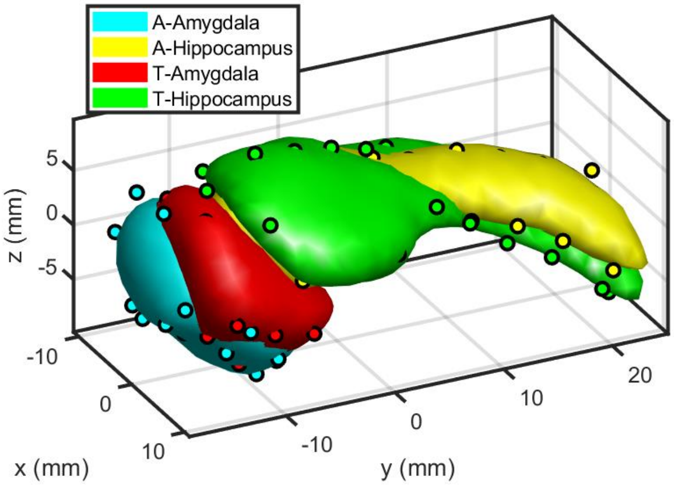
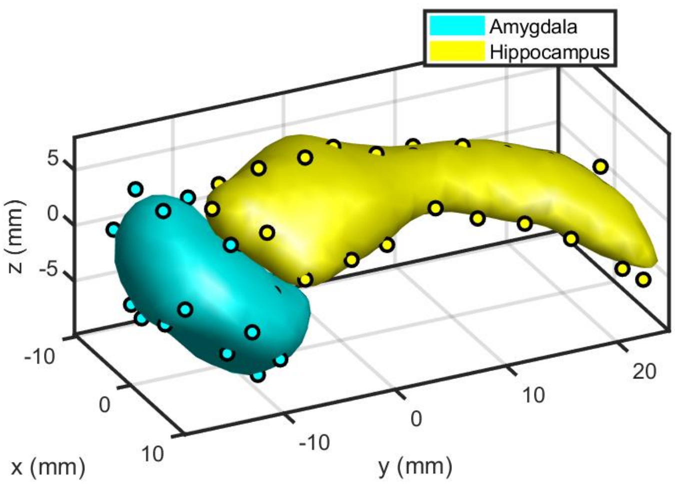
1)

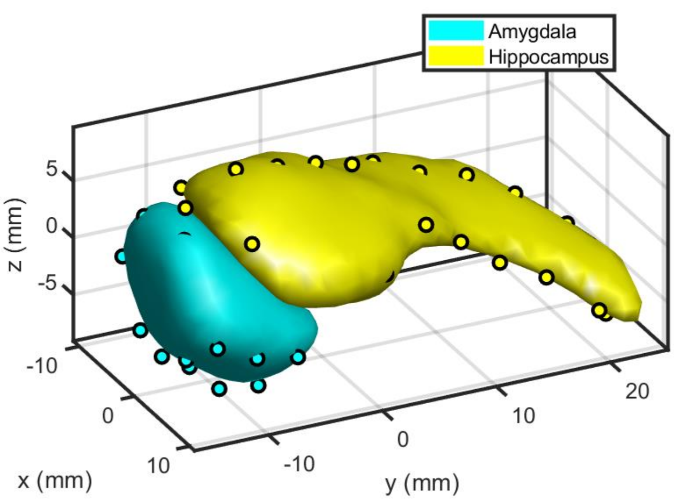
Overlaid No Transform



Atlas



Target



2) $E = \Sigma |AX(i) - Y(i)|^2$ $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

$$E = \Sigma \begin{vmatrix} a_{11}X_1(i) + a_{12}X_2(i) + a_{13}X_3(i) - Y_1(i) \\ a_{21}X_1(i) + a_{22}X_2(i) + a_{23}X_3(i) - Y_2(i) \\ a_{31}X_1(i) + a_{32}X_2(i) + a_{33}X_3(i) - Y_3(i) \end{vmatrix}$$

$$E = \Sigma\big((a_{11}X_1(i) + a_{12}X_2(i) + a_{13}X_3(i) - Y_1(i))^2 + (a_{21}X_1(i) + a_{22}X_2(i) + a_{23}X_3(i) - Y_2(i))^2$$
$$+ (a_{31}X_1(i) + a_{32}X_2(i) + a_{33}X_3(i) - Y_3(i))^2\big)$$

$$\partial E/\partial a_{11} = \Sigma 2(a_{11}X_1(i) + a_{12}X_2(i) + a_{13}X_3(i) - Y_1(i))X_1(i)$$
$$= \Sigma 2(a_{11}X_1(i)^2 + a_{12}X_1(i)X_2(i) + a_{13}X_1(i)X_3(i) - X_1(i)Y_1(i))$$
$$\partial E/\partial a_{12} = \Sigma 2(a_{11}X_1(i)X_2(i) + a_{12}X_2(i)^2 + a_{13}X_2(i)X_3(i) - X_2(i)Y_1(i))$$
$$\partial E/\partial a_{13} = \Sigma 2(a_{11}X_1(i)X_3(i) + a_{12}X_2(i)X_3(i) + a_{13}X_3(i)^2 - X_3(i)Y_1(i))$$
$$\partial E/\partial a_{21} = \Sigma 2(a_{21}X_1(i)^2 + a_{22}X_1(i)X_2(i) + a_{23}X_1(i)X_3(i) - X_1(i)Y_2(i))$$
$$\partial E/\partial a_{22} = \Sigma 2(a_{21}X_1(i)X_2(i) + a_{22}X_2(i)^2 + a_{23}X_2(i)X_3(i) - X_2(i)Y_2(i))$$
$$\partial E/\partial a_{23} = \Sigma 2(a_{21}X_1(i)X_3(i) + a_{22}X_2(i)X_3(i) + a_{23}X_3(i)^2 - X_3(i)Y_2(i))$$
$$\partial E/\partial a_{31} = \Sigma 2(a_{31}X_1(i)^2 + a_{32}X_1(i)X_2(i) + a_{33}X_1(i)X_3(i) - X_1(i)Y_3(i))$$
$$\partial E/\partial a_{32} = \Sigma 2(a_{31}X_1(i)X_2(i) + a_{32}X_2(i)^2 + a_{33}X_2(i)X_3(i) - X_2(i)Y_3(i))$$
$$\partial E/\partial a_{33} = \Sigma 2(a_{31}X_1(i)X_3(i) + a_{32}X_2(i)X_3(i) + a_{33}X_3(i)^2 - X_3(i)Y_3(i))$$

we set everything $=0$ & shift y term.

$$\Sigma(a_{11}X_1(i)^2 + a_{12}X_1(i)X_2(i) + a_{13}X_1(i)X_3(i)) = \Sigma(X_1(i)Y_1(i))$$
$$\Sigma(a_{11}X_1(i)X_2(i) + a_{12}X_2(i)^2 + a_{13}X_2(i)X_3(i)) = \Sigma(X_2(i)Y_1(i))$$
$$\Sigma(a_{11}X_1(i)X_3(i) + a_{12}X_2(i)X_3(i) + a_{13}X_3(i)^2) = \Sigma(X_3(i)Y_1(i))$$
$$\Sigma(a_{21}X_1(i)^2 + a_{22}X_1(i)X_2(i) + a_{23}X_1(i)X_3(i)) = \Sigma(X_1(i)Y_2(i))$$
$$\Sigma(a_{21}X_1(i)X_2(i) + a_{22}X_2(i)^2 + a_{23}X_2(i)X_3(i)) = \Sigma(X_2(i)Y_2(i))$$
$$\Sigma(a_{21}X_1(i)X_3(i) + a_{22}X_2(i)X_3(i) + a_{23}X_3(i)^2) = \Sigma(X_3(i)Y_2(i))$$
$$\Sigma(a_{31}X_1(i)^2 + a_{32}X_1(i)X_2(i) + a_{33}X_1(i)X_3(i)) = \Sigma(X_1(i)Y_3(i))$$
$$\Sigma(a_{31}X_1(i)X_2(i) + a_{32}X_2(i)^2 + a_{33}X_2(i)X_3(i)) = \Sigma(X_2(i)Y_3(i))$$
$$\Sigma(a_{31}X_1(i)X_3(i) + a_{32}X_2(i)X_3(i) + a_{33}X_3(i)^2) = \Sigma(X_3(i)Y_3(i))$$

We convert this to matrix form.

$$\sum \overbrace{\begin{bmatrix} x_1(i)^2 & x_1(i)x_2(i) & x_1(i)x_3(i) & 0 & 0 & 0 & 0 & 0 & 0 \\ x_1(i)x_2(i) & x_2(i)^2 & x_2(i)x_3(i) & 0 & 0 & 0 & 0 & 0 & 0 \\ x_1(i)x_3(i) & x_2(i)x_3(i) & x_3(i)^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & x_1(i)^2 & x_1(i)x_2(i) & x_1(i)x_3(i) & 0 & 0 & 0 \\ 0 & 0 & 0 & x_1(i)x_2(i) & x_2(i)^2 & x_2(i)x_3(i) & 0 & 0 & 0 \\ 0 & 0 & 0 & x_1(i)x_3(i) & x_2(i)x_3(i) & x_3(i)^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & x_1(i) & x_1(i)x_2(i) & x_1(i)x_3(i) \\ 0 & 0 & 0 & 0 & 0 & 0 & x_1(i)x_2(i) & x_2(i)^2 & x_2(i)x_3(i) \\ 0 & 0 & 0 & 0 & 0 & 0 & x_1(i)x_3(i) & x_2(i)x_3(i) & x_3(i)^2 \end{bmatrix}}^{B} \underbrace{\begin{bmatrix} a_{11} \\ a_{12} \\ a_{13} \\ a_{21} \\ a_{22} \\ a_{23} \\ a_{31} \\ a_{32} \\ a_{33} \end{bmatrix}}_{a}$$

$$= \sum \underbrace{\begin{bmatrix} x_1(i) & 0 & 0 \\ x_2(i) & 0 & 0 \\ x_3(i) & 0 & 0 \\ 0 & x_1(i) & 0 \\ 0 & x_2(i) & 0 \\ 0 & x_3(i) & 0 \\ 0 & 0 & x_1(i) \\ 0 & 0 & x_2(i) \\ 0 & 0 & x_3(i) \end{bmatrix} \begin{pmatrix} y_1(i) \\ y_2(i) \\ y_3(i) \end{pmatrix}}_{D}$$

$Ba = D$

$a = B^{-1} D$

We solve this in MATLAB

2) (cont.)

A =

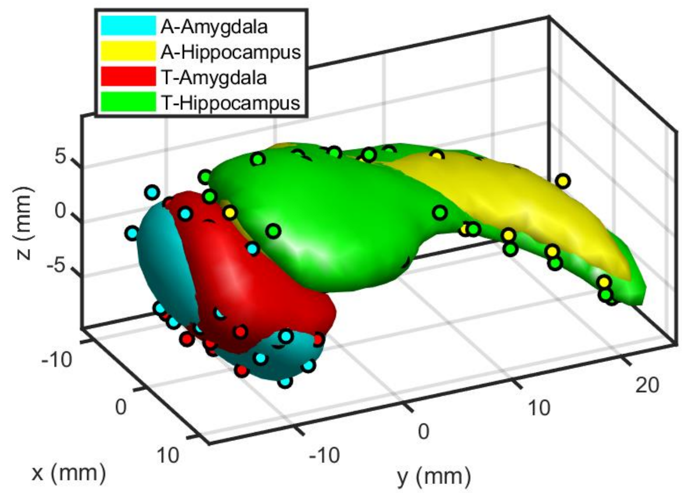
1.0435	-0.1166	0.0048
0.0976	0.9312	-0.0901
-0.0043	-0.0578	1.0498

Error Before = 471.4809

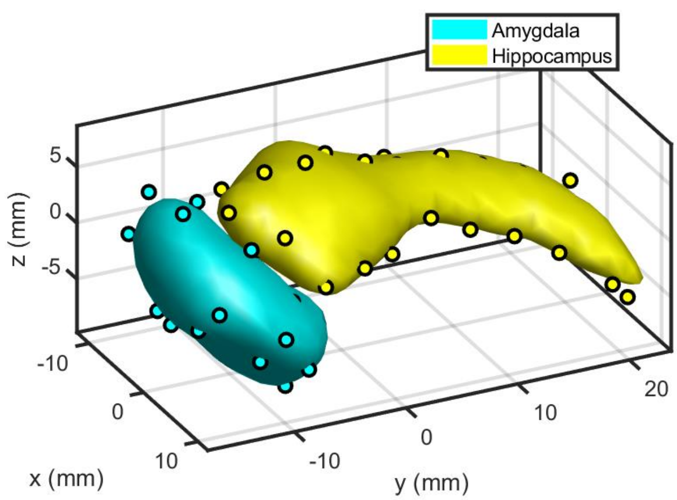
Error After = 221.9677

3)

Overlaid Linearly Transformed



Atlas Linearly Transformed



4) $K(x) = \exp(-1/2a^2 |x|^2)$

we know boundary conditions are

$v_i(X(j)) = Y_i(j) - X_i(j) = \sum_{k=1}^{58} \exp(-1/2a^2 |X(j) - X(k)|^2) p_i(k)$

for $i = 1:3$ and $j = 1:58$

Therefore, we solve these boundary conditions simultaneously in matrix form to get p.

$$v_1(x(1)) = \exp(-1/2a^2 |X(1) - X(1)|^2) p_1(1) + \dots + \exp(-1/2a^2 |X(1) - X(58)|^2) p_1(58)$$

$$\vdots$$

$$v_1(X(58)) = \exp(-1/2a^2 |X(58) - X(1)|^2) p_1(1) + \dots + \exp(-1/2a^2 |X(58) - X(58)|^2) p_1(58)$$

Thus in matrix form,

$$\underbrace{\begin{bmatrix} v_1(x(1)) \\ v_1(x(2)) \\ \vdots \\ v_1(x(58)) \end{bmatrix}}_{V_1} = \underbrace{\begin{bmatrix} 1 & \cdots & \exp(-1/2a^2 |X(1) - X(58)|^2) \\ \exp(-1/2a^2 |X(2) - X(1)|^2) & \cdots & \exp(-1/2a^2 |X(2) - X(58)|^2) \\ \vdots & \ddots & \vdots \\ \exp(-1/2a^2 |X(58) - X(1)|^2) & \cdots & 1 \end{bmatrix}}_{\hat{K}} \underbrace{\begin{bmatrix} p_1(1) \\ p_1(2) \\ \vdots \\ p_1(58) \end{bmatrix}}_{p_1}$$

Therefore $p_1 = \hat{K}^{-1} V_1$

Boundary conditions are similar for V_2 & V_3, so we can keep $\hat{K}$ and find p_2 & p_3 with

$$p_2 = \hat{K}^{-1} V_2 \qquad \text{where } V_i = \begin{bmatrix} V_i(x(1)) \\ V_i(x(2)) \\ \vdots \\ V_i(x(58)) \end{bmatrix} \text{ \& } p_i = \begin{bmatrix} p_i(1) \\ p_i(2) \\ \vdots \\ p_i(58) \end{bmatrix}$$

$$p_3 = \hat{K}^{-1} V_3$$

we solve this in MATLAB

4) (cont.)

P =

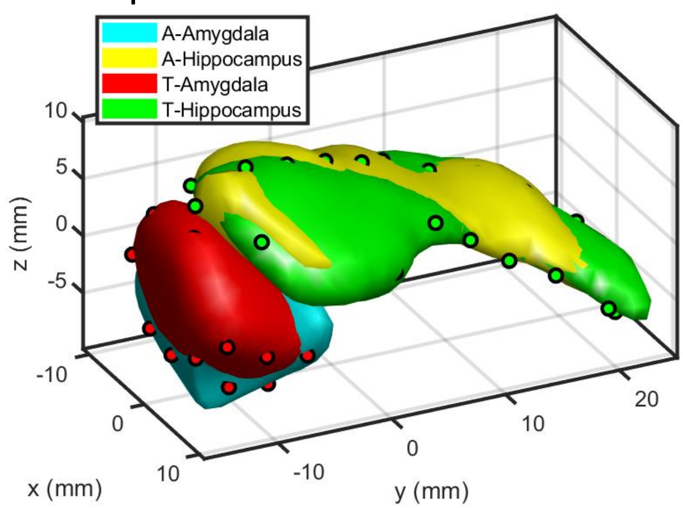
36.3930	3.4814	-7.3591
4.9496	0.0136	-16.5687
-14.6959	1.4380	2.4587
-30.0476	-2.7621	10.0830
4.8956	1.0144	3.4185
-12.2185	-8.5541	20.5845
14.9122	17.0509	-20.2620
9.8577	-8.1939	12.8816
-4.4493	-2.4001	-12.4893
-1.6206	-2.9781	-6.3578
1.6878	-1.4310	-4.6937
23.4973	20.5043	-61.7450
9.3774	46.2278	-77.9738
-6.6424	2.1145	0.0072
-7.3715	-1.3594	-1.7040
2.4844	0.8694	8.0825
0.7763	-2.5056	-2.0368
2.7947	12.5696	16.2275
-6.4765	7.3708	12.4672
10.6475	0.8588	8.2977
-25.9823	-42.6838	59.3782
-2.1744	0.6096	1.5547
-18.4181	-25.8745	54.0974
-22.5705	-19.1685	56.9545
15.6552	20.7136	-21.9417
7.1489	-8.7482	16.7215
19.2914	1.4616	-52.8252
8.8905	7.6500	-32.1501
-13.4886	-7.9237	4.9541
7.8020	-9.8015	2.0418
-22.7570	-10.0095	26.2676
-7.6673	-3.7864	20.4610
9.7956	4.0870	5.0780
-9.5267	8.3494	-1.0204
12.4112	5.3231	4.0478
7.2801	-0.9929	-21.4139
-2.4872	-2.6388	-6.7374
12.6966	-3.9567	0.3832
-8.6790	-4.0360	-11.8265
-5.2615	1.8980	18.3498
-3.7634	6.0044	15.7059
-4.7619	2.3971	-2.1231
-1.7441	3.8904	13.6478
1.5055	-1.1803	-9.9687
13.3203	-9.1914	-23.7663
3.8992	1.1768	8.7578

17.3280	-8.3744	-20.5300
4.8302	0.9785	4.8800
-25.8002	11.9211	24.0445
-1.4885	-3.2874	-11.8512
-11.0081	5.1087	14.1001
-15.8074	1.4416	0.9635
23.0732	-6.6766	-11.8490
0.2364	2.5111	7.5397
3.4874	-3.7448	-7.7023
6.6178	1.4740	3.3739
-7.6966	1.9264	-1.1753
1.3931	-0.9853	-3.5382

Error = 6.1922e-27

5)

Overlaid Spline Transformed



Atlas Spline Transformed