Lecture 3

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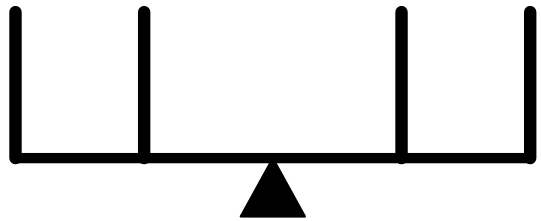
Expected values

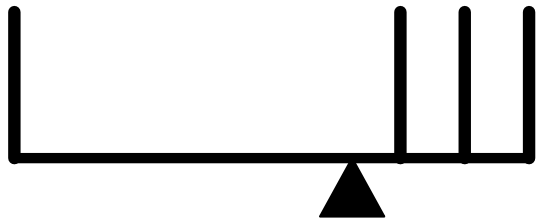
- The **expected value** or **mean** of a random variable is the center of mass of its distribution
- For discrete random variable X with PMF $p(x)$, it is defined as follows

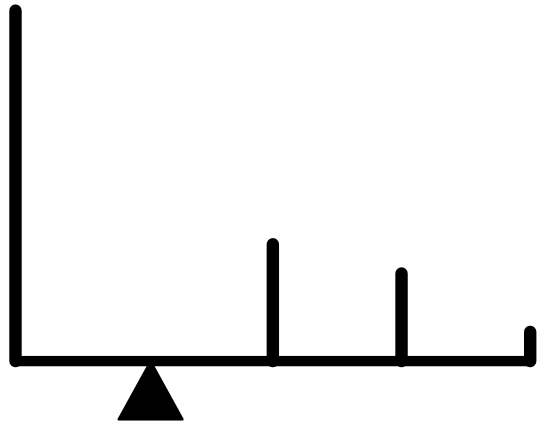
$$E[X] = \sum_x xp(x) = \sum_{\text{possible values of X}} xP(X = x)$$

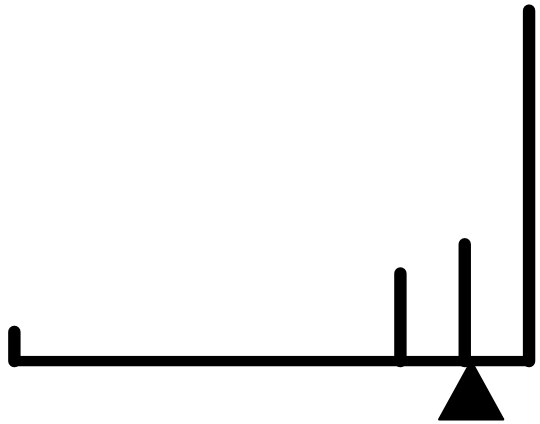
  where the sum is taken over the possible values of x
- $E[X]$ represents the center of mass of a collection of locations and weights, $\{x, p(x)\}$
- $E[X]$ is not necessarily among the values that the variable X takes

Example









Expected values

$$E[X] = \sum_x xp(x)$$

- $E(X_1) = (-4) \times 0.25 + (-3) \times 0.25 + 3 \times 0.25 + 4 \times 0.25 = 0$
- $E(X_2) = (-4) \times 0.25 + 1 \times 0.25 + 2 \times 0.25 + 3 \times 0.25 = 0.5$
- $E(X_3) = (-4) \times 0.60 + 1 \times 0.20 + 2 \times 0.15 + 3 \times 0.05 = -1.75$
- $E(X_4) = (-4) \times 0.05 + 1 \times 0.10 + 2 \times 0.15 + 3 \times 0.60 = 2$

Note that $\min(X) \leq E[X] \leq \max(X)$

Example: Fair coin

- Suppose a coin is flipped and X is declared 0 or 1 corresponding to a head or a tail, respectively
- What is the expected value of X?

$$E[X] = .5 \times 0 + .5 \times 1 = .5$$

- Note, if thought about geometrically, this answer is obvious; if two equal weights are spaced at 0 and 1, the center of mass will be .5

Example: Bernoulli

- Suppose that a person is infected with the flu virus with probability θ
- 1 is "person is infected", 0 is "person is not infected"
- X takes the value 1 with probability θ and 0 with probability $1 - \theta$
- What is the expected value of X?

$$E[X] = (1 - \theta) \times 0 + \theta \times 1 = \theta$$

- Interpretation: If one selects at random one person from a population of individuals with $100\theta\%$ infected people then the probability of this person to be infected is θ

Example: die rolls

- Suppose that a die is tossed and X is the number face up
- What is the expected value of X?
- X takes values in $\{1, 2, 3, 4, 5, 6\}$
- $P(X = 1) = \ldots = P(X = 6) = \frac{1}{6}$

$$E[X] = 1 \times \frac{1}{6} + 2 \times \frac{1}{6} + 3 \times \frac{1}{6} + 4 \times \frac{1}{6} + 5 \times \frac{1}{6} + 6 \times \frac{1}{6} = 3.5$$

- Again, the geometric argument makes this answer obvious without calculation
- Note that the mean **is not** the most likely value the variable takes. In this example $P\{X = E[X]\} = 0$

Example: die rolls

- Suppose that a die is tossed and X is the number face up
- What is the expected value of X^2?
- What is the expected value of $\sqrt{X}$?

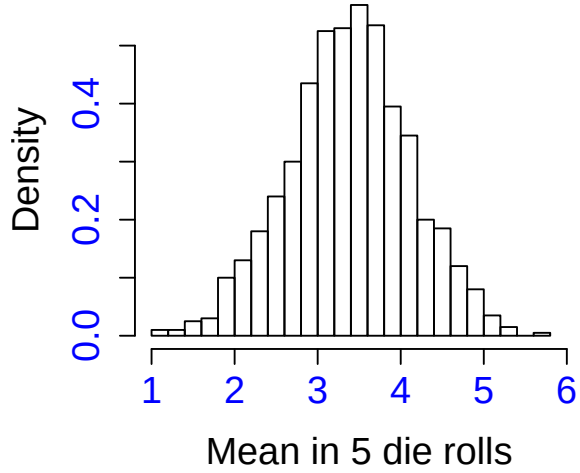
Using R to calculate the sample mean

Simulate the mean of n die rolls

```
mx5 <- rep ( 0, 1000 )
mx10=mx5
mx20=mx5
mx100=mx5


for ( i in 1:1000 )
   {mx5[i] <- mean(sample(1:6,5,replace=T))
   mx10[i] <- mean(sample(1:6,10,replace=T))
   mx20[i] <- mean(sample(1:6,20,replace=T))
   mx100[i] <- mean(sample(1:6,100,replace=T))}
```

Example


Density
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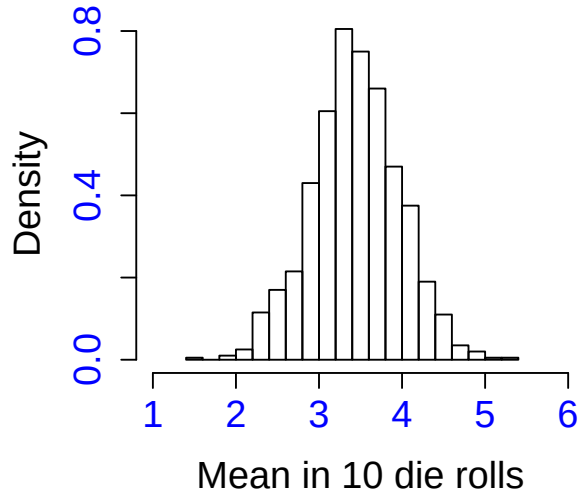
0.2
0.4
1
2
3
4
5
6
Mean in 5 die rolls


Density
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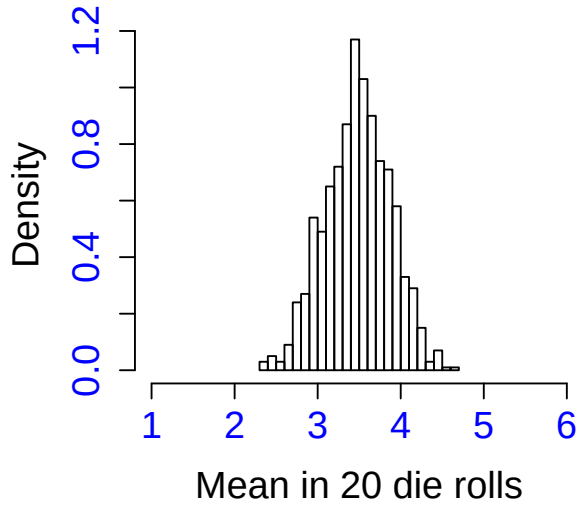
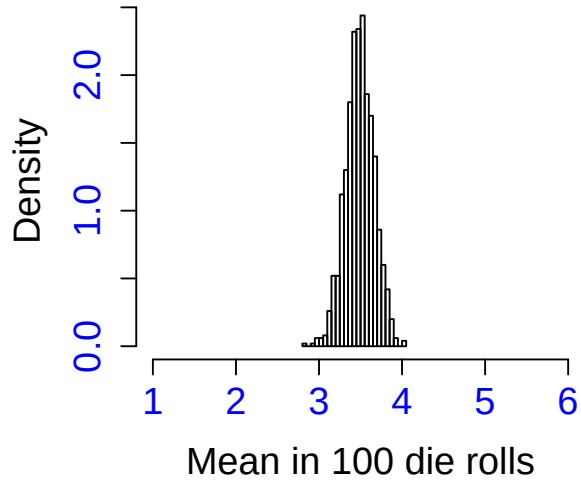
0.4
0.8
1
2
3
4
5
6
Mean in 10 die rolls


Density
0.0
0.4
0.8
1.2
1
2
3
4
5
6
Mean in 20 die rolls


Density
0.0
1.0
2.0
1
2
3
4
5
6
Mean in 100 die rolls

Example: mental health

- Suppose that one observes the mental health of people in a large ($n = 10,000$) cohort study
- For each subject one observes one of the outcomes "healthy", "mild", "moderate", "serious"

What is the random variable and what is its mean?

Continuous random variables

- For a continuous random variable, X, with density, $f(x)$, the expected value is defined as follows

$$E[X] = \int_{-\infty}^{\infty} xf(x)dx$$

- This definition borrows from the definition of center of mass for a continuous body

Example: uniform distribution

- Consider that the random variable X has a distribution with density $f(x) = 1$ for $x \in [0, 1]$ and 0 otherwise
- Plot $f(x)$
- This is called the uniform distribution and we write $X \sim U[0, 1]$
- Is this a valid density?
- Suppose that X follows this density; what is its expected value?

$$E[X] = \int_0^1 x dx = \frac{x^2}{2} \Big|_0^1 = 1/2$$

- What is the pdf of the Uniform distribution on $[a, b]$?
- What is its mean?

Example: uniform distribution

```
y1<-mean(runif(10000))
y2<-mean(runif(10000,0,10))
y3<-mean(runif(10000,-2,15))
```

- If $X \sim U(a, b)$ calculate $E[X]$
- If $X_1, \ldots, X_n$ are independent random variables with pdf $f(x)$

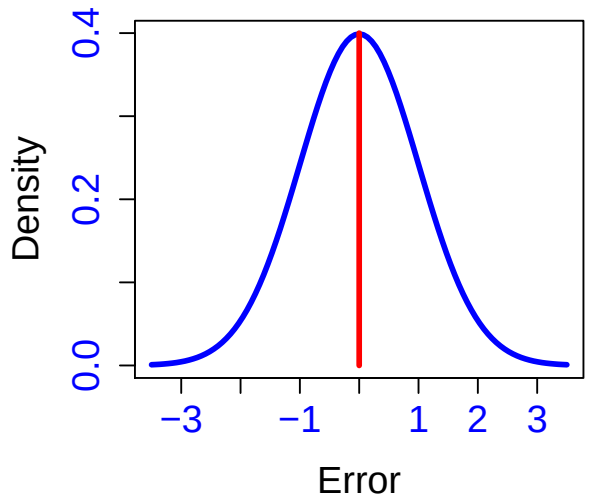
$$\overline{X}_n = \frac{1}{n}(X_1 + \ldots X_n) \approx E[X] = \int x f(x) dx$$

- When n is larger the approximation is better (aka the strong law of large numbers)

Distributions: the mean

- The mean is the "center of mass"; may not even be an acceptable value for experiment
- The mean is not typically the median
- For symmetric distributions (aka error distributions) the mean and the median are equal
- The mean can be heavily influenced by skewness, outliers
- Examples: medical expenditure in the US, average net worth, insurance claims
- **Remember**: interpretation, interpretation, interpretation

Example


Density
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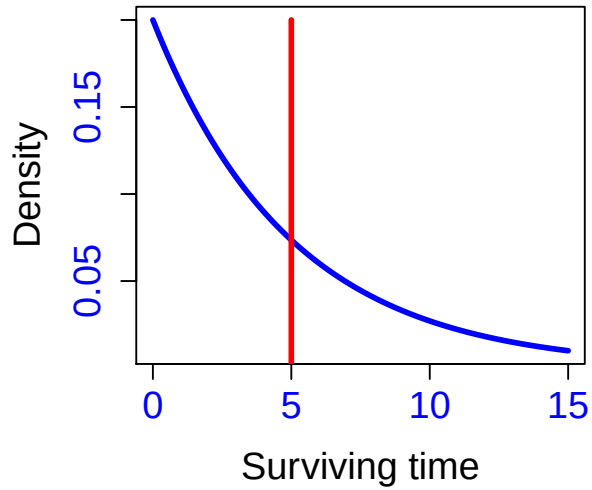
0.2
0.4
−3
−1
1
2
3
Error


Density
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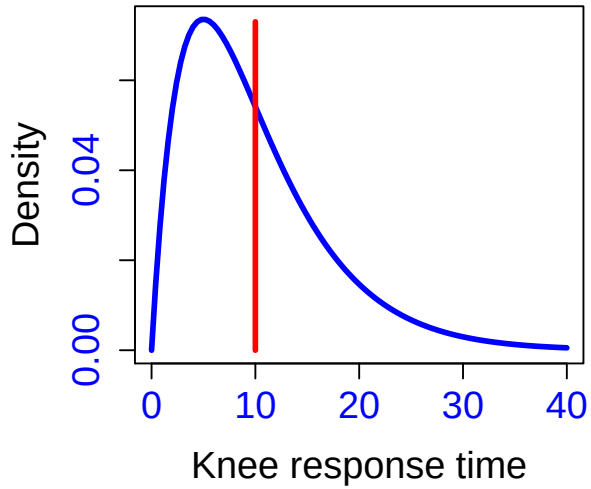
0.15
0
5
10
15
Surviving time


Density
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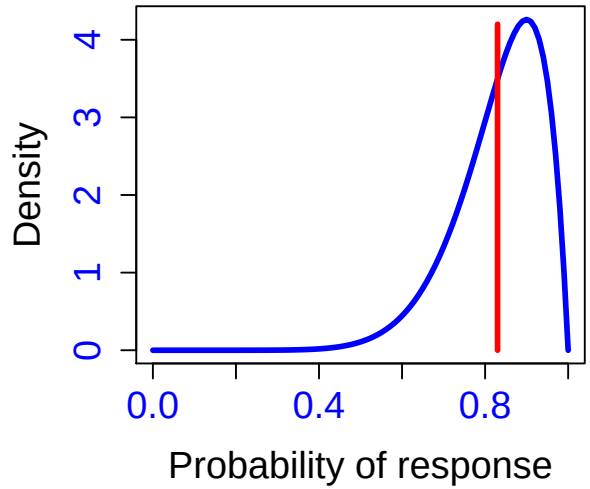
0.04
0
10
20
30
40
Knee response time


Density
0
1
2
3
4
0.0
0.4
0.8
Probability of response

Example: Gamma distribution

- Consider that the random variable X has a distribution with density

$$f(x) = \frac{1}{\beta^{\alpha}\Gamma(\alpha)} x^{\alpha-1} e^{-x/\beta}, \text{ for every } x \in (0, \infty)$$

- $\Gamma(\alpha) = \int_0^{\infty} x^{\alpha-1} e^{-x} dx$
- Is this a valid density?
- Suppose that X follows this density; what is $E[X]$?
- What is $E[X^{\pi}]$?

Rules about expected values

- The expected value is a linear operator
- If a and b are not random and X and Y are two random variables then
  - $E[aX + b] = aE[X] + b$
  - $E[X + Y] = E[X] + E[Y]$
  - $X \sim exp(5)$, $Y \sim U(20, 25)$. $E[3X + 2Y] =?$
- *In general* if g is a function that is not linear,

$$E[g(X)] \neq g(E[X])$$

- For example, in general, $E[X^2] \neq E[X]^2$, $E[\log(X)] \neq \log(E[X])$

Nonlinear transformation of the mean

$Var = E(X^2) - E^2(X) \geq 0$

- We show that $E[X]^2 \leq E[X^2]$ for every discrete random variable X
- This is actually true for every random variable
- We need to show $\{\sum_x xp(x)\}^2 \leq \sum_x x^2 p(x)$
- It is known that $(\sum_i a_i b_i)^2 \leq (\sum_i a_i^2)(\sum_i b_i^2)$
- Take $a_x = x\sqrt{p(x)}$, $b_x = \sqrt{p(x)}$
- Show that $(a_1 b_1 + a_2 b_2)^2 \leq (a_1^2 + a_2^2)(b_1^2 + b_2^2)$

In general, if $h(\cdot)$ is a convex function $h(E[X]) \leq E[h(X)]$

Example

- You flip a coin, X and simulate a uniform random number Y, what is the expected value of their sum?

$$E[X + Y] = E[X] + E[Y] = .5 + .5 = 1$$

- Another example, you roll a die twice. What is the expected value of the average?
- Let X_1 and X_2 be the results of the two rolls

$$E[(X_1 + X_2)/2] = \frac{1}{2}(E[X_1] + E[X_2]) = \frac{1}{2}(3.5 + 3.5) = 3.5$$

Example

X_1	X_2	$P(X)$	$\frac{X_1+X_2}{2}$
1	1	1/36	1
1	2	1/36	1.5
1	3	1/36	2
1	4	1/36	2.5
1	5	1/36	3
1	6	1/36	3.5
2	1	1/36	1.5
2	2	1/36	2
2	3	1/36	2.5
2	4	1/36	3
⋮	⋮	⋮	⋮

- What is the probability of getting $\frac{X_1+X_2}{2} = 3.5$?
- How many values does $\frac{X_1+X_2}{2}$ take?

Example

$\frac{X_1+X_2}{2}$	1	1.5	2	2.5	3	3.5	4	...
$P\left(\frac{X_1+X_2}{2}=x\right)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	...

$$E[X] = 1 \times \frac{1}{36} + 1.5 \times \frac{2}{36} + 2 \times \frac{3}{36} + \dots$$

How many values does $\overline{X}_n$ take?

- min is 1, max is 6
- all values in increments of $1/n$

```
length(seq(1,6,by=1/20))
length(seq(1,6,by=1/200))
```

Example

1. Let X_i for $i = 1, \ldots, n$ be a collection of random variables, each from a distribution with mean μ
2. Calculate the expected value of the sample average of the X_i

$$\begin{aligned} E\left[\frac{1}{n}\sum_{i=1}^{n} X_i\right] &= \frac{1}{n} E\left[\sum_{i=1}^{n} X_i\right] \\ &= \frac{1}{n}\sum_{i=1}^{n} E\left[X_i\right] \\ &= \frac{1}{n}\sum_{i=1}^{n} \mu = \mu. \end{aligned}$$

Remark

- Therefore, the expected value of the **sample mean** is the **population mean** that it's trying to estimate
- When the expected value of an estimator is what it is trying to estimate, we say that the estimator is **unbiased**
- An estimator is any function of the data, $U(X)$. A parameter is any unknown quantity in the model, θ
- $U(X)$ is unbiased for θ if $E[U(X)] = \theta$

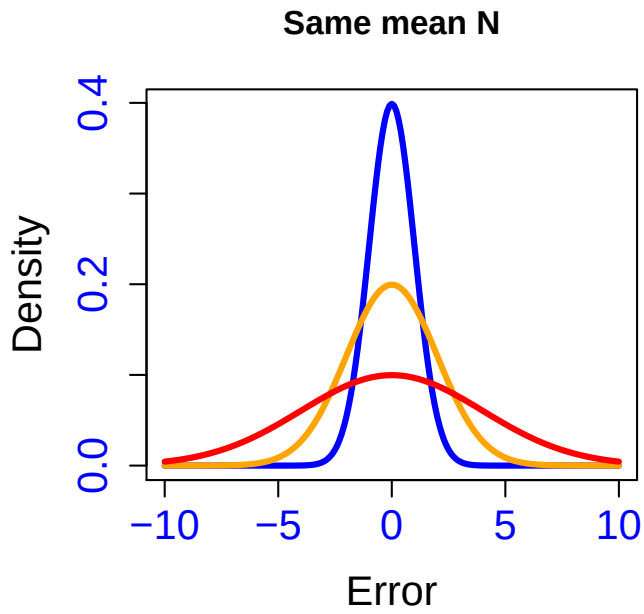
The variance

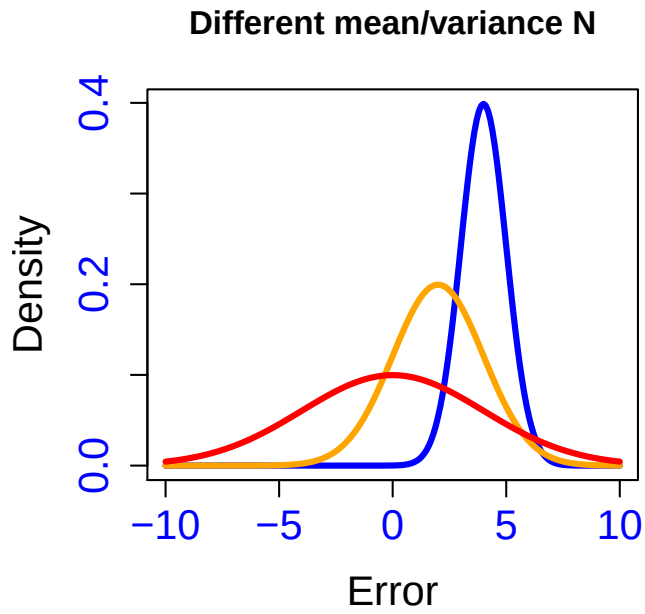
- The variance of a random variable is a measure of *spread*
- If X is a random variable with mean μ, the variance of X is defined as

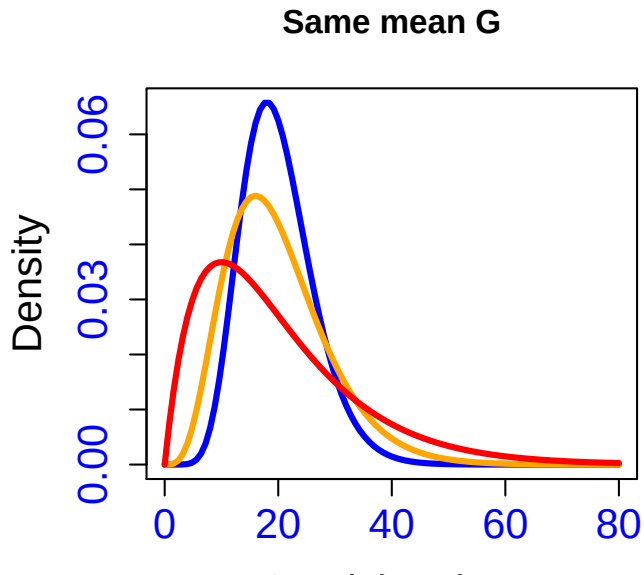
$$\mathrm{Var}(X) = E[(X - \mu)^2]$$

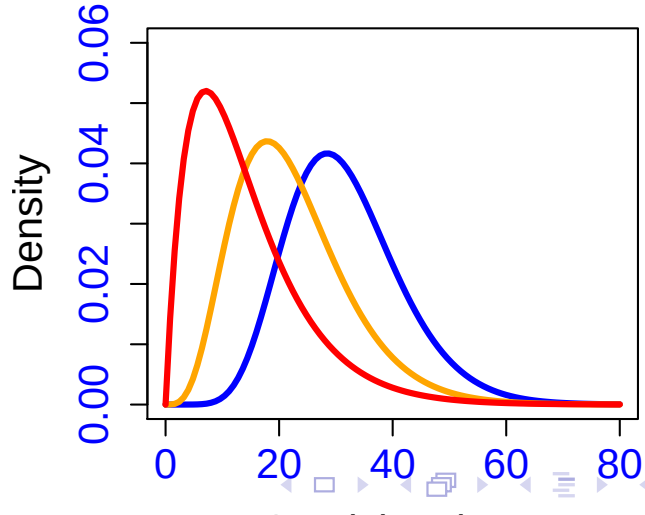
  the expected (squared) distance from the mean
- If X is a discrete random variable $\mathrm{Var}[X] = \sum_x (x - \mu)^2 p(x)$
- If X is a continuous random variable $\mathrm{Var}[X] = \int_{-\infty}^{\infty} (x - \mu)^2 f(x) dx$
- Densities with a higher variance are more spread out than densities with a lower variance

Example


Same mean N
Density
0.0
0.2
0.4
−10
−5
0
5
10
Error


Different mean/variance N
Density
0.0
0.2
0.4
−10
−5
0
5
10
Error


Same mean G
Density
0.00
0.03
0.06
0
20
40
60
80


Same variance G
Density
0.00
0.02
0.04
0.06
0
20
40
60
80

The variance

- Convenient computational form

$$\mathrm{Var}(X) = E[X^2] - E[X]^2$$

- If a is constant then $\mathrm{Var}(aX) = a^2\mathrm{Var}(X)$
- The square root of the variance is called the **standard deviation**
- $\mathrm{SD}(X) = \sqrt{\mathrm{Var}(X)}$
- The standard deviation has the same units as X
- If a is constant then $\mathrm{SD}(aX) = a\mathrm{SD}(X)$: scale invariant
- Prove these results

Variance: some R code

```
x=seq(-10,10,length=201)
y1<-dnorm(x)
y2<-dnorm(x,0,2)
y3<-dnorm(x,0,4)
plot(x,y1,type="l",col="blue",lwd=3)
lines(x,y2,col="orange",lwd=3)
lines(x,y3,col="red",lwd=3)
```

Calculating the empirical variance in R

```
y<-rnorm(100,0,4)
var(y)
sd(y)
```

Example

- What is the sample variance from the result of a toss of a die?
- $\mathrm{Var}(X) = E[X^2] - E[X]^2$
- What are the values that X^2 can take?

$Y = X^2$	1	4	9	16	25	36
$P(Y = y)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

Example

- What is the sample variance from the result of a toss of a die?
  - $E[X] = 3.5$
  - $E[X^2] = 1^2 \times \frac{1}{6} + 2^2 \times \frac{1}{6} + 3^2 \times \frac{1}{6} + 4^2 \times \frac{1}{6} + 5^2 \times \frac{1}{6} + 6^2 \times \frac{1}{6} = 15.17$
- $\mathrm{Var}(X) = E[X^2] - E[X]^2 \approx 2.92$

```
x=1:6
ex2=sum(x^2*rep(1/6,6))
ex=sum(x*rep(1/6,6))
varx=ex2-ex^2
```

Example

- What's the sample variance from the result of the toss of a coin with probability of heads (1) of p?
  - $E[X] = 0 \times (1-p) + 1 \times p = p$
  - $E[X^2] = E[X] = p$
- $\mathrm{Var}(X) = E[X^2] - E[X]^2 = p - p^2 = p(1-p)$

Example

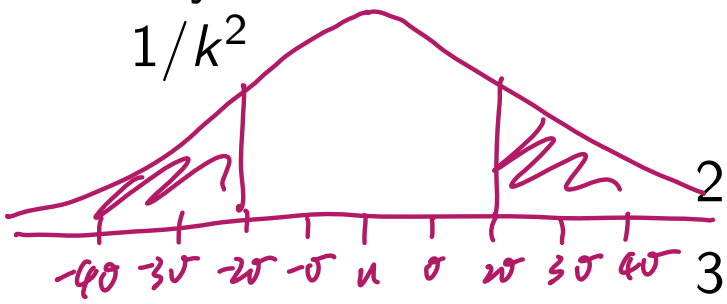
- Suppose that a random variable is such that $0 \leq X \leq 1$ and $E[X] = p$
- Note that $X^2 \leq X$ so that $E[X^2] \leq E[X] = p$
- $\mathrm{Var}(X) = E[X^2] - E[X]^2 \leq E[X] - E[X]^2 = p(1-p)$
- Therefore the Bernoulli variance is the largest possible for random variables bounded between 0 and 1
- Largest variance is attained at $p = 0.5$
- For every Beta distribution there exists a Bernoulli distribution with the same mean and larger variance

Interpreting variances

- Chebyshev's inequality is useful for interpreting variances
- This inequality states that

$$P(|X - \mu| \geq k\sigma) \leq \frac{1}{k^2}$$

- For example, the probability that a random variable lies beyond k standard deviations from its mean is less than $1/k^2$



$$2\sigma \rightarrow 25\%$$
$$3\sigma \rightarrow 11\%$$
$$4\sigma \rightarrow 6\%$$

- Note that this is only an upper bound; the actual probability might be quite a bit smaller

Proof of Chebyshev's inequality

$$
\begin{aligned}
P(|X-\mu| > k\sigma) &= \int_{\{x:|x-\mu|>k\sigma\}} f(x)dx \\
&\leq \int_{\{x:|x-\mu|>k\sigma\}} \frac{(x-\mu)^2}{k^2\sigma^2} f(x)dx \\
&\leq \int_{-\infty}^{\infty} \frac{(x-\mu)^2}{k^2\sigma^2} f(x)dx \\
&= \frac{1}{k^2}
\end{aligned}
$$

Comparing Chebyshev with parametric assumptions

$$P(|X - \mu| \geq k\sigma) \leq \frac{1}{k^2}$$

$k\sigma$	2	3	4	5
Any	0.250	0.111	0.063	0.040
t(3)	0.041	0.014	0.006	0.003
Gamma(2,2)	0.046	0.014	0.004	0.001
Normal	0.046	0.003	6.33×10^{-5}	5.73×10^{-7}

Comparing Chebyshev with parametric assumptions

```
k=2:5 # Multiples of SD

2*(1-pnorm(k)) # Normal

sdt3=sqrt(3) # SD of a t(3) distribution
2*(1-pt(k*sdt3,df=3)) # t(3)

sh=2 # shape of Gamma(sh,sc)
sc=2 # scale of Gamma(sh,sc)
m=sh*sc # mean of Gamma(sh,sc)
sdg = sqrt(sh*sc^2) # SD of Gamma(sh,sc)
pgamma(m-k*sdg,shape=sh,scale=sc)+
1-pgamma(m+k*sdg,shape=sh,scale=sc)
```

Example

- IQs are often said to be distributed with a mean of 100 and a sd of 15
- What is the probability of a randomly drawn person having an IQ higher than 160 or below 40?
- Thus we want to know the probability of a person being more than 4 standard deviations from the mean
- Thus Chebyshev's inequality suggests that this will be no larger than 6%
- IQs distributions are often cited as being bell shaped, in which case this bound is very conservative
- The probability of a random draw from a bell curve being 4 standard deviations from the mean is on the order of 10^{-5} (one thousandth of one percent)

Example

- A popular buzz phrase in industrial quality control is Motorola's"Six Sigma" whereby businesses are suggested to control extreme events or rare defective parts
- Chebyshev's inequality states that the probability of a "Six Sigma" event is less than $1/6^2 \approx 3\%$
- If a bell curve is assumed, the probability of a "six sigma" event is on the oder of 10^{-9} (one ten millionth of a percent)

Coefficient of variation

- For a variable $X > 0$

$$C_v(X) = \frac{\text{SD}(X)}{E[X]}$$

- Measures the amount of variability (as described by the standard deviation) relative to the mean
- C_v is the inverse of the signal-to-noise ratio
- It does not have units (mean and standard deviation are on the same scale)
- Strong link to Cohen's d used in sample size calculations

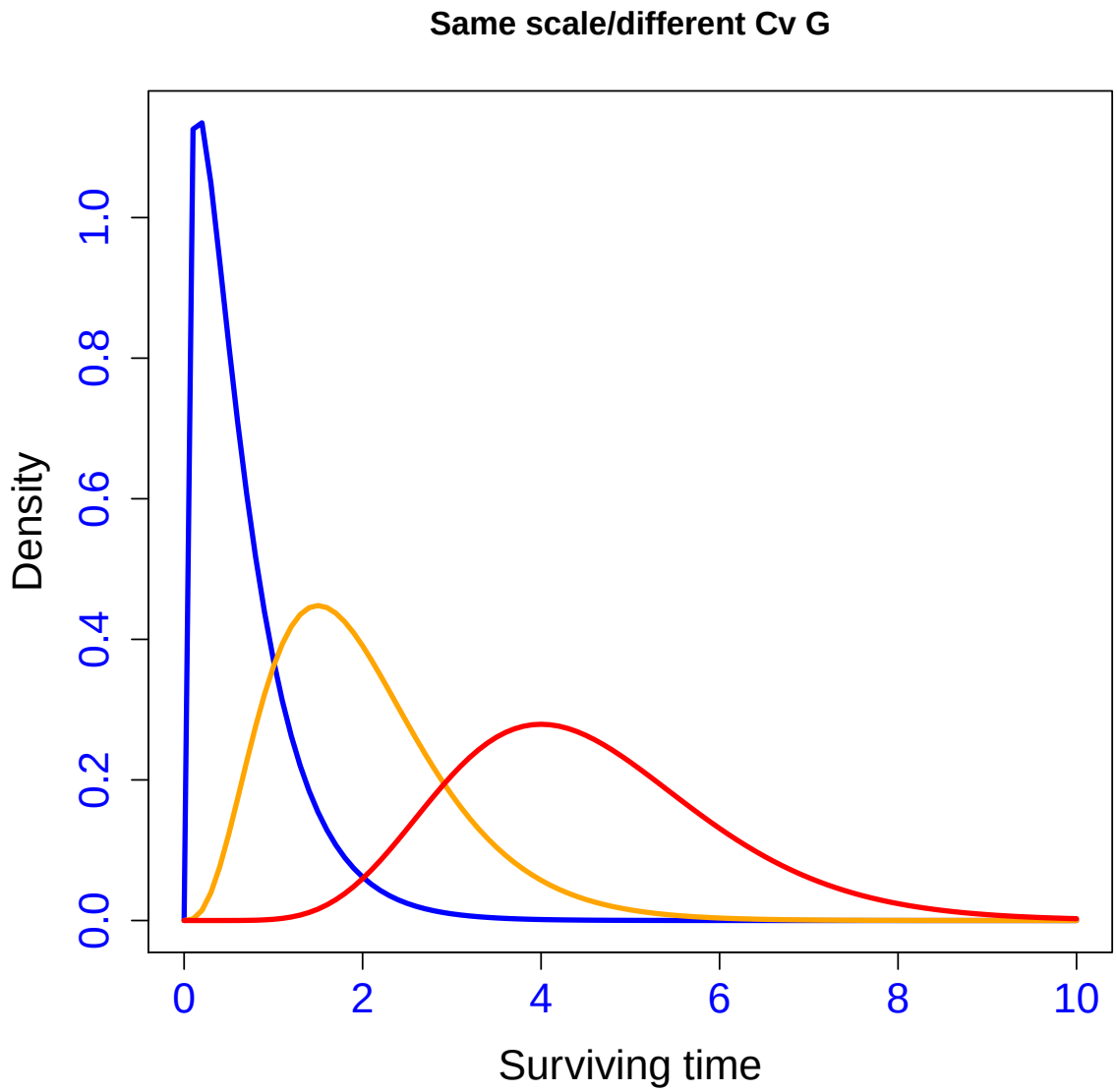
$$C_d = \frac{E[X_1] - E[X_2]}{\text{SD}}$$

Coefficient of variation

```
x=seq(0,10,length=101) # set the grid
# Gamma with C_v = 1/sqrt(1.3) = 0.88
y1<-dgamma(x,shape=1.3,scale=1/2)
# Gamma with C_v = 1/sqrt(4) = 0.50
y2<-dgamma(x,shape=4,scale=1/2)
# Gamma with C_v = 1/sqrt(9) = 0.33
y3<-dgamma(x,shape=9,scale=1/2)

plot(x,y1,type="l",col="blue",lwd=3)
lines(x,y2,col="orange",lwd=3)
lines(x,y3,col="red",lwd=3)
```

Varying C_v


Same scale/different Cv G
Density
0.0
0.2
0.4
0.6
0.8
1.0
0
2
4
6
8
10
Surviving time