One Dimensional Heat Equations

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One dimensional Heat equation

Let us consider a homogeneous rod of density ρ , area of cross section A, specific heat capacity c and thermal conductivity k. The sides of the rod are insulated so that the loss of heat from the sides by conduction or radiation is negligible.

Now, we assume the following assumptions:
1. Heat flows from higher to lower temperature.

2. The amount of heat required to produce a given temperature change in a body is proportional to the mass of the body and to the temperature change.

3. The rate at which heat flows through an area is proportional to the area and to the temperature gradient normal to the area.

Let us take one end of the rod as origin and direction of heat flows as positive $x - axis$. Let $u(x, t)$ be the temperature at any position x at time t on the rod.

Now consider an element between two parallel sections $BDEF$ and $LMNP$ at distances x and $x + \Delta x$ from the origin.
The mass of the element is

$$= A\Delta x\rho$$

$$[\because density = \frac{mass}{volume}]$$

By second assumption, rate of increase of heat in the element

$$\propto mass \times \frac{\partial u}{\partial t}$$

$$= cA\Delta x \rho . \frac{\partial u}{\partial t}$$

where the constant of proportionality c is specific heat capacity of material of rod.

By third assumption, the rate of flow of heat through the area is

$$\propto area \times \frac{\partial u}{\partial x}$$

Let R_1 and R_2 be the rates of inflow and outflow for the sections at $x = x$ and $x = x + \Delta x$ then

$$R_1 = -k.A. \left(\frac{\partial u}{\partial x} \right)_x$$

and

$$R_2 = -k.A. \left(\frac{\partial u}{\partial x} \right)_{x+\Delta x}$$

the negative sign indicates that heat flows from higher to lower temperature.

Now, in the element of the rod

$$rate \quad of \quad increase \quad of \quad heat = rate \quad of \quad inflow - rate \quad of \quad inflow$$

or

$$cA\Delta x\rho.\frac{\partial u}{\partial t} = R_1 - R_2$$

or

$$cA\Delta x\rho.\frac{\partial u}{\partial t} = -k.A.\left(\frac{\partial u}{\partial x}\right)_x + k.A.\left(\frac{\partial u}{\partial x}\right)_{x+\Delta x}$$

Dividing by $cA\Delta x\rho$. we get

$$\frac{\partial u}{\partial t} = \frac{k}{c\rho}\left\{\frac{\left(\frac{\partial u}{\partial x}\right)_{x+\Delta x} - \left(\frac{\partial u}{\partial x}\right)_{x}}{\Delta x}\right\}$$

Taking limit as $\Delta x \to 0$ we get

$$\frac{\partial u}{\partial t} = \frac{k}{c\rho} \lim_{\Delta x \to 0} \left\{ \frac{\left(\frac{\partial u}{\partial x}\right)_{x+\Delta x} - \left(\frac{\partial u}{\partial x}\right)_{x}}{\Delta x} \right\}$$

or,

$$\frac{\partial u}{\partial t} = \frac{k}{c\rho}\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)$$

Hence,

$$\frac{\partial u}{\partial t} = C^2 \frac{\partial^2 u}{\partial x^2}$$

which is required one dimensional heat equation, where $C^2 = \frac{k}{c\rho}$

Note that, the constant

$$C^2 = \frac{k}{c\rho}$$

is called the diffusivity of a substance.

Solution of one dimensional heat equation with homogeneous boundary condition::

If both ends of the rod are kept at zero temperature, the boundary conditions are called homogeneous otherwise they are called non homogeneous boundary condition.
One dimensional heat equation is

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} ...(1)$$

As the ends of rod are kept at zero temperature, the boundary conditions are

$$u(0, t) = 0$$

and

$$u(L, t) = 0$$

Let the initial temperature on the rod be

$$u(x, 0) = f(x) \quad (initial \quad temperature)$$

Let

$$u(x, t) = X(x).T(t)$$

where, $X(x)$ is function of x alone and $T(t)$ is function of t alone, be the solution of (1).
Then, differentiating partially with respect to x we get

$$\frac{\partial u}{\partial x} = \frac{dX}{dx}.T$$

Also, differentiating partially with respect to t we get

$$\frac{\partial u}{\partial t} = X.\frac{dT}{dt}$$

Again, differentiating partially w.r to x , we get

$$\frac{\partial^2 u}{\partial x^2} = \frac{d^2 X}{dx^2}.T$$

Now, from equation (1),

$$X.\frac{dT}{dt} = c^2 T.\frac{d^2X}{dx^2}$$

Dividing by c^2XT we get

$$\frac{1}{c^2T}\frac{dT}{dt} = \frac{1}{X}\frac{d^2X}{dx^2} = k(say)... \quad ...(2)$$

Now we have the following three cases.
Case I:
Let k is positive, say $k = \lambda^2$,then

$$\frac{1}{X}\frac{d^2X}{dx^2} = \lambda^2$$

and

$$\frac{1}{c^2T}\frac{dT}{dt} = \lambda^2$$

or,

$$\frac{d^2X}{dx^2} - \lambda^2 X = 0$$

and

$$\frac{dT}{T} = \lambda^2 c^2 dt$$

Solving we get

$$X = C_1 e^{\lambda x} + C_2 e^{-\lambda x}$$

and

$$\log T = \lambda^2 c^2 t + \log C_3$$

i.e,

$$T = C_3 e^{\lambda^2 c^2 t}$$

So,

$$u(x,t) = X.T$$

now becomes

$$u(x,t) = \left(C_1 e^{\lambda x} + C_2 e^{-\lambda x}\right)\left(C_3 e^{\lambda^2 c^2 t}\right) \ldots .(3)$$

Using the boundary condition $u(0, t) = 0$, we get

$$0 = \left(C_1 e^0 + C_2 e^0\right)\left(C_3 e^{\lambda^2 c^2 t}\right)$$

or

$$0 = (C_1 + C_2)\left(C_3 e^{\lambda^2 c^2 t}\right)$$

$$i.e. C_1 + C_2 = 0$$

since

$$C_3 e^{\lambda^2 c^2 t} \neq 0$$

Again, using the boundary condition $u(L,t) = 0$, we get

$$0 = \left(C_1 e^{\lambda L} + C_2 e^{-\lambda L}\right)\left(C_3 e^{\lambda^2 c^2 t}\right)$$

or

$$0 = C_1 e^{\lambda L} - C_1 e^{-\lambda L}$$

$$\left[\because C_1 + C_2 = 0 \quad \textit{and} \quad C_3 e^{\lambda^2 c^2 t} \neq 0\right]$$

$$or, e^{\lambda L} = e^{-\lambda L}$$

or,

$$e^{2\lambda L} = 1$$

or

$$e^{2\lambda L} = e^{0}$$

or,

$$2\lambda L = 0$$

or,

$$\lambda = 0, \quad \because 2L \neq 0$$

This is a contradiction to the fact that $k = \lambda^2$ is positive. So this solution is rejected.

Case II:
Let $k = 0$, then from (2)

$$\frac{1}{X}\frac{d^2X}{dx^2} = 0$$

and

$$\frac{1}{c^2T}\frac{dT}{dt} = 0$$

or

$$\frac{d^2X}{dx^2} = 0$$

and

$$\frac{dT}{dt} = 0$$

Integrating we get

$$\frac{dX}{dx} = A$$

and

$$T = C$$

where A and C are constants of integration.

Again, integrating we get

$$X = Ax + B$$

So,

$$u(x,t) = X.T$$

now becomes

$$u(x,t) = (Ax + B).C$$

which shows that temperature of the rod depends only on its position but not on time, which is physically impossible. So this solution is also rejected.

Case III:
Let k is negetive, say $k = -\lambda^2$,then

$$\frac{1}{X}\frac{d^2X}{dx^2} = -\lambda^2$$

and

$$\frac{1}{c^2T}\frac{dT}{dt} = -\lambda^2$$

or,

$$\frac{d^2X}{dx^2} + \lambda^2 X = 0$$

and

$$\frac{dT}{T} = -c^2\lambda^2 dt$$

Solving, we get

$$X = C_1 \cos \lambda x + C_2 \sin \lambda x$$

and

$$T = C_3 e^{-\lambda^2 c^2 t}$$

So,

$$u(x,t) = X.T$$

now becomes

$$u(x,t) = (C_1 \cos \lambda x + C_2 \sin \lambda x)(C_3 e^{-\lambda^2 c^2 t})... \quad ...(4)$$

Using the boundary condition $u(0, t) = 0$, we get

$$0 = (C_1.1 + C_2.0)(C_3 e^{-\lambda^2 c^2 t})$$

i.e.

$$C_1 = 0 \quad since \quad (C_3 e^{-\lambda^2 c^2 t}) \neq 0$$

Again, using the boundary condition $u(L, t) = 0$, we get

$$0 = (0 + C_2.\sin\lambda L)(C_3 e^{-\lambda^2 c^2 t})$$

i.e.

$$\sin\lambda L = 0 \quad \textit{since} \quad C_2 \neq 0 \quad \textit{and} \quad (C_3 e^{-\lambda^2 c^2 t}) \neq 0$$

$$or, \sin \lambda L = 0 = \sin n\pi$$

or

$$\lambda L = n\pi$$

or,

$$\lambda = \frac{n\pi}{L} \quad n = 1, 2, 3, 4, ...$$

Now, from equation (4), we get

$$u(x,t) = C_2 \sin \frac{n\pi}{L} x . C_3 e^{-\left(\frac{n\pi}{L}\right)^2 c^2 t}$$

i.e.

$$u_n(x,t) = a_n \sin \frac{n\pi}{L} x . e^{-\left(\frac{n\pi}{L}\right)^2 c^2 t}$$

where

$$a_n = C_2 C_3$$

Using principle of superposition, (i.e. adding all possible solutions) , we get

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t)$$

i.e.

$$u(x,t) = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi}{L} x . e^{-\left(\frac{n\pi}{L}\right)^2 c^2 t} .....(5)$$

Using the initial condition

$$u(x,0) = f(x)$$

we get from (5)

$$f(x) = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi}{L} x e^0$$

or

$$f(x) = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi}{L} x$$

which is a half range Fourier sine series, where

$$a_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi}{L} x \quad dx \quad ....(6)$$

The equation (5), together with the equation (6) gives the required general solution of one dimensional heat equation with the homogeneous boundary conditions.

Find the temperature in the bar of length 2 whose ends are kept at zero temperature and lateral surface are insulated if the initial temperature is given by

$$\sin \frac{\pi x}{2} + 3 \sin \frac{5\pi x}{2}$$

Sol^n :

It is the case of one dimensional heat equation with its equation

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} ...(1)$$

The boundary conditions are

$$u(0, t) = 0$$

and

$$u(L, t) = 0$$

Let the initial temperature on the rod is

$$u(x, 0) = f(x) = \sin \frac{\pi x}{2} + 3 \sin \frac{5\pi x}{2}$$

Its general solution is

$$u(x,t) = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi}{L} x . e^{-\left(\frac{n\pi}{L}\right)^2 c^2 t}$$

i.e,

$$u(x,t) = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi}{2} x . e^{-\left(\frac{n\pi}{2}\right)^2 c^2 t} .....(2)$$

Using the initial condition,

$$u(x,0) = f(x) = \sin\frac{\pi x}{2} + 3\sin\frac{5\pi x}{2}$$

we get

$$\sin\frac{\pi x}{2} + 3\sin\frac{5\pi x}{2} = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi}{2}x.e^0$$

or,

$$\sin\frac{\pi x}{2} + 3\sin\frac{5\pi x}{2} = a_1 \sin\frac{\pi}{2}x + ... + a_5 \sin\frac{5\pi}{2}x + ...$$

Compairing the similar terms, we get

$$a_1 = 1$$

,

$$a_5 = 3$$

and rest are zeros.
Substituting these values in equation (2), we get

$$
\begin{aligned}
u(x,t) &= a_1 \sin \frac{\pi}{2} x . e^{-\left(\frac{\pi}{2}\right)^2 c^2 t} + a_5 \sin \frac{5\pi}{2} x . e^{-\left(\frac{5\pi}{2}\right)^2 c^2 t} \\
\therefore u(x,t) &= \sin \frac{\pi}{2} x . e^{-\left(\frac{\pi}{2}\right)^2 c^2 t} + 3 \sin \frac{5\pi}{2} x . e^{-\left(\frac{5\pi}{2}\right)^2 c^2 t}
\end{aligned}
$$

which is required solution.

Solve the heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

with the boundary conditions $u(0, t) = 0$ and $u(l, t) = 0$. The initial condition is

$$u(x, 0) = 3 \sin n\pi x$$

Sol^n :

Given equation is

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \ldots (1)$$

The boundary conditions are

$$u(0, t) = 0$$

and

$$u(l, t) = 0$$

Let the initial temperature on the rod is

$$u(x, 0) = f(x) = 3 \sin n\pi x$$

Its general solution is

$$u(x,t) = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi}{l} x . e^{-\left(\frac{n\pi}{l}\right)^2 t} .....(2)$$

Using the initial condition

$$u(x, 0) = f(x) = 3 \sin n\pi x$$

we get

$$3 \sin n\pi x = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi}{l} x . e^0$$

or

$$3 \sin n\pi x = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi}{l} x$$

Comparing the similar coefficients, we get

$$a_n = 3 \quad l = 1$$

Hence, from (2), we get

$$u(x,t)=\sum_{n=1}^{\infty} 3\sin n\pi x . e^{-(n\pi)^2 t}$$

which is required solution of one dimensional heat equation.

A uniform rod of length L has its ends maintained at zero temperature and the initial temperature on the rod is

$$3\sin\frac{\pi x}{L}$$

. Find the temperature distribution on the rod.

A homogeneous rod of conducting material of length 100 cm has its ends kept at zero temperature and temperature initially is

$$f(x) = x \quad for 0 \leq x \leq 50$$

and

$$f(x) = 100 - x \quad for 50 \leq x \leq 100$$

. Find the temperature distribution on the rod.

Sol^n :
It is the case of one dimensional heat equation with its equation

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} ...(1)$$

The boundary conditions are

$$u(0, t) = 0$$

and

$$u(L, t) = 0$$

The initial temperature on the rod is

$$u(x, 0) = f(x) = x \quad \textit{for} \quad 0 \leq x \leq 50$$

and

$$u(x, 0) = f(x) = 100 - x \quad \textit{for} \quad 50 \leq x \leq 100$$

Its general solution is

$$u(x,t)=\sum_{n=1}^{\infty} a_n \sin\frac{n\pi}{L}x.e^{-\left(\frac{n\pi}{L}\right)^2 c^2 t}......(2)$$

Using the initial condition,

$$u(x,0) = f(x)$$

we get

$$u(x,0) = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi}{L} x.e^0$$

or,

$$f(x) = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi}{100} x$$

Which is a half range Fourier sine series, where

$$a_n = \frac{2}{L} \int_{0}^{L} f(x) \sin \frac{n\pi}{L} x \quad dx$$

$$
\begin{aligned}
&= \frac{2}{L}\int_{0}^{50} f(x)\sin\frac{n\pi}{100}x\,dx + \frac{2}{L}\int_{50}^{100} f(x)\sin\frac{n\pi}{100}x\,dx \\
&= \frac{2}{100}\int_{0}^{50} x\sin\frac{n\pi}{100}x\,dx + \frac{2}{100}\int_{50}^{100} (100-x)\sin\frac{n\pi}{100}x\,dx
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{2}{100}\left[x.\frac{100}{n\pi}(-)\cos\frac{n\pi}{100}x - 1.\left(\frac{100}{n\pi}\right)^2(-)\sin\frac{n\pi}{100}x\right]_0^{50} \\
&+ \frac{2}{100}\left[(100-x).\frac{100}{n\pi}(-)\cos\frac{n\pi}{100}x - (-1).\left(\frac{100}{n\pi}\right)^2(-)\sin\frac{n\pi}{100}x\right]_{50}^{100}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{50}\left[(-)50.\frac{100}{n\pi}\cos\frac{n\pi}{2} + \left(\frac{100}{n\pi}\right)^2 \sin\frac{n\pi}{2} - 0 + 0\right] \\
&+ \frac{1}{50}\left[0 - 0 - \left\{(-)50.\frac{100}{n\pi}\cos\frac{n\pi}{2} - \left(\frac{100}{n\pi}\right)^2 \sin\frac{n\pi}{2}\right\}\right]
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{50}.2.\left(\frac{100}{n\pi}\right)^2 \sin\frac{n\pi}{2} \\
&= \frac{400}{n^2\pi^2}\sin\frac{n\pi}{2}
\end{aligned}
$$

$$\therefore a_n = \begin{cases} \frac{400}{n^2\pi^2} \sin \frac{n\pi}{2} & \text{for} \quad n = odd \\ 0 & \text{for} \quad n = even \end{cases}$$

Hence, from equation (2), we get

$$u(x,t) = \frac{400}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \sin \frac{n\pi}{100} x . e^{-\left(\frac{n\pi}{100}\right)^2 c^2 t}$$

where $n = odd$

This is required temperature distribution on the given rod.