140.651 - Lab 11

Q5 and Q9 are nice exercises and Q7,8,10

HW5 Problem 5

Problem 5. Perform the following simulation. Randomly simulate $1,000$ sample means of size 16 from a normal distribution with means 5 and variances 1. Calculate $1,000$ test statistics for a test of $H_0 : \mu = 5$ versus $H_a : \mu < 5$. Using these test statistics calculate $1,000$ P-values for this test. Plot a histogram of the P-values. Note, this exercise demonstrates the interesting fact that the distribution of P-values is uniform.

- t-test formulation:

$$H_0 : \mu = \mu_0 \overset{here}{=} 5$$
$$H_1 : \mu < \mu_0 \overset{here}{=} 5$$

- Test statistics:

$$TS = \frac{\overline{X}_n - \mu_0}{S/\sqrt{n}} \sim \text{ Student's } t\text{-distribution with } n-1 \text{ degrees of freedom}$$

In our case: $n = 16$.

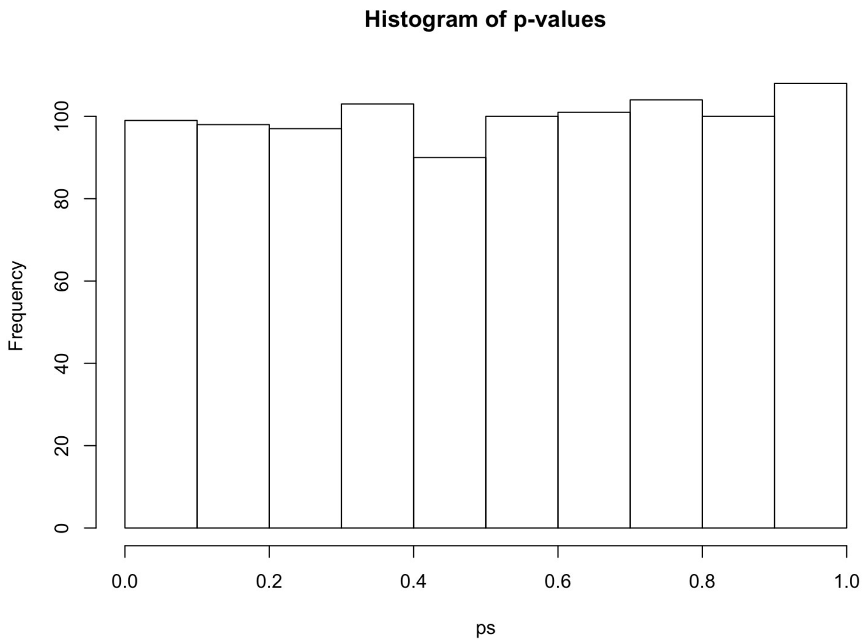
- Recall: p-value of a test is the probability of seeing a result at least as extreme as the one that we actually observe, assuming the null hypothesis is true. Our result can be summarized as t , that is, value of test statistic TS that we have actually observed in our sample data.
- Given we work with one-side t-test, the probability that the result is at least as extreme as the one we observe is

$$P(TS \leq t)$$

which, knowing that $TS \sim$ Student's t-distribution with $n-1$ degrees of freedom , can be computed by evaluating probability function for Student's t -distribution with $n-1$ degrees of freedom.

p - value

```
set.seed(123)

## Simulate matrix of 1000 rows and 16 columns
## Each row is sample of 16 random variables from N(5,1)
x <- matrix(rnorm(1000 * 16, mean = 5, sd = 1), nrow = 1000)

## For each out of 1,000 samples, compute mean of 16-element sample
means <- apply(x, 1, mean)

## For each out of 1,000 samples, compute SD of 16-element sample
sds <- apply(x, 1, sd)

## For each out of 1,000 samples, compute value of t
## (t := observed value of a test statistic)
ts <- (means - 5)/(sds/sqrt(16))

## For each out of 1,000 samples, compute P(TS <= t)
## that is, evaluate probability function for
## Student's $t$-distribution with $n-1$ degrees of freedom
ps <- pt(ts, df = 16-1)

## Histogram of obtaines values
hist(ps, main = "Histogram of p-values")
```



- Alternatively, the above could be computed using built-in t-test function

```
## For each out of 1,000 samples, compute p-value
ps_B <- apply(x, 1, function(x.row) {
  t.test.out <- t.test(x.row, alternative = "less", mu = 5)
  t.test.out$p.value
})

## Check if all elements of vector ps are equal to elements of vector ps_B
all(ps == ps_B)
```

```
## [1] TRUE
```

HW5 Problem 9

Problem 9. Researchers studying brain volume found that in a random sample of 16 sixty five year old subjects with Alzheimer's disease, the average loss in grey matter volume as a person aged four years was .1 mm^3 with a standard deviation of .04 mm^3.

a. Calculate and interpret a P-value for the hypothesis that there is no loss in grey matter volumes as people age. Show your work.

b. The researchers would now like to plan a similar study in 100 healthy adults to detect a four year mean loss of .01 mm^3. Motivate a general formula for power calculations in this setting and calculate the power for a test with $\alpha = .05$? Assume that the variation in grey matter loss will be similar to that estimated in the Alzheimer's study.

(a)

We assume loss in grey matter volume in this population is normally distributed.

- t-test formulation:

$$H_0 : \mu = \mu_0 \overset{here}{=} 0 \text{ (no loss in grey matter volumes)}$$
$$H_1 : \mu \neq \mu_0 \overset{here}{=} 0 \text{ (there is loss in grey matter volumes)}$$

- Test statistics:

$$TS = \frac{\overline{X}_n - \mu_0}{S/\sqrt{n}} \sim \text{Student's } t\text{-distribution with } n-1 \text{ degrees of freedom}$$

In our case: $n = 16$.

- Given we work with one-side t-test, the probability that the result is at least as extreme as the one we observe is

$$P(TS \geq |t|) + P(TS \leq -|t|) = 2 \cdot P(TS \geq |t|)$$

- We have from data: $\overline{X}_n = 0.1$, $\mu_0 = 0$, $S = 0.04$, $n = 16$, hence observed value of test statistic TS is

$$t = \frac{0.1 - 0}{0.04/\sqrt(16)} = 10$$

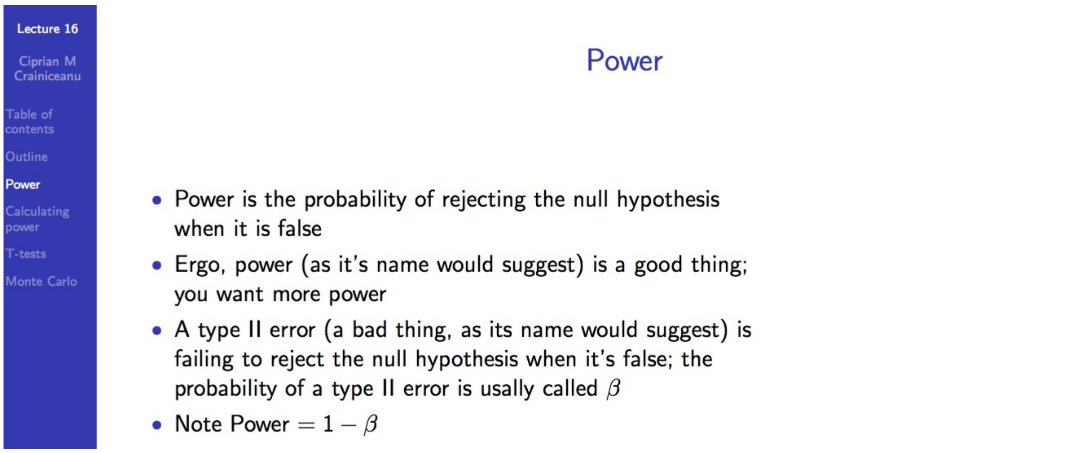
```
t <- (0.1) / (0.04 / sqrt(16))
2 * pt(abs(t), df = 16-1, lower.tail = FALSE)
```

```
## [1] 4.996898e-08
```

- Therefore we reject the null hypothesis at singificance level $\alpha = 0.05$ for this population and conclude that there is a statistically significant change in grey matter volume.

(b)

Relevant lecture notes:



- Consider μ_α – the likely value of true unknown parameter μ at which we want to evaluate the power; μ_α is chosen subjectively to reflect the likely value of μ from the researcher's prior knowledge (following how this (https://onlinecourses.science.psu.edu/stat500/node/46/) resurce describes it)
- In this problem:

$$\mu_0 = 0$$
$$\mu_\alpha = 0.01$$

t-test formulation:

$$H_0 : \mu = \mu_0 = 0$$
$$H_\alpha : \mu > \mu_0 = 0$$

Test statistic:

$$TS = \frac{\overline{X} - \mu_0}{S/\sqrt{n}}$$

Reject the null hypothesis at significance level $alpha$ when

$$TS \geq t_{1-\alpha, n-1}$$

(or: $TS \geq z_{1-\alpha}$ if it was said that n is quite large).

- Following the lecture notes (Lecture 17, slides 10, 11) we have:

power

$$
\begin{aligned}
&= P\left(TS > t_{1-\alpha,n-1} \,|\, \mu = \mu_a\right) \\
&= P\left(\frac{\overline{X} - \mu_0}{S/\sqrt{n}} > t_{1-\alpha,n-1} \,|\, \mu = \mu_a\right) \\
&= P\left(\sqrt{n}(\overline{X} - \mu_0) > t_{1-\alpha,n-1} \cdot S \,|\, \mu = \mu_a\right) \\
&= P\left(\frac{\sqrt{n}(\overline{X} - \mu_0)}{\sigma} > t_{1-\alpha,n-1} \cdot \frac{S}{\sigma} | \, \mu = \mu_a\right) \\
&= P\left(\frac{\sqrt{n}(\overline{X} - \mu_a + \mu_a - \mu_0)}{\sigma} > t_{1-\alpha,n-1} \cdot \frac{S}{\sigma} | \, \mu = \mu_a\right) \\
&= P\left(\frac{\sqrt{n}(\overline{X} - \mu_a)}{\sigma} + \frac{\sqrt{n}(\mu_a - \mu_0)}{\sigma} > t_{1-\alpha,n-1} \cdot \frac{S}{\sigma} | \, \mu = \mu_a\right) \\
&= P\left(\frac{\sqrt{n}(\overline{X} - \mu_a)}{\sigma} + \frac{\sqrt{n}(\mu_a - \mu_0)}{\sigma} > t_{1-\alpha,n-1} \cdot \sqrt{\frac{S^2}{\sigma^2}} \,|\, \mu = \mu_a\right) \\
&= P\left(\frac{\sqrt{n}(\overline{X} - \mu_a)}{\sigma} + \frac{\sqrt{n}(\mu_a - \mu_0)}{\sigma} > \frac{t_{1-\alpha,n-1}}{\sqrt{(n-1)}} \cdot \sqrt{\frac{(n-1)S^2}{\sigma^2}} \,|\, \mu = \mu_a\right) \\
&= P\left(Z + \frac{\sqrt{n}(\mu_a - \mu_0)}{\sigma} > \frac{t_{1-\alpha,n-1}}{\sqrt{(n-1)}} \cdot \sqrt{\chi^2_{n-1}} \,|\, \mu = \mu_a\right)
\end{aligned}
$$

where Z and χ^2_{n-1} are independent standard normal and chi-squared random variables, respectively.

We use the above general result to obtain power calculations in this problem after plugging $\mu_0 = 0$, $\mu_a = 0.01$, $n = 100$ and σ (that we should know).

HW5 Problem 7

Problem 7. Will a Student's T or Z hypothesis test for a mean with the data recorded in pounds always agree with the same test conducted on the same data recorded in kilograms? (explain)

- Since the standard deviations are in the same units as the data collected, the true population standard deviation units of a Z test or the sample standard deviation units in a T-test will cancel with the units of the given data. **Therefore, the T and Z test statistics are unit-less.** As a result, **the units on certain data will not affect the outcome of the test and will give the same test statistics**.
- Example: we have data $X_1, \ldots, X_n$ measured in kilograms with mean and variance μ_X , σ^2_X . If we let $Y_1, \ldots, Y_n$ be the same data data expressed in pounds, then

$$
\begin{aligned}
Y_i &= 2.2 \cdot X_i \\
\overline{Y}_n &= 2.2 \cdot \overline{X}_n \\
var(Y) &= (2.22)^2 \cdot \sigma^2_X \\
sd(Y) &= 2.22 \cdot \sigma_X
\end{aligned}
$$

HW5 Problem 8

Problem 8. A researcher consulting you is very concerned about falsely rejecting her null hypothesis. As a result the researcher decides to increase the sample size of her study. Would you have anything to say? (explain)

- An increase in sample size generally will make a study more powerful to detect a difference if one truly exists, but will not shield against false rejection of the null hypothesis (is other words, will not protect from rejecting null hypothesis if there is no difference).
- The **significance level α is the probability of falsely rejecting null hypothesis**, so if a researcher is concerned about falsely rejecting her null hypothesis, she should lower α .

HW5 Problem 10

Problem 10. A recent Daily Planet article reported on a study of a two week weight loss program. The study reported a 95% confidence interval for weight loss from baseline of [2 lbs, 6 lbs]. (There was no control group, all subjects were on the weight loss program.) The exact sample size was not given, though it was known to be over 200.

a. What can be said of a $\alpha = 5\%$ hypothesis test of whether or not there was any weight change from baseline? Can you determine the result of a $\alpha = 10\%$ test without any additional calculation or information? (explain your answer)

- Recall that the definition of a p-value is the probability of seeing a test statistic as or more extreme than the one you obseve, given your null hypothesis.
- If you reject the null hypothesis for $\alpha = 0.05$, this implies the probability of seeing a test statistic as or more extreme than the one you observe is less than 0.05 . It follows immediately that this probability is also less than 0.1 , leading us to reject the null hypothesis for $\alpha = 0.1$ whenever we reject for $\alpha = 0.05$.