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Ex 8.2

Q-N. (c) Find $u(x,t)$ of the string of length $L = \pi$ when $c^2 = 1$, the initial velocity is zero and initial deflection is $0.1x(\pi^2 - x^2)$

Solution: Here

$$\text{Initial velocity } = g(x) = 0$$

$$\text{initial deflection } = f(x) = 0.1x(\pi^2 - x^2)$$

$$L = \pi, c^2 = 1.$$

We know that the solution of 1-D wave eqn is

$$u(x,t) = \sum_{n=1}^{\infty} (B_n \cos nt + B_n^* \sin nt) \sin \frac{n\pi}{L} x \quad \dots (1)$$

where

$$B_n = \frac{2}{L} \int_0^L f(x) \cdot \sin \frac{n\pi}{L} x dx$$

$$B_n^* = \frac{2}{L} \int_0^L g(x) \cdot \sin \frac{n\pi}{L} x dx$$

Since $g(x) = 0$. So

$$B_n^* = \frac{2}{L} \int_0^L g(x) \sin \frac{n\pi}{L} x dx = 0$$

Then eqn (1) becomes

$$u(x,t) = \sum_{n=1}^{\infty} B_n \cos nt \cdot \sin \frac{n\pi}{L} x \quad \dots (ii)$$

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Now

$$B_n = \frac{2}{L} \int_0^L b(x) \cdot \sin \frac{n\pi}{L} x dx$$

$$= \frac{2}{\pi} \int_0^\pi 0.1x(\pi^2 - x^2) \sin \frac{n\pi}{\pi} x dx \quad [\text{put } L=\pi]$$

$$= \frac{0.2}{\pi} \int_0^\pi (\pi^2 x - x^3) \sin nx dx$$

$$= \frac{0.2}{\pi} \left[(\pi^2 x - x^3) \cdot -\frac{\cos nx}{n} \right]$$

$$+ (\pi^2 - 3x^2) \frac{\sin nx}{n^2} - 6x \frac{\cos nx}{n^3}$$

$$- 6 \frac{\sin nx}{n^4}$$

$$\begin{aligned} & \pi^2 x - x^3 \xrightarrow{\sin nx} \\ & \pi^2 - 3x^2 \xrightarrow{-\frac{\cos nx}{n}} \\ & -6x \xrightarrow{-\frac{\sin nx}{n^2}} \\ & -6 \xrightarrow{-\frac{\cos nx}{n^3}} \\ & 0 \xrightarrow{-\frac{\sin nx}{n^4}} \end{aligned}$$

$$= \frac{0.2}{\pi} \left[(\pi^3 - \pi^3) \cdot -\frac{\cos n\pi}{n} + (\pi^2 - 3\pi^2) \cdot \frac{\sin n\pi}{n^2} \right. \\ \left. - 6\pi \frac{\cos n\pi}{n^3} - 6 \frac{\sin n\pi}{n^4} \right]$$

$$= \frac{0.2}{\pi} \times -6\pi \frac{\cos n\pi}{n^3} = \frac{1.2 \times -1 \times (-1)^n}{n^3} = \frac{1.2 (-1)^{n+1}}{n^3}$$

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Eqn (ii) becomes

$$U(x, t) = \sum_{n=1}^{\infty} B_n \cos n\pi t \cdot \sin \frac{n\pi}{L} x$$

$$= \sum_{n=1}^{\infty} \frac{1 \cdot 2}{n^3} (-1)^{n+1} \cos \frac{cn\pi}{L} xt \cdot \sin \frac{n\pi}{\pi} x$$

$$[A_n = \frac{cn\pi}{L}]$$

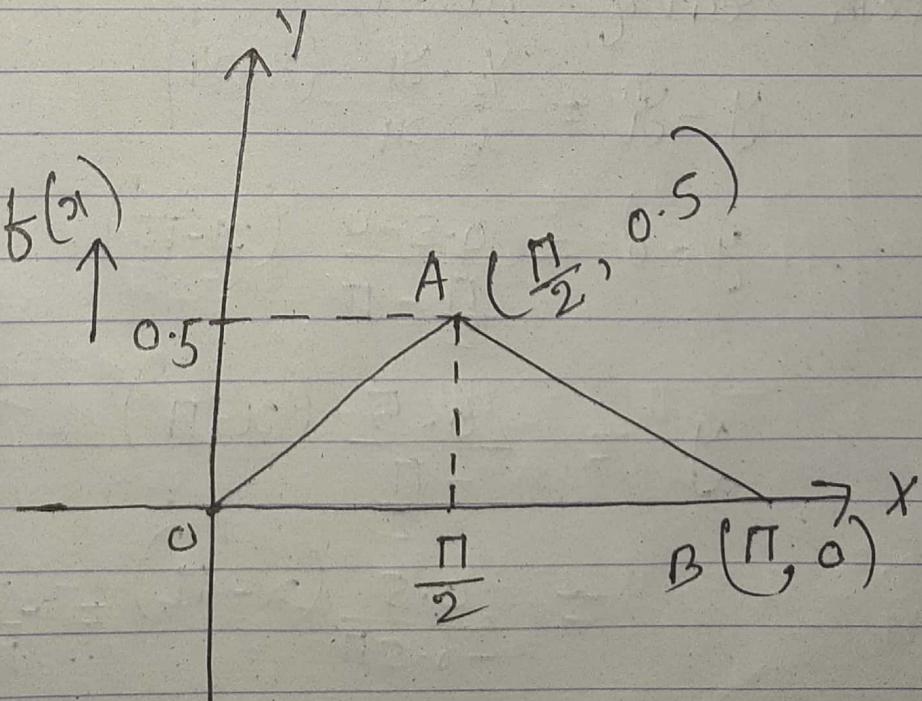
$$= 1 \cdot 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^3} \cos \frac{1 \times n\pi}{\pi} t \cdot \sin x \quad [L = \pi]$$

$$= 1 \cdot 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^3} \text{const.} \sin x$$

$$U(x, t) = 1 \cdot 2 \left(\cos t \sin x - \frac{1}{8} \cos 2t \cdot \sin 2x + \dots \right)$$

is the required ~~one~~ solution.

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Here, Length of string = $L = \pi$
 $c^2 = 1$

Initial Velocity = $g(x) = 0$

From the given figure, Initial deflection
 is OA and AB

Now Eqn of OA is given by

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1} (x - x_1)$$

$$y - 0 = \frac{0.5 - 0}{\frac{\pi}{2} - 0} (x - 0)$$

$$y = \frac{0.5}{0.5\pi} \times x = \frac{x}{\pi}$$

$$\text{i.e } f(x) = \frac{x}{4}$$

Again, Eqn of AB is

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1} (x - x_1)$$

$$y - 0 = \frac{0.5 - 0}{\frac{\pi}{2} - \pi} (x - \pi)$$

$$y = \frac{0.5}{-\frac{\pi}{2}} (x - \pi)$$

$$= \frac{0.5}{-0.5\pi} (x - \pi) = -\frac{1}{\pi} + 1$$

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Eqn of AB is

$$f(x) = 1 - \frac{x}{\pi}$$

Then initial deflection is

$$f(x) = \begin{cases} \frac{x}{\pi} & \text{for } 0 < x < \frac{\pi}{2} \\ 1 - \frac{x}{\pi} & \text{for } \frac{\pi}{2} < x < \pi \end{cases}$$

We know that the solution of 1-D wave equation is

$$u(x, t) = \sum_{n=1}^{\infty} (B_n \cos nt + B_n^* \sin nt) \sin \frac{n\pi}{L} x \quad (1)$$

Where $B_n = \frac{2}{L} \int_0^L f(x) \cdot \sin \frac{n\pi}{L} x dx$

$$B_n^* = \frac{2}{Cn\pi} \int_0^L g(x) \sin \frac{n\pi}{L} x dx = 0$$

Since $g(x) = 0$

Eqn (1) becomes

$$u(x, t) = \sum_{n=1}^{\infty} B_n \cos nt \cdot \sin \frac{n\pi}{L} x \quad (2)$$

Now, we have to find B_n

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$$\begin{aligned}
 B_n &= \frac{2}{\pi} \int_0^L f(x) \cdot \sin \frac{n\pi}{L} x \cdot dx \\
 &= \frac{2}{\pi} \int_0^\pi f(x) \sin \frac{n\pi}{\pi} x \cdot dx \\
 &= \frac{2}{\pi} \int_0^\pi f(x) \sin nx \cdot dx \\
 &= \frac{2}{\pi} \left[\int_0^{n/2} f(x) \sin nx \cdot dx + \int_{n/2}^{\pi} f(x) \sin nx \cdot dx \right] \\
 &= \frac{2}{\pi} \left[\int_0^{n/2} \frac{x}{\pi} \sin nx \cdot dx + \int_{n/2}^{\pi} \left(1 - \frac{x}{\pi}\right) \sin nx \cdot dx \right] \\
 &\approx \frac{2}{\pi^2} \int_0^{n/2} x \sin nx \cdot dx + \frac{2}{\pi^2} \int_{n/2}^{\pi} (\pi - x) \sin nx \cdot dx
 \end{aligned}$$

$$\begin{array}{c}
 x \quad \sin x \\
 1 \quad \downarrow -\frac{\cos x}{n} \\
 0 \quad \downarrow \frac{\sin x}{n^2}
 \end{array}$$

$$\begin{array}{c}
 \pi - x \quad \sin x \\
 -1 \quad \downarrow -\frac{\cos x}{n} \\
 0 \quad \downarrow \frac{\sin x}{n^2}
 \end{array}$$

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$$= \frac{2}{n^2} \left[-\frac{\pi \cos nx}{n} + \frac{\sin nx}{n^2} \right]_0^{\pi/2}$$

$$+ \left[(\pi - x) \cdot -\frac{\cos nx}{n} - \frac{\sin nx}{n^2} \right]_{\pi/2}^{\pi}$$

$$= \frac{2}{n^2} \left[-\frac{\pi \cos \frac{n\pi}{2}}{n} + \frac{\sin \frac{n\pi}{2}}{n^2} - 0 \right] = 0$$

$$+ (\pi - \pi) \cdot \cancel{-\frac{\cos n\pi}{n}} - \cancel{\frac{\sin n\pi}{n^2}}$$

$$- (\pi - \frac{\pi}{2}) \cdot \cancel{-\frac{\cos \frac{n\pi}{2}}{n}} + \cancel{\frac{\sin \frac{n\pi}{2}}{n^2}}$$

$$= \frac{4}{(n\pi)^2} \cdot \sin \frac{n\pi}{2}$$

Putting value of B_n in eqn (2)

$$v(x, t) = \sum_{n=1}^{\infty} \frac{4}{(n\pi)^2} \cdot \sin \frac{n\pi}{2} \cdot \cos nt \cdot \sin \frac{n\pi}{L} \cdot 2$$

$$= \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{\sin \frac{n\pi}{2}}{n^2} \cdot \cos \frac{cn\pi t}{L} \sin \frac{n\pi}{\pi} \cdot 2$$

$$= \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \cdot \sin \frac{n\pi}{2} \cdot \frac{\cos \frac{1 \cdot n \cdot \pi \cdot t}{\pi}}{\pi} \cdot \sin n\pi \cdot 2$$

$$= \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \sin \frac{n\pi}{2} \cdot \cos nt \cdot \sin n\pi \cdot 2$$

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$$u(x,t) = \frac{4}{\pi^2} \left(\cos t \sin x - \frac{1}{9} \cos 3t \sin 3x + \frac{1}{25} \cos 5t \sin 5x - \dots \right)$$

(5) Find the deflection of the string of length $L = \pi$, $c^2 = 1$ and its initial deflection is zero and initial velocity is

$$g(x) = \begin{cases} 0.01x & \text{if } 0 < x < \frac{\pi}{2} \\ 0.01(\pi-x) & \text{if } \frac{\pi}{2} < x < \pi \end{cases}$$

Solution: Solution of L-D is

$$u(x,t) = \sum_{n=1}^{\infty} (B_n \cos nt + B_n^* \sin nt) \sin \frac{n\pi}{L} x \quad (1)$$

where

$$B_n = \frac{2}{L} \int_0^L f(x) \cdot \sin \frac{n\pi}{L} x dx = C$$

being initial deflection is zero

$$B_n^* = \frac{2}{Cn\pi} \int_0^L g(x) \sin \frac{n\pi}{L} x dx =$$

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Eqn ① becomes

$$\text{Q} \quad u(x,t) = \sum_{n=1}^{\infty} B_n^* \sin nx \cdot \sin \frac{3\pi}{L} t, \quad (2)$$

Now

$$B_n^* = \frac{2}{cn\pi} \int_0^L g(x) \sin \frac{n\pi}{L} x \, dx$$

$$= \frac{2}{cn\pi} \int_0^{\pi} g(x) \sin \frac{n\pi}{\pi} x \, dx$$

$$= \frac{2}{cn\pi} \int_0^{\pi} g(x) \sin nx \, dx$$

$$= \frac{2}{cn\pi} \left[\int_0^{\pi/2} 0.01x \sin nx \, dx + \int_{\pi/2}^{\pi} 0.01(\pi-x) \sin nx \, dx \right]$$

$$= \frac{0.02}{cn\pi} \left[\int_0^{\pi/2} x \sin nx \, dx + \int_{\pi/2}^{\pi} (\pi-x) \sin nx \, dx \right]$$

Same as Q.N. 1(c)

$$= \frac{0.02}{1 \times n\pi} \times \frac{2 \sin \frac{n\pi}{2}}{n^2}$$

$$= \frac{0.04}{n^3 \pi}$$

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Putting value of B_n^* in eqn ②

$$v(x, t) = \sum_{n=1}^{\infty} B_n^* \sin n\pi t \sin \frac{n\pi}{L} x$$

$$= \sum_{n=1}^{\infty} \frac{0.04}{n^3 \pi} \sin \frac{n\pi}{2} \cdot \sin \frac{cn\pi}{\pi} \cdot t \cdot \sin \frac{n\pi}{\pi} x$$

$$= \sum_{n=1}^{\infty} \frac{0.04}{n^3 \pi} \cdot \sin \frac{n\pi}{2} \sin nt \sin nx$$

$$= \frac{0.04}{\pi} \sum_{n=1}^{\infty} \frac{1}{n^3} \sin \frac{n\pi}{2} \sin nt \sin nx$$

$$= \frac{0.04}{\pi} \left[1 \cdot \sin t \cdot \sin nx - \cancel{\frac{1}{3} \sin 3t \cdot \sin 3x} - \frac{1}{27} \sin 3t \cdot \sin 3x + \dots \right]$$

is the required solution.

- ⑥ A tightly stretched string of length L is drawn a side its mid point a distance $\frac{L}{2}$ perpendicular to the equilibrium position so that its initial position is given by

$$f(x) = \begin{cases} x & \text{if } 0 < x < \frac{L}{2} \\ (L-x) & \text{if } \frac{L}{2} < x < L \end{cases}$$

It is initially at rest and suddenly released
Find $v(x, t)$.

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Solution :- Here

$$\text{Length of string} = L$$

$$\text{Initial velocity} = g(x) = 0$$

Initial deflection is

$$f(x) = \int_{-L}^L \sin \frac{n\pi}{L} x dx$$

we know

$$u(x, t) = \sum_{n=1}^{\infty} (B_n \cos nt + B_n^* \sin nt) \sin \frac{n\pi}{L} x \quad (1)$$

$$\text{Since } g(x) = 0 \Rightarrow B_n^* = 0$$

Eqn (1) becomes

$$u(x, t) = \sum_{n=1}^{\infty} B_n \cos nt \cdot \sin \frac{n\pi}{L} x \quad (2)$$

$$\text{here } B_n = \frac{2}{L} \int_0^L f(x) \cdot \sin \frac{n\pi}{L} x dx$$

$$= \frac{2}{L} \left[\int_0^{L/2} x \sin \frac{n\pi}{L} x dx + \int_{L/2}^L (L-x) \sin \frac{n\pi}{L} x dx \right]$$

$$\begin{aligned} & \left[x \sin \frac{n\pi}{L} x \right]_0^{L/2} - \left[-\cos \frac{n\pi}{L} x \right]_0^{\frac{n\pi}{L}} \\ & \quad \downarrow \quad \downarrow \\ & \quad 0 \quad -\frac{\cos \frac{n\pi}{L} \cdot \frac{n\pi}{L}}{\frac{n\pi}{L}} \\ & \quad \downarrow \quad \downarrow \\ & \quad -\frac{\sin \frac{n\pi}{L} x}{(\frac{n\pi}{L})^2} \end{aligned}$$

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$$= \frac{2}{L} \left[\left| -\frac{\cos \frac{n\pi}{L} \cdot x}{n\pi} + \frac{\sin \frac{n\pi}{L} \cdot x}{(\frac{n\pi}{L})^2} \right|_{L/2} \right.$$

$$\left. + \left| (L-x) \cdot -\frac{\cos \frac{n\pi}{L} \cdot x}{n\pi} + \frac{-\sin \frac{n\pi}{L} \cdot x}{(\frac{n\pi}{L})^2} \right|_{L/2} \right]$$

$$= \frac{2}{L} \left[-\frac{L/2 \cdot L \cdot \cos \frac{n\pi}{L} \times L/2}{n\pi} + \frac{\sin \frac{n\pi}{L} \times L/2 \times \frac{L^2}{(n\pi)^2}}{(n\pi)^2} \right.$$

$$\left. - 0 + (L-L) \cdot \frac{-\cos \frac{n\pi}{L} \cdot L}{n\pi} \right]$$

$$+ \frac{L^2}{(n\pi)^2} \cdot \sin \frac{n\pi}{L} \cdot L + (L-\frac{L}{2}) \cdot \frac{\cos \frac{n\pi}{L} \cdot L \times L}{2(n\pi)}$$

$$+ \left. \frac{\sin \frac{n\pi}{L} \times L}{(\frac{n\pi}{L})^2} \right]$$

$$= \frac{2}{L} \left[-\frac{L^2}{2n\pi} \cancel{\cos \frac{n\pi}{2}} + \frac{\sin \frac{n\pi}{L} \cdot \frac{L^2}{(n\pi)^2}}{2(n\pi)^2} \right]$$

$$+ \frac{L^2}{(n\pi)^2} \cdot \cancel{\sin \frac{n\pi}{L}} + \frac{L^2}{2n\pi} \cancel{\cos \frac{n\pi}{2}} + \frac{L}{(n\pi)^2} \cdot \cancel{\sin \frac{n\pi}{L}}$$

$$B_n = \frac{4L}{(n\pi)^2} \cdot \sin \frac{n\pi}{2} \Rightarrow \text{Putting value in eqn (2)} \\ \text{which is required soln.}$$