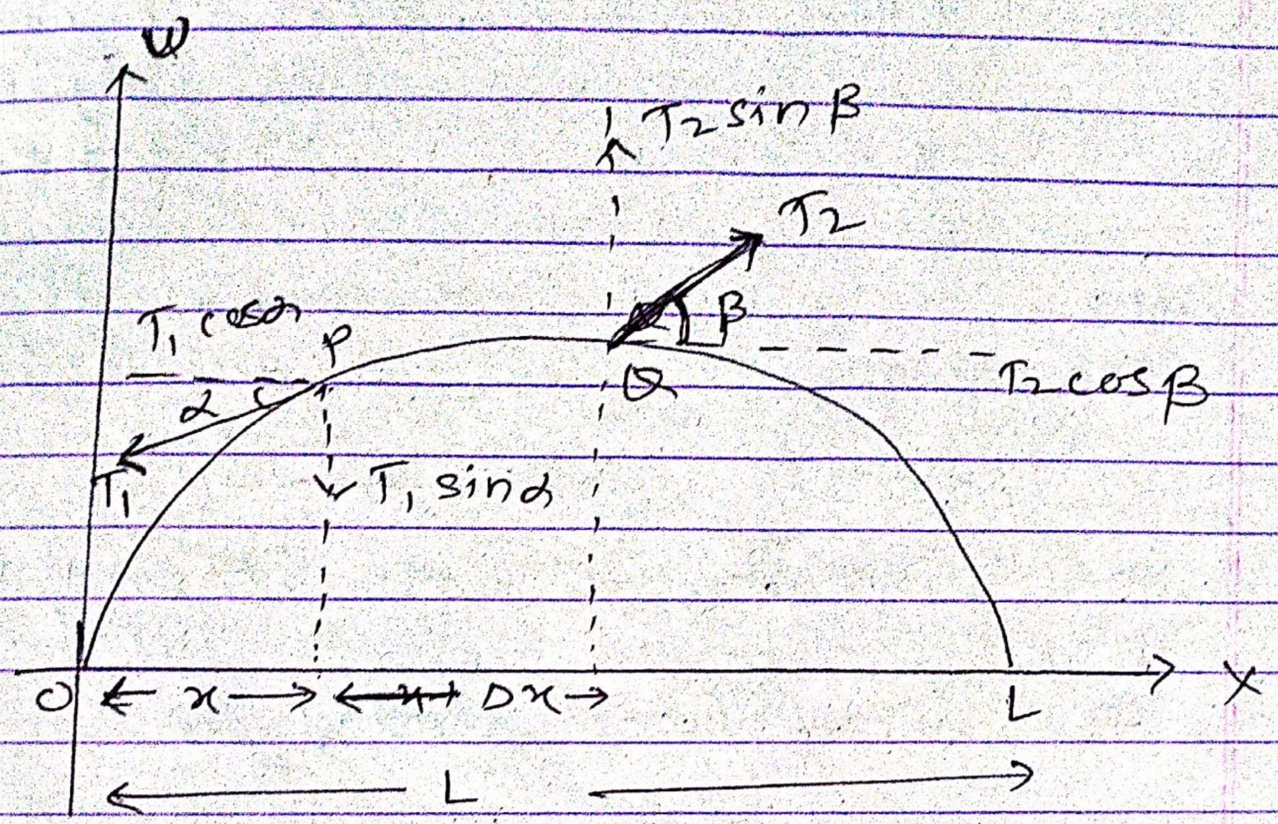
Q. Derive one dimensional wave equation of an elastic string which is tightly stretched to the length L and two ends are fixed.



consider a tightly stretched elastic string of length OL = L and fix it at the ends O and L. When the string vibrates each point of the string makes small vibration on u-axis which is shown in figure.

Let $P(x, u)$ and $Q(x+\Delta x, u+\Delta u)$ be two neighbouring points on the string. Let T_1 and T_2 be the tension at the points P and Q where the tensions makes angels

α and β with x-axis. Since the point of the string moves vertically there is no motion in the horizontal direction. Thus the horizontal components of the tension must be equal i.e. constant

$$\therefore T_1 \cos\alpha = T_2 \cos\beta = T \text{ (say)} \quad \text{—— (I)}$$

Also the vertical components of the force acting on this elements is

$$T_2 \sin\beta - T_1 \sin\alpha \quad \text{—— (II)}$$

Let $\rho \Delta x$ be the mass per unit length of string. Then by Newton's 2nd law of motion. force is equal to

$$F = ma$$

$$= \rho \Delta x \cdot \frac{\partial^2 u}{\partial t^2} \quad \text{—— (III)}$$

where ρ be the density, Δx is the length of portion string and $\frac{\partial^2 u}{\partial t^2}$ is acceleration.

From eqn (II) & (III)

$$T_2 \sin\beta - T_1 \sin\alpha = \rho \Delta x \frac{\partial^2 u}{\partial t^2} \quad \text{—— (IV)}$$

Now, dividing eqn (IV) by (I), we get

$$\frac{T_2 \sin\beta}{T_2 \cos\beta} - \frac{T_1 \sin\alpha}{T_1 \cos\alpha} = \frac{\rho \Delta x \frac{\partial^2 u}{\partial t^2}}{T}$$

$$\tan\beta - \tan\alpha = \frac{\rho \Delta x}{T} \cdot \frac{\partial^2 u}{\partial t^2} \quad \text{—— (V)}$$

we know that ~~the~~ $\tan\alpha$ and $\tan\beta$ are the slopes of the string at x and $x+\Delta x$ respectively.

where, $\tan\alpha = \left.\frac{\partial u}{\partial x}\right|_{at\ x}$

$\tan\beta = \left.\frac{\partial u}{\partial x}\right|_{at\ x+\Delta x}$

eqn (V) becomes, we get.

$$\left[\left.\frac{\partial u}{\partial x}\right|_{at\ x+\Delta x} - \left.\frac{\partial u}{\partial x}\right|_{at\ x}\right] = \frac{\rho \Delta x}{T} \cdot \frac{\partial^2 u}{\partial t^2}$$

Taking limit $\Delta x \to 0$ on both sides, we ge

$$\lim_{\Delta x \to 0} \frac{1}{\Delta x}\left[\left.\frac{\partial u}{\partial x}\right|_{at\ x+\Delta x} - \left.\frac{\partial u}{\partial x}\right|_{at\ x}\right] = \frac{\rho}{T} \frac{\partial^2 u}{\partial t^2}$$

$$\text{or, } \lim_{\Delta x \to 0} \left[\frac{\left.\frac{\partial u}{\partial x}\right|_{x+\Delta x} - \left.\frac{\partial u}{\partial x}\right|_{x}}{\Delta x}\right] = \frac{\rho}{T} \frac{\partial^2 u}{\partial t^2}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\rho}{T} \frac{\partial^2 u}{\partial t^2}$$

or, $$\frac{\rho}{T} \cdot \frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$$

or, $$\boxed{\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}} \quad \text{where } c^2 = \frac{T}{\rho}$$

This is the required one dimensionall wave equation.

Q.2 Find $u(x,t)$ from one dimensional wave equation $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$, with boundary condition $u(0,t) = u(L,t)$ initial deflection $f(x)$ and initial velocity $\left.\frac{\partial u}{\partial t}\right|_{t=0} = g(x)$.

Solution:-

We have one dimensional wave eqn is,

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \quad \text{———} \ (1)$$

where $u(x,t)$ is the deflection of the string with boundary condition

$u(0,t) = 0$ and $u(L,t) = 0$ for all t — (2)

Then its initial condition is,

$u(x,0) = f(x)$ —— (3) initial deflection

$\left.\frac{\partial u}{\partial t}\right|_{t=0} = g(x)$ — (4) initial velocity.

Suppose $u(x,t) = F(x) \cdot G(t)$ —— (5)

be the solution of eqn (1)

Differentiating eqn (5) w.r.to. x & t

$\frac{\partial^2 u}{\partial t^2} = F \cdot \ddot{G}$ and $\frac{\partial^2 u}{\partial x^2} = F'' G$.

where, dot $(\cdot)$ denotes derivatives w.r.to t & prime denotes derivatives w.r.to. x

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Putting these values in eqn (1), we get

$$F\ddot{G} = c^2 F'' G$$

$$\Rightarrow \frac{F''}{F} = \frac{\ddot{G}}{G c^2}$$

$$\Rightarrow \frac{F''}{F} = \frac{\ddot{G}}{G c^2} = k \text{ (say)}$$

which gives.

$$F'' - Fk = 0 \quad \text{———} \quad (6)$$

$$\ddot{G} - c^2 k G = 0 \quad \text{———} \quad (7)$$

These are second order ordinary differential equation. we have to determine F and G from eqn (6) & (7) under boundary conditions.

Also, we have

$$u(x,t) = F(x) \cdot G(t)$$

$$u(0,t) = F(0) \cdot G(t) = 0$$

$$u(L,t) = F(L) \cdot G(t) = 0$$

for all t.

For solving eqn (6) we have to generate boundary condition (b.c.)

If $G = 0$ then $u = 0$ which is no meaning or interest. so $G \neq 0$ Then we get,

$$F(0) = 0 = F(L) \quad \text{—— (8)}$$

case (i) Let $k = 0$, then eqn (6) becomes

$$F'' = 0$$

integrating

$$F' = a$$

again integrating

$$F = ax + b \quad \text{—— (9)}$$

where a & b are constants. we have to find a and b under boundary condition.

we have,

$$F(0) = 0 \quad \text{and} \quad F(L) = 0$$

$F(x) = ax + b$

or, $F(0) = a \cdot 0 + b$

or, $0 = b$

$\therefore b = 0$

Again

$F(L) = aL + b$

$0 = aL + b$

$\therefore a = 0$

we set $a = b = 0$.

putting these values of a and b in eqn (9)

$$F = 0x + 0 \Rightarrow F = 0$$

which is no interest

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case (ii) suppose that $k > 0$ i.e. k is positive.

let $k = \lambda^2$

Then eqn (6) becomes

$F'' - \lambda^2 F = 0$

its auxiliary eqn is,

$m^2 - \lambda^2 = 0$

$m = \pm \lambda$

its general eqn is,

$F(x) = Ae^{\lambda x} + Be^{-\lambda x}$ ——— (10)

where A and B are constants to be determine under boundary condition.

i.e. $F(0) = Ae^0 + Be^0$

$0 = A + B$ ——— (a)

Again, $F(L) = Ae^{\lambda L} + Be^{-\lambda L}$

$0 = Ae^{\lambda L} + Be^{-\lambda L}$ ——— (b)

solving (a) and (b), we get,

$A = B = 0$

eqn (10) becomes

$F = 0 \Rightarrow u = 0$

This is also no interest.

case (iii) :- Let $K < 0$

i.e. K is negative

Suppose $K = -p^2$

eqn ⑥ becomes

$$F'' + p^2 F = 0$$

auxiliary eqn is,

$$m^2 + p^2 = 0$$

$$m = \pm pi$$

its general solution is,

$F(x) = e^{ax}(A\cos bx + B\sin bx)$ where

a = real part

b = imaginary part

$$F(x) = e^{0\cdot x}(A\cos px + B\sin px)$$

$$F(x) = A\cos px + B\sin px \quad \text{——— ⑦}$$

using boundary condition

$$F(0) = A\cos 0 + B\sin 0$$

$$0 = A\cdot 1 + 0$$

$$\therefore A = 0$$

Also,

$$F(L) = A\cos pL + B\sin pL$$

$$0 = 0\cos pL + B\sin pL$$

$$\therefore B\sin pL = 0$$

Let $B \neq 0$

Then $\sin pL = 0$

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$\therefore \sin pL = \sin n\pi$

$\therefore pL = n\pi$

$p = \frac{n\pi}{L}$ where n is an integer

we have $B \neq 0$, setting $B = 1$

eqn. (1) becomes.

$F(x) = 0 \cos px + 1 \cdot \sin \frac{n\pi}{L} \cdot x$

$F(x) = \sin \frac{n\pi}{L} x$

we obtain infinitely many soln:

$F(x) = F_n(x)$

i.e. $F_n(x) = \sin \frac{n\pi}{L} x$ —— (12)

For $n = 1, 2, 3$ - - - —

similarly, for solution of eqn (7) i.e.

$\ddot{G} - c^2 k G = 0$

we have

$k = -p^2 = -\left(\frac{n\pi}{L}\right)^2$

Then eqn (7) becomes

$\ddot{G} + c^2 \left(\frac{n\pi}{L}\right)^2 G = 0$

$\therefore \ddot{G} + \left(\frac{cn\pi}{L}\right)^2 G = 0$

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$G'' + \lambda_n G = 0$ where $\lambda_n = \frac{cn\pi}{L}$

its auxiliary eqn is,

$$m^2 + \lambda_n^2 = 0$$

$$\therefore m = \pm \lambda_n i \quad \text{here } a = 0,\ b = \lambda_n$$

its general soln is,

$$G(t) = e^{pt}(B_n \cos\lambda_n t + B_n^* \sin\lambda_n t)$$

$$G(t) = e^0 (B_n \cos\lambda_n t + B_n^* \sin\lambda_n t)$$

$$G(t) = B_n \cos\lambda_n t + B_n^* \sin\lambda_n t$$

i.e. $G_n(t) = B_n \cos\lambda_n t + B_n^* \sin\lambda_n t$.

Therefore required solution of eqn (1) is,

$$u_n(x,t) = F_n(x) \cdot G_n(t)$$

$$= \sin\frac{n\pi}{L} x \, (B_n \cos\lambda_n t + B_n^* \sin\lambda_n t)$$

By fundamental theorem the solution of given wave eqn is the infinitely many u_n. Thus.

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t)$$

$$\Rightarrow u(x,t) = \sum_{n=1}^{\infty} (B_n \cos\lambda_n t + B_n^* \sin\lambda_n t) \sin\frac{n\pi x}{L} \quad \text{---(12)}$$

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From eqn (3) (initial condition)

$u(x,0) = f(x)$

Then eqn (13) becomes

$$u(x,0) = \sum_{n=1}^{\infty} (B_n \cos 0 + B_n^* \sin 0) \sin \frac{n\pi}{L} x$$

i.e. $$f(x) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi}{L} x$$

which is fourier sine series and its co-efficient B_n is given by

$$B_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi}{L} x \, dx$$

Also, we have

$$\left.\frac{\partial u}{\partial t}\right|_{t=0} = g(x)$$

Eqn (13) becomes

$$\left[\sum_{n=1}^{\infty} \left(-B_n \lambda_n \sin \lambda_n t + B_n^* \lambda_n \cos \lambda_n t \right) \sin \frac{n\pi}{L} x \right]_{t=0}$$

$$= g(x)$$

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put $t=0$

$$\Rightarrow \sum_{n=1}^{\infty} B_n^* \lambda_n \sin \frac{n\pi}{L} x = g(x)$$

or, $$g(x) = \sum_{n=1}^{\infty} B_n^* \lambda_n \sin \frac{n\pi x}{L}$$

which is also fourier sine series of $g(x)$ its co-efficient is given by

$$B_n^* \lambda_n = \frac{2}{L} \int_0^L g(x) \sin \frac{n\pi}{L} x \, dx.$$

$$B_n^* = \frac{\cancel{2}\, 2}{\lambda_n L} \int_0^L g(x) \sin \frac{n\pi}{L} x \, dx.$$

$$= \frac{2}{\frac{cn\pi}{L} \times L} \int_0^L g(x) \sin \frac{n\pi}{L} x \, dx.$$

where $\lambda_n = \frac{cn\pi}{L}$

$$= \frac{2}{cn\pi} \int_0^L g(x) \sin \frac{n\pi}{2} x \, dx$$

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Therefore we get required solution of one dimensional wave eqn is,

$$u(x,t) = \sum_{n=1}^{\infty} \left(B_n \cos\lambda_n t + B_n^{*} \sin\lambda_n t\right) \sin\frac{n\pi}{L}x$$

where,

$$B_n = \frac{2}{L}\int_0^L f(x) \sin\frac{n\pi}{L}x\, dx$$

$$B_n^{*} = \frac{2}{cn\pi}\int_0^L g(x) \sin\frac{n\pi}{L}x\, dx.$$

where $f(x)$ and $g(x)$ are initial deflection and initial velocity of the string respectively.

Q3. Find the solution of one dimensional wave equation corresponding to the triangular initial deflection.

$$f(x) = \begin{cases} \frac{2k}{L}x, & \text{if } 0 < x < \frac{L}{2} \\ \frac{2k}{L}(L-x), & \text{if } L/2 < x < L \end{cases}$$

with its initial velocity zero and $c = 1$.

Soln:- we know solution of one dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \text{ is}$$

$$u(x,t) = \sum_{n=1}^{\infty} (B_n \cos \lambda_n t + B_n^* \sin \lambda_n t) \sin \frac{n\pi}{L} x \qquad \text{—— (1)}$$

where $\lambda_n = \frac{cn\pi}{L}$ $\quad B_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi}{L} x \, dx$

where $f(x)$ is initial deflection

and $\quad B_n^* = \frac{2}{\lambda_n L} \int_0^L g(x) \sin \frac{n\pi}{L} x \, dx$.

where $g(x)$ is initial velocity.

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But we have,

$$f(x) = \begin{cases} \frac{2k}{L} x & \text{if } 0 < x < L/2 \\ \frac{2k}{L}(L-x) & \text{if } \frac{L}{2} < x < L \end{cases}$$

and $g(x) = 0$

Therefore, we get,

$$B_n = \frac{2}{L}\left[\int_0^{L/2} \frac{2k}{L} \sin\frac{n\pi}{L} x\, dx + \int_{L/2}^{L} \frac{2k}{L}(L-x) \sin\frac{n\pi}{L} x\, dx\right]$$

$$B_n = \frac{4k}{L^2}\left[\int_0^{L/2} x \sin\frac{n\pi}{L} x\, dx + \int_{L/2}^{L} (L-x) \sin\frac{n\pi}{L} x\, dx\right] \quad \text{--- (2)}$$

Here, $$\int_0^{L/2} x \sin\frac{n\pi}{L} x\, dx = \left[\frac{x \cos\frac{n\pi}{L} x}{-\frac{n\pi}{L}} + \frac{\sin\frac{n\pi}{L} x}{\left(\frac{n\pi}{L}\right)^2}\right]_0^{L/2}$$

$$= \left[\frac{L/2 \cos\frac{n\pi}{2} x}{-\frac{n\pi}{L}} + \frac{\sin\frac{n\pi}{2} x}{\left(\frac{n\pi}{L}\right)^2}\right] = \frac{\sin n\pi/2}{\left(\frac{n\pi}{2}\right)^2}$$

Also, $\int_{L/2}^{L} (L-x)\sin\frac{n\pi}{L}x\,dx$

$$= \left[\frac{(L-x)\cos\frac{n\pi}{L}x}{-\frac{n\pi}{L}} - \frac{\sin\frac{n\pi}{L}x}{\left(\frac{n\pi}{2}\right)^2} \right]_{L/2}^{L}$$

$$= \left[0 - \frac{\frac{L}{2}\cos\frac{n\pi}{2}}{-\frac{n\pi}{2}} + \frac{\sin\frac{n\pi}{2}}{\left(\frac{n\pi}{2}\right)^2} \right] = \frac{\sin\frac{n\pi}{2}}{\left(\frac{n\pi}{L}\right)^2}$$

Therefore from eqn ②, we get

$$B_n = \frac{4K}{L^2} \frac{2\sin\frac{n\pi}{2}}{\left(\frac{n\pi}{2}\right)^2}, \quad B_n = \frac{8K}{(n\pi)^2}\sin\frac{n\pi}{2}$$

Also, $B_n^* = 0$ since $g(x) = 0$

Therefore from eqn ①, u

$$u(x,t) = \sum_{n=1}^{\infty} \left(\frac{8K}{(n\pi)^2} \sin\frac{n\pi}{2} \cos\lambda_n t \right) \sin\frac{n\pi}{L}x$$

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$$\frac{8K}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\frac{n\pi}{2}\cos\frac{cn\pi}{L}t$$

$$u(x,t)=\frac{8K}{\pi^2}\left[\sin\frac{\pi}{L}x\cos\frac{\pi c}{L}t+-\frac{1}{3^2}\sin\frac{3\pi}{L}.\cos\frac{3\pi c}{L}t+\cdots\right]$$

This is the required solution of one dimensional wave equation with given initial velocity and initial deflection.

Q. 4 Find the solution of one dimensional wave equation with initial deflection $1/2\sin 3x+\sin x$ and initial velocity is zero.

Soln,-

we have one dimensional wave eqn is,

$$\frac{\partial^2 u}{\partial t^2}=c^2\frac{\partial^2 u}{\partial x^2}$$

And its solution is,

$$u(x,t)=\sum_{n=1}^{\infty}(B_n\cos\lambda_n t+B_n^{*}\sin\lambda_n t)\sin\frac{n\pi}{L}x$$

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where, $B_n = \frac{2}{L}\int_0^L f(x) \sin\frac{n\pi}{L}x\,dx$

& $B_n^* = \frac{2}{cn\pi}\int_0^L g(x) \sin\frac{n\pi}{L}x\,dx$.

where $f(x)$ = initial deflection &
$g(x)$ = initial velocity.

we have, $f(x) = \frac{1}{2}\sin 3x + \sin x$

& $g(x) = 0$, so.

$B_n^* = \frac{2}{cn\pi}\int_0^L g(x) \sin\frac{n\pi}{L}x\,dx = 0$.

eqn (1) becomes:

$u(x,t) = \sum_{n=1}^{\infty} B_n \cos\lambda_n t \cdot \sin\frac{n\pi}{L}x$

$u(x,t) = \sum_{n=1}^{\infty} B_n \cos\lambda_n t \sin\frac{n\pi}{\pi}x$ $[\because L=\pi]$

$\therefore u(x,t) = \sum_{n=1}^{\infty} B_n \cos\lambda_n t \sin nx$ ——— (11)

put $t=0$ in eqn (11), we get

$$U(x,0) = \sum_{n=1}^{\infty} B_n \cos 0 \cdot \sin nx$$

$$U(x,0) = \sum_{n=1}^{\infty} B_n \sin nx$$

$$U(x,0) = B_1 \sin x + B_2 \sin 2x + B_3 \sin 3x + \ldots \quad \text{— (iii)}$$

Also, we have

$$U(x,0) = \frac{1}{2} \sin 3x + \sin x \quad \text{— (iv)}$$

Equating (iii) & (iv), we get

$$\frac{1}{2} \sin 3x + \sin x = B_1 \sin x + B_2 \sin 2x + B_3 \sin 3x + \ldots$$

Equating coefficients, we get

$B_1 = 1$, $B_2 = 0$, $B_3 = 1/2$,

$B_4 = B_5 = 0 \ldots = 0$ (others all are zero)

From eqn (ii), we have

$$U(x,t) = \sum_{n=1}^{\infty} B_n \cos \lambda_n t \cdot \sin nx$$

$$U(x,t) = B_1 \cos \lambda_1 t \sin x + B_2 \cos \lambda_2 t \sin 2x + B_3 \cos \lambda_3 t \sin 3x + \ldots$$

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putting values of $B_1, B_2, B_3 \ldots$

$u(x,t) = \cos\lambda_1 t \sin x + 0 + \frac{1}{2}\cos\lambda_3 t \sin 3x$

$u(x,t) = \cos\lambda_1 t \sin x + \frac{1}{2}\cos\lambda_3 t \sin 3x$

we have,

$$\lambda_n = \frac{cn\pi}{L}$$

$$\lambda_n = \frac{1 \cdot n\pi}{\pi} \quad [\because L = \pi]$$

$$\therefore \lambda_n = n$$

i.e., $\lambda_1 = 1$, $\lambda_2 = 2$; $\lambda_3 = 3 \ldots$

Then.

$u(x,t) = \cos t \sin x + \frac{1}{2}(\cos 3t \sin 3x)$

is the required soln.

(6) A tightly stretched with fixed ends at $x = 0$ and $x = L$ is initially at rest in the equilibrium position (initial deflection is zero). Find $u(x,t)$ if it is set vibrating by giving to each of its point a velocity is $3(Lx - x^2)$.

⇒ Soln:-

Given,

initial deflection, $f(x) = 0$.
initial velocity, $g(x) = 3(Lx - x^2)$.

we know that the soln of 1-D wave eqn is

$$u(x,t) = \sum_{n=1}^{\infty} \left(B_n \cos \lambda_n t + B_n^* \sin \lambda_n t \right) \sin \frac{n\pi}{L} x \quad \text{--- (1)}$$

Where,

$$B_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} \, dx.$$

$$\& \quad B_n^* = \frac{2}{cn\pi} \int_0^L g(x) \sin \frac{n\pi}{L} x \, dx$$

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$\frac{cn\pi}{x}$

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since, $f(x) = 0$, so,

$$B_n = 0.$$

Then eqn ① becomes,

$$u(x,t) = \sum_{n=1}^{\infty} B_n^* \sin \lambda_n t \sin \frac{n\pi}{L} x \quad \text{——}①$$

Now,

$$B_n^* = \frac{2}{cn\pi} \int_0^L g(x) \sin \frac{n\pi}{L} x \, dx.$$

$$= \frac{2}{cn\pi} \int_0^L 3(Lx - x^2) \sin \frac{n\pi}{L} x \, dx.$$

$$= \frac{6}{cn\pi} \int_0^L (Lx - x^2) \sin \frac{n\pi}{L} x \, dx.$$

$$
\left[\begin{array}{ccc}
Lx - x^2 & & \sin\frac{n\pi}{L}x \\
L - 2x & + & \dfrac{-\cos\frac{n\pi}{L}x}{\frac{n\pi}{L}} \\
-2 & - & \dfrac{-\sin\frac{n\pi}{L}x}{\left(\frac{n\pi}{L}\right)^2} \\
0 & + & \dfrac{\cos\frac{n\pi}{L}x}{\left(\frac{n\pi}{L}\right)^3}
\end{array}\right]
$$

$$
\frac{6}{cn\pi}\left[-(Lx - x^2)\frac{\cos\frac{n\pi}{L}x}{\frac{n\pi}{L}} + (L-2x)\frac{\sin\frac{n\pi}{L}x}{\left(\frac{n\pi}{L}\right)^2} - \frac{2\cos\frac{n\pi}{L}x}{\left(\frac{n\pi}{L}\right)^3}\right]_0^L
$$

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$$\frac{6}{cn\pi}\left[-\cancel{(L^2-L^2)}^{\,0}\frac{\cos\frac{n\pi}{L}L}{\frac{n\pi}{L}}+(L-2L)\frac{\sin\frac{n\pi}{L}.L}{\left(\frac{n\pi}{L}\right)^2}\right.$$

$$\left.-2\frac{\cos\frac{n\pi}{L}.L}{\left(\frac{n\pi}{L}\right)^3}\right]$$

$$=\frac{6}{cn\pi}\left[0+\cancel{L\frac{\sin n\pi}{\left(\frac{n\pi}{L}\right)^2}}^{\,0}-2\frac{\cos n\pi}{\left(\frac{n\pi}{L}\right)^3}\right]$$

$$=\frac{6}{cn\pi}\left[-2\frac{\cos n\pi}{\left(\frac{n\pi}{L}\right)^3}\right]$$

$$=\frac{12L^3}{cn^4\pi^4}\left[-\cos n\pi\right]$$

Eqn (11) becomes

$$u(x,t)=\sum_{n=1}^{\infty}\frac{12L^3}{cn^4\pi^4}(-\cos n\pi)\sin\frac{n\pi}{L}x\cdot\sin\lambda_n t$$

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$$= \sum_{n=1}^{\infty} \frac{12L^3}{cn^4\pi^4} (-\cos n\pi) \cdot \sin \frac{cn\pi}{L} \times t \cdot \sin \frac{n\pi}{L} x$$

$$= \sum_{n=1}^{\infty} \frac{12L^3}{cn^4\pi^4} (-\cos n\pi) \sin \frac{cn\pi}{L} t \cdot \sin \frac{n\pi}{L} x \cdot$$

$\left[\because \lambda_n = \frac{cn\pi}{L} \right.$ [redacted] $\left. \right]$

$$= \sum_{n=1}^{\infty} \frac{12L^3}{cn^4\pi^4} (-\cos n\pi) \sin \frac{cn\pi t}{L} \cdot \sin \frac{n\pi}{L} x \cdot$$

$$\frac{12L^3}{c\pi^4} \sum_{n=1}^{\infty} \frac{(-\cos n\pi)}{n^4} \cdot \sin \frac{cn\pi t}{L} \cdot \sin \frac{n\pi}{L} x$$

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5) Find u(x,t) of eqn $xu_{xy}+2yu=0$ by using separation of variable method.

soln:-

Given eqn is,

$$xu_{xy}+2yu=0 \quad \text{—— (1)}.$$

Suppose $u(x,y)=F(x)\cdot G(y)$ —— (11)

be the solution of eqn (1) where F is the func^n of x and G is func^n of y.

diff. eqn (11),

$$u_x = F'G \;\&\; u_y = F\cdot \dot{G}$$

$$u_{xy} = F'\dot{G}$$

$$u = F\cdot G.$$

putting values in eqn (1)

$$x F'\dot{G} + 2yF\cdot G = 0.$$

$$\text{or, } xF'\dot{G} = -2yF\cdot G.$$

$$\text{or, } x\frac{F'}{F} = -2y\frac{G}{\dot{G}}$$

$$\text{or, } x\frac{F'}{F} = -2y\frac{G}{\dot{G}} = k \text{ (say)}.$$

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Then,

$$x\frac{F'}{F} = k$$

$$\frac{F'}{F} = \frac{k}{x}$$

integrating both sides we get.

$$\log F = k \log x + \log c$$

$$\log F = \log(x^k . c)$$

$$F = x^k . c$$

Again.

$$\frac{G'}{G} = \frac{-2y}{k}$$

Integrating both sides we get.

$$\log G = \frac{-2y^2}{2k} + c$$

$$G = e^{-y^2/k + c}$$

Now,

$$u(x, y) = F(x) . G(y)$$

$$= x^k . c . e^{-y^2/k + c}$$

$$= x^k . c . e^{-y^2/k} . e^c$$

$$= c . e^c x^k e^{-y^2/k}$$

$$= R x^k e^{-y^2/k} \quad \text{where } R = ce^c$$

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