Folding a 3D Euclidean Space

J. C. Lucero

Abstract: *This paper considers an extension of origami geometry to the case of "folding" a three dimensional (3D) space along a plane. First, all possible incidence constraints between given points, lines and planes are analyzed by using the geometry of reflections. Next, a set of 3D elementary fold operations is defined, which satisfy specific combinations of constraints with a finite number of solutions. The set consists of 47 valid fold operations, and solutions to some of them are explored to determine their number and conditions of existence.*

1   Introduction

Origami is the traditional Japanese art of creating figures by folding a sheet of paper. In modern days, the term has been extended to denote constructions techniques based on folding operations, and a number of applications to science and technology have been proposed, e.g., in aerospace and automotive technology [Cipra 01], civil engineering [Filipov et al. 15], biology [Mahadevan and Rica 05], robotics [Felton et al. 14], and other areas. Its geometry has been studied through the so-called "axioms" of origami, which are elementary fold operations that satisfy specific incidence constraints between given points and lines on a plane. The axioms were introduced over three decades ago [Justin 86] and, since then, have been expressed under a variety of forms [Alperin 00, Alperin and Lang 06, Ghourabi et al. 13]. In their application, the folded medium (i.e., the "sheet") is usually regarded as a 2D Euclidean plane, and extensions to the sphere and the hyperbolic plane have also been considered [Kawasaki 11, Alperin 11].

What if, instead of folding a 2D medium, folds in higher dimensions are considered? This question has been considered from a topological point of view [Robertson 78]; here, an axiomatic approach is adopted. In regular 2D origami, folding operations take place within a 3D space. If we increase one dimension, then we have a 3D medium folded within a 4D (or higher) hyperspace. Further, instead of folding along a line, the 3D folds take place along a plane. In this article, the 3D case is explored by extending the folding axioms to an Euclidean space and considering incidence constraints between points, lines and planes. Following a previous study [Lucero 17], the analysis is based on the geometry of reflections.
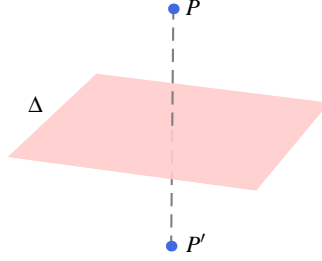
Figure 1: *Reflection of point P in plane Δ.*

2 Reflections in 3D

Points are denoted by capital letters (*P*, *Q* etc.), lines by small Latin letters (*m*, *n* etc.), and planes by small Greek letters (*π*, *τ* etc.) except the fold plane which is denoted by the special symbol Δ. The notations $P \in m$ and $P \in \pi$ mean that point *P* is on line *m* and plane *π*, respectively. Further, $m \subset \pi$ means that line *m* is contained in plane *π*.

2.1 Reflection of a point

Following the definition of reflection in a line [Martin 98], reflection in a plane is defined as (see Fig. 1):

Definition 1. Given a plane Δ, the reflection $\mathscr{F}_\Delta$ in Δ is the mapping on the set of points in the 3D space such that for point *P*

$$\mathscr{F}_\Delta(P) = \begin{cases} P & \text{if } P \in \Delta, \\ P' & \text{if } P \notin \Delta \text{ and } \Delta \text{ is the perpendicular bisector plane of segment } \overline{PP'}, \end{cases}$$

It is easy to see that $\mathscr{F}_\Delta(P) = P'$ iff $\mathscr{F}_\Delta(P') = P$.
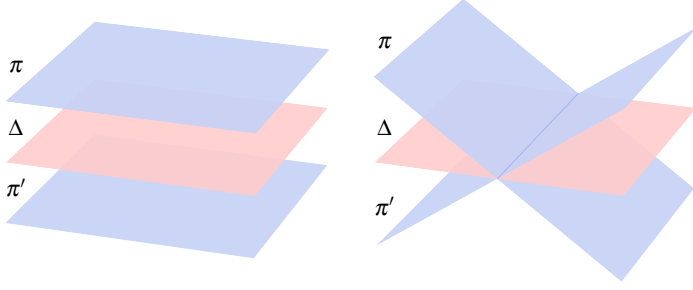
2.2 Reflection of a line

Reflection of a line *m* in Δ is obtained by reflecting every point in *m*. Therefore, $\mathscr{F}_\Delta(m) = \{\mathscr{F}_\Delta(P) | P \in m\}$.

Let $m' = \mathscr{F}_\Delta(m)$. Then, lines *m* and *m'* are coplanar, and the plane defined by them is perpendicular to Δ. Consider the following cases:

1. If *m* and Δ are parallel ($m \parallel \Delta$), then $m \parallel m'$ (Fig. 2, left).
2. If *m* and Δ are not parallel ($m \nparallel \Delta$) then the intersection of Δ with the plane spanned by *m* and *m'* is a bisector of the angle between *m* and *m'* (Fig. 2, right).
3. If $m \subset \Delta$, then every point $P \in m$ is its own reflection (i.e., $\mathscr{F}_\Delta(P) = P$), and therefore $m' = m$.
4. If *m* and Δ are perpendicular ($m \perp \Delta$), then the reflection of every point of *m* is also on *m*; therefore, $m = m'$. Also, Δ divides *m* into two half-lines (rays), and each half-line is reflected onto the other.
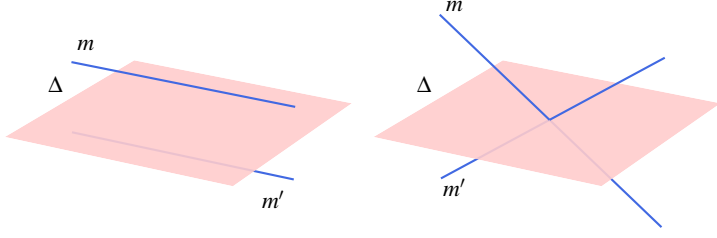
Figure 2: *Reflection of line m in plane Δ. Left: m ∥ Δ. Right: m ∦ Δ.*

Figure 3: *Reflection of plane π in plane Δ. Left: π ∥ Δ. Right: π ∦ Δ.*

2.3 Reflection of a plane

Reflection of a plane π in Δ is similarly defined by $\mathscr{F}_\Delta(\pi) = \{\mathscr{F}_\Delta(P) | P \in \pi\}$.
   Let $\pi' = \mathscr{F}_\Delta(\pi)$. Then:

1. If π and Δ are parallel, then $\pi \parallel \pi'$ (Fig. 3, left).
2. If π and Δ are not parallel then Δ is a bisector of the dihedral angle between π and π' (Fig. 3, right).
3. If $\pi = \Delta$, then every point $P \in \pi$ is its own reflection, and therefore $\pi' = \pi$.
4. If π and Δ are perpendicular, then the reflection of every point of π is also on π; therefore, $\pi = \pi'$. Also, Δ divides π into two half-planes, and each half-plane is reflected onto the other.

3 Incidence constraints

Each 3D constraint involves an object α and the image $\mathscr{F}_\Delta(\beta)$ of an object β by reflection in the fold plane Δ. The symmetry of the reflection mapping implies that all incidence relations are symmetric.

   A total of 12 different incidence constraints may be defined, as follows. For convenience, incidences involving distinct objects (i.e., $\alpha \neq \beta$) are defined separately from those involving the same object (i.e., $\alpha = \beta$).

3.1 Incidence I_1: $\mathscr{F}_\Delta(P) = Q$, with $P \neq Q$

In this incidence, the reflection of a given point P coincides with another given point Q. Its solution is the unique fold plane Δ which is the perpendicular bisector of segment $\overline{PQ}$ (Fig. 1, with $Q = P'$).

3.2 Incidence I_2: $\mathscr{F}_\Delta(m) = n$, with $m \neq n$

In this incidence, the reflection of a given line m coincides with another given line n. The following cases are possible:

1. If m and n are coplanar and $m \nparallel n$, there are two possible fold planes. The fold planes are perpendicular between them and their intersections with the plane spanned by m and n are bisectors of the angles between those lines (Fig. 2 , right, with $n = m'$, shows one of the possible fold planes).
2. If $m \parallel n$, there is only one solution, which is a fold plane perpendicular to the plane spanned by m and n and equidistant to them (Fig. 2, left, with $n = m'$).
3. If m and n are not coplanar, then the incidence does not have a solution.

3.3 Incidence I_3: $\mathscr{F}_\Delta(m) \cap n \neq \emptyset$, with $m \cap n = \emptyset$

In this incidence, the reflection of a given line m intersects another line n, and the case in which m and n already intersect is excluded.

Assume first that m and n are not coplanar and consider a Cartesian system of coordinates x, y, z such that m lies in the yz coordinate plane and passes through point $R(0,0,1)$, and n lies in the $z = -1$ plane and passes through point $Q(0,0,-1)$. Let $P(0,t,1)$ be an arbitrary point in m, where t is a parameter, and its reflection in n be $P'(s\cos\delta, s\sin\delta, -1)$, where $\delta \neq \pi/2$ is the angle with the x-axis, and s is a parameter.

A fold plane Δ must be normal to segment $\overline{PP'}$ and pass through its midpoint. Then, Δ has an equation

$$sx\cos\delta + y(s\sin\delta - t) - 2z = \frac{s^2 - t^2}{2}. \tag{1}$$

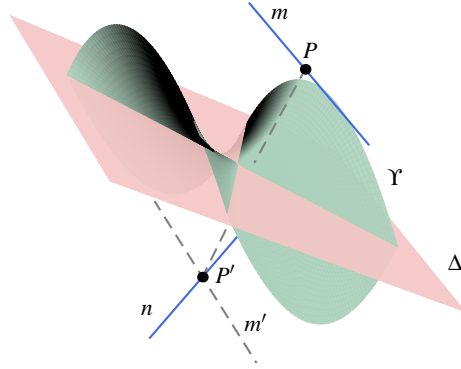This equation defines a family of fold planes in two parameters (t and s). Its envelope is the quadric surface

$$x^2\cos^2\delta + 2xy\sin\delta\cos\delta - y^2\cos^2\delta - 4z = 0. \tag{2}$$

Eq. (2) may be put into normal form by rotating the coordinate system around the z-axis so as to eliminate the term in xy. A rotation by an angle $\delta/2$ produces

$$u^2\cos\delta - v^2\sin\delta - 4z = 0, \tag{3}$$

where

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \cos\delta/2 & \sin\delta/2 \\ -\sin\delta/2 & \cos\delta/2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}. \tag{4}$$

Figure 4: *Example of fold plane for incidence I_3 in the case that m are not coplanar. Plane Δ reflects m onto m', which intersects n at P', and is tangent to the hyperbolic paraboloid Υ.*

Eq. (3) describes an hyperbolic paraboloid that opens up along the u axis and down along the v axis. An example is shown in Fig. 4.

If m and n are coplanar and do not intersect, then they are parallel. Letting $\delta = \pi/2$ in Eq. (1) produces

$$y(t-s) + 2z = \frac{t^2 - s^2}{2}. \tag{5}$$

This equation describes any plane parallel to the x-axis that intersects the y-axis (including the case that the fold plane is the xy plane itself) and is not perpendicular to it.

If m and n already intersect at a point P and $m \neq n$, then there are two families of possible fold planes:

1. Any plane Δ that passes through P reflects P onto itself, and therefore m' and n intersect at the same point. This case may be included within incidence I_7 (Section 3.7).

2. Any plane Δ that is perpendicular to the plane spanned by m and n and either Δ contains the angle bisector of m and n, denoted as ℓ, or Δ does not contain ℓ and is not parallel to it. This case may be included within incidence I_{10} (Section 3.10), which is solved by any fold plane perpendicular to a given plane π. Plane π is the plane spanned by m and n, and the case in which the fold plane is parallel to ℓ must be excluded.

If $m = n$ then any fold plane solves the incidence, with the exception of planes parallel to m and not containing it. This case does not constitute a valid constraint (the family of fold planes still has three degrees of freedom; see Section 4.1) and may be disregarded.

3.4 Incidence I_4: $\mathscr{F}_\Delta(\pi) = \tau$, with $\pi \neq \tau$

In this incidence, the reflection of a given plane π coincides with another given plane τ. Two cases are possible:

1. If $\pi \nparallel \tau$, there are two possible fold planes. The fold planes are perpendicular and they bisect the dihedral angles between π and τ (Fig. 3, right, with $\tau = \pi'$, shows one of the possible fold planes).

2. If $\pi \parallel \tau$, there is only one solution, which is a fold plane parallel and equidistant to both π and τ (Fig. 3, left, with $\tau = \pi'$).

3.5 Incidence I_5: $\mathscr{F}_\Delta(P) \in m$, with $P \notin m$

In this incidence, the reflection of a given point P is on a given line m, and the case in which P is already on m is excluded.

Choose a Cartesian system of coordinates x, y, z so that point P is at $(0,0,1)$, and line m passes through point $(0,0,-1)$ and is parallel to the y-axis. Let $P' = \mathscr{F}_\Delta(P) \in m$ be at $(0,t,-1)$, where t is a free parameter. Computing the fold plane Δ as in the previous incidence produces

$$2ty - 4z = t^2, \tag{6}$$

which describes a one-parameter family of planes perpendicular to the yz coordinate plane. Its envelope is

$$z = \frac{y^2}{4}. \tag{7}$$

Eq. (7) describes a parabolic cylinder generated by a parabola in the yz plane, with focus at P and directrix m, when translated in direction parallel to the x axis. An example is shown in Fig. 5.
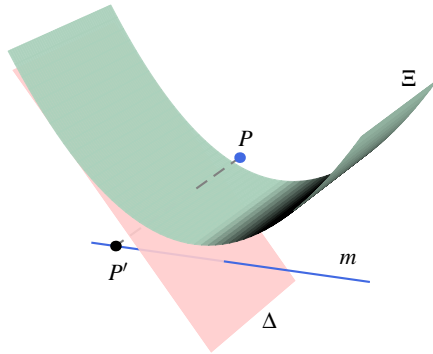
Figure 5: *Example of fold plane for incidence I_5. Plane Δ reflects P onto $P' \in m$ and is tangent to the parabolic cylinder Ξ.*
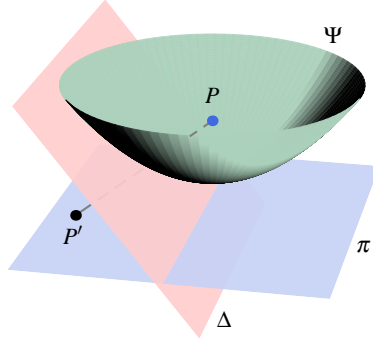
Figure 6: *Example of fold plane for incidence I_6. Plane Δ reflects P onto $P' \in \pi$ and is tangent to the paraboloid Ψ.*

If $P \in m$, then any plane containing P or perpendicular to m reflects P onto m. Those two cases are contemplated by incidences I_8 (Section 3.8) and I_9 (Section 3.9), respectively.

3.6 Incidence I_6: $\mathscr{F}_\Delta(P) \in \pi$, with $P \notin \pi$

In this incidence, the reflection of a given point P is on a given plane π, and the case in which P is already on π is excluded.

The analysis is similar to the previous incidence. In the same Cartesian system, let plane π be described by $z = -1$, and $P' = \mathscr{F}_\Delta(P) \in \pi$ be located at $(s, t, -1)$, where s and t are free parameters. An equation for Δ is

$$2sx + 2ty - 4z = s^2 + t^2, \tag{8}$$

which describes a family of planes with two parameters (s and t). Its envelope is

$$x^2 + y^2 - 4z = 0. \tag{9}$$

Eq. (9) describes a paraboloid generated by rotation around the z axis of a parabola with vertex at $(0,0,0)$ and axis coincident with the z axis. The parabola has focus at P and its directrix is any line in π that intersects the z axis. An example is shown in Fig. 6.

If $P \in \pi$, then any plane containing P or perpendicular to π reflects P onto π. Those two cases are contemplated by incidences I_8 (Section 3.8) and I_{11} (Section 3.11), respectively.

3.7 Incidence I_7: $\mathscr{F}_\Delta(m) \subset \pi$, with $m \not\subset \pi$

In this incidence, the reflection of a given line m is on a given plane π, and the case in which m is already on π is excluded.

Assume first $m \nparallel \pi$, and choose a Cartesian system of coordinates x, y, z so that the point of intersection of m and π, denoted by O, is located at $(0, 0, 0)$, plane π is the xy coordinate plane, and m is a line in yz plane forming an angle $\theta \neq \pi/2$ with the z-axis. An arbitrary point $P \in m$ is located at $(0, \sin\theta, \cos\theta)t$, where t is a parameter. Its reflection $P' = \mathscr{F}_\Delta(P)$ is in π at $(\cos\delta, \sin\delta, 0)t$, where δ is the angle with the x-axis.

An equation for Δ is

$$x\cos\delta + y(\sin\delta - \sin\theta) - z\cos\theta = 0, \tag{10}$$

which describes a family of fold planes in one parameter (δ). Its envelope is

$$x^2 + y^2\cos^2\theta - 2yz\sin\theta\cos\theta - z^2\cos^2\theta = 0. \tag{11}$$

Eq. (11) describes a quadric surface, and the equation may be put into normal form by rotating the coordinate system around the x-axis so as to eliminate the term in yz. A rotation by an angle $-\theta/2$ produces

$$\frac{x^2}{\cos\theta} + u^2 - v^2 = 0, \tag{12}$$

where

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \cos\theta/2 & -\sin\theta/2 \\ \sin\theta/2 & \cos\theta/2 \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix}. \tag{13}$$

Eq. (12) describes an elliptical cone that opens along the v-axis. An example is shown in Fig. 7.
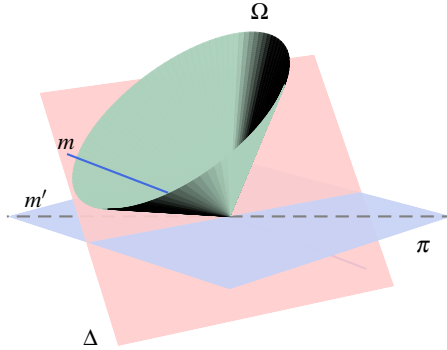
Figure 7: *Example of fold plane for incidence I_7 in the case of $m \nparallel \pi$. Plane Δ reflects m onto $m' \in \pi$ and is tangent to the elliptical cone Ω. For clarity of the figure, only the upper nappe of Ω is shown.*

Next, consider the case $m \parallel \pi$. Let π be the plane $z = -1$, and m be a line in the yz coordinate plane passing through point R at $(0, 0, 1)$ and parallel to the y-axis. The reflection of m is $m' = \mathscr{F}_\Delta(m) \subset \pi$, with $m' \parallel m$, and passes through
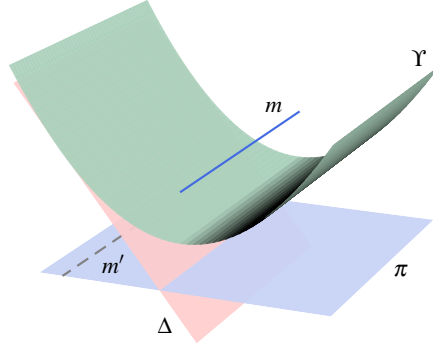
Figure 8: *Example of fold plane for incidence I_7 in the case of $m \parallel \pi$. Plane Δ reflects m onto $m' \in \pi$ and is tangent to the parabolic cylinder Υ.*

$R' = \mathscr{F}_\Delta(R)$ at $(k, 0, -1)$, where k is a parameter. An arbitrary point $P \in m$ is located at $(0, t, 1)$, where t is a parameter, and its reflection $P' = \mathscr{F}(P)$ is at $(k, t, -1)$.

An equation for Δ is

$$-2kx + 4z = -k^2, \tag{14}$$

which describes a family of fold planes in parameter k. Its envelope is

$$x^2 - 4z = 0. \tag{15}$$

Eq. (15) describes a parabolic cylinder generated by a parabola in the xz plane when translated in direction parallel to the y-axis. The parabola has focus in R and its directrix is a line parallel to the x-axis and passing through the point $(0, 0, -1)$. An example is shown in Fig. 8.

If $m \subset \pi$, then any plane containing m or perpendicular to π reflects m onto π. Those two cases are contemplated by incidences I10 (Section 3.10) and I11 (Section 3.11), respectively.

3.8   Incidence I_8: $\mathscr{F}_\Delta(P) = P$

In this incidence, the reflection of a given point P coincides with itself. It is satisfied by any fold plane Δ passing through P. An arbitrary normal direction for Δ may be defined, e.g., by the azimuthal angle θ in the xy and the polar angle ϕ from the z-axis. Therefore, the solution to the incidence is a family of fold planes with two free parameters, namely, θ and ϕ.

3.9   Incidence I_9: $\mathscr{F}_\Delta(m) = m$, and $\exists P \in m, \mathscr{F}_\Delta(P) \neq P$

Both this and the next incidence consider the reflection of a line m to itself. In the current incidence, half of m, defined from an arbitrary point $R \in m$, is reflected upon the opposite half.

The position of point R may be specified by its distance s from a particular point $P_0 \in m$. A fold plane is perpendicular to m and passes through point R. Therefore, the solution to the incidence is a family of fold planes with one free parameter (s).

3.10 Incidence I_{10}: $\mathscr{F}_\Delta(m) = m$, and $\forall P \in m, \mathscr{F}_\Delta(P) = P$

This is the second case of reflection of line m to itself. In this incidence, each point $P \in m$ is reflected to itself, and therefore the incidence is satisfied by a fold plane containing m. There is an infinite number of such planes, an all of them have m as intersection. Each plane may be specified by the dihedral angle φ from a reference plane; therefore, the solution to the incidence is a family of fold planes with one free parameter (φ).

3.11 Incidence I_{11}: $\mathscr{F}_\Delta(\pi) = \pi$, and $\exists P \in \pi, \mathscr{F}_\Delta(P) \neq P$

Both this and the next incidence consider the reflection of a plane π to itself. In the current case, half of π, defined from an arbitrary line $m \subset \pi$, is reflected upon the opposite half. Any fold plane perpendicular to π satisfies the incidence, and it may be specified by an equation of the form $\mathbf{n} \cdot \mathbf{x} = k$, where $\mathbf{n}$ is a normal vector and k a parameter. Vector $\mathbf{n}$ is parallel to π, and an arbitrary direction may be specified by its angle δ from a reference direction parallel to π. Therefore, the solution to the incidence is a family of fold planes with two free parameter, namely, k and δ.

3.12 Incidence I_{12}: $\mathscr{F}_\Delta(\pi) = \pi$, and $\forall P \in \pi, \mathscr{F}_\Delta(P) = P$

This is the second case of reflection of plane π to itself. In this case, each point $P \in \pi$ is reflected to itself, and therefore the fold plane is $\Delta = \pi$

4 Elementary fold operations

4.1 Definition

A plane in 3D space is an object with three degrees of freedom. When an incidence constraint is set for the fold plane Δ, satisfying the constraint consumes a number of degrees of freedom, and that number is called the codimension of the constraint. Incidences I_1, I_2, I_4 and I_{12} have either a unique or a finite number of solutions; therefore, they define constraints of codimension 3. Incidences I_5, I_7, I_9 and I_{10} have a family of solutions with one free parameter and therefore they define constraints of codimension 2. Finally, incidences I_3, I_6, I_8 and I_{11} have a family of solutions with two free parameter and therefore they define constraints of codimension 1.

An elementary fold operation may be defined as the resolution of a minimal set of incidence constraints with a finite number of solutions [Alperin and Lang 06]. Let us recall that the total codimension of the combination is *at most* the sum of the individual codimensions. Therefore, that sum must not be smaller than 3.

Each of the incidences I_1, I_2, I_4 and I_{12} already define an elementary operation. Here, it may be argued that incidence I_{12} should be disregarded because it does

not create a new plane. However, completeness of the set of possible operations demands its inclusion. The other constraints must be applied in combinations, as follows:

1. Two constraints of respective codimensions 1 and 2. There are four constraints of each type, and so the total number of combinations is $4 \times 4 = 16$.
2. Two constraints both of codimension 2. The number N of possible combinations when selecting r objects from a set of n objects and allowing repetitions is given by [Rosen 12]

$$N = \binom{n+r-1}{r} = \frac{(n+r-1)!}{r!(n-1)!}. \tag{16}$$

   In the present case, two constraints ($r = 2$) are selected from a total of four ($n = 4$), which results in 10 combinations.
3. Three constraints of codimension 1. Again, using Eq. (16) with $n = 4$ and $r = 3$ produces a number of 20 combinations.

However, incidence I_{11} may not be applied three times. If it is, then the fold plane Δ must be perpendicular to three given planes. If two of them are parallel, then the associated constraints are mutually redundant and therefore the total codimension is less than 3, with an infinite number of solutions. If all given planes are mutually nonparallel, then a fold plane does not exist (in an Euclidean space). Hence, this combination does not define a valid folding operation. For similar reasons, incidence I_{11} may not be combined with incidence I_9, and also incidence I_9 may not be used twice.

The remaining combinations may not necessarily result in distinct operations. Thus, a total of at most 47 valid operations may be defined, and some of them are discussed in the next subsections. A notation in the format $a_1 I_{b_1} + \cdots + a_k I_{b_k}$ is used, meaning that incidences $I_{b_1}, \ldots, I_{b_k}$ are combined, and each one is used $a_1, \ldots, a_k$ times, respectively.

4.2 Operation $I_5 + I_6$

This operation combines one constraint of codimension 2 (I_5) and one of codimension 1 (I_6), and it may be stated as: *Given a line m, a plane π, a point P not on m, and a point Q not on π, fold along a plane to place P onto m and Q onto π.* Its solution is a fold plane tangent to the parabolic cylinder generated by P and m (Fig. 5) and the paraboloid generated by Q and π (Fig. 6, with Q in place of P).

Assume P and m as in Section 3.5, point Q at (x_q, y_q, z_q) and Q' at (x_q', y_q', z_q'). The fold plane Δ is given by Eq. (6) and it must be normal to segment $\overline{QQ'}$ and pass through its midpoint. Those conditions produce $x_q = x_q'$ and

$$2(y_q - y_q')^2 = -(y_q^2 - y_q'^2)(z_q - z_q') - (z_q^2 - z_q'^2)(z_q - z_q'). \tag{17}$$

For a given plane π described by $ax + by + cz + d = 0$, where a, b, c and d are constants, the coordinates of Q' satisfy (using $x_q = x_q'$)

$$ax_q + by_q' + cz_q' + d = 0. \tag{18}$$

Eqs. (17) and (18) may be solved for y'_q and z'_q and three cases are possible:

1. If $P = Q$ then $y_q = 0$, $z_q = 1$. Further, if π is the plane $z = -1$ (i.e., π contains m and is perpendicular to the plane spanned by P and m), then $z'_q = -1$. In this case, Eq. (17) is satisfied by any value of y'_q and therefore the operation has an infinite number of solutions.

2. If π is a plane parallel to the xy coordinate plane, then $a = b = 0$, $c \neq 0$, and $z'_q = -d/c$ (constant). At the same time, if conditions of item 1 above are not satisfied ($z'_q \neq -1$ or $P \neq Q$), then Eq. (17) is quadratic in y'_q and may have none to two solutions.

3. In any other case, Eq. (17) results in a cubic for y'_q or z'_q and it may have one to three solutions.

We conclude that the operation is well defined iff $P \neq Q$ or $m \not\subset \pi$ or π is not perpendicular to the plane spanned by P and m, and it may have none to three solutions. An example is shown in Fig. 9.

4.3   Operation $I_5 + I_9$

This operation combines two different constraints of codimension 2, and it may be stated as: *Given lines m and n, and a point P not on m, fold along a plane to place P onto m and to reflect half of n onto its other half.* Its solution is a fold plane tangent to the parabolic cylinder generated by P and m (Fig. 5) and normal to n.

The operation has a solution iff line n is parallel to the plane spanned by P and m, and n is not normal to m, and that solution is unique. An example is shown in Fig. 10.

4.4   Operation $I_6 + I_8 + I_{11}$

This operation combines three different constraints of codimension 1, and it may be stated as: *Given planes π and τ, a point P not on π and a point Q, fold along a plane passing through Q to place P onto π and to reflect half of τ onto its other half.* Its solution is a fold plane passing through Q, tangent to the paraboloid generated by P and π (Fig. 6), and normal to τ.

Assume P and π as in Section 3.6, point Q at (x_q, y_q, z_q) and τ described by $ax + by + cz + d = 0$. The fold plane Δ is given by Eq. (8) and it must contain Q, therefore

$$2sx_q + 2ty_q - 4z_q = s^2 + t^2. \tag{19}$$

The normal vectors of Δ, $(2s, 2t, -4)$, and τ, (a, b, c), must be mutually normal and so they satisfy

$$2sa + 2tb - 4c = 0. \tag{20}$$

If $a \neq 0$ or $b \neq 0$ then the Eq. (20) may be solved for parameter s or t, respectively, and substituting in Eq. (19) produces a quadratic equation on the other parameter. Thus, the operation may have none to two solutions. If $a = 0$ and $b = 0$ (i.e., plane τ is parallel to π) then $c \neq 0$ and Eq. (20) is not satisfied. An example is shown in Fig. 11.
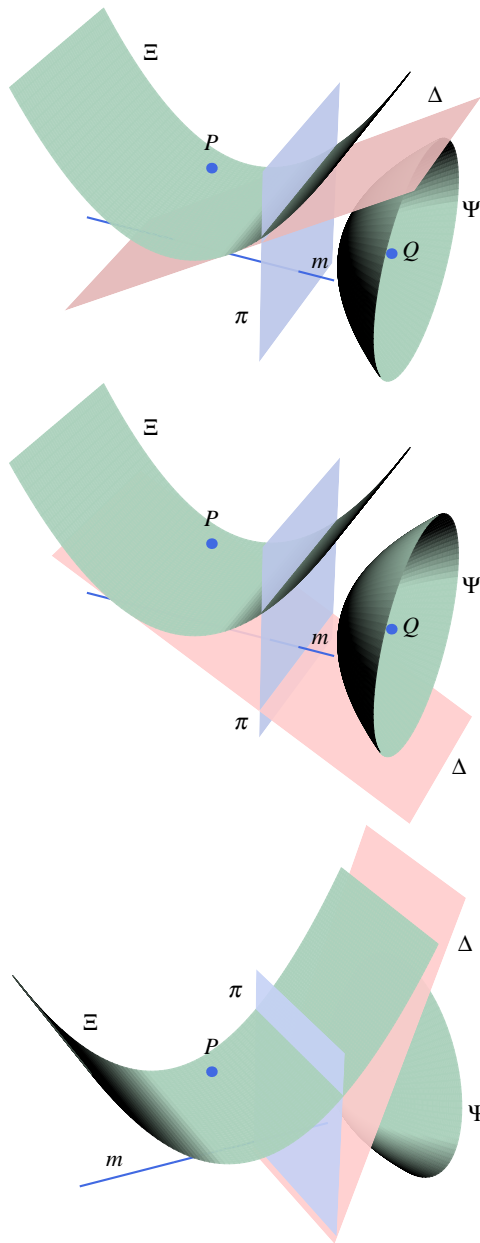
Figure 9: *Example of operation $I_5 + I_6$ with three solutions (the bottom figure has been rotated to improve clarity). The fold plane Δ is tangent to the parabolic cylinder Ξ, with focus P and directrix line m, and the paraboloid Ψ, with focus Q and directrix plane π.*
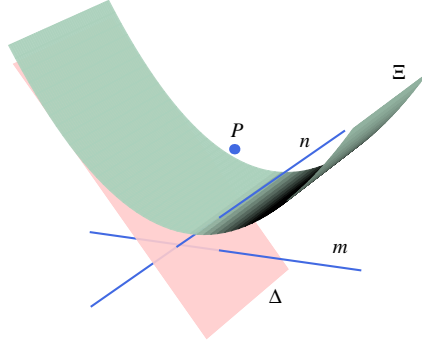
Figure 10: *Example of operation $I_5 + I_9$. The fold plane Δ is tangent to the parabolic cylinder Ξ, with focus P and directrix line m, and normal to line n.*
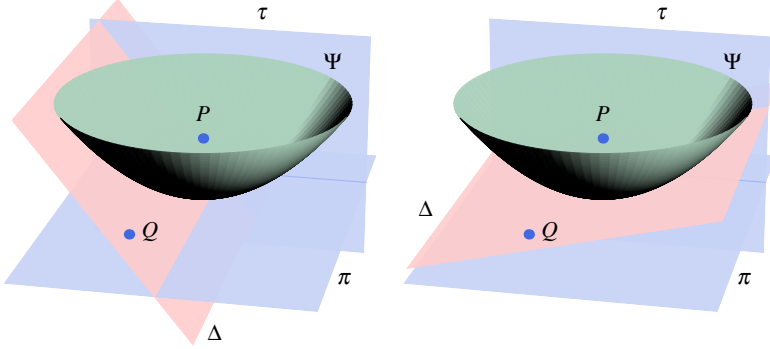
Figure 11: *Example of operation $I_6 + I_8 + I_{11}$ with two solutions. The fold plane Δ is tangent to paraboloid Ψ, with focus P and directrix plane π, contains point Q and is normal to plane τ.*

4.5 Operation $3I_6$

This operation also combines three constraints of codimension 1, and it may be stated as: *Given planes π, τ, and ρ, a point P not on π, a point Q not on τ, and a point R not on ρ, fold along a plane to place P onto π, Q onto τ, and R onto ρ.* Its solution is a fold plane tangent to the three paraboloids defined by each of the incidences.

Assume P and π as in Section 3.6, point Q at (x_q, y_q, z_q) and Q' at (x'_q, y'_q, z'_q). The fold plane Δ is given by Eq. (8); it must be normal to segment $\overline{QQ'}$ and pass through its midpoint. Those conditions produce

$$2(y_q - y'_q)^2 + 2(x_q - x'_q)^2 = \\ -(x_q^2 - x'^2_q)(z_q - z'_q) - (y_q^2 - y'^2_q)(z_q - z'_q) - (z_q^2 - z'^2_q)(z_q - z'_q). \quad (21)$$

Similarly, letting point R be at (x_r, y_r, z_r) and R' at (x'_r, y'_r, z'_r) produces

$$2(y_r - y'_r)^2 + 2(x_r - x'_r)^2 =$$
$$- (x_r^2 - x'^2_r)(z_r - z'_r) - (y_r^2 - y'^2_r)(z_r - z'_r) - (z_r^2 - z'^2_r)(z_r - z'_r). \quad (22)$$

Segments $\overline{QQ'}$ and $\overline{RR'}$ are parallel (they are both normal to Δ), therefore,

$$(x'_q - x_q, y'_q - y_q, z'_q - z_q) = \ell(x'_r - x_r, y'_r - y_r, z'_r - z_r). \quad (23)$$

The coordinates of Q' and R' must also satisfy the equations for planes τ and ρ, of the form $a_\tau x + b_\tau y + c_\tau z + d_\tau = 0$ and $a_\rho x + b_\rho y + c_\rho z + d_\rho = 0$, respectively, where a_τ, b_τ, c_τ, d_τ, a_ρ, b_ρ, c_ρ, and d_ρ are constants.

The above equations form a system in ℓ and the coordinates of Q' and R'. An analysis using Gröbner basis computations [Sturmfels 02] in a Computer Algebra System reveals nine solutions at most. However, it may be shown that two of the solutions are always complex, which leaves a maximum of seven possible solutions for the fold operation (calculations are lengthy and are therefore omitted here).

5 Discussion

A 3D extension of origami may be defined based on a set of at most 47 elementary folding operations. For comparison, let us recall that 2D origami is based on seven elementary operations [Alperin and Lang 06]. 3D folding takes place on a plane, and each operation satisfies a set of incidence constraints between given points, lines and planes with a finite number of solutions. Some of the operations have been analyzed, and it has been shown that one of them may have up to seven solutions. Again, for comparison, let us recall that the maximum number of solutions to 2D operations is 3. As a next step, a complete analysis of all 3D operations to determine conditions for solutions and their number would be desirable.

This analysis may found application to studies of Universe structure in cosmology. The concept of 3D origami tessellations has been used to model the large-scale distribution of matter in filaments, clusters and voids [Neyrinck 14]. In that model, a 3D "sheet" of dark matter folds within a 6D Eulerian space (three spacial dimensions plus three velocity dimensions). Superposed regions of the folded 3D "sheet" form walls, streams and nodes, which approximates the observed distribution of galaxies. The present results might add to the mathematical formalization of such model and facilitate computer simulations.

References

[Alperin and Lang 06] Roger C. Alperin and Robert J. Lang. "One-, two-, and multi-fold origami axioms." In *Origami 4 - Fourth International Meeting of Origami Science, Mathematics and Education*, edited by Robert J. Lang, pp. 371–393. A. K. Peters, 2006.

[Alperin 00] Roger C. Alperin. "A mathematical theory of origami constructions and numbers." *New York Journal of Mathematics* 6 (2000), 119–133.

[Alperin 11]   Roger C. Alperin.   "Origami alignments and constructions in the hyperbolic plane."   In *Origami 5 - Fifth International Meeting of Origami Science, Mathematics, and Education*, edited by Patsy Wang-Iverson, Robert J. Lang, and Mark Yim, pp. 553–573. CRC Press, 2011.

[Cipra 01]   Barry Cipra. "In the fold: Origami meets mathematics." *SIAM News* 34 (2001), 1–4.

[Felton et al. 14]   S. Felton, M. Tolley, E. Demaine, D. Rus, and R. Wood. "A method for building self-folding machines." *Science* 345 (2014), 644–646.

[Filipov et al. 15]   Evgueni T. Filipov, Tomohiro Tachi, and Glaucio H. Paulino. "Origami tubes assembled into stiff, yet reconfigurable structures and metamaterials." *Proceedings of the National Academy of Sciences* 112 (2015), 12321–12326.

[Ghourabi et al. 13]   Fadoua Ghourabi, Asem Kasem, and Cezary Kaliszyk. "Algebraic analysis of Huzita's origami operations and their extensions." In *Automated Deduction in Geometry*, edited by Tetsuo Ida and Jacques Fleuriot, pp. 143–160. Springer, 2013.

[Justin 86]   Jaques Justin. "Résolution par le pliage de l'équation du troisième degré et applications géométriques." *L'Ouvert* 42 (1986), 9–19.

[Kawasaki 11]   Toshikazu Kawasaki. "A note on operations of spherical origami constructions." In *Origami 5 - Fifth International Meeting of Origami Science, Mathematics, and Education*, edited by Patsy Wang-Iverson, Robert J. Lang, and Mark Yim, pp. 543–551. CRC Press, 2011.

[Lucero 17]   Jorge C. Lucero. "On the elementary single-fold operations of origami: reflections and incidence constraints on the plane." *Forum Geometricorum* 17 (2017), 207–221.

[Mahadevan and Rica 05]   L. Mahadevan and S. Rica. "Self-organized origami." *Science* 307 (2005), 1740–1740.

[Martin 98]   George E. Martin. *Geometric Constructions*. Springer, 1998. Chapter 10: Paperfolding.

[Neyrinck 14]   Mark C. Neyrinck. "An origami approximation to the cosmic web." *Proceedings of the International Astronomical Union* 11 (2014), 97–102.

[Robertson 78]   S. A. Robertson. "Isometric folding of Riemannian manifolds." *Proceedings of the Royal Society of Edinburgh* 79 (1978), 275–284.

[Rosen 12]   Kenneth H. Rosen. *Discrete Mathematics and Its Applications*. MacGraw-Hill, 2012. Chapter 6: Counting.

[Sturmfels 02]   Bernd Sturmfels. *Solving Systems of Polynomial Equations*, CBMS Regional Conference Series in Mathematics, 97. American Mathematical Society, 2002.

Jorge C. Lucero

Dept. Computer Science, University of Brasília, Brasília DF 70910-900, Brazil, e-mail: lucero@unb.br