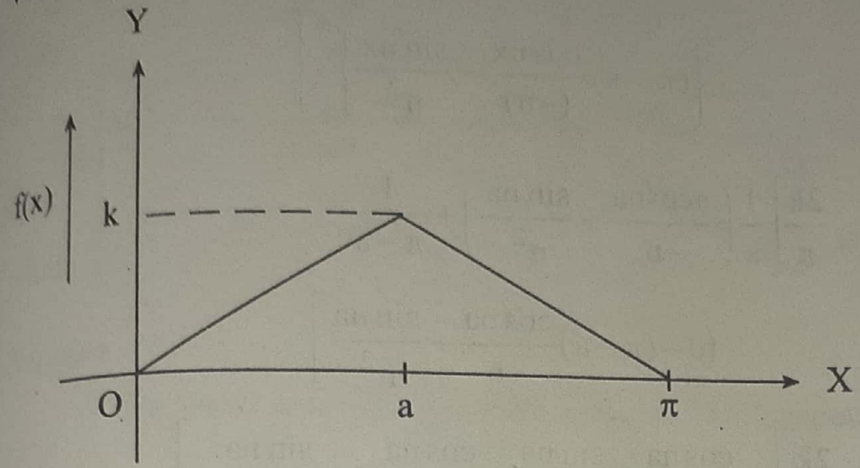
$$u(x, t) = \sum_{n=1}^{\infty}\frac{0.4}{\pi}\frac{(1 - \cos n\pi)}{n^3}\cos nt\,\sin nx$$

since $L = \pi$, $c^2 = 1$

This is the required solutions of one dimensional wave equation with given initial velocity and initial defection.

Example

Find u(x, t) of the string of length $L = \pi$ when $c^2 = 1$, the initial velocity is zero and the initial deflection is shown in the figure.



Solution

We know solution of one dimensional wave equation is

$$u(x, t) = \sum_{n=1}^{\infty} (B_n \cos\lambda_n t + B_n^* \sin\lambda_n t) \sin\frac{n\pi}{L} x$$

Here $B_n^* = 0$, since initial velocity is zero.

$$\therefore \quad u(x, t) = \sum_{n=1}^{\infty} B_n \cos\lambda_n t \sin\frac{n\pi}{L} x \quad \ldots\ldots\ldots\ldots\ldots(1)$$

$$\text{where } B_n = \frac{2}{L}\int_0^L f(x) \sin\frac{n\pi}{L} x \, dx \quad \ldots\ldots\ldots\ldots\ldots(2)$$

We have from figure,

$$f(x) = \begin{cases} \dfrac{k}{a} a & \text{if} \quad 0 \le x \le a \\ \dfrac{k}{\pi - a}(\pi - x) & \text{for} \quad a \le x \le \pi \end{cases}$$

Then, we get,

$$B_n = \frac{2}{\pi}\int_0^{\pi} f(x) \sin nx \, dx, \qquad \text{Since } L = \pi$$

$$= \frac{2}{\pi}\left[\int_0^a \frac{k}{a} x \sin nx \, dx + \int_a^{\pi} \frac{k}{(\pi - a)}(\pi - x) \sin nx \, dx\right]$$

$$= \frac{2k}{\pi}\left[\frac{1}{a}\int_0^a x \sin nx \, dx + \frac{1}{\pi - a}\int_a^{\pi}(\pi - x) \sin nx \, dx\right]$$

$$= \frac{2k}{\pi}\left[\frac{1}{a}\left\{x\frac{\cos nx}{-n}+\frac{\sin nx}{n^2}\right\}_0^a+\frac{1}{\pi-a}\right.$$

$$\left.\left\{(\pi-x)\frac{\cos nx}{(-n)}-\frac{\sin nx}{n^2}\right\}_a^\pi\right]$$

$$= \frac{2k}{\pi}\left[\frac{1}{a}\left(\frac{a\cos na}{-n}+\frac{\sin na}{n^2}\right)+\frac{1}{\pi-a}\right.$$

$$\left.(0-(\pi-a)\frac{\cos na}{-n}+\frac{\sin na}{n^2}\right]$$

$$= \frac{2k}{\pi}\left[-\frac{\cos na}{n}+\frac{\sin na}{an^2}+\frac{\cos na}{n}+\frac{\sin na}{n^2(\pi-a)}\right]$$

$$= \frac{2k}{\pi}\frac{\sin na}{an^2}\left[\frac{1}{a}+\frac{1}{\pi-a}\right]$$

$$= \frac{2k}{\pi}\frac{\sin na}{an^2}\cdot\frac{\pi}{a(\pi-a)}$$

$$\therefore \quad B_n = \frac{2k}{a(\pi-a)}\frac{\sin na}{n^2}$$

Therefore from equation (1), we get

$$u(x, t) = \sum_{n=1}^{\infty}\frac{2k}{a(\pi a)}\frac{\sin na}{n^2}\cos\lambda_n t\sin\frac{n\pi}{L}x$$

Put $L = \pi$, $c^2 = 1$, we get

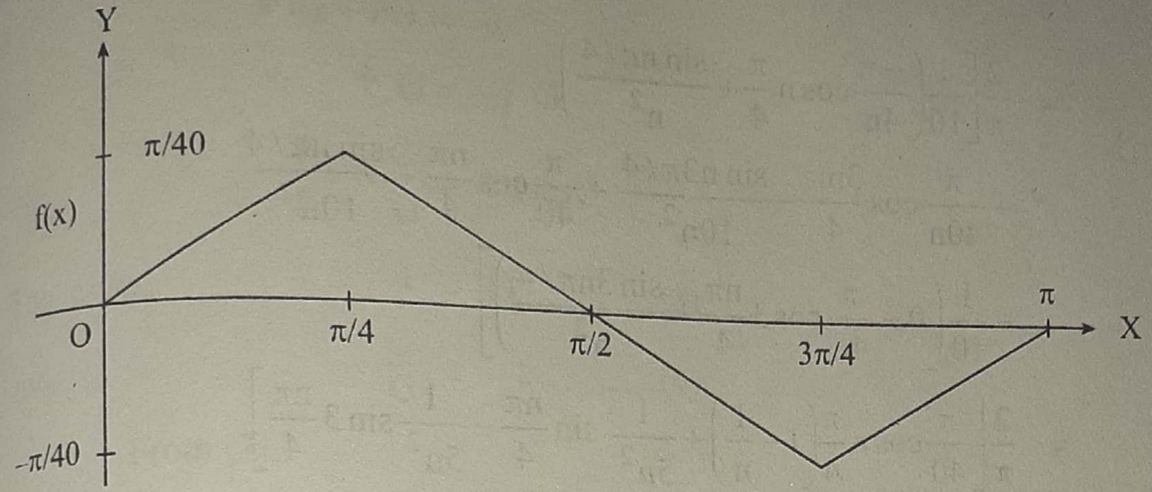
$$u(x, t) = \sum_{n=1}^{\infty}\frac{2k}{a(\pi a)}\frac{\sin na}{n^2}\cos nt\sin nx$$

$$u(x, t) = \frac{2k}{a(\pi a)}\sum_{n=1}^{\infty}\frac{\sin na}{n^2}\cos nt\sin nx$$

This is the required solution of the one dimensional wave equation with given conditions.

Example

Find u(x,t) of the string of length $L = \pi$ when $c^2 = 1$, the initial velocity is zero and its initial defection is shown in the figure.



Solution

We know solution of one dimensional wave equation is

$$u(x, t) = \sum_{n=1}^{\infty} B_n \cos\lambda_n t \sin\frac{n\pi}{L}x \qquad \text{.........(1)}$$

if initial velocity is zero.

$$\text{where } B_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi}{L}x\,dx$$

But we have $L = \pi$, $c = 1$ and from figure, we get

$$f(x) = \begin{cases} \dfrac{x}{40} & \text{if} \quad 0 \le x \le \dfrac{\pi}{4} \\ \left(\dfrac{\pi}{20} - \dfrac{x}{10}\right) & \text{if} \quad \dfrac{\pi}{4} \le x \le \dfrac{3\pi}{4} \\ \left(\dfrac{x-\pi}{10}\right) & \text{if} \quad \dfrac{3\pi}{4} \le x \le \pi \end{cases}$$

Thus,

$$B_n = \frac{2}{\pi}\left[\int_0^{\pi/4}\frac{x}{10}\sin nx\,dx + \int_{\pi/4}^{3\pi/4}\left(\frac{\pi}{20} - \frac{x}{10}\right)\sin nx\,dx + \int_{3\pi/4}^{\pi}\left(\frac{x-\pi}{10}\right)\sin nx\,dx\right]$$

$$= \frac{2}{\pi}\left[\frac{1}{10}\left(x\frac{\cos nx}{-n} + \frac{\sin nx}{n^2}\right)_0^{\pi/4} + \left\{\left(\frac{\pi}{20} - \frac{x}{10}\right)\frac{\cos nx}{-n} - \frac{\sin nx}{10n^2}\right\}_{\pi/4}^{3\pi/4} + \frac{1}{10}\left\{(x-\pi)\frac{\cos nx}{-n} + \frac{\sin nx}{n^2}\right\}_{3\pi/4}^{\pi}\right]$$

$$= \frac{2}{\pi}\left[\frac{1}{10}\left(\frac{-\pi}{4n}\cos n\frac{\pi}{4}+\frac{\sin n\pi/4}{n^2}\right)\right.$$

$$+\frac{\pi}{40n}\cos\frac{3n\pi}{4}-\frac{\sin n3\pi/4}{10n^2}+\frac{\pi}{40}\cos\frac{n\pi}{4}+\frac{\sin n\pi/4}{10n^2}$$

$$\left.+\frac{1}{10}\left(0-\frac{\pi}{4n}\cos 3\frac{n\pi}{4}-\frac{\sin 3n\pi/4}{n^2}\right)\right]$$

$$= \frac{2}{\pi}\left[\frac{\pi}{40}\cos n\frac{\pi}{4}\left(1-\frac{1}{n}\right)+\frac{1}{5n^2}\sin\frac{n\pi}{4}-\frac{1}{5n^2}\sin 3\frac{n\pi}{4}\right]$$

Therefore

$$B_n = \frac{2}{\pi}\left[\frac{\pi}{40}\left(1-\frac{1}{n}\right)\cos\frac{n\pi}{4}+\frac{1}{5n^2}\left(\sin\frac{n\pi}{4}-\sin\frac{3n\pi}{4}\right)\right]$$

From equation (1), we get

$$u(x,t) = \sum_{n=1}^{\infty}\frac{2}{\pi}\left[\frac{\pi}{40}\left(1-\frac{1}{n}\right)\cos\frac{n\pi}{4}+\frac{1}{5n^2}\left(\sin\frac{n\pi}{4}-\sin\frac{3n\pi}{4}\right)\right]\cos nt\sin nx$$

This is the required solution of the one dimensional wave equations with given conditions.

Example

Solve $u_x + u_y = 0$ by using separating variables.

Solution

We have given differential equation is

$$u_x + u_y = 0 \qquad \text{.........(1)}$$

Its solution is function of x and y.

Let $\qquad u(x, y) = F(x)\,G(y) \qquad$ ....(2)

where F is function of x only and G is function of y only.

be the solution of (1), then it must satisfy (1).

Thus, $\qquad u_x = \frac{\partial u}{\partial x} = \dot{F}\,G$

and $\qquad u_y = \frac{\partial u}{\partial y} = F\,G'$

Where dot and dash denotes partial differentiation with respect to x and y respectively.

Putting these values in equation (1), we get

$$\dot{F}G + FG' = 0$$

$$\Rightarrow \quad \dot{F}G = -FG'$$

$$\Rightarrow \quad \frac{\dot{F}}{F} = -\frac{G'}{G} = k \text{ (say)}$$

This gives, $\dot{F} = Fk$ ....(3)

and $\frac{G'}{G} = -k$ ....(4)

From (3),

$$\frac{\dot{F}}{F} = k$$

on integration with respect to x, we get,

$$\log F = kx + A, \; F = e^{kx}.D$$

From (4), $\frac{G'}{G} = -k$

on integration, with respect to y, we get

$$\log G = -ky + c, \; G = e^{-ky}.E$$

Therefore from equation (2), we get

$$u = FG = De^{kx} \; e^{-ky} E = DE \; e^{k(x-y)}$$

$$u = M \; e^{k(x-y)}$$

where M is a constant. This is the required solution of the given differential equation.

Example

Find the solution of the following differential equation $y^2 u_x - x^2 u_y = 0$ by using separating of variables.

Solution

We have gives differential equation is

$$y^2 u_x - x^2 u_y = 0 \qquad \text{.........(1)}$$

$$\text{Let } u(x, y) = F(x)\,G(y) \qquad \text{.........(2)}$$

be the solution of the given differential equation, where F and G are function of x and y respectively.

Here $u_x = \frac{\partial u}{\partial x}$, $u_x = \dot{F}G$ and $u_y = FG'$

where dot and dashes denotes differentiation with respect to x and y respectively. Putting these values in equation (1), we get,

$$y^2 \dot{F} G - x^2 FG' = 0$$

By separating variables, we get

$$\frac{\dot{F}}{Fx^2} = \frac{G'}{Gy^2} = k \text{ (say)}$$

This gives $$\frac{\dot{F}}{Fx^2} = k \qquad \ldots\ldots\ldots(3)$$

and $$\frac{G'}{Gy^2} = k \qquad \ldots\ldots\ldots(4)$$

From (3), we get

$$\frac{\dot{F}}{F} = kx^2$$

On integration with respect to x on both side we get

$$\log F = \frac{kx^3}{3} + C$$

$$F = e^{\frac{kx^3}{3}} D, \text{ where D is a constant.}$$

Also, from equation (4), we get

$$\frac{G'}{G} = ky^2$$

Taking integration with respect to y on both side, we get

$$\log G = \frac{ky^3}{3} + A$$

$$G = e^{\frac{ky^3}{3}} E$$

where A, E are constants. Putting these values in equation (2) we get

$$u(x, y) = D\ e^{\frac{kx^3}{3}} . E\ e^{\frac{ky^3}{3}}$$

$$u(x, y) = DE\ .\ e^{\frac{k}{3}(x^3+y^3)}$$

$$u(x, y) = M\ e^{\frac{k}{3}(x^3+y^3)},$$

where k and M is a constant.

This is the required solution of the given differential equation.

Example

Solve $u_{xx}+u_{yy} = 0$ by using separating of variables.

Solution

We have given differential equation is

$$u_{xx}+u_{yy} = 0 \qquad \text{.........(1)}$$

$$\text{Let } u(x, y) = F(x)\,G(y) \qquad \text{.........(2)}$$

be the solution of the given differential equation, where F and G are function of x and y respectively. Differentiating we get

$$u_x = \dot{F}\,G,\ u_{xx} = \ddot{F}\,G \text{ and } u_{yy} = F\,G''$$

where dot and dash denotes the differentiation of function with respect to x and y respectively.

Putting these values in equation (1) we get,

$$\ddot{F}G + FG'' = 0$$

$$\Rightarrow \quad \frac{\ddot{F}}{F} = \frac{G''}{-G} = k \text{ (say)}$$

$$\Rightarrow \quad \ddot{F} = kF \qquad \text{.........(3)}$$

$$\text{and } G'' = -kG \qquad \text{.........(4)}$$

From equation (3)

$$\ddot{F} - kF = 0$$

Its auxiliary equation is $m^2 - k = 0$, $m = \pm\sqrt{k}$

Then its solution is $F = A\,e^{\sqrt{k}x} + Be^{-\sqrt{k}x}$

where A and B are constants.

Also from equation (4), we get $G'' + kG = 0$

Its auxiliary equation is

$$m^2 + k = 0$$

$$m = \pm i\sqrt{k}$$

Its solution is

$$G = C\cos\sqrt{k}\,y + D\sin\sqrt{k}\,y$$

where C and D are constants.

Substituting these values in equation (2) we get,

$$u = \left[Ae^{\sqrt{k}x} + Be^{-\sqrt{k}x}\right]\left[C\cos\sqrt{k}y + D\sin\sqrt{k}y\right]$$

where A, B, C, D are constants.

This is the required solution of the given differential equation.

Example

Solve $u_{xx} - u_{yy} = 0$ by using separating of variables.

Solution

We have given differential equation is

$$u_{xx} - u_{yy} = 0 \qquad \text{.........(1)}$$

$$\text{Let } u = F(x)\, G(y) \qquad \text{.........(2)}$$

be the solution of (1).

$$u_{xx} = \ddot{F}\, G \text{ and } u_{yy} = F\, G''$$

where dot and dash denotes differentiation with respect to x and y respectively.

Putting these values in equation (1), we get

$$\ddot{F}\, G - F\, G'' = 0$$

$$\Rightarrow \quad \frac{\ddot{F}}{F} = \frac{G''}{G} = k(\text{say})$$

This gives

$$\ddot{F} = kF \qquad \text{.........(3)}$$

and

$$G'' = kG \qquad \text{.........(4)}$$

From equation (3), we get

$\ddot{F} - kF = 0$, which is ordinary differential equation of F with respect to x.

Its auxiliary equation is

$$m^2 - k = 0$$

$$\Rightarrow \quad m = \pm\sqrt{k}$$

Then its solution is

$$F = A\, e^{\sqrt{k}x} + Be^{-\sqrt{k}x}$$

where A and B are constants.

Also from equation (4), we get

$$G'' - k\,G = -0$$

$$m = \pm\sqrt{k}$$

$\Rightarrow$ and its solution is

$$G = Ce^{\sqrt{k}x} + De^{-\sqrt{k}x}$$

where C and D are constants. From equation (2), we get

$$u = (A\,e^{\sqrt{k}x} + Be^{-\sqrt{k}x})\,(Ce^{\sqrt{k}y} + De^{-\sqrt{k}y})$$

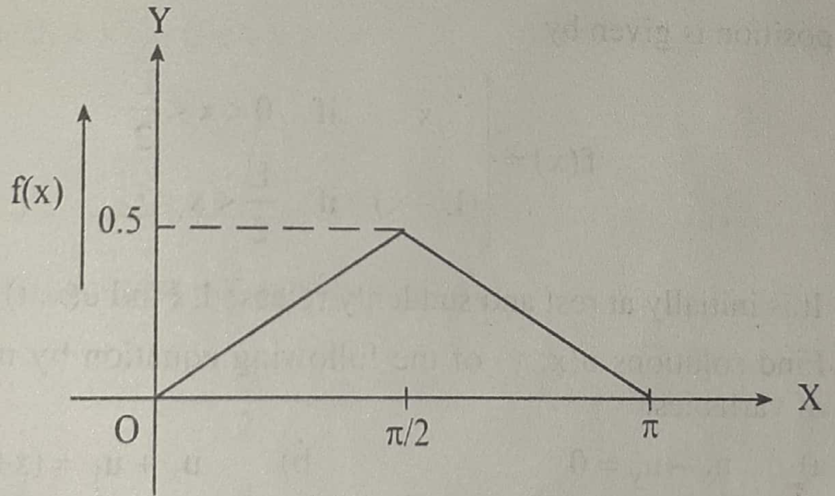
which is the required solution of the given differential equation.

Exercise 8.2

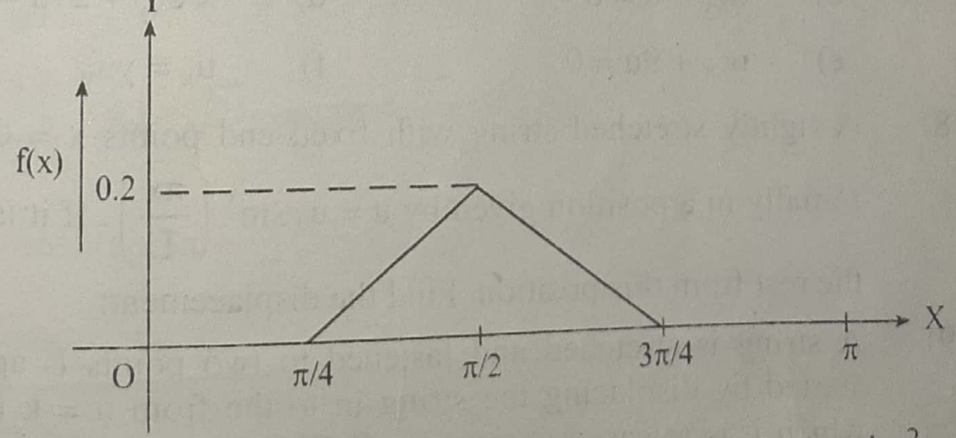
1. Find u(x, t) of the string of length $L = \pi$ when $c^2 = 1$, the initial velocity is zero and the initial deflection is

(a) $k\left(\sin x - \frac{1}{2}\sin 2x\right)$ (b) $0.1\,x\,(\pi^2 - x^2)$

(c)



(d)



2. Find the deflection u(x, t) of a vibrating string of length π and $c^2 = 4$ for zero initial velocity and initial deflection sin5x.

3. A tightly stretched string with fixed ends at $x = 0$ and $x = L$ is initially at rest in its equilibrium position. Find $u(x, t)$ if it is set vibrating by giving to each of its points a velocity $3(Lx - x^2)$.

4. If the string be fixed at both ends, find the solution with the following initial conditions.

   The initial displacement $u(x, 0) = u_0 \sin \frac{\pi}{L} x$ and initial velocity is zero.

5. Find the deflection $u(x, t)$ of the vibrating string of length $L = \pi$, $c^2 = 1$ and its initial deflection is zero and initial velocity is

$$\begin{cases} 0.01x & \text{if } 0 < x < \frac{\pi}{2} \\ 0.01(\pi - x) & \text{if } \frac{\pi}{2} < x < \pi \end{cases}$$

6. A tightly stretched string of length L is drawn a side at its midpoint a distance $\frac{L}{2}$ perpendicular to the equilibrium position so that its initial position is given by

$$f(x) = \begin{cases} x & \text{if } 0 < x < \frac{L}{2} \\ (L - x) & \text{if } \frac{L}{2} < x < L \end{cases}$$

   It is initially at rest and suddenly released. Find $u(x, t)$.

7. Find solutions $u(x, y)$ of the following equation by using separating of variables:

   a) $u_x - u_y = 0$ b) $u_x + u_y = (x+y)\,u$

   c) $u_{xy} - u = 0$ d) $x\,u_{xy} + 2yu = 0$

   e) $u_{xx} + 9u = 0$ f) $u_x = yu_y$

(8) A tightly stretched string with fixed end points $x = 0$ and $x = L$ is initially in a position given by $u = u_0 \sin^3\left(\frac{\pi x}{L}\right)$. If it is released from the rest from this position. Find the displacement.

(9) A string is stretched and fastened to two points L apart. Motion is started by displacing the string in to the from $u = k\,(Lx - x^2)$ from which it is released at time $t = 0$. Find the displacement of any point on the string at a distance of x from one end at time t.

(10) The vibration of an elastic string is governed by the partial differential equation $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$ with $c^2 = 1$. The length of string is π and ends are fixed. The initial velocity is zero and initial deflection is $f(x) = 2(\sin x + \sin 3x)$. Find the deflection $u(x, t)$ of the vibrating string.

(11) The vibration of an elastic string is governed by the partial differential equation $\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$ with length π and ends are fixed. The initial deflection $u(x, 0) = 0$ and initial velocity

$$\left.\frac{\partial u}{\partial t}\right|_{t=0} = u_t(x, 0) = \begin{cases} kx, \text{ for } 0 < x < \pi/2 \\ k(\pi - x), \text{ for } \pi/2 < x < \pi \end{cases}$$

(12) Find the solution of one dimensional wave equation of the tight string of length π under the condition

$u = (0,t) = 0, \quad u(\pi, t) = 0, \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = 0$

and $u(x, 0) = x$ for $0 < x < \pi$.

Answers

1. a) $k\left(\cos t \sin x - \frac{1}{2}\cos 2t \sin 2x\right)$

b) $1.2\left[\cos t \sin x - \frac{1}{2^3}\cos 2t \sin 2x + \frac{1}{3^3}\cos 3t \sin 3x \ldots\right]$

c) $\frac{4}{\pi^2}\left[\cos t \sin x - \frac{1}{9}\cos 3t \sin 3x + \frac{1}{25}\cos 5t \sin 5x \ldots\right]$

d) $\frac{1.6}{\pi^2}\left[(2-\sqrt{2})\cos t \sin x - \frac{1}{9}(2+\sqrt{2})\right.$

$\left.\cos 3t \sin 3x + \frac{1}{25}(2+\sqrt{2})\cos 5t \sin 5x \ldots\ldots\right]$

2. $-\cos 10t \sin 5x$

3. $\frac{24L^3}{c\pi^4}\sum_{n=1}^{\infty}\left\{\frac{1}{(2n-1)^4}\sin\frac{(2n-1)\pi x}{L}\sin\frac{(2n-1)\pi ct}{L}\right\}$

4. $\frac{u_0}{\pi}\sin\frac{\pi x}{L}\cos\frac{\pi ct}{L}$

5. $\frac{0.04}{\pi}\sum_{n=1}^{\infty}\frac{1}{n^3}\sin\frac{n\pi}{2}$ sinnx sinnt.

6. $\sum_{n=1}^{\infty}\frac{(1)^{n+1}4L}{(2n-1)^2\pi^2}\sin\frac{(2n-1)\pi x}{L}\cos\frac{(2n-1)\pi x}{L}$

7. a) $u = ce^{k(x+y)}$ b) $u = c\, e^{\left[\frac{1}{2}(x^2+y^2)+k(x-y)\right]}$

c) $u = c\, e^{\left(kx+\frac{y}{k}\right)}$ d) $u = cx^k\, e^{-y^2/k}$

e) $u = A\cos 3x + B\sin 3x$ f) $u = cy^k e^{kx}$

8. $u(x, t) = \frac{u_0}{4}\left[\cos\frac{c\pi t}{L}\sin\frac{\pi x}{L} - \cos\frac{3c\pi t}{L}\sin\frac{3\pi x}{L}\right]$

9. $u(x, t) = \sum_{n=1}^{\infty}\frac{4L^2k}{n^3\pi^3}\left[1-(-1)^n\right]\sin\frac{n\pi x}{L}\cos\frac{cn\pi t}{L}$

10. $u(x, t) = 2(\cos t\sin x + \cos 3t\sin 3x)$

11. $u(x, t) = \frac{4k}{\pi}\sum_{n=1}^{\infty}\sin\frac{n\pi}{2}\sin nt\sin nx$

12. $u(x, t) = 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin nx\cos nct$

D'Alembert's Solution of the Wave Equation

We have one dimensional wave equation is

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2} \qquad \ldots\ldots\ldots(1)$$

where $c^2 = \frac{T}{\rho}$

Let $v = x+ct$ and $z = x - ct$. Then u is a function of v and z. Therefore,

$$u_x = u_v v_x + u_z z_x$$

$$u_x = u_v + u_z$$

and $u_{xx} = (u_v + u_z)_x$