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550.620
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Assignment 3
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550.620 Homework #3 (to turn in)

(a) Let $E_1, E_2, \dots$ be an arbitrary sequence of events satisfying

$$\text{(i)} \lim_n P(E_n) = 0 \qquad \text{and} \qquad \text{(ii)} \sum_n P(E_n \cap E_{n+1}^c) < \infty.$$

Prove that $P(E_n \text{ i.o.}) = 0$.

(b) Show that the result of part (a) strengthens the first Borel–Cantelli Lemma by showing that it implies the first Borel–Cantelli Lemma.

(c) Deduce that the result of part (a) *strictly* strengthens the first Borel–Cantelli Lemma by providing an explicit example of a probability space $(\Omega, \mathscr{F}, P)$ and a sequence of events $E_1, E_2, \dots$ such that $\sum_n P(E_n) = \infty$ but the result of part (a) allows us to conclude that $P(E_n \text{ i.o.}) = 0$.

Solution:

(a) First we prove a lemma that

$$\limsup_n A_n - \liminf_n A_n = \limsup_n (A_n \cap A_{n+1}^c)$$

Proof of lemma: Calculating, it is clear that

$$\begin{aligned}
\limsup_n A_n - \liminf_n A_n &= (\limsup_n A_n) \cap (\liminf_n A_n)^c \\
&= (\bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} A_n) \cap (\bigcup_{m=1}^{\infty} \bigcap_{n=m}^{\infty} A_n)^c \\
&= (\bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} A_n) \cap (\bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} A_n^c) \text{ by DeMorgan's laws}
\end{aligned}$$

Our task is to show that this set is the same as

$$\limsup_n (A_n \cap A_{n+1}^c) = (\bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} A_n \cap A_{n+1}^c)$$

Now we show set containment in both directions. Suppose $\omega \in (\cap_{m=1}^{\infty} \cup_{n=m}^{\infty} A_n \cap A_{n+1}^c)$. Hence, for all integer $m \geq 1$, there exists integer $n \geq m$ such that $\omega \in A_n$ and $\omega \in A_{n+1}^c$. That is equivalent to the definition of ω occuring infinitely often, or that $\omega \in \limsup_n A_n = \cap_{m=1}^{\infty} \cup_{n=m}^{\infty} A_n$ and $\omega \in \limsup_n A_{n+1}^c = \cap_{m=1}^{\infty} \cup_{n=m}^{\infty} A_{n+1}^c = \cap_{m=1}^{\infty} \cup_{n=m}^{\infty} A_n^c$. We can change the index on A_{n+1} to A_n because the lim sup has all outcomes that occur infinitely often. Those events happen in the diminishing tail union. Therefore, $\omega \in (\cap_{m=1}^{\infty} \cup_{n=m}^{\infty} A_n) \cap (\cap_{m=1}^{\infty} \cup_{n=m}^{\infty} A_n^c)$. Now for set containment in the other direction. Suppose $\omega \in (\cap_{m=1}^{\infty} \cup_{n=m}^{\infty} A_n) \cap (\cap_{m=1}^{\infty} \cup_{n=m}^{\infty} A_n^c)$. Thus $\omega \in \cap_{m=1}^{\infty} \cup_{n=m}^{\infty} A_n$ and $\omega \in \cap_{m=1}^{\infty} \cup_{n=m}^{\infty} A_n^c = \limsup_n A_n^c$. By definition, for all integer $M \geq 1$ there exists integer $m > M$ such that $\omega \in A_m^c$. By the well-ordering principle, there exists a least integer m' such that $\omega \in A_{m'}^c$. It follows that $\omega \notin A_{m'-1}^c$ and $\omega \in A_{m'-1}$. Let $n = m' - 1$. Hence $\omega \in A_n \cup A_{n+1}^c$, and it is clear that $\omega \in \limsup_n A_n \cup A_{n+1}^c$. Thus we have shown set containment in both directions, and we conclude that

$$\limsup_n A_n - \liminf_n A_n = \limsup_n (A_n \cap A_{n+1}^c)$$

to complete the lemma.

Now we prove problem (a). By the lemma, $\limsup_n E_n - \liminf_n E_n = \limsup_n (E_n \cap E_{n+1}^c)$. Taking probabilities, we have $P(\limsup_n A_n - \liminf_n A_n) = P(\limsup_n (A_n \cap A_{n+1}^c))$. Since $\liminf_n E_n \subset \limsup_n E_n$, it follows that $P(\limsup_n E_n - \liminf_n E_n) = P(\limsup_n E_n) - P(\liminf_n E_n)$. By Mini-Fatou's Lemma and assumption (i) in the statement of the problem, we know that

$$0 = \lim_n P(E_n) = \liminf_n P(E_n) \geq P(\liminf_n E_n) \geq 0.$$

Clearly, $P(\liminf_n E_n) = 0$. Since the series in condition (ii) converges, the summand converges to 0.

But now we can't bound $P(\limsup_n (A_n \cap A_{n+1}^c))$. This proof is flawed.