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Ex 8.2

7 (b) $U_x + U_y = (x+y)U$

Solution: Given differential eqn is

$$U_x + U_y = (x+y)U \quad \cdots (I)$$

Let $u(x,y) = F(x) \cdot G(y)$ (II) be the solution of eqn (I)

Differentiating eqn (II) w.r.to x and y we get

$$\frac{\partial U}{\partial x} = U_x = F'G$$

$$\frac{\partial U}{\partial y} = U_y = F\dot{G}$$

Putting values in eqn (I)

$$F'G + F\cdot\dot{G} = (x+y)F\cdot G$$

Dividing both sides by FG

or $$\frac{F'G + F\cdot\dot{G}}{FG} = \frac{(x+y)F\cdot G}{FG}$$

or, $$\frac{F'G}{FG} + \frac{F\cdot\dot{G}}{F\cdot G} = x+y$$

or, $$\frac{F'}{F} + \frac{\dot{G}}{G} = x+y$$

or, $$\frac{F'}{F} - x = -\frac{\dot{G}}{G} + y$$

or, $\frac{F'}{F} - x = -\frac{G'}{G} + y = k$ (say)

Then

$$\frac{F'}{F} - x = k$$

or $\frac{F'}{F} = k + x$

integrating w.r.to x

$$\log F = kx + \frac{x^2}{2} + C$$

or, $F = e^{\frac{x^2}{2} + kx + C}$

or $F = e^{\frac{x^2}{2} + kx} \cdot e^{C}$

or, $F = A \cdot e^{\frac{x^2}{2} + kx}$

i.e $F(x) = A \cdot e^{\frac{x^2}{2} + kx}$

Again we have

$$-\frac{G'}{G} + y = k$$

or $\frac{G'}{G} = y - k$

Integrating w.r.to y

$$\log G = \frac{y^2}{2} - ky + D$$

or $G = e^{\frac{y^2}{2} - ky + D}$

or $G = e^{\frac{y^2}{2} - ky} \cdot e^{D}$

or $G = B \cdot e^{\frac{y^2}{2} - ky}$ where $e^{D} = B$

$\therefore G(y) = B \cdot e^{\frac{y^2}{2} - ky}$

Finally $u(x, y) = F(x) \cdot G(y)$

or $u(x, y) = A \cdot e^{\frac{x^2}{2} + kx} \cdot B \cdot e^{\frac{y^2}{2} - ky}$

$= AB\, e^{\frac{x^2}{2} + kx + \frac{y^2}{2} - ky}$

$\therefore u(x, y) = R \cdot e^{\frac{x^2 + y^2}{2} + k(x - y)}$

is the solution of given partial differential eqn.

(c) $U_{xy} - U = 0$

Solution:- Given eqn is

$$U_{xy} - U = 0 \quad \text{(I)}$$

Suppose $u(x,y) = F(x) \cdot G(y)$ - (II)

be the Solution of eqn (I) where F is function of x and G is function of y.

Diff eqn (II)

$$U_x = F'G \text{ and } U_y = F \cdot \dot{G}$$

$$U_{xy} = F'\dot{G}$$

Putting values in eqn (I)

or, $F'\dot{G} - FG = 0$

or $F'\dot{G} = FG$

or $\frac{F'}{F} = \frac{G}{\dot{G}}$

or $\frac{F'}{F} = \frac{G}{\dot{G}} = k \text{ (say)}$

Then $\frac{F'}{F} = K$

integrating

$\log F = Kx + C$

$F = e^{Kx+C}$

$F = e^{Kx} \cdot e^{C}$

$\therefore F(x) = A \cdot e^{Kx}$ where $e^{C} = K$

Again $\frac{G}{\dot{G}} = K$

or $\frac{\dot{G}}{G} = \frac{1}{K}$

or $\log G = \frac{y}{K} + D$

or, $G = e^{\frac{y}{K} + D}$

or, $G = e^{\frac{y}{K}} \cdot e^{D}$

or $G(y) = B \cdot e^{\frac{y}{K}}$

Finally we have

$$u(x,y) = F(x)\cdot G(y)$$

$$= A\cdot e^{kx}\cdot B\cdot e^{\frac{y}{k}}$$

$$= AB\cdot e^{kx+\frac{y}{k}}$$

$$= R\cdot e^{\left(kx+\frac{y}{k}\right)}$$

is the required Solution

(d) $x\,u_{xy} + 2yu = 0$

$\Rightarrow$ class work

Q.N 1 (a) Find $u(x,t)$ of the string of length $L=\pi$ and $c^2=1$ the initial velocity is zero and the initial deflection is

$$k\left(\sin x - \frac{1}{2}\sin 2x\right)$$

Solution

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Solution ÷ Here

Length of string $(L) = \pi$

$c^2 = 1$

Initial velocity $= g(x) = 0$

Initial deflection $= f(x) = u(x,0) = k(\sin x - \frac{1}{2}\sin 2x)$

i.e $u(x,0) = k\sin x - \frac{k}{2}\sin 2x$

We know that the solution of 1-D wave eqn is

$$u(x,t) = \sum_{n=1}^{\infty} (B_n \cos\lambda_n t + B_n^* \sin\lambda_n t) \cdot \sin\frac{n\pi}{L}x \quad \text{--- (1)}$$

Where $B_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi}{L}x\,dx$

$$B_n^* = \frac{2}{cn\pi}\int_0^L g(x)\sin\frac{n\pi}{L}x\,dx$$

Since $g(x) = 0$, So

$$B_n^* = \frac{2}{cn\pi}\int_0^L g(x)\sin\frac{n\pi}{L}x\,dx = 0$$

Eqn ① becomes

$$u(x,t) = \sum_{n=1}^{\infty} B_n \cos \lambda_n t \cdot \sin \frac{n\pi}{L} x$$

$$u(x,t) = \sum_{n=1}^{\infty} B_n \cos \lambda_n t \sin \frac{n\pi}{\pi} x \quad [L = \pi]$$

$$u(x,t) = \sum_{n=1}^{\infty} B_n \cos \lambda_n t \cdot \sin nx \quad \cdots (II)$$

put $t = 0$ in eqn (II), we get

$$u(x,0) = \sum_{n=1}^{\infty} B_n \cos 0 \cdot \sin nx$$

$$u(x,0) = \sum_{n=1}^{\infty} B_n \sin nx$$

$$u(x,0) = B_1 \sin 1x + B_2 \sin 2x + B_3 \sin 3x + \cdots \quad \cdots (III)$$

we have

$$u(x,0) = K \sin x - \frac{K}{2} \sin 2x \quad \cdots (IV)$$

Equating eqn (III) and (IV)

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We get

$$K\sin x - \frac{K}{2}\sin 2x = B_1 \sin x + B_2 \sin 2x + B_3 \sin 3x + B_4 \sin 4x + \cdots$$

Equating corresponding coefficients

$B_1 = K$, $B_2 = -\frac{K}{2}$,

$B_3 = B_4 = B_5 = \cdots\cdots = 0$ (others all are zero)

From eqn (11) we have

$$u(x,t) = \sum_{n=1}^{\infty} B_n \cos \lambda_n t \cdot \sin nx$$

$$u(x,t) = B_1 \cos \lambda_1 t \sin x + B_2 \cos \lambda_2 t \sin 2x + B_3 \cos \lambda_3 t \cdot \sin 3x + B_4 \cos \lambda_4 t \sin 4x + \cdots$$

Putting values of $B_1, B_2, B_3 \cdots$

$$= K \cos \lambda_1 t \sin x - \frac{K}{2} \cos \lambda_2 t \sin 2x + 0 + 0 + 0 \cdots$$

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(10)

$$u(x,t) = K\cos\lambda_1 t\,\sin x - \frac{K}{2}\cos\lambda_2 t\,\sin 2x$$

We have $\lambda_n = \frac{cn\pi}{L}$

$$\lambda_n = \frac{1\cdot n\cdot\pi}{\pi} \quad [L = \pi]$$

$$\therefore \lambda_n = n$$

ie $\lambda_1 = 1$, $\lambda_2 = 2$, $\lambda_3 = 3$ ...

Then

$$u(x,t) = K\cos t\cdot\sin x - \frac{K}{2}\cos 2t\cdot\sin 2x$$

$$u(x,t) = K\left(\cos t\cdot\sin x - \frac{1}{2}\cos 2t\cdot\sin 2x\right)$$

is the required solution. #

classwork

(2) $L = \pi$, $c^2 = 4$

$f(x) = u(x,0) = \sin 5x$,

$g(x) = 0$, $u(x,t) = ?$