Fourier Sine and Cosine Transforms

Chhabi Siwakoti

Nepal college of information and Technology,Balkumari,Lalitpur

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Recall: Find fourier transform of

$$f(x) = \begin{cases} 1 & \text{if} \quad |x| < 1 \\ 0 & \text{if} \quad |x| > 1 \end{cases}$$

And hence evaluate

$$\int_0^\infty \frac{\sin x}{x} dx$$

Sol^n : We have,

$$f(x) = \begin{cases} 1 & \text{if} \quad -1 < x < 1 \\ 0 & \text{if} \quad Otherwise \end{cases}$$

Its fourier transform is

$$\mathcal{F}(f) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} f(t).e^{-i\omega t} dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} f(t).e^{-i\omega t} dt + 0$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} e^{-i\omega t} dt$$

$$= \frac{1}{\sqrt{2\pi}} \cdot \left[\frac{e^{-i\omega t}}{-i\omega} \right]_{-1}^{1}$$

$$= \frac{1}{\sqrt{2\pi}} \left(\frac{e^{-i\omega} - e^{i\omega}}{-i\omega} \right)$$

$$= \frac{1}{\sqrt{2\pi}}.2\left(\frac{e^{i\omega} - e^{-i\omega}}{2i.\omega}\right)$$

$$\therefore \mathcal{F}(f) = \sqrt{\frac{2}{\pi}}\left(\frac{\sin\omega}{\omega}\right)$$

$$\left[\because \frac{e^{i\omega} - e^{-i\omega}}{2i} = \sin\omega\right]$$

which is required fourier transform of given function.

Next, taking inversion formula for fourier transform, we get

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \{\mathcal{F}(f)\} e^{i\omega x} d\omega$$

$$i.e. f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left\{ \sqrt{\frac{2}{\pi}} \frac{\sin \omega}{\omega} \right\} e^{i\omega x} d\omega$$

Put $x = 0$, then

$$1 = \frac{1}{\pi} \int\limits_{-\infty}^{\infty} \left(\frac{\sin \omega}{\omega} \right) e^{0} d\omega$$

$$\left[\because f(x) = \begin{cases} 1 & \text{if} \quad -1 < x < 1 \\ 0 & \text{if} \quad Otherwise \end{cases} \right]$$

$$or, \int_{-\infty}^{\infty} \frac{\sin \omega}{\omega} d\omega = \pi$$

Replacing ω by x we get

$$or, \int_{-\infty}^{\infty} \frac{\sin x}{x} dx = \pi$$

$$or, 2\int_{0}^{\infty} \frac{\sin x}{x} dx = \pi$$

$$\therefore \int_{0}^{\infty} \frac{\sin x}{x} dx = \pi/2$$

Fourier Cosine and Sine Transforms

The fourier cosine integral of a function $f(x)$ is

$$f(x) = \int_0^\infty A(\omega) \cos \omega x d\omega ...... \quad (1)$$

, where

$$A(\omega) = \frac{2}{\pi} \int_0^\infty f(t) \cos \omega t dt$$

So, from equation (1), we get

$$f(x) = \int_0^\infty \left(\frac{2}{\pi} \int_0^\infty f(t) \cos \omega t dt \right) \cos \omega x d\omega$$

$$i.e. f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \left(\sqrt{\frac{2}{\pi}} \int_0^\infty f(t) \cos \omega t dt \right) \cos \omega x d\omega .... \quad ..(2)$$

The expression in the bracket of this equation is called fourier cosine transform of given function $f(x)$. It is denoted by symbol $\mathcal{F}_c(f)$.

$$i.e. \mathcal{F}_c(f) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(t) \cos \omega t dt ... \quad ... \quad (3)$$

And the given function $f(x)$ itself is called inverse transform of $\mathcal{F}_c$ So from (2)

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \{\mathcal{F}_c(f)\} \cos \omega x d\omega ... \quad ...(4)$$

Which is the inversion formula for fourier cosine transform.

Also, the fourier sine integral of a function $f(x)$ is

$$f(x) = \int_0^\infty B(\omega)\sin\omega x d\omega ...... \quad (1)$$

, where

$$B(\omega) = \frac{2}{\pi}\int_0^\infty f(t)\sin\omega t dt$$

So, from equation (1), we get

$$f(x) = \int_0^\infty \left(\frac{2}{\pi} \int_0^\infty f(t) \sin \omega t dt \right) \sin \omega x d\omega$$

$$i.e.f(x) = \sqrt{\frac{2}{\pi}} \int\limits_0^\infty \left(\sqrt{\frac{2}{\pi}} \int\limits_0^\infty f(t) \sin \omega t dt \right) \sin \omega x d\omega .... \quad ..(2)$$

The expression in the bracket of this equation is called fourier sine transform of given function $f(x)$. It is denoted by symbol $\mathcal{F}_s(f)$.

$$i.e.\mathcal{F}_s(f) = \sqrt{\frac{2}{\pi}} \int\limits_0^\infty f(t) \sin \omega t dt ... \quad ... \quad (3)$$

And the given function $f(x)$ itself is called inverse transform of $\mathcal{F}_s$ So from (2)

$$f(x) = \sqrt{\frac{2}{\pi}} \int\limits_0^\infty \{\mathcal{F}_s(f)\} \sin \omega x d\omega ... \quad ...(4)$$

Which is the inversion formula for fourier
sine transform.

Find fourier cosine transform of $f(x) = e^{-mx}$ where , $m > 0$

Solution: The fourier cosine transform of given fucntion is

$$\mathcal{F}_c(f) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(t) \cos \omega t dt$$

$$= \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} e^{-mt} \cos \omega t dt$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{e^{-mt}}{m^2 + \omega^2} \left(-m \cos \omega t + \omega \sin \omega t \right) \right]_0^\infty$$

$$= \sqrt{\frac{2}{\pi}} \left[0 - \frac{e^0}{m^2 + \omega^2} \left(-m.1 + 0 \right) \right]$$

$$\mathcal{F}_c(f) = \sqrt{\frac{2}{\pi}}\left(\frac{m}{m^2+\omega^2}\right)$$

which is required fourier cosine transform of given function.