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Assignment 4
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Homework #4 (to turn in)

Define an "inverse" $F^{\sim}$ to F by the recipe

$$F^{\sim}(t) := \inf\{x : F(x) \geq t\},\ 0 < t < 1$$

Show that $F^{\sim}$ is

(a) increasing

(b) *left* continuous

(c) a map of $(0, 1)$ into $\mathbf{R}$.

Solution:

(a) We show $F^{\sim}$ is increasing (or more precisely, non-decreasing). Let $s, t \in (0, 1)$ and assume $s \leq t$. Define $A = \{x : F(x) \geq s\}$ and $B = \{x : F(x) \geq t\}$. Clearly, if $x \in B$, then $F(x) \geq t \geq s$. It follows that $x \in A$. Thus $B \subset A$. Thus $\inf A$ is a lower bound of B. Since $\inf B$ is the largest lower bound, $\inf B \geq \inf A$. This is equivalent to $F^{\sim}(t) \geq F^{\sim}(s)$. Therefore $t \geq s$ implies $F^{\sim}(t) \geq F^{\sim}(s)$, showing $F^{\sim}$ is increasing. $\square$

(b) Here we show $F^{\sim}$ is left continuous. Let $\{t_n\} \subset (0, 1)$ with $t_n \uparrow t$, by which we mean that $\{t_n\}$ is a monotone increasing sequence with limit t. Since $F^{\sim}$ is increasing by (a), it follows that $\{F^{\sim}(t_n)\}$ is an increasing sequence with $F^{\sim}(t_n) \leq F^{\sim}(t)$ for all integer n. Note $F^{\sim}(t)$ is finite by (c). Furthermore, since $\{F^{\sim}(t_n)\}$ is a bounded and monotone sequence, it has a limit, call it m, according to the Monotone Convergence Theorem. Taking the limit of $F^{\sim}(t_n) \leq F^{\sim}(t)$, we have

$$\begin{aligned} \lim_n F(t_n) &\leq \lim_n F^{\sim}(t) \\ m &\leq F^{\sim}(t). \end{aligned}$$

We now show $F^{\sim}(t) \leq m$ to conclude $F^{\sim}(t) = m = \lim_n F^{\sim}(t_n)$. Equation (2) of the handout states

$$F(F^{\sim}(t)) \geq t.$$

Hence for all n, $F(F^{\sim}(t_n)) \geq t_n$. Because $F^{\sim}$ is increasing, $F^{\sim}(t_n) \uparrow m$, and it follows that for all n, $F^{\sim}(t_n) \leq m$. Furthermore, since F is increasing, $F(F^{\sim}(t_n)) \leq F(m)$. Linking the two inequalities together, it is clear $t_n \leq F(F^{\sim}(t_n)) \leq F(m)$. Taking the limit we have

$$\begin{aligned} \lim_n t_n &\leq \lim_n F(m) \\ t &\leq F(m). \end{aligned}$$

Hence $m \in \{x : F(x) \geq t\}$, and $\inf\{x : F(x) \geq t\} = F^{\sim}(t) \leq m$. Therefore, we conclude $\lim_n F^{\sim}(t_n) = F^{\sim}(t)$, and we have demonstrated left continuity for $F^{\sim}$. $\square$

(c) Finally we show $F^{\sim}$ is a map of $(0,1)$ into $\mathbf{R}$. It is obvious that $F^{\sim}$ is either a real number or infinite. Let $t \in (0,1)$. Since F is normalized, it is clear that there exists x_1 such that $0 < F(x_1) < t$. Since F is increasing, x_1 is a lower bound to $\{x : F(x) \geq t\}$. Hence $x_1 \leq \inf\{x : F(x) \geq t\} = F^{\sim}(t)$. On the other hand, there exists x_2 such that $1 > F(x_2) > t$ because F is normalized. Clearly, $x_2 \in \{x : F(x) \geq t\}$, and $x_2 \geq \inf\{x : F(x) \geq t\} = F^{\sim}(t)$. Combining these inequalities, we have $x_1 \leq F^{\sim}(t) \leq x_2$, which shows that $F^{\sim}(t)$ is finite. Hence $F^{\sim}$ is a map of $(0,1)$ into $\mathbf{R}$. $\square$

From the text, "one has the important switching relation $t \leq F(x) \Leftrightarrow F^{\sim}(t) \leq x$ ... Supply the details." In other words, we prove that $t \leq F(x)$ if and only if $F^{\sim}(t) \leq x$.

Solution: First we prove the forward implication. Assume $t \leq F(x)$. Automatically $x \in \{y : F(y) \geq t\}$. Thus $x \geq \inf\{y : F(y) \geq t\} = F^{\sim}(t)$, completing the proof of the forward implication. Now we prove the backward implication. Assume $F^{\sim}(t) \leq x$. Since F is increasing, we have $F(F^{\sim}(t)) \leq F(x)$. By (2) in the handout, we have $F(F^{\sim}(t)) \geq t$. Thus $t \leq F(F^{\sim}(t)) \leq F(x)$. This completes the proof of the biconditional. $\square$