James K. Pringle
140.751
Dr. Brian Caffo
Assignment 1
28 September 2012, Friday

2.10 The (lower) incomplete gamma function is defined as $\Gamma(k,c) = \int_0^c x^{k-1}e^{-x}dx$. By convention $\Gamma(k,\infty)$, the complete gamma function, is written $\Gamma(k)$. Consider a density

$$\frac{1}{\Gamma(\alpha)}x^{\alpha-1}e^{-x} \text{ for } x > 0$$

where α is a known number.

a. Argue that this is a valid density.

Solution: Listed above in the statement of problem a is a density. Call it $f(x)$. To show that it is a valid density we need to show that $f(x) > 0$ and $\int_{-\infty}^{\infty} f(x)dx = 1$. First, we show that $f(x)$ is nonnegative. We examine each factor of $f(x)$ and show they are nonnegative. The function $\Gamma(k)$ is nonnegative because its integrand, $x^{k-1}e^{-x}$, is nonnegative when evaluated on nonnegative x values. Thus the other two factors of $f(x)$, which are precisely $x^{\alpha-1}e^{-x}$, are nonnegative. Thus each factor of $f(x)$ is positive. Now for the integral condition. It follows that

$$\int_{-\infty}^{\infty} f(x)dx = \int_0^{\infty} \frac{1}{\Gamma(\alpha)}x^{\alpha-1}e^{-x}dx = \frac{\int_0^{\infty} x^{\alpha-1}e^{-x}dx}{\Gamma(\alpha)} = \frac{\Gamma(\alpha)}{\Gamma(\alpha)} = 1$$

And hence $f(x)$ is a density.

b. Write out the survival function associated with this density using gamma functions.

Solution: The survival function $S(x) = 1 - F(x)$, and $F(x) = \int_0^x \frac{1}{\Gamma(\alpha)}x^{\alpha-1}e^{-x}dx$ for $x > 0$. Thus $S(x) = 1 - \int_0^x \frac{1}{\Gamma(\alpha)}x^{\alpha-1}e^{-x}dx = \frac{1}{\Gamma(\alpha)}\int_0^x x^{\alpha-1}e^{-x}dx = \frac{1}{\Gamma(\alpha)}\Gamma(\alpha,x)$.

c. Let β be a known number; argue that

$$\frac{1}{\beta^{\alpha}\Gamma\alpha}x^{\alpha-1}e^{-x/\beta} \text{ for } x > 0$$

is a valid density. This is known as the **gamma density**.

Solution: By the similar reasoning as in a, we know that the new density function is nonnegative. From the above work, and using a u-substition, $u = x/\beta$ and $u\beta = x$ we have

$$\int_0^\infty \frac{1}{\beta^\alpha\Gamma(\alpha)} x^{\alpha-1}e^{-x/\beta}dx = \int_0^\infty \frac{1}{\beta^\alpha\Gamma(\alpha)}(u\beta)^{\alpha-1}e^{-u}\beta du$$
$$= \int_0^\infty \frac{1}{\Gamma(\alpha)} u^{\alpha-1}e^{-u}du = 1$$

Thus we see that this new density is just a transformation of the original gamma density.

d. Plot the density for different values of α and β.

2.15 Let U be a uniform $(0,1)$ random variable. Calculate the distribution function and density of U^p where p is a power. What is the name of this density?

Solution: Let $V = U^p$. The distribution function of V is $F(x) = P(V \leq x) = P(U^p \leq x) = P(U \leq x^{1/p})$. It is given that the distribution of U is

$$G(x) = \begin{cases} 0 & \text{if } x < 0, \\ x & \text{if } 0 \leq x \leq 1, \\ 1 & \text{if } x > 1 \end{cases}$$

To find $F(x)$ we looke at $G(x^{1/p})$. Hence if p is positive, we have

$$F(x) = \begin{cases} 0 & \text{if } x < 0, \\ x^{1/p} & \text{if } 0 \leq x^{1/p} \leq 1, \text{ or } 0 \leq x \leq 1 \\ 1 & \text{if } x > 1 \end{cases}$$

and the density function is $f(x) = \frac{1}{p}x^{1/p-1}$ for $x \in (0,1)$ and 0 elsewhere.

4.4 A particularly sadistic warden has three prisoners, A, B and C. He tells prisoner C that the sentences are such that two prisoners will be executed and let one free, though he will not say who has what sentence. Prisoner C convinces the warden to tell him the identity of one of the prisoners to be executed. The warden has the following strategy, which prisoner C is aware of. If C is sentenced to be let free, the warden flips a coin to pick between A and B and tells prisoner C that persons sentence. If C is sentenced to be executed he gives the identity of whichever of A or B is also sentenced to be executed.

a. Does this new information about one of the other prisoners give prisoner C any more information about his sentence?

Solution: Let A, B, C be the event that prisoner A, B, C, respectively, is condemned, and the complement be the event that the prisoner is set free. It doesn't matter which prisoner C is told about, so let's say A. Hence, if C learns something about A, we can model that with conditional probability. $P(C|A) = P(C \cap A)/P(A) = (1/3)/(2/3) =$

1/2. Also, $P(C|A^C) = P(C \cap A^C)/P(A^C) = (1/3)/(1/3) = 1$. Hence if A is condemned, then we would expect C has a one half chance of dying, and if A is set free, then C is surely going to die. Thus this knowledge doesn't really add any information.

b. The warder offers to let prisoner C switch sentences with the other prisoner whose sentence he has not identified. Should he switch?

   Solution: Of course he should switch if A is condemned to die. In that case the probability B lives is $P(B^C|A) = P(B \cap A)/P(A) = 1/2$ which is a higher probability than the 1/3 that C has initially. If A is set free, then it doesn't matter what C does because he will surely die.

7.5 Suppose that DBPs drawn from a certain population are normally distributed with a mean of 90 mmHg and standard deviation of 5 mmHg. Suppose that 1,000 people are drawn from this population.

a. If you had to guess the number of people having DBPs less than 80 mmHg what would you guess?

   Solution: People with less than 80mmHg DBP have a z-score less than -2. If I had to guess, I would guess $P(z < -2) = \texttt{pnorm}(-2) = 0.02275$.

b. You draw 25 people from this population. Whats the probability that the sample average is larger than 92 mmHg?

   Solution: $\bar{X} \sim N(\mu, \sigma^2/n)$. The z-score for such a sample average is $z = \frac{\hat{\mu}-\mu}{\sigma/\sqrt{n}} = \frac{92-90}{5/5} = 2$. And $P(z > 2) = P(z < -2) = 0.02275$.

c. You select 5 people from this population. Whats the probability that 4 or more of them have a DBP larger than 100 mmHg?

   Solution: This is precisely the sum of the probabilities that four of the five have a DBP larger than 100 and that five of the five have DBP larger than 100. Since 100 mmHg is a z-score of 2, $P(DBP > 100) = P(Z > 2) = 0.02275$. Thus the sum of two probabilities above is $P(\text{five of five DBP} > 100) + P(\text{four of five DBP} > 100) = P(z > 2)^5 + \binom{5}{1}P(z > 2)^4(1 - P(z > 2)) = 5 \cdot 0.02275^4 \cdot (1 - 0.02275) = 1.314978e - 06$.

8.6 Often infection rates per time at risk are modelled as Poisson random variables. Let X be the number of infections and let t be the person days at risk. Consider the Poisson mass function $(t\lambda)^x e^{-t\lambda}/x!$. The parameter λ is called the population incident rate.

A. Derive the ML estimate for λ.

   Solution: First we take the log of the mass function, then we differentiate with respect to λ since we want to maximize the mass function by varying λ. Calculating, $\log((t\lambda)^x e^{-t\lambda}/x!) = x\log(t\lambda) - t\lambda - \log x!$. Differentiating and setting it equal to 0, then solving for λ we have

$$\begin{aligned} x(1/(t\lambda))t - t &= 0 \\ x(1/(t\lambda)) &= 1 \\ x/t &= \lambda \end{aligned}$$

B. Suppose that 5 infections are recorded per 1000 person-days at risk. Plot the likelihood.

Solution: We will plot $f(\lambda) = (1000\lambda)^5 e^{-1000\lambda}/5!$.

C. Suppose that five independent hospitals are monitored and that the infection rate (λ) is assumed to be the same at all five. Let X_i, t_i be the count of the number of infections and person days at risk in hospital i. Derive the ML estimate of λ.

Solution: Because the hospitals are assumed independent, the joint mass density function is

$$\prod_{i=1}^{5} \frac{(t_i\lambda)^{x_i} e^{-t_i\lambda}}{x_i}$$

As before, we will take the log, then the derivative with respect to λ. The log of the density is

$$\log(\prod_{i=1}^{5} \frac{(t_i\lambda)^{x_i} e^{-t_i\lambda}}{x_i}) = \sum_{i=1}^{5}(x_i \log(t_i\lambda) - t_i\lambda - \log(x_i!))$$

Setting the derivative equal to 0, we have

$$\begin{aligned} \frac{d}{d\lambda}(\sum_{i=1}^{5}(x_i \log(t_i\lambda) - t_i\lambda - \log(x_i!))) &= 0 \\ \sum_{i=1}^{5}(x_i/\lambda - t_i) &= 0 \\ \sum_{i=1}^{5} x_i / \sum_{i=1}^{5} t_i &= \lambda \end{aligned}$$

is the ML estimate of λ.