Application of partial differential equation

– 20 marks

– 10 × 2 = 20

– 5 × 4 = 20

Sure question, Theorem and Solution:
1. One-dimensional wave equation
2. One-dimensional heat equation
3. Two-dimensional heat equation, Laplace equation
4. Laplace equation in polar form

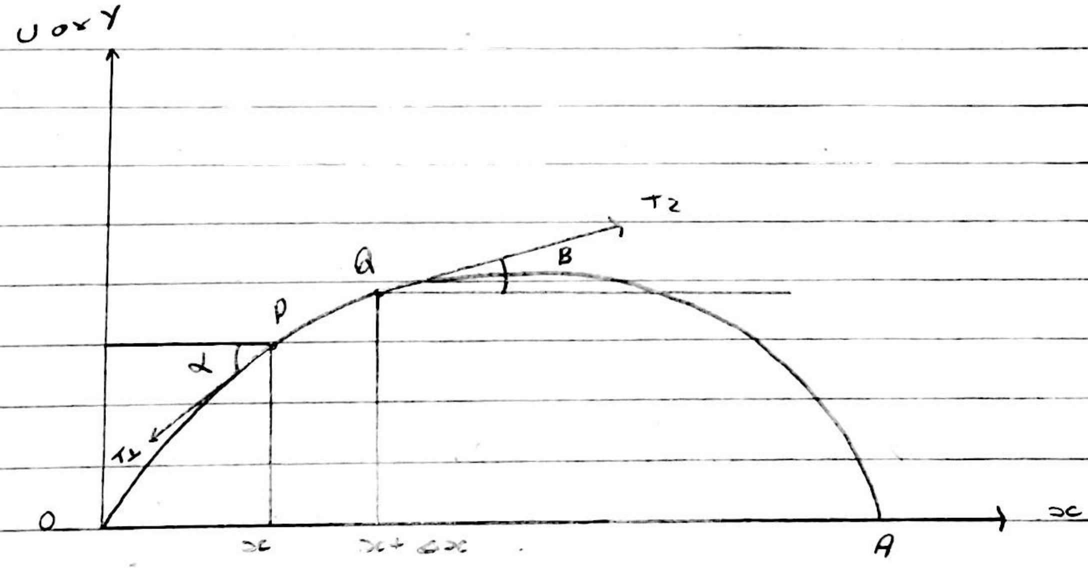
10 → Derivation and its solution
10 → Example

Imp

Derivation of one-dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \left(\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \right) \quad u \to \text{displacement}$$

=)



Let us consider a elastic string of length 'l' fixed at point O and A & displaced slightly from its equlibrium position OA. Let us take end O as origin, OA as x-axis and line perⁿ. to OA through O as u or Y-axis. We find the displacement 'u' as function of distance 'x' and time 't'.

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Let force acting to small element PQ of string. Let tension at P and Q are T_1 and T_2 respectively. Let T_1 and T_2 make an angle α and β with x-axis. Since, points of string moves only vertical direction so that horizontal components of string must be constant.

So,

$$T_1 \cos\alpha = T_2 \cos\beta = 'T' = \text{Constant (say)} \quad \text{--- (i)}$$

In vertical direction, vertical components are $-T_1 \sin\alpha$ and $T_2 \sin\beta$ of T_1 and T_2 respectively. Here, negative sign appears because vertical component at P acts downward.

Resultant force acting to element PQ is $T_2 \sin\beta - T_1 \sin\alpha$.

Then, by newton's second law of motion,

Resultant force is equal to product of mass times acceleration

i.e. $$T_2 \sin\beta - T_1 \sin\alpha = \rho \Delta x \frac{\partial^2 u}{\partial t^2} \quad \text{--- (ii)}$$

where 'ρ' be mass per unit length and Δx be length of element PQ.

Dividing eqⁿ (ii) by eqⁿ (i),

$$\frac{T_2 \sin\beta}{T_2 \cos\beta} - \frac{T_1 \sin\alpha}{T_1 \cos\alpha} = \frac{\rho \Delta x}{T} \frac{\partial^2 u}{\partial t^2}$$

i.e. $$\tan\beta - \tan\alpha = \frac{\rho \Delta x}{T} \frac{\partial^2 u}{\partial t^2} \quad \text{--- (iii)}$$

We know, $\tan\alpha$ and $\tan\beta$ are slope of string at P and Q.

$$\therefore \tan\alpha = \left(\frac{\partial u}{\partial x}\right)_x \quad \text{and} \quad \tan\beta = \left(\frac{\partial u}{\partial x}\right)_{x+\Delta x}$$

So,

Eq^n. (iii) becomes.

$$\left(\frac{\partial u}{\partial x}\right)_{x+\Delta x} - \left(\frac{\partial u}{\partial x}\right)_{x} = \frac{\rho \Delta x}{T}\frac{\partial^2 u}{\partial t^2}$$

or, $$\frac{\left(\frac{\partial u}{\partial x}\right)_{x+\Delta x} - \left(\frac{\partial u}{\partial x}\right)_{x}}{\Delta x} = \frac{\rho}{T}\frac{\partial^2 u}{\partial t^2}$$

Taking limit $\Delta x \to 0$

$$\lim_{\Delta x \to 0} \frac{(\partial u/\partial x)_{x+\Delta x} - (\partial u/\partial x)_{x}}{\Delta x} = \frac{\rho}{T}\frac{\partial^2 u}{\partial t^2}$$

or, $$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\rho}{T}\frac{\partial^2 u}{\partial t^2}$$

or, $$\frac{\partial^2 u}{\partial x^2} = \frac{\rho}{T}\frac{\partial^2 u}{\partial t^2}$$

or, $$\frac{\partial^2 u}{\partial t^2} = \frac{T}{\rho}\frac{\partial^2 u}{\partial x^2}$$

$\therefore$ $$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \quad \text{where } c^2 = T/\rho$$

which is one-dimensional wave eq^n.

Solution of one-dimensional wave eq^n :

One dimensional wave eq^n is

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \quad \text{——— (i)}$$

Let $u = XT$ be general sol^n. of eq^n (i) where X be the function of 'x' only and T be the function of 't' only.

So, eq^n (i) becomes

$$\frac{\partial^2 (XT)}{\partial t^2} = c^2 \frac{\partial^2 (XT)}{\partial x^2}$$

or, $XT'' = c^2 TX''$

or, $\frac{X''}{X} = \frac{T''}{c^2 T}$

or, $\frac{X''}{X} = \frac{T''}{c^2 T} = k$ (constant) say

case I Let $k < 0$ and $k = -p^2$.

Then,

$\frac{X''}{X} = -p^2$ and $\frac{T''}{c^2 T} = -p^2$

i.e. $\frac{X''}{X} = -p^2$

$\Rightarrow X'' + p^2 X = 0$

$\Rightarrow \frac{d^2X}{dx^2} + p^2 X = 0$

$\Rightarrow (D^2 + p^2) X = 0$

Its auxiliary eqn. is

$m^2 + p^2 = 0$

$\therefore m = \pm pi$

$\therefore X(x) = C_1 \cos px + C_2 \sin px$

Also,

$\frac{T''}{c^2 T} = -p^2$

$\Rightarrow T'' + c^2 p^2 T = 0$

$\Rightarrow \frac{d^2T}{dt^2} + c^2 p^2 T = 0$

$\Rightarrow (D^2 + c^2 p^2) T = 0$

Its A.E. is

$m^2 + c^2 p^2 = 0$

Flair $\therefore m = \pm cpi$

case II

$\therefore T(t) = C_3 \cos cpt + C_4 \sin cpt$

$\therefore u(x,t) = XT$

$= (C_1 \cos px + C_2 \sin px)(C_3 \cos cpt + C_4 \sin cpt)$

be general soln. for $k < 0$.

case II Let $k > 0$ and $k = p^2$.

Then,

$$\frac{X''}{X} = p^2 \quad \text{and} \quad \frac{T''}{c^2 T} = p^2$$

i.e. $\frac{X''}{X} = p^2$

$\Rightarrow X'' - p^2 X = 0$

$\Rightarrow \frac{d^2X}{dx^2} - p^2 X = 0$

$\Rightarrow (D^2 - p^2) X = 0$

Its AE is

$m^2 - p^2 = 0$

$\therefore m = \pm p$

$\therefore X(x) = C_5 e^{-px} + C_6 e^{px}$

Also,

$$\frac{T''}{c^2 T} = p^2$$

$\Rightarrow T'' - p^2 c^2 T = 0$

$\Rightarrow \frac{d^2T}{dt^2} - p^2 c^2 T = 0$

$\Rightarrow (D^2 - c^2 p^2) T = 0$

Its AE is

$m^2 - c^2 p^2 = 0$

$\therefore m = \pm cp$

$\therefore T(t) = C_7 e^{-cpt} + C_8 e^{cpt}$

$\therefore u(x,t) = \left(C_5 e^{-px} + C_6 e^{px}\right)\left(C_7 e^{-cpt} + C_8 e^{+cpt}\right)$
be general soln. for $k > 0$.

Case III Let $k = 0$.

Then,

$$\frac{X''}{X} = 0 \quad \text{and} \quad \frac{T''}{c^2 T} = 0$$

i.e. $X'' = 0$

$\Rightarrow \dfrac{d^2 X}{dx^2} = 0$

$\Rightarrow D^2 X = 0$

Its AE is

$m^2 = 0$

$\therefore m = 0$

$\therefore X(x) = (C_9 x + C_{10}) e^0$

$= C_9 x + C_{10}$

Also,

$$\frac{T''}{c^2 T} = 0$$

i.e. $T'' = 0$

$\Rightarrow \dfrac{d^2 T}{dt^2} = 0$

$\Rightarrow D^2 T = 0$

Its AE is

$m^2 = 0$

$\therefore m = 0$

$\therefore T(t) = (c_{11}t + c_{12})e^{0}$
$= c_{11}t + c_{12}$

$\therefore u(x,t) = (c_9 x + c_{10})(c_{11}t + c_{12})$ be general sol^n. for $k=0$.

The three possible solutions are

(i) $u(x,t) = (c_1 \cos px + c_2 \sin px)(c_3 \cos cpt + c_4 \sin cpt)$
(ii) $u(x,t) = (c_5 e^{-px} + c_6 e^{px})(c_7 e^{-cpt} + c_8 e^{cpt})$
(iii) $u(x,t) = (c_9 x + c_{10})(c_{11}t + c_{12})$

of these three solution we choose that solution which is constant with physical nature problem. Since, we are dealing with problem on vibration, 'u' must be periodic function of x and t. So, solution must contain trigonometric term.

$\therefore u(x,t) = (c_1 \cos px + c_2 \sin px)(c_3 \cos cpt + c_4 \sin cpt)$ be feasible solution of one dimensional wave eq^n.

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Example :- A tightly stretched flexible string of length 'l' fixed at $x=0$ and $x=l$. At time $t=0$ the string is given a shape defined by $f(x) = \lambda x(l-x)$ where λ is constant and then released. Find the displacement at any time $t>0$.

sol^n :- One dimensional wave eq^n is

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \quad \text{——— (i)}$$

The consistent sol^n of one dimensional wave eq^n is

$u(x,t) = (c_1 \cos px + c_2 \sin px)(c_3 \cos cpt + c_4 \sin cpt)$ — (ii)

Boundary conditions. $u(0,t) = 0$ and $u(l,t) = 0$

Initial Conditions. $\left(\frac{\partial u}{\partial t}\right)_{t=0} = 0$ (initial velocity)

$u(x,0) = \lambda x(l-x)$ (initial deflection)

Using the condition $u(0,t) = 0$ in eqⁿ (ii),

$$0 = c_1 (c_3 \cos cpt + c_4 \sin cpt)$$

Since, $c_3 \cos cpt + c_4 \sin cpt \neq 0$ otherwise solution is trivial (zero solⁿ). So, $c_1 = 0$.

So,

Eqⁿ. (ii) becomes,

$$u(x,t) = c_2 \sin px (c_3 \cos cpt + c_4 \sin cpt) \quad \text{--- (iii)}$$

Using the condition $u(l,t) = 0$ in eqⁿ (iii),

$$0 = c_2 \sin pl (c_3 \cos cpt + c_4 \sin cpt)$$

Since, $c_2 (c_3 \cos cpt + c_4 \sin cpt) \neq 0$

So,

$$\sin pl = 0 = \sin n\pi, \quad n = 1, 2, 3, \ldots$$

$$\Rightarrow pl = n\pi$$

$$\Rightarrow p = \frac{n\pi}{l}$$

So,

Eqⁿ. (iii) becomes,

$$u(x,t) = c_2 \sin \frac{n\pi x}{l} \left(c_3 \cos \frac{cn\pi t}{l} + c_4 \sin \frac{cn\pi t}{l} \right)$$

$$= \left(A \cos \frac{cn\pi t}{l} + B \sin \frac{cn\pi t}{l} \right) \sin \frac{n\pi x}{l}$$

where $A = c_2 c_3$

$B = c_2 c_4$

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By the principle of superposition, the most general solution is

$$u(x,t) = \sum_{n=1}^{\infty}\left(A_n \cos\frac{cn\pi t}{l} + B_n \sin\frac{cn\pi t}{l}\right)\sin\frac{n\pi x}{l} \quad \text{— (iv)}$$

Now, diff. eq^n (iv) partially w.r.t 't', we get

$$\frac{\partial u}{\partial t} = \sum_{n=1}^{\infty}\left(-A_n \frac{cn\pi}{l}\sin\frac{cn\pi t}{l} + B_n\frac{cn\pi}{l}\cos\frac{cn\pi t}{l}\right)\sin\frac{n\pi x}{l}$$

Using the condition $\frac{\partial u}{\partial t} = 0$ at $t = 0$,

$$0 = \sum_{n=1}^{\infty} B_n \frac{cn\pi}{l}\sin\frac{n\pi x}{l}$$

$$\therefore B_n = 0$$

So, eq^n (iv) becomes,

$$u(x,t) = \sum_{n=1}^{\infty} A_n \cos\frac{cn\pi t}{l}\sin\frac{n\pi x}{l} \quad \text{— (v)}$$

Now,

Using the condition $u(x,0) = \lambda x(l-x)$ in eq^n (v),

$$\lambda x(l-x) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{l}$$

which is half range Fourier sine series of function $f(x) = \lambda x(l-x)$ on the interval $0 < x < l$.

So,

$$A_n = \frac{2}{l}\int_0^l \lambda x(l-x)\sin\frac{n\pi x}{l}\,dx$$

Here,

$u = \lambda x(l-x) = \lambda l x - \lambda x^2$ $\qquad v_1 = \sin\frac{n\pi x}{l}$

$u' = \lambda l - 2\lambda x$ $\qquad v_2 = -\cos\frac{n\pi x}{l} \Big/ \frac{n\pi}{l}$

$u'' = -2\lambda$ $\qquad v_3 = -\sin\frac{n\pi x}{l} \Big/ \frac{n^2\pi^2}{l^2}$

$u''' = 0$ $\qquad v_4 = \cos\frac{n\pi x}{l} \Big/ \frac{n^3\pi^3}{l^3}$

Then,

$$A_n = \frac{2}{l}\left[\lambda x(l-x)\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - (\lambda l - 2\lambda x)\left(\frac{-\sin\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right) + (-2\lambda)\,\frac{\cos\frac{n\pi x}{l}}{\frac{n^3\pi^3}{l^3}} \right]_0^l$$

$$= \frac{2\lambda}{l}\left[-(lx - x^2)\frac{\cos\frac{n\pi x}{l}}{n\pi/l} + (l-2x)\frac{\sin\frac{n\pi x}{l}}{n^2\pi^2/l^2} - 2\,\frac{\cos n\pi x/l}{n^3\pi^3/l^3} \right]_0^l$$

$$= \frac{2\lambda}{l} \times \left[-2\,\frac{\cos n\pi}{n^3\pi^3/l^3} + 2\,\frac{\cos 0}{n^3\pi^3/l^3} \right]$$

$$= \frac{4\lambda}{l} \times \frac{1}{n^3\pi^3/l^3}\left(-\cos n\pi + \cos 0\right)$$

$$= \frac{4\lambda l^2}{n^3\pi^3}\left(1 - (-1)^n\right) \quad = 0 \text{ when } n \text{ is even}$$

$$= \frac{8\lambda l^2}{n^3\pi^3} \text{ when } n \text{ is odd i.e. } n.$$

So,

From eq^n ②.

$$u(x,t) = \sum_{n=1}^{\infty} \frac{4\lambda l^2}{n^3\pi^3}\left(1 - (-1)^n\right)\cos\frac{cn\pi t}{l}\,\sin\frac{n\pi x}{l}$$

Tutorial 3 Solve the equation $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$ corresponding to the initial deflection $u(x,0) = \begin{cases} \frac{2kx}{l} & , 0 < x < l/2 \\ \frac{2k}{l}(l-x) & , \frac{l}{2} < x < l \end{cases}$ and initial velocity $\left(\frac{\partial u}{\partial t}\right)_{t=0} = g(x) = 0$.

$u(x,0) = u_0 \sin^3 \pi x/l$
$u(0,t) = 0$
$u(l,t) = 0$

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Example A tightly stretched string with fixed ends $x = 0$ and $x = l$ is initially in position given by $u = u_0 \sin^3 \frac{\pi x}{l}$. If it is released from rest from this position, find the displacement at time $t > 0$. $(\partial u/\partial t)_{t=0} = 0$

Soln: One dimensional wave eqn is

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \quad \text{—— (i)}$$

The ~~~~ consistent soln. of one dimensional wave eqn (i) is

$$u(x,t) = (c_1 \cos px + c_2 \sin px)(c_3 \cos cpt + c_4 \sin cpt) \quad \text{—— (ii)}$$

Boundary conditions, $u(0,t) = 0$
$u(l,t) = 0$

Initial condition, $\left(\frac{\partial u}{\partial t}\right)_{t=0} = 0$ (initial velocity)

$$u(x,0) = u_0 \sin^3 \frac{\pi x}{l}$$

Using the condition $u(0,t)$ in eqn (ii),

$$0 = c_1(c_3 \cos cpt + c_4 \sin cpt)$$

Since, $c_3 \cos cpt + c_4 \sin cpt \neq 0$

$$\therefore c_1 = 0$$

So,

Eqn (ii) becomes,

$$u(x,t) = c_2 \sin px (c_3 \cos cpt + c_4 \sin cpt) \quad \text{—— (iii)}$$

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Using the condition $u(l,t) = 0$ in eqn (iii).

$$0 = c_2 \sin pl \,(c_3 \cos cpt + c_4 \sin cpt)$$

Since. $c_2(c_3 \cos cpt + c_4 \sin cpt) \neq 0$, so

$$\sin pl = 0 = \sin n\pi \quad , \quad n = 1, 2, 3, \ldots$$

$$\Rightarrow \quad pl = n\pi$$

$$\Rightarrow \quad p = \frac{n\pi}{l}$$

Then.

Eqn (iii) becomes,

$$u(x,t) = c_2 \sin \frac{n\pi x}{l} \left(c_3 \cos \frac{cn\pi t}{l} + c_4 \sin \frac{cn\pi t}{l} \right)$$

$$= \sin \frac{n\pi x}{l} \left(A \cos \frac{cn\pi t}{l} + B \sin \frac{cn\pi t}{l} \right)$$

where $A = c_2 c_3$

$B = c_2 c_4$

By the principle of superposition, the most general soln. is

$$u(x,t) = \sum_{n=1}^{\infty} \left(A_n \cos \frac{cn\pi t}{l} + B_n \sin \frac{cn\pi t}{l} \right) \sin \frac{n\pi x}{l} \quad \text{—}$$

Now, diff eqn (iv) partially w.r.t 't', we get

$$\frac{\partial u}{\partial t} = \sum_{n=1}^{\infty} \left(-A_n \frac{cn\pi}{l} \sin \frac{cn\pi t}{l} + B_n \frac{cn\pi}{l} \cos \frac{cn\pi t}{l} \right) \sin$$

Using the condition $\partial u/\partial t = 0$ at $t = 0$

$$0 = \sum_{n=1}^{\infty} B_n \frac{cn\pi}{l} \sin \frac{n\pi x}{l}$$

$$\therefore \quad B_n = 0$$

So,

Eqn (iv) becomes,

$$u(x,t) = \sum_{n=1}^{\infty} A_n \cos \frac{cn\pi t}{l} \sin \frac{n\pi x}{l} \quad \text{———} \quad (v)$$

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$\sin 3A = 3\sin A - 4\sin^3 A$

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Now, using the condition $u(x,0) = U_0 \sin^3 \pi x/l$ in eq^n (v), we get

$$U_0 \sin^3 \frac{\pi x}{l} = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{l}$$

i.e. $U_0\left(\frac{3}{4}\sin\frac{\pi x}{l} - \frac{1}{4}\sin\frac{3\pi x}{l}\right) = A_1 \sin\frac{\pi x}{l} + A_2 \sin\frac{2\pi x}{l}$

$+ A_3 \sin\frac{3\pi x}{l} + \cdots\cdots$

Comparing the coefficient, we get

$A_1 = \frac{3U_0}{4}$

$A_2 = 0$

$A_3 = \frac{1}{4} U_0$

$A_4 = A_5 = A_6 = \cdots\cdots = 0$.

So,

Eq^n (v) becomes,

$$u(x,t) = \sum_{n=1}^{\infty} A_n \cos\frac{cn\pi t}{l} \sin\frac{n\pi x}{l}$$

(iv) $$= \frac{3U_0}{4}\cos\frac{c\pi t}{l}\sin\frac{\pi x}{l} + \frac{U_0}{4}\cos\frac{3c\pi t}{l}\sin\frac{3\pi x}{l}$$

is required solution.

$\frac{n\pi x}{l}$

Tutorial 4 The vibration of elastic string is governed by PDE

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \quad (c^2 = 1)$$

The length of string is π and ends are fixed. The initial velocity is zero and initial deflection $u(x,0) = 2(\sin x + \sin 3x)$. Find the displacement $u(x,t)$ at $t>0$.

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Soln :- Here.

The vibration of elastic string is governed by PDE

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \quad (c^2 = 1) \quad \text{—— (i)}$$

The consistent soln. of eqn (i) is

$$u(x,t) = (c_1 \cos px + c_2 \sin px)(c_3 \cos cpt + c_4 \sin cpt) \quad \text{—— (ii)}$$

Boundary condition, $u(0,t) = 0$

$u(\pi, t) = 0$

Initial condition. $\left(\frac{\partial u}{\partial t}\right)_{t=0} = 0$ (initial velocity)

$u(x,0) = 2(\sin x + \sin 3x)$

Using the condition $u(0,t)$ in eqn (ii).

$0 = c_1 (c_3 \cos cpt + c_4 \sin cpt)$

Since, $c_3 \cos cpt + c_4 \sin cpt \neq 0$

$\therefore c_1 = 0$

So,

Eqn (ii) becomes,

$$u(x,t) = c_2 \sin px \,(c_3 \cos cpt + c_4 \sin cpt) \quad \text{—— (iii)}$$

Using the condition $u(\pi, t) = 0$ in eqn (iii).

$0 = c_2 \sin p\pi \,(c_3 \cos cpt + c_4 \sin cpt)$

Since. $c_2 (c_3 \cos cpt + c_4 \sin cpt) \neq 0$, so

$\sin p\pi = 0 = \sin n\pi \;, \; n = 1, 2, 3, \ldots$

$\Rightarrow p\pi = n\pi$

$\Rightarrow p = n$

Then,

Eqn (iii) becomes,

$$u(x,t) = c_2 \sin nx \,(c_3 \cos cnt + c_4 \sin cnt)$$

$$= \sin nx \,(A \cos cnt + B \sin cnt)$$

where $A = c_2 c_3$

$B = c_2 c_4$

By the principle of superposition, the most general soln. is

$$u(x,t) = \sum_{n=1}^{\infty} \left(A_n \cos cnt + B_n \sin cnt \right) \sin nx \quad \text{—— (v)}$$

Now, diff. eqn. (iv) partially w.r.t 't', we get

$$\frac{\partial u}{\partial t} = \sum_{n=1}^{\infty} \left(-A_n cn \sin cnt + B_n cn \cos cnt \right) \sin nx$$

using the condition $\partial u/\partial t = 0$ at $t = 0$

$$0 = \sum_{n=1}^{\infty} B_n cn \sin nx$$

$$\therefore B_n = 0$$

So,

Eqn. (iv) becomes.

$$u(x,t) = \sum_{n=1}^{\infty} A_n \cos cnt \sin nx \quad \text{—— (v)}$$

Now, using the condition $u(x,0) = 2(\sin x + \sin 3x)$ in eqn. (v), we get

$$2(\sin x + \sin 3x) = \sum_{n=1}^{\infty} A_n \sin nx$$

or, $2\sin x + 2\sin 3x = A_1 \sin x + A_2 \sin 2x + A_3 \sin 3x + \cdots\cdots$

Comparing the coeff., we get

$A_1 = 2$

$A_2 = 0$

$A_3 = 2$

$A_4 = A_5 = A_6 = \cdots = 0$

So,

Eqn. (v) becomes.

$$u(x,t) = \sum_{n=1}^{\infty} A_n \cos cnt \sin nx$$

$$= 2\cos ct \sin x + 2\cos 3ct \sin 3x$$

$$= 2(\cos ct \sin x + \cos 3ct \sin 3x) \quad \text{—— } (*)$$

Here.

It is given that.

$$c^2 = 1$$
$$\therefore c = \pm 1$$

Substituting $c = 1$ in eqⁿ ⊛. we get
$$u(x,t) = 2(\cos t \sin x + \cos 3t \sin 3x) \quad \#$$

Tutorial 3 =) The given eqⁿ. is
$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \quad \text{——} \quad (i)$$

The consistent solⁿ. of eqⁿ (i) is
$$u(x,t) = (c_1 \cos px + c_2 \sin px)(c_3 \cos cpt + c_4 \sin cpt) \quad \text{——} \quad (ii)$$

Boundary conditions. $u(0,t) = 0$ and $u(l,t) = 0$

Initial conditions. $\left(\frac{\partial u}{\partial t}\right)_{t=0} = 0$ (initial velocity)

$$u(x,0) = \begin{cases} \frac{2kx}{l}, & 0 < x < \frac{l}{2} \\ \frac{2k}{l}(l-x), & \frac{l}{2} < x < l \end{cases} \quad \text{(initial deflecti}$$

Using the condition $u(0,t) = 0$ in eqⁿ. (ii).
$$0 = c_1(c_3 \cos cpt + c_4 \sin cpt)$$
Since. $c_3 \cos cpt + c_4 \sin cpt \neq 0$ otherwise solution is trivial (zero solⁿ). So, $c_1 = 0$.

So,

Eqⁿ. (ii) becomes.
$$u(x,t) = c_2 \sin px (c_3 \cos cpt + c_4 \sin cpt) \quad \text{——} \quad (iii)$$

Using the condition $u(l,t) = 0$ in eqⁿ. (iii),
$$0 = c_2 \sin pl (c_3 \cos cpt + c_4 \sin cpt)$$

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Since. $c_2(c_3 \cos cpt + c_4 \sin cpt) \neq 0$

So,

$$\sin pl = 0 = \sin n\pi \quad , n = 1, 2, 3, \dots$$

$$\Rightarrow pl = n\pi$$

$$\Rightarrow p = \frac{n\pi}{l}$$

So,

Eq^n. (iii) becomes,

$$u(x,t) = c_2 \sin \frac{n\pi x}{l} \left(c_3 \cos \frac{cn\pi t}{l} + c_4 \sin \frac{cn\pi t}{l} \right)$$

$$= \left(A \cos \frac{cn\pi t}{l} + B \sin \frac{cn\pi t}{l} \right) \sin \frac{n\pi x}{l}$$

where $A = c_2 c_3$

$B = c_2 c_4$

By the principle of superposition, the most general solution is

$$u(x,t) = \sum_{n=1}^{\infty} \left(A_n \cos \frac{cn\pi t}{l} + B_n \sin \frac{cn\pi t}{l} \right) \sin \frac{n\pi x}{l} \quad \text{—— (iv)}$$

n)

Now, diff. eq^n. (iv) partially w.r.t 't', we get

$$\frac{\partial u}{\partial t} = \sum_{n=1}^{\infty} \left(-A_n \frac{cn\pi}{l} \sin \frac{cn\pi t}{l} + B_n \frac{cn\pi}{l} \cos \frac{cn\pi t}{l} \right) \sin \frac{n\pi x}{l}$$

Using the condition $\partial u / \partial t = 0$ at $t = 0$,

$$0 = \sum_{n=1}^{\infty} B_n \frac{cn\pi}{l} \sin \frac{n\pi x}{l}$$

$$\therefore B_n = 0$$

So,

Eq^n. (iv) becomes,

$$u(x,t) = \sum_{n=1}^{\infty} A_n \cos \frac{cn\pi t}{l} \sin \frac{n\pi x}{l} \quad \text{—— (v)}$$

Now.

Using the condition $u(x,0)$ in eqn ⊙.

$$u(x,0) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{l}$$

which is half range Fourier sine series on the interval $0 < x < l$.

So,

$$A_n = \frac{2}{l}\int_0^l u(x,0) \sin\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{l}\int_0^{l/2} \frac{2kx}{l} \sin\frac{n\pi x}{l}\,dx + \frac{2}{l}\int_{l/2}^{l} \frac{2k}{l}(l-x) \sin\frac{n\pi x}{l}\,dx$$

$$= \frac{4k}{l^2}\int_0^{l/2} x \sin\frac{n\pi x}{l}\,dx + \frac{4k}{l^2}\int_{l/2}^{l} (l-x)\sin\frac{n\pi x}{l}\,dx$$

$$= I_1 + I_2 \quad (\text{say}) \quad \text{———} \quad ⊛$$

where, $I_1 = \frac{4k}{l^2}\int_0^{l/2} x \sin\frac{n\pi x}{l}\,dx$

$$= \frac{4k}{l^2}\left[-x \cdot \cos\frac{n\pi x}{l} * \frac{l}{n\pi} + 1 \cdot \sin\frac{n\pi x}{l} * \frac{l^2}{n^2\pi^2}\right]$$

$$= \frac{4k}{l^2}\left\{-\frac{l}{2}\cos\frac{n\pi}{2} * \frac{l}{n\pi} + \sin\frac{n\pi}{2} * \frac{l^2}{n^2\pi^2} + 0 + 0\right\}$$

$$= \frac{4k}{l^2}\left\{-\frac{l^2}{2n\pi}\cos\frac{n\pi}{2} + \frac{l^2}{n^2\pi^2}\sin\frac{n\pi}{2}\right\}$$

$$I_2 = \frac{4k}{l^2}\int_{l/2}^{l} (l-x)\sin\frac{n\pi x}{l}\,dx$$

Flair

$$= \frac{4k}{l^2}\left[-(l-x)\cos\frac{n\pi x}{l} * \frac{l}{n\pi} + (-1)\sin\frac{n\pi x}{l} * \frac{l^2}{n^2\pi^2}\right]_{l/2}^{l}$$

$$= \frac{4k}{l^2}\left[-(l-x)\cos\frac{n\pi x}{l} * \frac{l}{n\pi} - \sin\frac{n\pi x}{l} * \frac{l^2}{n^2\pi^2}\right]_{l/2}^{l}$$

$$= \frac{4k}{l^2}\left\{0 - \sin n\pi * \frac{l^2}{n^2\pi^2} + \left(\frac{l}{2}\cos\frac{n\pi}{2} * \frac{l}{n\pi}\right) + \sin\frac{n\pi}{2} * \frac{l^2}{n^2\pi^2}\right\}$$

$$= \frac{4k}{l^2}\left\{0 - 0 + \frac{l^2}{2n\pi}\cos\frac{n\pi}{2} + \sin\frac{n\pi}{2} * \frac{l^2}{n^2\pi^2}\right\}$$

$$= \frac{4k}{l^2}\left\{\frac{l^2}{2n\pi}\cos\frac{n\pi}{2} + \frac{l^2}{n^2\pi^2}\sin\frac{n\pi}{2}\right\}$$

Then,

From eq^n (*).

$$A_n = \frac{4k}{l^2}\left\{-\frac{l^2}{2n\pi}\cos\frac{n\pi}{2} + \frac{l^2}{n^2\pi^2}\sin\frac{n\pi}{2}\right\}$$

$l/2$

$$+ \frac{4k}{l^2}\left\{\frac{l^2}{2n\pi}\cos\frac{n\pi}{2} + \frac{l^2}{n^2\pi^2}\sin\frac{n\pi}{2}\right\}$$

0

$$= \frac{8k}{l^2}\sin\frac{n\pi}{2} * \frac{l^2}{n^2\pi^2} = \frac{8k}{n^2\pi^2}\sin\frac{n\pi}{2}$$

Then,

From eq^n. (v).

$$u(x,t) = \sum_{n=1}^{\infty}\frac{8k}{n^2\pi^2}\sin\frac{n\pi}{2}\cos\frac{cn\pi t}{l}\sin\frac{n\pi x}{l}$$

$$= \frac{8k}{\pi^2}\left[\frac{1}{1^2}\cos\frac{c\pi t}{l}\sin\frac{\pi x}{l} + 0 - \frac{1}{3^2}\cos\frac{3c\pi t}{l}\sin\frac{3\pi x}{l} + \dots\dots\right]$$

is the required eq^n.

Flair

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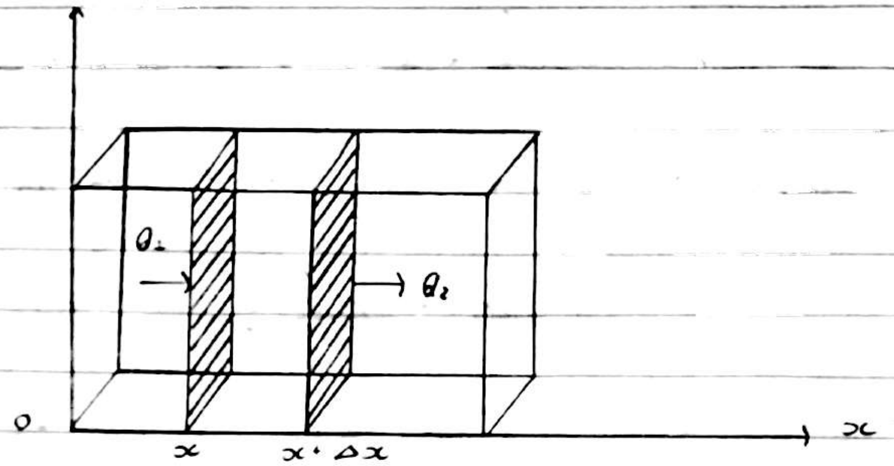
One dimensional heat equation :

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad u \rightarrow \text{temperature}$$



Consider the flow of heat by conduction in a uniform metalic bar. It is assumed that sides of bar are insulated and loss of heat by conduction is negligible. Let us take one end of bar as origin and direction of flow of heat as positive x - axis. The temperature 'u' at any point depends on the ~~direction~~ distance 'x' and time 't'.

The quantity of heat crossing to any section of bar per second depends on the cross - sectional area A, conductivity k of material of bar and temperature gradient $\partial u/\partial x$ i.e. ratio of change of temperature w.r.t distance normal to area.

Q_1, the quantity of heat flowing into a section at a distance of x from end O $= -kA \left(\frac{\partial u}{\partial x}\right)_x$ per sec

Here, -ve sign is ~~attact~~ attached because as distance increases 'u' decreases.

Flair

Q_2. the quantity of flowing out of section at a distance of $x + \Delta x = -kA \left(\frac{\partial u}{\partial x}\right)_{x+\Delta x}$ per sec

$\therefore$ The amount of heat retained by a slab of thickness Δx is

$$Q_1 - Q_2 = kA \left[\left(\frac{\partial u}{\partial x}\right)_{x+\Delta x} - \left(\frac{\partial u}{\partial x}\right)_{x} \right] \text{ per sec} \quad — ①$$

Also,

The rate of increase of heat in a slab $= s \times S \times A \times \Delta x \times \frac{\partial u}{\partial t}$ — ②

$\partial Q = m \times s \times \partial u$

$= S \times V \times s \times \partial u$

$\frac{\partial Q}{\partial t} = S \times A \times \Delta x \times s \times \frac{\partial u}{\partial t}$

where s be the specific heat capacity and S be the density of material of box.

So,

From eqn ① & ②.

$$sSA\Delta x \frac{\partial u}{\partial t} = kA \left[\left(\frac{\partial u}{\partial x}\right)_{x+\Delta x} - \left(\frac{\partial u}{\partial x}\right)_{x} \right]$$

$$\Rightarrow sS \frac{\partial u}{\partial t} = k \frac{\left[\left(\frac{\partial u}{\partial x}\right)_{x+\Delta x} - \left(\frac{\partial u}{\partial x}\right)_{x} \right]}{\Delta x}$$

Now,

Taking limit $\Delta x \to 0$

$$sS \frac{\partial u}{\partial t} = k \lim_{\Delta x \to 0} \frac{\left[(\partial u/\partial x)_{x+\Delta x} - (\partial u/\partial x)_x \right]}{\Delta x}$$

$$\Rightarrow sS \frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}$$

$$\Rightarrow \frac{\partial u}{\partial t} = \frac{k}{sS} \frac{\partial^2 u}{\partial x^2}$$

$$\therefore \frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \quad \left[c^2 = k/sS \right]$$

which is one dimensional heat equation.

Solution of one dimensional heat equation :- One dimensional heat equation is

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \quad \text{——} \quad (1)$$

Let $u = XT$ be the general solution of eqn (1) where X be the function of x only and T be the function of t only.

So, eqn (1) becomes,

$$\frac{\partial XT}{\partial t} = c^2 \frac{\partial^2 XT}{\partial x^2}$$

$$\Rightarrow XT' = c^2 TX''$$

Seperating the variables,

$$\frac{X''}{X} = \frac{T'}{c^2 T} = k \text{ (constant) say}$$

$$\Rightarrow \frac{X''}{X} = k \quad \text{and} \quad \frac{T'}{c^2 T} = k$$

Case I: Let $k < 0$ and $k = -p^2$

So,

$$\frac{X''}{X} = -p^2 \quad \text{and} \quad \frac{T'}{c^2 T} = -p^2$$

$$\Rightarrow X'' + p^2 X = 0$$

$$\Rightarrow \frac{d^2 X}{dx^2} + p^2 X = 0$$

$$\Rightarrow (D^2 + p^2) X = 0$$

Its AE is

$$m^2 + p^2 = 0$$

$$\Rightarrow m = \pm pi$$

$$\therefore X(x) = c_1 \cos px + c_2 \sin px$$

$$\Rightarrow T' + p^2 c^2 T = 0$$

$$\Rightarrow \frac{dT}{dt} + p^2 c^2 T = 0$$

$$\therefore T(t) = c_3 e^{-p^2 c^2 t}$$

$\therefore u(x,t) = (c_1 \cos px + c_2 \sin px)\, c_3 e^{-p^2c^2t}$ be general soln. for $k < 0$.

Let $k = 0$.

Then,

$$\frac{X''}{X} = 0 \quad \text{and} \quad \frac{T'}{c^2 T} = 0$$

$\Rightarrow X'' = 0$ $\qquad$ $\Rightarrow T' = 0$

$\Rightarrow \frac{d^2X}{dx^2} = 0$ $\qquad$ $\Rightarrow \frac{dT}{dt} = 0$

$\Rightarrow D^2X = 0$ $\qquad$ $\Rightarrow dT = 0$

Its AE is $\qquad$ on integration, we get

$m^2 = 0$ $\qquad$ $T = C_6$

$\therefore m = 0$

$\therefore T(t) = C_6$

$\therefore X(x) = (C_4x + C_5)e^0$

$= C_4x + C_5$

$\therefore u(x,t) = (C_4x + C_5)C_6$ be general soln. for $k = 0$.

Let $k > 0$ and $k = p^2$

So,

$$\frac{X''}{X} = p^2 \quad \text{and} \quad \frac{T'}{c^2 T} = p^2$$

$\Rightarrow X'' - p^2X = 0$ $\qquad$ $\Rightarrow T' - p^2c^2T = 0$

$\Rightarrow \frac{d^2X}{dx^2} - p^2X = 0$ $\qquad$ $\Rightarrow \frac{dT}{dt} - p^2c^2T = 0$

$\Rightarrow (D^2 - p^2)X = 0$

Its AE is $\qquad$ $\therefore T(t) = C_9 e^{p^2c^2t}$

$m^2 - p^2 = 0$

$\therefore m = \pm p$

$\therefore X(x) = C_7 e^{-px} + C_8 e^{px}$

$\therefore u(x,t) = (C_7e^{-px} + C_8e^{px}) \cdot C_9e^{p^2c^2t}$ be general soln. for $k > 0$.

Flair

Three possible $soln^n$. of one dimensional heat eq^n

1 $u(x,t) = (c_1 \cos px + c_2 \sin px) c_3 e^{-c^2p^2t}$

2 $u(x,t) = (c_4 e^{px} + c_5 e^{-px}) c_6 e^{c^2p^2t}$

3 $u(x,t) = (c_8 x + c_9) c_{10}$

Thus, of the three sol^n. we choose that sol^n. which is consistent with physical nature of problem. In one dimensional heat eq^n. as 't' increases 'u' decreases. so, feasible sol^n. of one dimensional heat eq^n. is

$$u(x,t) = (c_1 \cos px + c_2 \sin px) c_3 e^{-c^2p^2t} \quad \#$$

Q A homogenous rod of conducting material of length 100 cm has its ends kept zero temperature and initial temp. is given by $u(x,0) = \begin{cases} x & 0 \le x \le 50 \\ 100 - x & 50 \le x \le 100 \end{cases}$

Find the temp. at any time $t > 0$.

sol^n:- One dimensional heat eq^n. is

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \quad \text{—— (i)}$$

The consistent sol^n. of one dimensional heat eq^n. is

$$u(x,t) = (c_1 \cos px + c_2 \sin px) c_3 e^{-c^2p^2t} \quad \text{—— (ii)}$$

Boundary condition,

$$u(0,t) = 0\,°c$$
$$u(100,t) = 0\,°c$$

Initial condition,

$$u(x,0) = \begin{cases} x & 0 \le x \le 50 \\ 100 - x & 50 \le x \le 100 \end{cases}$$

Flair

Using the condition $u(0,t) = 0$ in eq^n. (ii),

$$0 = c_1 \left(c_3 e^{-c^2p^2t} \right)$$

Since, $c_3 e^{-c^2p^2t} \neq 0$ otherwise the sol^n. is trivial, so $c_1 = 0$

Eq^n (ii) becomes.

$$u(x,t) = c_2 \sin px * c_3 e^{-c^2p^2t} \quad \text{——— (iii)}$$

Using the condition $u(100,t) = 0$ in eq^n. (iii),

$$0 = c_2 \sin 100p * c_3 e^{-c^2p^2t}$$

Since, $c_2 c_3 e^{-c^2p^2t} \neq 0$, so

$$\sin 100p = 0 = \sin n\pi \quad , n = 1, 2, 3, \ldots\ldots$$

$$\Rightarrow \quad 100p = n\pi$$

$$\Rightarrow \quad p = \frac{n\pi}{100}$$

Putting the value of p in eq^n. (iii),

$$u(x,t) = c_2 \sin \frac{n\pi x}{100} * c_3 e^{-c^2\left(\frac{n\pi}{100}\right)^2 t}$$

$$= A \sin \frac{n\pi x}{100} \, e^{-(cn\pi/100)^2 t} \quad \text{where } A = c_2 c_3$$

The most general sol^n. is

$$u(x,t) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{100} \, e^{-\left(\frac{cn\pi}{100}\right)^2 t} \quad \text{——— (iv)}$$

Using the condition $u(x,0) = \begin{cases} x & 0 \leq x \leq 50 \\ 100 - x & 50 \leq x \leq 100 \end{cases}$

in eq^n (iv),

$$u(x,0) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{100}$$

which is half range Fourier sine series of function $u(x,0)$ in the interval $0 \leq x \leq 100$.

Flair

So,

$$A_n = \frac{2}{100}\int_0^{100} u(x,0)\sin\frac{n\pi x}{100}\,dx$$

$$= \frac{2}{100}\int_0^{50} x\sin\frac{n\pi x}{100}\,dx + \frac{2}{100}\int_{50}^{100}(100-x)\sin\frac{n\pi x}{100}\,dx$$

$$= \frac{1}{50}\left[-x\cos\frac{n\pi x}{100}\Big/\frac{n\pi}{100} + \frac{\sin\frac{n\pi x}{100}}{(n\pi/100)^2}\right]_0^{50}$$

$$+ \frac{1}{50}\left[-(100-x)\frac{\cos\frac{n\pi x}{100}}{\frac{n\pi}{100}} - \frac{\sin\frac{n\pi x}{100}}{(n\pi/100)^2}\right]_{50}^{100}$$

$$= \frac{1}{50}\left\{-50\cos\frac{n\pi 50}{100}\Big/\frac{n\pi}{100} + \frac{\sin\frac{50n\pi}{100}}{(n\pi/100)^2}\right\}$$

$$- \frac{1}{50}\left\{0+0\right\} +$$

$$\frac{1}{50}\left\{0 - \frac{\cancelto{0}{\sin n\pi}}{(n\pi/100)^2}\right\} - \frac{1}{50}\left\{-50\frac{\cos\frac{n\pi}{2}}{n\pi/100} - \frac{\sin\frac{n\pi}{2}}{(n\pi/100)^2}\right\}$$

$$= \frac{1}{50}\left\{-50\frac{\cos n\pi/2}{n\pi/100} + \frac{\sin n\pi/2}{(n\pi/100)^2}\right\} - 0 + 0$$

$$- \frac{1}{50}\left\{-50\frac{\cos n\pi/2}{n\pi/100} - \frac{\sin n\pi/2}{(n\pi/100)^2}\right\}$$

$$= \frac{1}{25}\frac{\sin n\pi/2}{n^2\pi^2/100^2}$$

$$= \frac{400}{n^2\pi^2}\sin\frac{n\pi}{2}$$

Then,

From eq^n. (iv),

$$u(x,t) = \sum_{n=1}^{\infty}\frac{400}{n^2\pi^2}\sin\frac{n\pi}{2}\sin\frac{n\pi x}{100}e^{-\left(\frac{cn\pi}{100}\right)^2 t}$$

#

Flair

Q Solve one dimensional heat eqn. $\partial u/\partial t = c^2 \frac{\partial^2 u}{\partial x^2}$ with boundary condition $\partial u/\partial x = 0$ at $x = 0$ and $x = l$ and initial condition $u(x, 0) = x$.

soln:: We have one dimensional heat eqn. is

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \quad \text{--- (i)}$$

The consistent solution of one dimensional heat eqn. is

$$u(x, t) = (c_1 \cos px + c_2 \sin px)(c_3 e^{-c^2p^2t}) \quad \text{--- (ii)}$$

Boundary condition, $\frac{\partial u}{\partial x} = 0$ at $x = 0$ and $x = l$

Initial condition, $u(x, 0) = x$

Diff. eqn (ii) partially w.r.t x, we get

$$\frac{\partial u}{\partial x} = (-c_1 p \sin px + c_2 p \cos px)(c_3 e^{-c^2p^2t})$$

Now, using the condition $\partial u/\partial x = 0$ at $x = 0$

$$0 = c_2 p (c_3 e^{-c^2p^2t})$$

Since, $p \neq 0$ and $c_3 e^{-c^2p^2t} \neq 0$ otherwise soln. is trivial.

$$\therefore c_2 = 0$$

Then,

Eqn. (ii) becomes,

$$u(x, t) = c_1 \cos px (c_3 e^{-c^2p^2t}) \quad \text{--- (iii)}$$

Diff. eqn (iii) partially w.r.t x, we get

$$\frac{\partial u}{\partial x} = -c_1 p \sin px (c_3 e^{-c^2p^2t})$$

Using the condition $\partial u/\partial x = 0$ at $x = l$

$$0 = -c_1 p \sin pl (c_3 e^{-c^2p^2t})$$

Since, $c_1 \neq 0$, $p \neq 0$, $c_3 e^{-c^2p^2t} \neq 0$, so

$$\sin pl = 0 = \sin n\pi, \quad n = 0, 1, 2, 3, \ldots$$

or, $pl = n\pi$

$\therefore \quad p = \dfrac{n\pi}{l}$

Then,

Eq^n (iii) becomes,

$$u(x,t) = c_1 \cos \frac{n\pi x}{l} \left(c_3 e^{-c^2 \frac{n^2\pi^2}{l^2} t} \right)$$

$$= A \cos\left(\frac{n\pi x}{l}\right) e^{-\left(\frac{cn\pi}{l}\right)^2 t} \quad \text{where } c_1 c_3 = A$$

The most general solution is

$$u(x,t) = \sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi x}{l}\right) e^{-\left(\frac{cn\pi}{l}\right)^2 t}$$

Without loss of generality, the solution can be written as

$$u(x,t) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{l}\right) e^{-\left(\frac{cn\pi}{l}\right)^2 t} \quad \text{— (iv)}$$

Now using the initial condition $u(x,0) = x$ in eq^n (iv),

$$x = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{l}\right)$$

which is half range cosine series of the function $f(x) = x$ in the interval $(0, l)$. So, by using Euler's formula,

$$A_0 = \frac{2}{l} \int_0^l f(x)\, dx$$

$$= \frac{2}{l} \int_0^l x\, dx$$

$$= \frac{2}{l} * \frac{l^2}{2}$$

$$= l$$

Flair

$$A_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{l}\int_0^l x\cos\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{l}\left[x\sin\frac{n\pi x}{l}\Big/\frac{n\pi}{l} + \cos\frac{n\pi x}{l}\Big/\frac{n^2\pi^2}{l^2}\right]_0^l$$

$$= \frac{2}{l}\left\{0 + \frac{(-1)^n}{n^2\pi^2/l^2} - \frac{1}{n^2\pi^2/l^2}\right\}$$

$$= \frac{2}{l n^2\pi^2} * l^2\left((-1)^n - 1\right)$$

$$= \frac{2l}{n^2\pi^2}\left\{(-1)^n - 1\right\}$$

Putting the value of A_0 and A_n in eqn. (iv),

$$u(x,t) = \frac{l}{2} + \sum_{n=1}^{\infty}\frac{2l}{n^2\pi^2}\left\{(-1)^n - 1\right\}\cos\left(\frac{n\pi x}{l}\right)e^{-\left(\frac{cn\pi}{l}\right)^2 t}$$

$$= \frac{l}{2} + \frac{2l}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\left\{(-1)^n - 1\right\}\cos\left(\frac{n\pi x}{l}\right)e^{-\left(\frac{cn\pi}{l}\right)^2 t}$$

which is the required solution. #

Date / /
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Q Solve one dimensional heat equation $\partial u/\partial t = c^2 \, \partial^2 u/\partial x^2$ under the condition

① u is not infinite as $t \to \infty$

② $\partial u/\partial x = 0$ at $x = 0$ and $x = l$

③ $u(x,0) = (lx - x^2)$ for $t = 0$. between $x = 0$ and $x = l$

Soln:: We have one dimensional heat equation is

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \quad \text{——— ①}$$

The consistent solution of one dimensional heat eqn. is

$$u(x,t) = (c_1 \cos px + c_2 \sin px)(c_3 e^{-c^2p^2t}) \quad \text{——— ②}$$

Boundary condition, $\partial u/\partial x = 0$ at $x = 0$ and $x = l$

Initial condition, $u(x,0) = (lx - x^2)$

Diff. eqn ② partially w.r.t x, we get

Same as before

Now using the initial condition $u(x,0) = (lx - x^2)$ in eqn. ④,

$$lx - x^2 = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{l}\right)$$

which is half range cosine series of the function $f(x) = lx - x^2$ in the interval $(0, l)$. So, by using Euler's formula

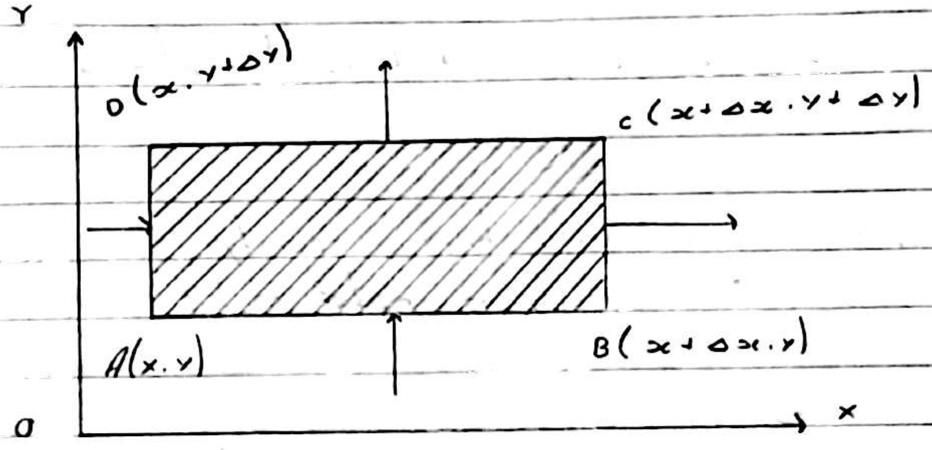
$$A_0 = \frac{2}{l}\int_0^l f(x)\,dx$$

$$= \frac{2}{l}\int_0^l (lx - x^2)\,dx$$

do it, its like before

☆ Two-dimensional heat equation $\frac{\partial u}{\partial t} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$.

=)



Consider the flow of heat in a metal plate in the XOY plane. If the temperature of any point of plate is independent of z-coordinate and depends only x, y and t then flow is called two dimensional and heat flow lies in XOY plane and is zero normal to the XOY plane.

Take a rectangular element ABCD of plate with sides Δx, Δy and small thickness 'α'. The quantity of heat that enters from sides AB and AD per sec are $-k\alpha\Delta x\left(\frac{\partial u}{\partial y}\right)_y$ and $-k\alpha\Delta y\left(\frac{\partial u}{\partial x}\right)_x$ respectively. Also, the quantity of heat that flows out from sides BC and CD per sec are $-k\alpha\Delta y\left(\frac{\partial u}{\partial x}\right)_{x+\Delta x}$ and $-k\alpha\Delta x\left(\frac{\partial u}{\partial y}\right)_{y+\Delta y}$ respectively.

The quantity of heat ~~retained~~ retained by the elementary plate ABCD per sec $= k\alpha\Delta y\left(\frac{\partial u}{\partial x}\right)_{x+\Delta x} + k\alpha\Delta x\left(\frac{\partial u}{\partial y}\right)_{y+\Delta y}$

$$- k\alpha\Delta x\left(\frac{\partial u}{\partial y}\right)_y - k\alpha\Delta y\left(\frac{\partial u}{\partial x}\right)_x$$

$$= k\alpha \Delta x \Delta y \left[\frac{(\partial u/\partial x)_{x+\Delta x} - (\partial u/\partial x)_x}{\Delta x} + \frac{(\partial u/\partial y)_{y+\Delta y} -}{\Delta y} \right.$$

Also, the rate of gain of heat by element ABCD $= s \times S \times \alpha \times \Delta x \times \Delta y \times \frac{\partial u}{\partial t}$

where s = Specific heat capacity
S = density of ~~metal~~ metal plate

From eqn ① & ⑪,

$$S s \alpha \Delta x \Delta y \frac{\partial u}{\partial t} = k\alpha \Delta x \Delta y \left[\frac{(\partial u/\partial x)_{x+\Delta x} - (\partial u/\partial x)_x}{\Delta x} + \frac{(\partial u/\partial y)_{y+\Delta y} - (\partial u/\partial y)_y}{\Delta y} \right]$$

$$\Rightarrow \frac{\partial u}{\partial t} = \frac{k}{Ss} \left[\frac{(\partial u/\partial x)_{x+\Delta x} - (\partial u/\partial x)_x}{\Delta x} + \frac{(\partial u/\partial y)_{y+\Delta y} -}{\Delta y} \right.$$

Now taking limit $\Delta x \to 0$, $\Delta y \to 0$

$$\frac{\partial u}{\partial t} = \frac{k}{Ss} \left[\lim_{\Delta x \to 0} \frac{(\partial u/\partial x)_{x+\Delta x} - (\partial u/\partial x)_x}{\Delta x} + \lim_{\Delta y \to 0} \frac{(\partial u/\partial y)_{y+\Delta y} - (\partial u/\partial y)_y}{\Delta y} \right]$$

$$= c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right), \quad c^2 = \frac{k}{Ss}$$

which is two dimensional heat equation.

Note :- In steady state condition 'u' is independent of time 't'. So $\frac{\partial u}{\partial t} = 0$. Hence, two dimensional heat equation becomes $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ which is called Laplace's equation in two dimensions.

$— ①$

Solution of Laplace equations :-

We have Laplace eqn. in two dimensions is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{——} \quad ①$$

Let $u = XY$ be the general soln. of eqn ① where X be the function of x only and Y be the function of y only.

So.

From eqn ①.

$$\frac{\partial^2 (XY)}{\partial x^2} + \frac{\partial^2 (XY)}{\partial y^2} = 0$$

$$\text{or,} \quad YX'' + XY'' = 0$$

Seperating the variables,

$$\frac{X''}{X} = \frac{-Y''}{Y} = k \text{ (constant) (say)}$$

Let $k < 0$ and $k = -p^2$

Then $\frac{X''}{X} = -p^2$ and $-\frac{Y''}{Y} = -p^2$

i.e. $\frac{X''}{X} = -p^2$

$$\Rightarrow \quad X'' + p^2 X = 0$$

$$\Rightarrow \quad \frac{d^2 X}{dx^2} + p^2 X = 0$$

Its AE is $m^2 + p^2 = 0$

$\therefore m = \pm pi$

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$$\therefore\ X(x) = c_1 \cos px + c_2 \sin px$$

Also,

$$-\frac{Y''}{Y} = -p^2$$

or, $$\frac{Y''}{Y} = p^2$$

or, $$Y'' = p^2 Y$$

or, $$Y'' - p^2 Y = 0$$

or, $$\frac{d^2Y}{dY^2} - p^2 Y = 0$$

Its AE is $m^2 - p^2 = 0$

$$\therefore\ m = \pm p$$

$$\therefore\ Y(y) = c_3 e^{py} + c_4 e^{-py}$$

[redacted] $$\therefore\ u(x,y) = (c_1 \cos px + c_2 \sin px)(c_3 e^{py} + c_4 e^{-py})$$

be general sol^n. for $k < 0$.

Case II Let $k > 0$ and $k = p^2$.

Then,

$$\frac{X''}{X} = p^2, \quad -\frac{Y''}{Y} = p^2$$

i.e. $$\frac{X''}{X} = p^2$$

$$\Rightarrow X'' - p^2 X = 0$$

$$\Rightarrow \frac{d^2X}{dx^2} - p^2 X = 0$$

Its AE is $m^2 - p^2 = 0$

$$\therefore\ m = \pm p$$

$$\therefore\ X(x) = c_5 e^{px} + c_6 e^{-px}$$

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Also,

$$-\frac{Y''}{Y} = p^2$$

or, $$\frac{Y''}{Y} = -p^2$$

or, $$Y'' + p^2 Y = 0$$

or, $$\frac{d^2Y}{dy^2} + p^2 Y = 0$$

Its AE is $m^2 + p^2 = 0$

$\therefore m = \pm pi$

$\therefore Y(y) = C_7 \cos py + C_8 \sin py$

$\therefore u(x,y) = (C_5 e^{px} + C_6 e^{-px})(C_7 \cos py + C_8 \sin py)$
be general soln. for $k > 0$.

case III let $k = 0$.
Then,

$$\frac{X''}{X} = 0 \quad \text{and} \quad -\frac{Y''}{Y} = 0$$

$\Rightarrow X'' = 0$ $\qquad\qquad$ $\Rightarrow Y'' = 0$

$\Rightarrow \frac{d^2X}{dx^2} = 0$ $\qquad\qquad$ $\Rightarrow \frac{d^2Y}{dy^2} = 0$

Its AE is $m^2 = 0$ $\qquad\qquad$ Its AE is $m^2 = 0$

$\therefore m = 0, 0$ $\qquad\qquad$ $\therefore m = 0, 0$

$\therefore X(x) = (C_9 x + C_{10}) e^{0 \cdot x}$ $\qquad\qquad$ $\therefore Y(x) = (C_{11} y + C_{12}) e^{0 \cdot y}$

$= C_9 x + C_{10}$ $\qquad\qquad$ $= C_{11} y + C_{12}$

$\therefore u(x,y) = (C_9 x + C_{10})(C_{11} y + C_{12})$
be general soln. for $k = 0$.

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Q Solve one dimensional heat eqn. under the boundary condition $\frac{\partial u}{\partial x} = 0$ at $x = 0$ and $x = \pi$ and initial condition

$u(x,0) = x^2$.

Imp Q Solve $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ which satisfies the conditions

$u(0,y) = u(l,y) = u(x,0) = 0$ and $u(x,a) = \sin\left(\frac{n\pi x}{l}\right)$.

Soln:- We have Laplace eqn. is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Let $u(x,y) = XY$ be general soln. where X be the function of x only and Y be the function of y only.

So, eqn. ① becomes

$$\frac{\partial^2 XY}{\partial x^2} + \frac{\partial^2 XY}{\partial y^2} = 0$$

$$\text{i.e. } YX'' + XY'' = 0$$

Seperating variables,

$$\frac{X''}{X} = -\frac{Y''}{Y}$$

$$\text{Let } \frac{X''}{X} = -\frac{Y''}{Y} = k \text{ (constant)}$$

Case I Let $k > 0$ and $k = p^2$.

Then,

$$\frac{X''}{X} = p^2 \quad \text{and} \quad -\frac{Y''}{Y} = p^2$$

i.e. $X'' - p^2 X = 0$

or, $(D^2 - p^2) X = 0$

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Its AE is

$$m^2 - p^2 = 0$$

$$\Rightarrow m = \pm p$$

$$\therefore X(x) = c_1 e^{px} + c_2 e^{-px}$$

Also.

$$\frac{-Y''}{Y} = p^2$$

$$\Rightarrow Y'' + p^2 Y = 0$$

$$\Rightarrow (D^2 + p^2) Y = 0$$

Its AE is

$$m^2 + p^2 = 0$$

$$\Rightarrow m = \pm pi$$

$$\therefore Y(y) = c_3 \cos py + c_4 \sin py$$

Using condition $X(0) = 0$,

$$c_1 + c_2 = 0$$

$$\Rightarrow c_1 = -c_2 \qquad \therefore X(x) = c_1 e^{px} - c_1 e^{-px}$$

Also,

Using $X(l) = 0$,

$$0 = c_1 e^{pl} - c_1 e^{-pl}$$

or. $0 = c_1 (e^{pl} - e^{-pl})$

Since, $e^{pl} - e^{-pl} \neq 0 \Rightarrow c_1 = 0$

$$\therefore X(x) = 0$$

$\therefore u(x, y) = XY = 0$ which is impossible so case I is rejected.

case II Let $k = 0$

So, $\frac{X''}{X} = 0$ and $\frac{-Y''}{Y} = 0$

i.e. $X'' = 0$ and $Y'' = 0$

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Integrating sucessively,

$$X(x) = C_5 x + C_6 \quad \text{and} \quad Y(y) = C_7 y + C_8$$

Using $X(0) = 0$,

$$0 = C_6$$

So,

$$X(x) = C_5 x$$

Also,

Using $X(l) = 0$.

$$0 = C_5 l$$

$$\Rightarrow C_5 = 0$$

$$\therefore X(x) = 0$$

$\therefore u(x, y) = XY = 0$, which is impossible so case II is rejected.

case III let $k < 0$ and $k = -p^2$, then

$$\frac{X''}{X} = -p^2 \quad \text{and} \quad -\frac{Y''}{Y} = -p^2$$

$\vdots \qquad \vdots$

$$\therefore u(x, y) = (c_9 \cos px + c_{10} \sin px)(c_{11} e^{py} + c_{12} e^{-py}) \quad \text{--- (i)}$$

Using the condition $u(0, y) = 0$ in eqn (i),

$$0 = c_9 (c_{11} e^{py} + c_{12} e^{-py})$$

Since, $c_{11} e^{py} + c_{12} e^{-py} \neq 0$

$$\therefore c_9 = 0$$

So,

Eqn (i) becomes,

$$u(x, y) = c_{10} \sin px (c_{11} e^{py} + c_{12} e^{-py}) \quad \text{--- (ii)}$$

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Using $u(l,y) = 0$ in eq^n 2,

$$0 = c_{10} \sin pl \left(c_{11} e^{py} + c_{12} e^{-py} \right)$$

Since, $c_{10}\left(c_{11} e^{py} + c_{12} e^{-py}\right) \neq 0$,

$$\therefore \sin pl = 0 = \sin n\pi$$

$$\Rightarrow pl = n\pi$$

$$\Rightarrow p = \frac{n\pi}{l}$$

So. eq^n (ii) becomes

$$u(x,y) = c_{10} \sin \frac{n\pi x}{l} \left(c_{11} e^{\frac{n\pi y}{l}} + c_{12} e^{-\frac{n\pi y}{l}} \right) \quad \text{— (iii)}$$

Using condition $u(x,0) = 0$,

$$0 = c_{10} \sin \frac{n\pi x}{l} \left(c_{11} + c_{12} \right)$$

Since. $c_{10} \sin {}^{n\pi x/l} \neq 0$

$$\therefore c_{11} + c_{12} = 0$$

$$\Rightarrow c_{11} = -c_{12}$$

So. eq^n (iii) becomes,

$$u(x,y) = c_{10} \sin \frac{n\pi x}{l}\, c_{11} \left(e^{\frac{n\pi y}{l}} - e^{-\frac{n\pi y}{l}} \right)$$

$$= A \sin \frac{n\pi x}{l} \left(e^{\frac{n\pi y}{l}} - e^{-\frac{n\pi y}{l}} \right) \quad \text{— (iv)}$$

Using $u(x,a) = \sin \frac{n\pi x}{l}$ in eq^n (iv)

$$\sin \frac{n\pi x}{l} = A \sin \frac{n\pi x}{l} \left(e^{\frac{n\pi a}{l}} - e^{-\frac{n\pi a}{l}} \right)$$

$$\therefore A = \frac{1}{e^{\frac{n\pi a}{l}} - e^{-\frac{n\pi a}{l}}}$$

$$= \frac{2}{\sinh \frac{n\pi a}{l}}$$

So. eq^n (iv),

$$u(x,y) = \frac{2}{\sinh \frac{n\pi a}{l}} \sin \frac{n\pi x}{l} \left(e^{\frac{n\pi a}{l}} - e^{-\frac{n\pi a}{l}} \right)$$

$$= \frac{\sin\left({}^{n\pi x}/l\right) \sinh \frac{n\pi y}{l}}{\sinh \frac{n\pi a}{l}}$$ is req. sol^n. #

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Imp Change the Laplace equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ into polar form.

=) We have,

The Laplace eqn. in two dimensions is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

To change the Laplace eqn. in polar form put

$$\left.\begin{array}{ll} x = r\cos\theta & y = r\sin\theta \\ x^2 + y^2 = r^2 \text{ and } & \theta = \tan^{-1}(y/x) \end{array}\right\} \text{---} \; (1)$$

Diff. eqn (1) partially w.r.t x, we get

$$2x = 2r \frac{\partial r}{\partial x}$$

$$\therefore \frac{\partial r}{\partial x} = \frac{x}{r} \quad \boxed{= r_x}$$

$$\theta_x = \frac{\partial \theta}{\partial x}$$

$$= \frac{1}{1 + (y/x)^2}.$$

$$= \frac{-y}{x^2 + y^2}$$

$$= \frac{-y}{r^2}$$

Also,

$$r_{xx} = \frac{\partial^2 r}{\partial x^2} \quad \text{and} \quad \theta_{xx} = \frac{\partial^2 \theta}{\partial x^2}$$

$$= \frac{\partial}{\partial x}\left(\frac{x}{r}\right)$$

$$= \frac{r - x \cdot \partial r/\partial x}{r^2}$$

$$= \frac{r - x \cdot x/r}{r^2}$$

$$= \frac{r^2 - x^2}{r^3}$$

$$= \frac{y^2}{r^3}$$

$$\theta_{xx} = \frac{\partial}{\partial x}\left(\frac{-y}{r^2}\right)$$

$$= (-y) \frac{\partial}{\partial r}\left(\frac{1}{r^2}\right).$$

$$= (-y)\left(\frac{-2}{r^3}\right) \cdot \frac{x}{r}$$

$$= \frac{2xy}{r^4}$$

$\therefore \; r_x = x/r \qquad \theta_x = -y/r^2$

$r_{xx} = y^2/r^3 \qquad \theta_{xx} = 2xy/r^4$

Similarly, diff. eqⁿ ① partially w.r.t y, we get

$r_y = y/r \qquad \theta_y = x/r^2$

$r_{yy} = x^2/r^3 \qquad \theta_{yy} = -2xy/r^4$

$u = f(x, y)$, $x = r, \theta$

$y = r, \theta$

We have,

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r} \cdot \frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta} \cdot \frac{\partial \theta}{\partial x}$$

i.e. $U_x = U_r \cdot r_x + U_\theta \cdot \theta_x$

and $$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r} \cdot \frac{\partial r}{\partial y} + \frac{\partial u}{\partial \theta} \cdot \frac{\partial \theta}{\partial y}$$

i.e. $U_y = U_r \cdot r_y + U_\theta \cdot \theta_y$

Also, diff. U_x partially w.r.t x by using chain rule, we get

$$U_{xx} = \frac{\partial^2 u}{\partial x^2}$$

$$= \frac{\partial}{\partial x}(U_x)$$

$$= \frac{\partial}{\partial x}(U_r \cdot r_x + U_\theta \cdot \theta_x)$$

$$= r_x\left(\frac{\partial}{\partial x} U_r\right) + U_r\left(\frac{\partial}{\partial x} r_x\right)$$

$$+ \theta_x\left(\frac{\partial}{\partial x} U_\theta\right) + U_\theta\left(\frac{\partial}{\partial x} \theta_x\right)$$

$$= r_x(U_{rr} \cdot r_x + U_{r\theta} \cdot \theta_x) + U_r \cdot r_{xx}$$

$$+ \theta_x(U_{\theta r} \cdot r_x + U_{\theta\theta} \cdot \theta_x) + U_\theta \cdot \theta_{xx}$$

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$f_{xy} = f_{yx}$

$$\frac{\partial^2 u}{\partial x^2} = U_{rr}(r_x)^2 + 2U_{r\theta}\cdot r_x\cdot \theta_x + U_r\cdot r_{xx} + U_{\theta\theta}(\theta_x)^2$$
$$+ U_\theta\cdot\theta_{xx} \quad [\because U_{r\theta} = U_{\theta r}] \quad \text{—} \ (*)$$

$$= U_{rr}\left(\frac{x}{r}\right)^2 + 2U_{r\theta}\cdot\frac{x}{r}\cdot\frac{(-y)}{r^2} + U_r\cdot\frac{y^2}{r^3}$$

$$+ U_{\theta\theta}\left(\frac{-y}{r^2}\right)^2 + U_\theta\cdot\frac{2xy}{r^4}$$

$$\therefore \frac{\partial^2 u}{\partial x^2} = \frac{x^2}{r^2}U_{rr} - \frac{2xy}{r^3}U_{r\theta} + \frac{y^2}{r^3}U_r + \frac{y^2}{r^4}U_{\theta\theta}$$

$$+ \frac{2xy}{r^4}U_\theta \quad \text{—} \ (ii)$$

Similarly, diff. U_y w.r.t y, we get

$$\frac{\partial^2 u}{\partial y^2} = U_{rr}\cdot(r_y)^2 + 2U_{r\theta}\cdot r_y\cdot\theta_y + U_r\cdot r_{yy} + U_{\theta\theta}\cdot(\theta_y)^2$$
$$+ U_\theta\cdot\theta_{yy}$$

$$= U_{rr}\cdot\frac{y^2}{r^2} + 2U_{r\theta}\cdot\frac{y}{r}\cdot\frac{x}{r^2} + U_r\cdot\frac{x^2}{r^3} + U_{\theta\theta}\cdot\frac{x^2}{r^4}$$

$$+ U_\theta\cdot\left(\frac{-2xy}{r^4}\right)$$

$$= \frac{y^2}{r^2}U_{rr} + \frac{2xy}{r^3}U_{r\theta} + \frac{x^2}{r^3}U_r + \frac{x^2}{r^4}U_{\theta\theta}$$

$$- \frac{2xy}{r^4}U_\theta \quad \text{—} \ (iii)$$

$$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$$\Rightarrow \left(\frac{x^2+y^2}{r^2}\right)U_{rr} + \left(\frac{x^2+y^2}{r^3}\right)U_r + \left(\frac{x^2+y^2}{r^4}\right)U_{\theta\theta} = 0$$

$$\Rightarrow U_{rr} + \frac{1}{r}U_r + \frac{1}{r^2}U_{\theta\theta} = 0$$

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i.e. $r^2 U_{rr} + rU_r + U_{\theta\theta} = 0$

$$\therefore \quad r^2 \frac{\partial^2 u}{\partial r^2} + r\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial \theta^2} = 0$$

which is laplace eqn. in polar form.

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* Solution of Laplace eqn. in polar form :: We have Laplace eqn. in polar form is

$$r^2 \frac{\partial^2 u}{\partial r^2} + r \frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial \theta^2} = 0 \quad \text{——} \quad (i)$$

Let $u(r,\theta) = R(r)\,\phi(\theta)$ be the general soln. of eqn (i) where R be the function of r only and ϕ be the function of θ only.

So.

Eqn (i) becomes.

$$r^2 \frac{\partial^2 (R\phi)}{\partial r^2} + r \frac{\partial (R\phi)}{\partial r} + \frac{\partial^2 (R\phi)}{\partial \theta^2} = 0$$

$$\Rightarrow r^2 \phi R'' + r\phi R' + R\phi'' = 0$$

$$\Rightarrow (r^2 R'' + rR')\phi + R\phi'' = 0$$

Seperating the variables,

$$\frac{r^2 R'' + rR'}{R} = -\frac{\phi''}{\phi}$$

Let $\frac{r^2 R'' + rR'}{R} = -\frac{\phi''}{\phi} = k$ (constant)

case I : Let $k > 0$ and $k = p^2$.

Then,

$$\frac{r^2 R'' + rR'}{R} = p^2$$

$$\Rightarrow r^2 R'' + rR' - p^2 R = 0$$

$$\Rightarrow r^2 \frac{d^2 R}{dr^2} + r\frac{dR}{dr} - p^2 R = 0 \quad \text{——} \quad (ii)$$

which is homogenous eqn. So, put $r = e^z$

i.e. $\log r = z$

Then,

$$\frac{1}{r} = \frac{dz}{dr}$$

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Also,

$$\frac{dR}{dx} = \frac{dR}{dz}\cdot\frac{dz}{dx}$$

$$\frac{1}{x}\frac{d}{dx}\left(dR/dz\right) = \frac{1}{x}\frac{d}{dz}\left(dR/dz\right)\cdot dz/dx$$

$$= \frac{1}{x}\frac{dR}{dz}$$

$$\therefore\ x\frac{dR}{dx} = \frac{dR}{dz}$$

Also,

$$\frac{d^2R}{dx^2} = \frac{d}{dx}\left(\frac{1}{x}\frac{dR}{dz}\right)$$

$$= \frac{dR}{dz}\left(-\frac{1}{x^2}\right) + \frac{1}{x}\frac{d^2R}{dz^2}\cdot\frac{dz}{dx}$$

$$= -\frac{1}{x^2}\frac{dR}{dz} + \frac{1}{x^2}\frac{d^2R}{dz^2}$$

$$\Rightarrow \frac{d^2R}{dx^2} = \frac{1}{x^2}\left(\frac{d^2R}{dz^2} - \frac{dR}{dz}\right)$$

$$\Rightarrow x^2\frac{d^2R}{dx^2} = \frac{d^2R}{dz^2} - \frac{dR}{dz}$$

So, eqn (iii) becomes

$$\frac{d^2R}{dz^2} - \frac{dR}{dz} + \frac{dR}{dz} - p^2R = 0$$

$$\Rightarrow \frac{d^2R}{dz^2} - p^2R = 0$$

$$\Rightarrow (D^2 - p^2)R = 0 \quad , \quad D = d/dz$$

Its AE is $m^2 - p^2 = 0$

$$\Rightarrow m = \pm p$$

$$\therefore R = c_1e^{pz} + c_2e^{-pz}$$

i.e. $R(x) = c_1x^p + c_2x^{-p}$

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Also.

$$-\frac{\phi''}{\phi} = p^2$$

$\Rightarrow \quad \phi'' + p^2\phi = 0$

$\Rightarrow \quad \frac{d^2\phi}{d\theta^2} + p^2\phi = 0$

$\Rightarrow \quad (D^2 + p^2)\phi = 0 \quad , \quad D = d/d\phi$

Its AE is $m^2 + p^2 = 0$

$\Rightarrow m = \pm pi$

$\therefore \phi(\theta) = c_3 \cos p\theta + c_4 \sin p\theta$

$\therefore u(r,\theta) = (c_1 r^p + c_2 r^{-p})(c_3 \cos p\theta + c_4 \sin p\theta)$ be general soln. for $k > 0$.

case II Let $k < 0$ and $k = -p^2$.

Then.

$$\frac{r^2R'' + rR'}{R} = -p^2$$

$\Rightarrow \quad r^2R'' + rR' + p^2R = 0$

$\Rightarrow \quad r^2\frac{d^2R}{dr^2} + r\frac{dR}{dr} + p^2R = 0$ ——— (iii)

which is homogenous eqn.

So, put $r = e^z$

i.e. $\log r = z$

Then,

$$\frac{1}{r} = \frac{dz}{dr}$$

Also,

$$r\frac{dR}{dr} = \frac{dR}{dz} \quad \text{and} \quad r^2\frac{d^2R}{dr^2} = \frac{d^2R}{dz^2} - \frac{dR}{dz}$$

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So, eq^n (iii) becomes,

$$\frac{d^2R}{dz^2} - \frac{dR}{dz} + \frac{dR}{dz} + p^2R = 0$$

$$\Rightarrow \frac{d^2R}{dz^2} + p^2R = 0$$

$$\Rightarrow (D^2 + p^2)R = 0 \quad , D = d/dz$$

Its AE is $m^2 + p^2 = 0$

$$\therefore m = \pm pi$$

$$\therefore R = C5 \cos pz + C6 \sin pz$$

$$= C5 \cos\{p\log r\} + C6 \sin\{p\log r\}$$

Also,

$$-\frac{\phi''}{\phi} = -p^2$$

$$\Rightarrow \phi'' - p^2\phi = 0$$

$$\Rightarrow \frac{d^2\phi}{d\theta^2} - p^2\phi = 0$$

$$\Rightarrow (D^2 - p^2)\phi = 0 \quad , D = d/d\phi$$

Its AE is $m^2 - p^2 = 0$

$$\therefore m = \pm p$$

$$\therefore \phi(\theta) = C7 e^{p\theta} + C8 e^{-p\theta}$$

$$\therefore u(r,\theta) = \{C5 \cos(p\log r) + C6 \sin(p\log r)\}\{C7 e^{p\theta} + C8 e^{-p\theta}\}$$

Let $k = 0$.

Then,

$$\frac{r^2R'' + rR'}{R} = 0$$

$$\Rightarrow r^2R'' + rR' = 0$$

$$\Rightarrow r^2\frac{d^2R}{dr^2} + r\frac{dR}{dr} = 0 \quad \text{—— (iv)}$$

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which is homogenous eqn.

So, put $x = e^{z}$

i.e. $\log x = z$

Then,

$$\frac{1}{x} = \frac{dz}{dx}$$

Also,

$$x\frac{dR}{dx} = \frac{dR}{dz} \quad \text{and} \quad x^2\frac{d^2R}{dx^2} = \frac{d^2R}{dz^2} - \frac{dR}{dz}$$

So, eqn. (iv) becomes,

$$\frac{d^2R}{dz^2} - \frac{dR}{dz} + \frac{dR}{dz} = 0$$

$$\therefore \frac{d^2R}{dz^2} = 0$$

$$\therefore R = c_9 + c_{10}z$$

$$= c_9 + c_{10}\log x$$

Also,

$$-\frac{\phi''}{\phi} = 0$$

$$\therefore \phi'' = 0$$

i.e. $\frac{d^2\phi}{d\theta^2} = 0$

$$\therefore \phi = c_{11} + c_{12}\theta$$

$$\therefore u(r,\theta) = (c_9 + c_{10}\log r)(c_{11} + c_{12}\theta)$$

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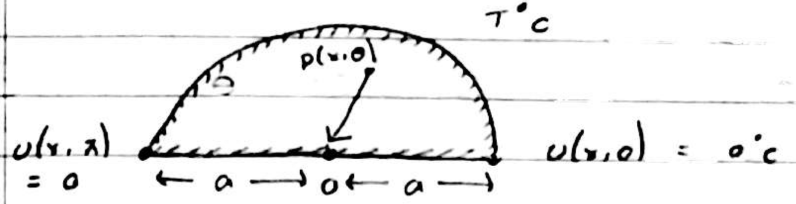
Q The diameter of a semi-circular plate of radius 'a' is kept at 0°c and the temperature of semi-circular boundary is T°c. Show that the steady state temperature in the plate is given by

$$u(r,\theta) = \frac{4T}{\pi} \sum_{n=1}^{\infty} \frac{1}{(2n-1)} \left(\frac{r}{a}\right)^{2n-1} \sin(2n-1)\theta$$

$u(a,\theta) = T°c$



=) We have Laplace eq^n. in polar form :

$$r^2 \frac{\partial^2 u}{\partial r^2} + r \frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial \theta^2} = 0 \quad \text{—— (i)}$$

Boundary condition,

$$u(r,0) = 0°c$$
$$u(r,\pi) = 0°c$$
$$u(a,\theta) = T°c$$

Since $u \to 0$ as $r \to 0$, so the feasible sol^n. is

$$u(r,\theta) = (c_1 r^p + c_2 r^{-p})(c_3 \cos p\theta + c_4 \sin p\theta) \quad \text{—— (ii)}$$

Using the condition $u(r,0) = 0$ in eq^n (ii),

$$0 = (c_1 r^p + c_2 r^{-p})\, c_3$$

Since, $c_1 r^p + c_2 r^{-p} \neq 0$, otherwise sol^n. is trivial, so

$$c_3 = 0$$

Eq^n. (ii) becomes,

$$u(r,\theta) = (c_1 r^p + c_2 r^{-p})(c_4 \sin p\theta) \quad \text{—— (iii)}$$

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Also, using the condition $u(r,\pi) = 0°c$ in eqn (iii)

$$0 = (c_1 r^p + c_2 r^{-p}) c_4 \sin p\pi$$

Since, $(c_1 r^p + c_2 r^{-p}) c_4 \neq 0$

$$\therefore \sin p\pi = 0 = \sin n\pi, \ n \in I$$

$$\Rightarrow p\pi = n\pi$$

$$\Rightarrow p = n$$

Putting $p = n$ in eqn (iii),

$$u(r,\theta) = (c_1 r^p + c_2 r^{-p}) c_4 \sin n\theta \quad \text{—— (iv)}$$

Since, $u \to 0$ as $r \to 0$ so we must have

$$c_2 = 0$$

$$\therefore u(r,\theta) = (c_1 r^n) c_4 \sin n\theta$$
$$= A r^n \sin n\theta$$

The most general soln. is

$$u(r,\theta) = \sum_{n=1}^{\infty} A_n r^n \sin n\theta \quad \text{—— (v)}$$

Now,

Using the condition $u(a,\theta) = T°c$ in eqn (v).

$$T = \sum_{n=1}^{\infty} A_n a^n \sin n\theta$$

which is half range Fourier sine series in $(0,\pi)$.

So, by euler's formula,

$$A_n a^n = \frac{2}{\pi} \int_0^{\pi} T \sin n\theta \, d\theta$$

$$A_n = \frac{2T}{\pi a^n} \int_0^{\pi} \sin n\theta \, d\theta$$

$$= \frac{2T}{\pi a^n} \left(-\frac{\cos n\pi}{n} \right)_0^{\pi}$$

$$= \frac{2T}{n\pi a^n} (1 - \cos n\pi)$$

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$$= \frac{2T}{n\pi a^n}\left\{1-(-1)^n\right\}$$

$$= \begin{cases} 4T/n\pi a^n & \text{for } n \text{ is odd} \\ 0 & \text{for } n \text{ is even} \end{cases}$$

Putting A_n in eqⁿ ⑤.

$$u(r,\theta) = \sum_{n=1}^{\infty} \frac{4T}{n\pi a^n} r^n \sin n\theta$$

$$\therefore u(r,\theta) = \frac{4T}{\pi}\sum_{n=1,3,5,\ldots}^{\infty} \frac{1}{(2n-1)}\left(\frac{r}{a}\right)^{2n-1} \sin(2n-1)\theta$$

Proved

Tut. 21 → Same as this

13 =) ~~xx~~

Flair