Partial Differential Equations

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October 16, 2020

Differential Equation

The equation involving dependent variable, independent variable and the derivative of dependent variable with respect to the independent variable is called a differential equation. For example:

1. $$\frac{dy}{dx} = e^x + 1$$

2. $$\frac{\partial u}{\partial t} = c^2 \left(\frac{\partial^2 x}{\partial^2 t} + \frac{\partial^2 y}{\partial^2 t} \right)$$

etc.

Dijfferential equations are mainly divided into two types.

1. Ordinary differential equation(ODE): The differential equation containing only one independent variable is called ODE.
2. Partial differential equation(PDE): The differential equation containing two or more independent variables is called a PDE.

Method of separation of variables:

It assumes the solution of the PDE as the product of functions of its independent variables. In this method , the PDE is first converted into ODE and the ODE is then solved to obtain the values of functions of independent variables. After substituting these values in our supposition, we get required solution.

Solve:

$$u_{xx} - u_{yy} = 0$$

, by using the method of separating the variables.

Sol^n : Given equation is

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = 0.... \quad ...(1), \qquad u = u(x, y)$$

Let

$$u(x, y) = X(x).Y(y)$$

be the solution of (1).
Then, differentiating with respect to x we get

$$\frac{\partial u}{\partial x} = \frac{dX}{dx}.Y$$

Also,

$$\frac{\partial^2 u}{\partial x^2} = \frac{d^2 X}{dx^2}.Y$$

Again, differentiating partially w.r to y , we get

$$\frac{\partial u}{\partial y} = X.\frac{dY}{dy}$$

Also,

$$\frac{\partial^2 u}{\partial y^2} = X.\frac{d^2 Y}{dy^2}$$

So, from equation (1), we get

$$\frac{d^2X}{dx^2}.Y - X.\frac{d^2Y}{dy^2} = 0$$

$$or, \frac{d^2X}{dx^2}.Y = X.\frac{d^2Y}{dy^2}$$

dividing by XY, we get

$$\frac{1}{X}\frac{d^2X}{dx^2} = \frac{1}{Y}\frac{d^2Y}{dy^2} = k(say)$$

Then,

$$\frac{1}{X}\frac{d^2X}{dx^2} = k$$

and

$$\frac{1}{Y}\frac{d^2Y}{dy^2} = k$$

or,

$$\frac{d^2X}{dx^2} - kX = 0... \quad (2)$$

and

$$\frac{d^2Y}{dy^2} - kY = 0... \quad (3)$$

The auxiliary equation of (2) is

$$m^2 - k = 0$$

$$or, m^2 = k$$

$$i.e. m = \pm\sqrt{k}$$

which are two different real values. So,

$$X = C_1 e^{\sqrt{k}x} + C_2 e^{-\sqrt{k}x}$$

Also, the auxiliary equation of (3) is

$$m^2 - k = 0$$

$$or, m^2 = k$$

$$i.e. m = \pm\sqrt{k}$$

which are two different real values. So,

$$Y = C_3 e^{\sqrt{k}y} + C_4 e^{-\sqrt{k}y}$$

Hence,

$$u(x, y) = X.Y$$

becomes

$$u = \left(C_1 e^{\sqrt{k}x} + C_2 e^{-\sqrt{k}x}\right)\left(C_3 e^{\sqrt{k}y} + C_4 e^{-\sqrt{k}y}\right)$$

which is required solution.

Solve:

$$u_{xx} + u_{yy} = 0$$

, by using the method of separating the variables.

Sol^n : Given equation is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.... \quad ...(1), \qquad u = u(x, y)$$

Let

$$u(x, y) = X(x).Y(y)$$

be the solution of (1).
Then, differentiating with respect to x we get

$$\frac{\partial u}{\partial x} = \frac{dX}{dx}.Y$$

Also,

$$\frac{\partial^2 u}{\partial x^2} = \frac{d^2 X}{dx^2}.Y$$

Again, differentiating partially w.r to y , we get

$$\frac{\partial u}{\partial y} = X.\frac{dY}{dy}$$

Also,

$$\frac{\partial^2 u}{\partial y^2} = X.\frac{d^2 Y}{dy^2}$$

So, from equation (1), we get

$$\frac{d^2X}{dx^2}.Y + X.\frac{d^2Y}{dy^2} = 0$$

$$or, \frac{d^2X}{dx^2}.Y = -X.\frac{d^2Y}{dy^2}$$

dividing by XY, we get

$$\frac{1}{X}\frac{d^2X}{dx^2} = -\frac{1}{Y}\frac{d^2Y}{dy^2} = k(say)$$

Then,

$$\frac{1}{X}\frac{d^2X}{dx^2} = k$$

and

$$-\frac{1}{Y}\frac{d^2Y}{dy^2} = k$$

or,

$$\frac{d^2X}{dx^2} - kX = 0... \quad (2)$$

and

$$\frac{d^2Y}{dy^2} + kY = 0... \quad (3)$$

The auxiliary equation of (2) is

$$m^2 - k = 0$$

$$or, m^2 = k$$

$$i.e. m = \pm\sqrt{k}$$

which are two different real values. So,

$$X = C_1 e^{\sqrt{k}x} + C_2 e^{-\sqrt{k}x}$$

Also, the auxiliary equation of (3) is

$$m^2 + k = 0$$

$$or, m^2 = -k$$

$$i.e. m = \pm\sqrt{k}i$$

which are two imaginary values. So,

$$Y = C_3 \cos\sqrt{k}y + C_4 \sin\sqrt{k}y$$

Hence,

$$u(x, y) = X.Y$$

becomes

$$u = \left(C_1 e^{\sqrt{k}x} + C_2 e^{-\sqrt{k}x}\right)\left(C_3 \cos\sqrt{k}y + C_4 \sin\sqrt{k}y\right)$$

which is required solution.

Solve:

$$u_{xx} + 9u_{yy} = 0$$

, by using the method of separating the variables.

Sol^n : Given equation is

$$\frac{\partial^2 u}{\partial x^2} + 9\frac{\partial^2 u}{\partial y^2} = 0.... \quad ...(1), \qquad u = u(x, y)$$

Let

$$u(x, y) = X(x).Y(y)$$

be the solution of (1).
Then, differentiating with respect to x we get

$$\frac{\partial u}{\partial x} = \frac{dX}{dx}.Y$$

Also,

$$\frac{\partial^2 u}{\partial x^2} = \frac{d^2 X}{dx^2}.Y$$

Again, differentiating partially w.r to y , we get

$$\frac{\partial u}{\partial y} = X.\frac{dY}{dy}$$

Also,

$$\frac{\partial^2 u}{\partial y^2} = X.\frac{d^2 Y}{dy^2}$$

So, from equation (1), we get

$$\frac{d^2X}{dx^2}.Y + 9X.\frac{d^2Y}{dy^2} = 0$$

$$or, \frac{d^2X}{dx^2}.Y = -9X.\frac{d^2Y}{dy^2}$$

dividing by $9XY$, we get

$$\frac{1}{9X}\frac{d^2X}{dx^2} = -\frac{1}{Y}\frac{d^2Y}{dy^2} = k(say)$$

Then,

$$\frac{1}{9X}\frac{d^2X}{dx^2} = k$$

and

$$-\frac{1}{Y}\frac{d^2Y}{dy^2} = k$$

or,

$$\frac{d^2X}{dx^2} - 9kX = 0... \quad (2)$$

and

$$\frac{d^2Y}{dy^2} + kY = 0... \quad (3)$$

The auxiliary equation of (2) is

$$m^2 - 9k = 0$$

$$or, m^2 = 9k$$

$$i.e. m = \pm 3\sqrt{k}$$

which are two different real values. So,

$$X = C_1 e^{3\sqrt{k}x} + C_2 e^{-3\sqrt{k}x}$$

Also, the auxiliary equation of (3) is

$$m^2 + k = 0$$

$$or, m^2 = -k$$

$$i.e. m = \pm\sqrt{k}i$$

which are two imaginary values. So,

$$Y = C_3 \cos\sqrt{k}y + C_4 \sin\sqrt{k}y$$

Hence,

$$u(x, y) = X.Y$$

becomes

$$u = \left(C_1 e^{3\sqrt{k}x} + C_2 e^{-3\sqrt{k}x}\right)\left(C_3 \cos\sqrt{k}y + C_4 \sin\sqrt{k}y\right)$$

which is required solution.

Solve the following by using method of separating the variables:

1. $$u_{xx} + 9u = 0$$
2. $$u_{xy} - u = 0$$
3. $$xu_{xy} + 2yu = 0$$
4. $$u_x + u_y = (x + y)u$$
5. $$u_x = yu_y$$