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550.620
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Assignment 3
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550.620 Homework #3 (to turn in)

(a) Let $E_1, E_2, \ldots$ be an arbitrary sequence of events satisfying

$$\text{(i)} \lim_n P(E_n) = 0 \qquad \text{and} \qquad \text{(ii)} \sum_n P(E_n \cap E_{n+1}^c) < \infty.$$

Prove that $P(E_n \text{ i.o.}) = 0$.

(b) Show that the result of part (a) strengthens the first Borel–Cantelli Lemma by showing that it implies the first Borel–Cantelli Lemma.

(c) Deduce that the result of part (a) *strictly* strengthens the first Borel–Cantelli Lemma by providing an explicit example of a probability space $(\Omega, \mathscr{F}, P)$ and a sequence of events $E_1, E_2, \ldots$ such that $\sum_n P(E_n) = \infty$ but the result of part (a) allows us to conclude that $P(E_n \text{ i.o.}) = 0$.

Solution:

(a) First we prove a lemma that

$$\bigcup_{i=n}^{m} E_n = \left(\bigcup_{i=n}^{m-1} (E_i \cap E_{i+1}^c) \right) \bigcup E_m$$

by strong induction, starting the index at arbitrary positive integer n. The base case is for $m = n+1$, and the statement is

$$\cup_{i=n}^{n+1} E_n = \cup_{i=n}^{n} (E_i \cap E_{i+1}^c) \cup E_{n+1}$$

Calculating on the right-hand side,

$$\begin{aligned} \cup_{i=n}^{n} (E_i \cap E_{i+1}^c) \cup E_{n+1} &= (E_n \cap E_{n+1}^c) \cup E_{n+1} \\ &= (E_n \setminus E_{n+1}) \cup E_{n+1} \\ &= E_n \cup E_{n+1} \\ &= \cup_{i=n}^{n+1} E_n \end{aligned}$$

as desired. Now we move to the inductive step. Suppose the statement is true for all integers m with $n \le m \le k$. We show it is true for $m = k+1$. In particular, we show

$$\cup_{i=n}^{k+1} E_n = \cup_{i=n}^{k} (E_i \cap E_{i+1}^c) \cup E_{k+1} \ .$$

Working with the right-hand side, we have

$$\begin{aligned} \cup_{i=n}^{k} (E_i \cap E_{i+1}^c) \cup E_{k+1} &= \cup_{i=n}^{k-1} (E_i \cap E_{i+1}^c) \cup (E_k \cap E_{k+1}^c) \cup E_{k+1} \\ &= \cup_{i=n}^{k-1} (E_i \cap E_{i+1}^c) \cup E_k \cup E_{k+1} \\ &= (\cup_{i=n}^{k-1} (E_i \cap E_{i+1}^c) \cup E_k) \cup E_{k+1} \\ &= (\cup_{i=n}^{k} E_i) \cup E_{k+1} \text{ by the inductive hypothesis} \\ &= \cup_{i=n}^{k+1} E_i \end{aligned}$$

as desired. Thus by strong induction, the statement

$$\bigcup_{i=n}^{m} E_n = \left(\bigcup_{i=n}^{m-1} (E_i \cap E_{i+1}^c)\right) \bigcup E_m$$

is true for all integers $m > n$, and our lemma is concluded.

Now we return to problem (a). Applying the probability function to both sides of the equation from the lemma and calculating a little, we have

$$\begin{aligned} P(\cup_{i=n}^{m} E_n) &= P((\cup_{i=n}^{m-1} (E_i \cap E_{i+1}^c)) \cup E_m) \\ &\leq P(\cup_{i=n}^{m-1} (E_i \cap E_{i+1}^c)) + P(E_m) \text{ by subadditivity} \\ &\leq \left(\sum_{i=n}^{m-1} P(E_i \cap E_{i+1}^c)\right) + P(E_m) \text{ by subadditivity} \\ &\leq \left(\sum_{i=n}^{\infty} P(E_i \cap E_{i+1}^c)\right) + P(E_m)\ . \end{aligned}$$

Now we take the limit as m approaches infinity.

$$\begin{aligned} P(\cup_{i=n}^{\infty} E_n) = \lim_{m\to\infty} P(\cup_{i=n}^{m} E_n) &\leq \lim_{m\to\infty} [\left(\sum_{i=n}^{\infty} P(E_i \cap E_{i+1}^c)\right) + P(E_m)] \\ &= \sum_{i=n}^{\infty} P(E_i \cap E_{i+1}^c) + \lim_{m\to\infty} P(E_m) \\ &= \sum_{i=n}^{\infty} P(E_i \cap E_{i+1}^c) \text{ by condition (i).} \end{aligned}$$

By condition (ii), the tail sum $\sum_{i=n}^{\infty} P(E_i \cap E_{i+1}^c)$ tends down to 0 as n approaches infinity. Taking the limit as n approaches infinity where we left off, we have

$$\begin{aligned} 0 = \lim_{n\to\infty} \sum_{i=n}^{\infty} P(E_i \cap E_{i+1}^c) &\geq \lim_{n\to\infty} P(\cup_{i=n}^{\infty} E_n) \\ &= \lim_n \downarrow P(\sup_{k\geq n} E_k) \\ &= P(\lim_n \downarrow \sup_{k\geq n} E_k) \end{aligned}$$

by monotone sequential continuity from above. Since $0 \geq P(\lim_n \downarrow \sup_{k\geq n} E_k) = P(\limsup_n E_n) = P(E_n \text{ i.o.}) \geq 0$, we have $P(E_n \text{ i.o.}) = 0$. □

(b) The result of (a) has already been proven. Let $E_1, E_2, \ldots$ be an arbitrary sequence of events satisfying $\sum_n P(E_n) < \infty$. It follows that the summand tends to 0, or in particular $\lim_n P(E_n) = 0$. Thus condition (i) is satisfied. Furthermore, for all n, $E_n \cap E_{n+1}^c \subset E_n$, and $P(E_n \cap E_{n+1}^c) \leq P(E_n)$. Therefore,

$$\sum_n P(E_n \cap E_{n+1}^c) \leq \sum_n P(E_n) < \infty\ .$$

Hence condition (ii) is satisfied. By (a), we have that $P(A_n \text{ i.o.}) = 0$, which is also the conclusion of the first Borel-Cantelli Lemma. Clearly then, the result of (a) implies the first Borel-Cantelli Lemma. □

(c) Let $\Omega = (0, 1]$, $\mathscr{F} = \mathscr{B}$, the Borel set on the unit interval, and P be Lebesgue measure. This defines our probability space, $(\Omega, \mathscr{F}, P)$. Define A_n to be the open interval $(0, 1/n)$. Clearly $\lim_n P(A_n) =$

$\lim_n(1/n) = 0$. Hence condition (i) holds. $A_n \cap A_{n+1}^c = (0, 1/n] \cap (0, 1/(n+1)]^c = (0, 1/n] \cap (1/(n+1), 1] = (1/(n+1), 1/n]$. It follows that

$$\begin{aligned}
\sum_{n=1}^{\infty} P(A_n \cap A_{n+1}^c) = \sum_{n=1}^{\infty} P((1/(n+1), 1/n]) &= P((1/2, 1]) + P((1/3, 1/2]) + \cdots \\
&= 1 - 1/2 + 1/2 - 1/3 + \cdots \\
&= 1 \ .
\end{aligned}$$

Since the intervals are all disjoint and probability is Lebesgue measure, it is clear that the sum is equal to 1. Hence condition (ii) holds. Notice that $\sum_n P(A_n) = \sum_n (1/n) = \infty$. However, $P(A_n \text{ i.o.}) = 0$ by (a). Therefore (a) strictly strengthens the first Borel-Cantelli Lemma. $\square$