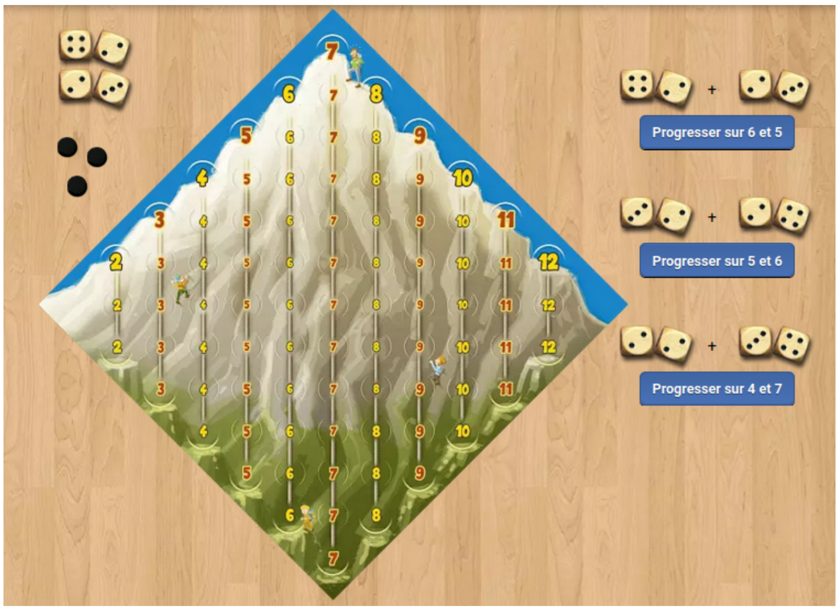
Beating Can't Stop

February 12, 2021



1 Rules of the game

Can't stop is a simple boardgame played on the board you can see in the image. It is a *push your luck* game, where a player have to choose between stopping and taking his gains, or trying to earn more at the risk of loosing all he had.
The board is made of 11 columns from 2 to 12. Each player have two type of markers :

- definitive markers (one for each column)
- tentitative markers (3 at most)

The objective of the game is to be the first to close three column. Closing a column means putting its definitive marker on the top of the column.
Players play *turns* in succession. During a *turn* a player can make multiple throw determining how his markers will advance. If a throw is successful the player as the choice to stop and capitalize, continue to win more - and risk to loose more.
A *throw* consists in throwing 4 dices and choosing how to make two pair (there is thus 3 choices of two pairs). Choose one of the pair. The sum n determine on which column you shall move your tentative marker. If a tentative maker is on a column we say that the column is *open*.

During a *turn* at most three column can be *open* (as there is only 3 tentitative markers). If n is an open column, move the tentitative marker up by one (unless the tentitative is already at the top of the column, in which case the move is illegal). If n is not opened yet, (and not *closed* either - see further) and you still have available tentitative markers (that is at most two other columns are open) put your tentitative marker one rank above your current definitive marker on this column (or at position 1 if there is none yet). If after one *throw* there is no legal move, then your turn is ended and the tentitative marker lost. We say that the throw is *failed*. After one *throw*, if you have moved at least one tentitative marker you can *stop*, changing your tentitative markers into definitive markers. If your tentitative marker was at the top of one column it is now closed and no other tentitative marker can be placed on it. If you closed at least 3 column since the beginning of the game you won.
The goal of the project is to find the *best policy* possible for playing the game. We are now going to define more precisely what "best" mean in the different questions.

1.1 An example

Player A start the game. All of it's definitive marker are on 0. The first throw show 4, 2, 2 and 3. Thus the possible pairing are $(6, 5)$, $(5, 6)$ and $(4, 7)$. The two first pairing are actually identical. All of them are admissible (no column are terminated yet, and moving does not put the tentitative marker above the column limit). Player A select $(4, 7)$, opening column 4 and 7, and placing a tentitative marker on the first step for each.
She has now the choice to stop, turning her tentitative markers in definitive one, or continue with a new throw. Choosing to continue she draw 1, 2, 4, 5. The pairings are $(3, 9)$, $(5, 7)$ and $(6, 6)$. Player A has thus 4 choices : i) opening column 3, ii) opening column 9, iii) opening column 5 and moving up on column 7, or iv) opening column 6 by moving twice the tentitative marker. She choose the third option. She now have three column open and cannot open any more during this turn. Her tentitative position is $(4 : 1, 6 : 2, 7 : 1)$. She choose to continue.
She draws 1, 1, 3, 6. The pairings are $(2, 9)$, $(4, 7)$ and $(7, 4)$. The first pairing is not admissible (three columns are open, none of them are 2 or 9). She has to choose one of the other two equivalent choices, moving up her tentitative marker to $(4 : 2, 6 : 2, 7 : 2)$. She choose to continue.
She draws 4, 5, 5, 6. The pairing are $(9, 11)$, $(10, 10)$ and $(9, 11)$. None of them are admissible. The throw is failure. She loose all of her tentitative markers, and end her turn.

1.2 A few clarifications

A throw offer three different pairing. Each pairing gives two potential column. Thus the result of a throw can be represented by a triplet of possible move couple $((i_1, i_2), (j_1, j_2), (k_1, k_2))$, with potential repetition.
During a throw:

- at most two tentitative marker are moved;
- one tentitative marker can be moved twice if both pair have the same sum;
- if both pair have the same sum but there is only one left before reaching the top, then the first pair move the tentitative marker by one and the second is discarded;
- once both pair are chosen you have to move the marker if possible, meaning that you cannot willingly discard a legal move

2 A simple push your luck game

We start with some simplified version of the problem that might be of use to determine the best possible strategy.
Assume that you play a very simple "push your luck" game, where you have a succession of *turns* made of *throws*. A throw i consists in drawing a Bernoulli variable X_i of parameter p (independant of all others), if $X_i = 1$, then you add one to your tentative gain, if $X_i = 0$ you loose all your tentatives gain and have to end your turn. If $X_i = 1$ you can *stop*, thus adding your tentitative gain to your current definitive gain, or continue, starting a new *throw*.

2.1 Question 1

What is the policy maximizing the expected gain of a turn ? An analytic answer is possible. What is the expected optimal gain ? What is, under the optimal policy, the law of the gain ? In particular what is the probability ending without any tentative gain ?

2.2 Question 2

Consider the problem of minimizing the expected number of turn required to reach a given total gain G_{max}. Write a dynamic programming equation to solve this problem. Write a function computing the optimal policy. Intuitively, under which conditions is this policy close to the optimal policy of the previous question ? Numerically verify your assertion.

3 Single player game

We now assume that you play a single player game.

3.1 Question 3

Assume that only columns (i, j, k) are open and unbounded. Compute the probability p of a throw of the 4 dices to have at least one legal move. Compute the policy maximizing the expected number of move (on whichever column) before stopping. Compute the probability, under the optimal strategy, of ending the turn without moving any definitive marker (that is of having a failed throw).

3.2 Question 4

Assume that only columns (i, j, k) are open. Compute by Dynamic Programming the minimum expected number of turn before closing the three columns. Check by simulation.

3.3 Question 5

Write a heuristic policy for playing the single player game with the objective of minimizing the expected number of turn before having closed any three columns. Numerically estimate the expected number of turns required, and the proportion of turn ending with a failed throw.

4 Two players game

The two player game is harder than the one player game. Indeed, the other player can close a given column before you, thus rendering any progress on this colon useless. Furthermore if he close his third column you loose. Consequently you might want to be more aggressive if your adversary is close to winning.

4.1 Question 6

Write a heuristic policy for playing the two player game with the objective of beating your adversary. Explain your reasoning behind your heuristic.

5 Multiplayer game

We call multiplayer game a game with 3 or 4 player on the board. The game ending as soon as any of them have closed 3 columns.

5.1 Question 7

Write a heuristic policy for playing the multiplayer game with the objective of having the maximum expected number of closed column at the end.

5.2 Question 8

Write a heuristic policy for playing the multiplayer game with the objective of being first.