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Problem 13.3

(13.3) Suppose that $f : \Omega \to R^1$. Show that f is measurable $T^{-1}\mathscr{F}'$ if and only if there exists a map $\varphi : \Omega' \to R'$ such that φ is measurable $\mathscr{F}'$ and $f = \varphi T$. *Hint:* First consider simple functions and then use Theorem 13.5.

Proof. First we prove the backward implication. Let $\varphi : \Omega' \to R'$ such that φ is measurable $\mathscr{F}'$ and $f = \varphi T$. By construction T is measurable $T^{-1}\mathscr{F}'/\mathscr{F}'$ and by assumption, φ is measurable $\mathscr{F}'/R^1$. Hence, by theorem 13.1 (ii), $f = \varphi T$ is measurable $T^{-1}\mathscr{F}'/R^1$ as desired.

Now we show the forward implication. Let f be measurable $T^{-1}\mathscr{F}'$. The rest of this proof comes from "Notes on the Problems" in Billingsley. By Theorem 13.5, there exist simple functions f_n, measurable $T^{-1}\mathscr{F}'$, such that $f_n(\omega) \to f(\omega)$ for each ω. Since f_n is simple and measurable $T^{-1}\mathscr{F}'$, we can write $f_n = \sum_i x_{ni} I_{A_{ni}}$ with $A_{ni} \in T^{-1}\mathscr{F}'$. Take $A'_{ni} \in \mathscr{F}'$ so that $A_i = T^{-1}A'_{ni}$ and set $\varphi_n = \sum_i x_{ni} I_{A'_{ni}}$. By construction φ_n is measurable $\mathscr{F}'$. It follows that $f_n = \varphi_n T$ for all n. Taking the limit of both sides, we have $f = \varphi T$. Let C' be the set of ω' for which $\varphi_n(\omega')$ has a finite limit, and define $\varphi(\omega') = \lim_n \varphi_n(\omega')$ for $\omega' \in C'$ and $\varphi(\omega') = 0$ for $\omega' \notin C'$. Since $f_n \to f$ for all $\omega \in \Omega$, it follows that $T(\Omega) \subset C'$. Finally, $C' \in \mathscr{F}'$, therefore, φ is measurable $\mathscr{F}'$ by Theorem 13.4 (iii). $\square$