# Existence of a Solution for Beloch's Fold[*]

Jorge C. Lucero[†]

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**Abstract**

A fundamental operation in geometric constructions by paper folding is: given two points and two lines on a sheet of paper, fold the paper so that each point is placed onto one of the lines. This article analyzes conditions of existence of the fold operation and shows that it may be realized iff the distance between the given points is not smaller than the distance between the given lines.

Almost a century ago, Italian mathematician Margharita Piazolla Beloch studied the following fold operation and its associated geometry [3, 9]:

**Beloch's fold.** Given points $P$ and $Q$ and lines $m$ and $n$ on a sheet of paper, fold the paper along a straight line so that $P$ is placed onto $m$ and $Q$ onto $n$ simultaneously (Fig. 1).
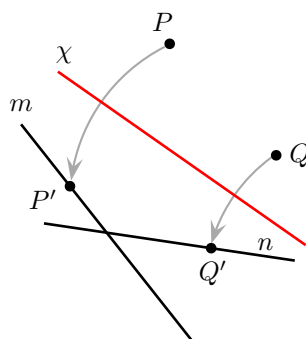


**Figure 1:** Beloch's fold along line $\chi$.

In her work, Beloch showed that the fold could be applied to solve arbitrary cubic equations. The operation is considered today as one of the so-called "axioms" of origami (the Japanese art of paper folding), which are combinations of alignments between given points and lines on a sheet of paper that may be achieved with a single fold [1, 2, 4, 7, 9]. For reference, a standard version of the complete set of axioms is listed in Table 1 [2]. It has been shown that Beloch's fold is the most powerful of such axioms, in the sense that it includes all the others as particular cases [7]. Due to its association with a cubic equation, the operation may have up to three solutions. In fact, it has been applied to solve construction problems related to cubic equations, such as trisecting arbitrary angles [8], duplicating the cube [12] and constructing heptagons [5].

[†]Dept. Computer Science, University of Brasília, Brazil. E-mail: lucero@unb.br

| Number | Axiom |
|--------|-------|
| 1 | Given two points P and Q, we can fold a line connecting them. |
| 2 | Given two points P and Q, we can fold P onto Q. |
| 3 | Given two lines m and n, we can fold m onto n. |
| 4 | Given a point P and a line m, we can make a fold perpendicular to m passing through P. |
| 5 | Given two points P and Q and a line m, we can make a fold that places P onto m and passes through Q. |
| 6 | Given two points P and Q and two lines m and n, we can make a fold that places P onto m and places Q onto n. |
| 7 | Given a point P and two lines m and n, we can make a fold perpendicular to m that places P onto n. |

**Table 1:** "Axioms" of origami constructions [2].

As an example of the utility of Beloch's fold, Fig. 2 shows how to find the cubic root of an arbitrary positive real number $a$ [4, 9]: (1) Set a Cartesian system of coordinates $x, y$ and mark points $P(0, -1)$ and $Q(-a, 0)$. (2) Trace (e.g., by folding) lines $y = 1$ and $x = a$, denoted by $m$ and $n$ respectively. (3) Fold along a line $\chi$ so as to place P onto $m$ and Q onto $n$. (4) The $x$ intercept of line $\chi$ (point R) is $\sqrt[3]{a}$.

Note that QOS, POR and SOR in Fig. 2 are similar right triangles. Therefore,

$$\frac{|\overline{QO}|}{|\overline{SO}|} = \frac{|\overline{SO}|}{|\overline{RO}|} = \frac{|\overline{RO}|}{|\overline{PO}|}.$$

Eliminating $|\overline{SO}|$ yields

$$|\overline{RO}|^3 = |\overline{QO}||\overline{PO}|^2,$$

and replacing $|\overline{QO}| = a$, $|\overline{PO}| = 1$, produces $|\overline{RO}| = \sqrt[3]{a}$.
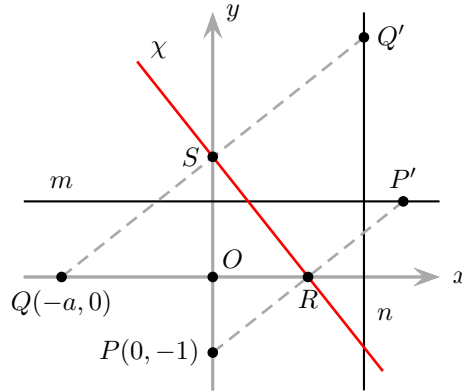


**Figure 2:** C                                                      $a > 0$.

However, depending on the specific configuration of points and lines, the fold operation may be impossible to perform. Therefore, it is sometimes stated in the form "...whenever possible, fold the paper so that..." or similar [1, 9]. Naturally, the question arises: When is Beloch's fold possible? Further, it is regularly assumed that P and Q are distinct, as well as lines m

and $n$, and that the points are not initially located on the respective lines. Then, one may also ask what happens if those conditions are not met, does the operation still have a solution? A complete answer to the previous questions does not seem reported in the literature, and this article provides a simple and general one: a solution exists iff the distance between $P$ and $Q$ is not smaller than the distance between $m$ and $n$. Here, the distance between a pair of lines is taken as the distance between the corresponding point sets. Hence, if $m$ and $n$ are nonparallel then their distance is zero and the fold is always possible. This fact was already deduced by Martin [11] under the restriction that $P$ and $Q$ are distinct. In the same work, he also indicated that if $m$ and $n$ are parallel then the fold may not have a solution. In addition, the particular cases of $P = Q$ and $m = n$ were discussed by Geretschläger [4, 6], who also argued that a solution always exists if $m$ and $n$ intersect.

The present analysis provides a general treatment encompassing all possible configurations of the given points and lines.

# 1  Analysis of solutions of Beloch's fold

## 1.1  Reflection of points

Folding a sheet of paper along a straight line superposes the paper on one side of the fold line to the other side, and the superposition may be modeled as a reflection across the fold line (Fig. 3) [11].

**Definition 1.** Given a line $\chi$, the reflection $\mathcal{F}_\chi$ in $\chi$ is the mapping on the set of points in the plane such that for point $P$

$$\mathcal{F}_\chi(P) = \begin{cases} P & \text{if } P \in \chi, \\ P' & \text{if } P \notin \chi \text{ and } \chi \text{ is the perpendicular bisector of segment } \overline{PP'} \end{cases}$$
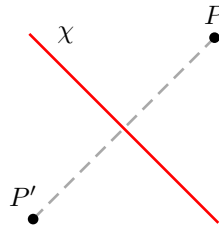


**Figure 3:** Reflection of point $P$ in line $\chi$.

## 1.2  Degenerate cases

The following lemmas are related to degenerate cases of Beloch's fold; i.e., when the given points or lines are equal, or one of the points is already on a line (the relation is explained in Theorem 1). Demonstrations of the lemmas may be found in the literature; e.g., [1, 7, 10, 11]. Nevertheless, they are repeated here for clarity of the analysis.

**Lemma 1.** *Given a point* $P$ *and a line* $m$, *with* $P \notin m$, *a fold line reflects* $P$ *onto* $m$ *iff the line is tangent to a parabola with focus* $P$ *and directrix* $m$.

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*Proof.* Without loss of generality, choose a Cartesian system of coordinates $x, y$ so that $P$ is located at $(0, k)$ and line $m$ is $y = -k$, where $k$ is a constant (Fig. 4). Also, let $P' = \mathcal{F}_\chi(P)$ be located at $(t, -k)$, where $t$ is a free parameter.
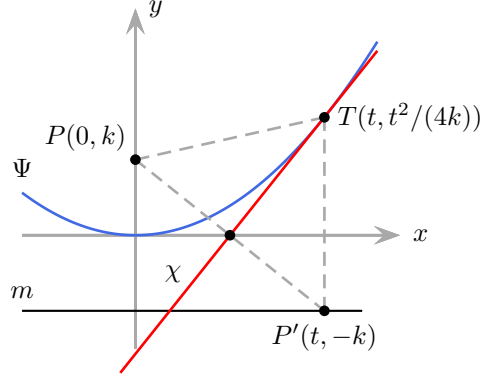


**Figure 4:** Reflection of ~~point~~ ... ~~line, and~~ $\Psi$ is a parabola with focus $P$ and directrix $m$.

The fold line $\chi$ is perpendicular to $\overline{PP'}$ and therefore has a slope of $t/(2k)$. Further, $\chi$ passes through the midpoint of $\overline{PP'}$, located at $(t/2, 0)$. Then, $\chi$ has an equation

$$y = \frac{t}{2k}\left(x - \frac{t}{2}\right). \tag{1}$$

Next, consider point $T$ located at the intersection of $\chi$ with a vertical line through $P'$. Its coordinates are given by Eq. (1) with $x = t$, which produces $(t, t^2/(4k))$. Those coordinates describe parametrically a parabola $\Psi$ with focus at $(0, k)$ and directrix $y = -k$. Further, the slope of a tangent to $\Psi$ at point $T$ is $y'(t) = t/(2k)$, which is the same slope of $\chi$. Therefore, $\chi$ is a line tangent to $\Psi$ at point $T$.

Also, Eq. (1) describes any tangent to $\Psi$ at an arbitrary point $T$; therefore, any tangent is a fold line that reflects $P$ onto $m$. $\qquad\square$

**Lemma 2.** *Given a point $P$ and a line $m$, with $P \in m$, any fold line that passes through $P$ or is perpendicular to $m$ reflects $P$ onto $m$.*

*Proof.* The lemma follows from Definition 1. Any fold line $\chi$ through $P$ reflects $P$ onto itself; therefore, $\mathcal{F}_\chi(P) = P \in m$. Further, any fold line $\xi$ perpendicular to $m$ is a perpendicular bisector of a segment $\overline{PQ}$, where $Q$ is a point in $m$. Therefore, $\mathcal{F}_\xi(P) = Q \in m$. $\qquad\square$

**Lemma 3.** *Given points $P$, $Q$, and a line $m$, with $P \notin m$, a fold line reflects $P$ onto $m$ and passes through $Q$ iff the distance between $Q$ and $P$ is not smaller than the distance between $Q$ and $m$.*

*Proof.* By Lemma 1, any fold line is tangent to a parabola with focus $P$ and directrix $m$. Assume the same parabola $\Psi$ of Fig. 4 and a point $Q$ at the position $(x_q, y_q)$. Replacing the coordinates of $Q$ into Eq. (1) produces the quadratic equation

$$t^2 - 2x_q t + 4y_q = 0. \tag{2}$$

The discriminant of Eq. (2) is $\Delta = 4x_q^2 - 16y_q$, and $\Delta = 0$ yields $y_q = x_q^2/4$, which implies $Q \in \Psi$. Since $\Psi$ is the location of points that are equidistant from $P$ and $m$, we conclude that the

fold operation has a unique solution when Q is equidistant to P and m, two solutions when Q is closer to m (i.e., $\Delta > 0$), and no solution when Q is closer to P (i.e., $\Delta < 0$). Fig. 5 shows an example for the case of two solutions. □
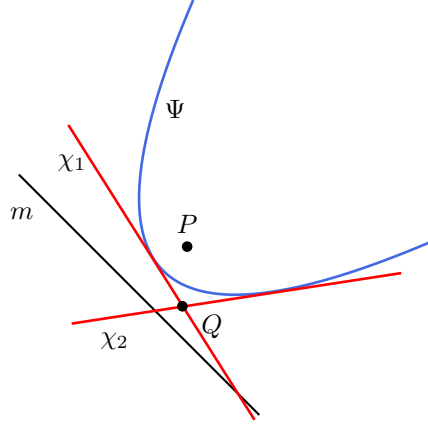


**Figure 5:** The fold lines $\chi_1$ and $\chi_2$ reflect P onto m and pass through Q. Curve $\Psi$ is the parabola with focus P and directrix m.

**Lemma 4.** *Given point P and lines m and n, with $P \notin m$, a fold line perpendicular to n reflects P onto m iff $m \nparallel n$.*

*Proof.* If $P \notin m$, according to Lemma 1 any fold line must be tangent to a parabola with focus P and directrix m. Again, assume the same parabola $\Psi$ of Fig. 4, and a line n with equation $ax + by + c = 0$. The fold line $\chi$ has a slope $t/(2k)$, and two cases are possible:

1. If $m \nparallel n$, then $a \neq 0$. Therefore,
$$\frac{t}{2k} = -\frac{b}{a}$$
   which has a unique solution for t (Fig. 6).

2. If $m \parallel n$ then n is parallel to the directrix of $\Psi$ and cannot be perpendicular to any tangent to the parabola. □

## 1.3 Main case

This case is the regular assumed configuration of given points and lines in Beloch's fold.

**Lemma 5.** *Given points P and Q and lines m and n, with $P \notin m$, $Q \notin n$, and $P \neq Q$ or $m \neq n$, a fold line places P onto m and Q onto n iff the distance between m and n is smaller than the distance between P and Q.*

*Proof.* According to Lemma 1, any fold line must be tangent to both a parabola $\Psi$ with focus P and directrix m, and a parabola $\Theta$ with focus Q and directriz n. Assume the same parabola $\Psi$ of Fig. 4, a point Q is located at $(x_q, y_q)$, and its reflection $Q' = \mathcal{F}_\chi(Q)$ at $(x'_q, y'_q)$. Then, segment $\overline{QQ'}$ has a slope $(y_q - y'_q)/(x_q - x'_q)$.
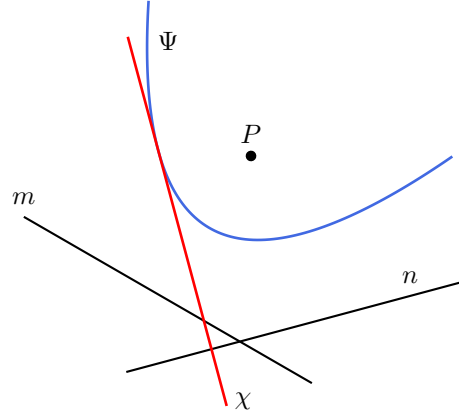
**Figure 6:** The fold line $\chi$ reflects P onto $m$ and is perpendicular to $n$. Curve $\Psi$ is the parabola with focus P and directrix $m$.

The fold line $\chi$, given by Eq. (1), is perpendicular to $\overline{QQ'}$. Therefore,

$$\frac{t}{2k} = -\frac{x_q - x_q'}{y_q - y_q'}. \tag{3}$$

Further, $\chi$ passes through the midpoint of $\overline{QQ'}$, which is located at $((x_q + x_q')/2, (y_q + y_q')/2)$. Replacing those coordinates into Eq. (1) produces

$$2k(y_q + y_q') = t\left(x_q + x_q' - t\right). \tag{4}$$

Finally, eliminating $t$ from Eqs. (3) and (4) produces

$$(y_q + y_q')(y_q - y_q')^2 = -(x_q^2 - x_q'^2)(y_q - y_q') - 2k(x_q - x_q')^2. \tag{5}$$

For a given line $n$, the coordinates of $Q'$ satisfy an equation of the form

$$ax_q' + by_q' + c = 0, \tag{6}$$

where $a$, $b$ and $c$ are constants.

Eqs. (5) and (6) may be solved for $x_q'$ and $y_q'$. Substituting this solution into Eq. (3) gives $t$, which defines the fold line $\chi$ in Eq. (1). Two cases are possible:

1. If $m \parallel n$, then $Q'$ is on a horizontal line and so $y_q' = -c/b$. In that case, Eq. (5) is quadratic in $x_q'$ and may have zero to two solutions.

2. If $m \nparallel n$, solving Eq. (6) for $x_q'$ or $y_q'$ and replacing in Eq. (5) produces a cubic equation with one to three solutions. An example for the latter case is shown in Fig. 7.

Thus, the operation may not have a solution only in the case of $m \parallel n$. Re-arranging Eq. (5) produces

$$(2k - y_q + y_q')(x_q - x_q')^2 + 2x_q(y_q - y_q')(x_q - x_q') + (y_q + y_q')(y_q - y_q')^2 = 0, \tag{7}$$

which is a quadratic equation in $(x_q - x_q')$. The discriminant is

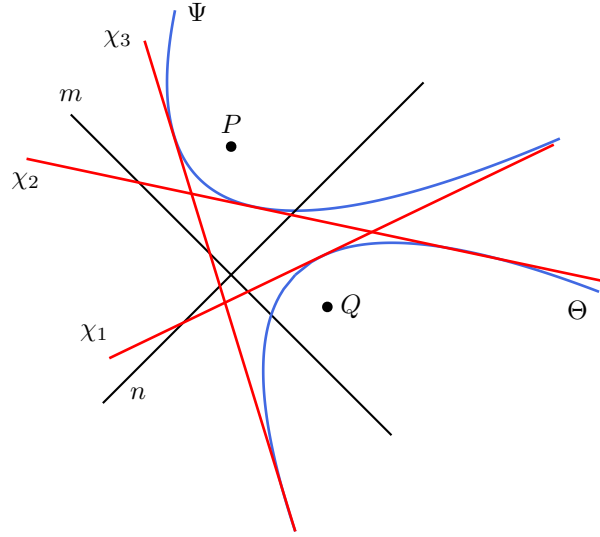$$\Delta = 4x_q^2(y_q - y_q')^2 - 4(2k - y_q + y_q')(y_q + y_q')(y_q - y_q')^2, \tag{8}$$

**Figure 7:** The fold lines $\chi_1$, $\chi_2$ and $\chi_3$ reflect P and Q onto m and n, respectively. Curves $\Psi$ and $\Theta$ are the parabolas with focus P, directrix m, and focus Q, directriz n, respectively.

and letting $\Delta = 0$ produces

$$x_q^2 + (y_q - k)^2 = (y_q' + k)^2. \tag{9}$$

The left side of Eq. (9) is the squared distance between P and Q, and the right side is the squared distance between m and n. Therefore, the operation has a solution iff the former is not smaller that the latter, which results in $\Delta \geqslant 0$. That condition also includes the case of $m \nparallel n$, in which the distance between m and n is zero. □

## 2 Main theorem

**Theorem 1.** *Given points* P, Q, *and lines* m, n, *a fold line that places* P *on* m *and* Q *on* n *exists iff the distance between* P *and* Q *is not smaller than the distance between* m *and* n.

*Proof.* Let us consider exhaustively all possible cases. To simplify the explanation, the distance between objects $\alpha$ and $\beta$ (points or lines) is denoted as $d(\alpha, \beta)$.

1. $P \notin m$ and $Q \notin n$:

   (a) $P \neq Q$ or $m \neq n$: This is the main case treated by Lemma 5. A solution exists iff $d(P, Q) \geqslant d(m, n)$.

   (b) $P = Q$ and $m = n$: This case reduces to reflecting point P onto line m, treated by Lemma 1, which always has a solution. Note that $d(P, Q) = d(m, n) = 0$.

2. $P \notin m$ and $Q \in n$: By combining Lemmas 1 and 2, this case reduces to reflecting P onto m with a fold line through Q or perpendicular to n.

   (a) $P \neq Q$ or $m \neq n$:

      i. $m \nparallel n$: By Lemma 4, a solution perpendicular to n always exists. Since $d(m, n) = 0$, then $d(P, Q) \geqslant d(m, n)$.

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    ii. $m \parallel n$ (including the case $m = n$): By Lemma 4, there is no solution perpendicular to $n$. However, by Lemma 3 there is a solution that reflects $P$ onto $m$ and passes through $Q$ iff $d(P, Q) \geqslant d(m, Q) = d(m, n)$.

   (b) $P = Q$ and $m = n$: The conditions of this case form a contradiction and therefore it is impossible.

3. $P \in m$ and $Q \notin n$: This case is similar to Case 2, with the roles of $P$ and $Q$, and $m$ and $n$ exchanged.

4. $P \in m$ and $Q \in n$: By Lemma 2, a fold line passing through both $P$ and $Q$ reflects $P$ onto $m$ and $Q$ onto $n$. Such a line always exists and therefore this case always has a solution. If $m \parallel n$ (including $m = n$), then $d(P, Q) = d(m, n)$. If If $m \nparallel n$ then $d(P, Q) \geqslant d(m,n) = 0$.

As detailed above, the fold has a solution in all cases that $d(P, Q) \geqslant d(m,n)$, and it does not have a solution when that condition is not satisfied. $\qquad\square$

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**Jorge C. Lucero** (MR Author ID 630437) graduated from the National University of Córdoba, Argentina, in 1987, and received a PhD in Engineering from Shizuoka University, Japan, in 1993. Currently, he is Professor of computer science at the University of Brasília, Brazil, specialist in applied and computational mathematics. In his spare time, he enjoys folding origami.