Part 3 - Question 1

$$\frac{dg}{dt} = -K_4 g - K_6 i + A(t) \quad - (i)$$

$$\frac{di}{dt} = K_3 g - K_1 i + B(t) \quad - (ii)$$

$$\frac{d^2g}{dt^2} = -K_4 \frac{dg}{dt} - K_6 \frac{di}{dt} + \frac{d}{dt} A(t)$$

Let $A(t) = a.u(t)$ then from (i)

$$\frac{d^2g}{dt^2} = -K_4 \frac{dg}{dt} - K_6 \frac{di}{dt} + a.\frac{du(t)}{dt}$$

Let $B(t) = 0$ and substitute $\frac{di}{dt}$ from (ii)

$$\frac{d^2g}{dt^2} = -K_4 \frac{dg}{dt} - K_6 \left[K_3 g - K_1 i + 0 \right] + a.\frac{du(t)}{dt}$$

$$= -K_4 \frac{dg}{dt} - K_3 K_6 g + K_1 K_6 i + a\frac{du(t)}{dt}$$

substitute for $K_6 i$ from (i)

$$\frac{d^2g}{dt^2} = -K_4 \frac{dg}{dt} - K_3 K_6 g + K_1 \left[-K_4 g + a.u(t) - \frac{dg}{dt} \right] + a.\frac{du(t)}{dt}$$

$$= -(K_4 + K_1)\frac{dg}{dt} - (K_3 K_6 + K_1 K_4) g - K_1 a u(t) + a\frac{du(t)}{dt}$$

$$\frac{d^2g}{dt^2} + (K_1 + K_4)\frac{dg}{dt} + (K_1 K_4 + K_3 K_6) g = K_1 a + a\frac{du(t)}{dt}$$

Subtitute typical values

$k_1 = 0.8\ h^{-1}$ $\quad k_3 = 0.2\ \text{l/h/g}$ $\quad K_4 = 2\ h^{-1}$ $\quad K_6 = 5\ \text{g/h/kc}$

$a = 1\ \text{g/l/h}$ $\quad \frac{du(t)}{dt} = 0$ for $t > 0$

$$\frac{d^2g}{dt^2} + (0.8+2)\frac{dg}{dt} + (0.8\times 2 + 0.2\times 5)g = 0.8\times 1 + 1\times 0$$

$$\frac{d^2g}{dt^2} + 2.8\frac{dg}{dt} + 2.6g = 0.8$$

$$g(t) = g_c(t) + g_p(t)$$

complementary soln:

$$m^2 + 2.8m + 2.6 = 0 \Rightarrow m = -1.4 \pm 0.8i$$

$$g_c(t) = c_1 e^{(-1.4+0.8i)t} + c_2 e^{(-1.4-0.8i)t}$$

$$= e^{-1.4t}\left[A\cos(0.8t) + B\sin(0.8t)\right]$$

Particular soln:

$$g_p(t) = k$$

$$0 + 0 + 2.6k = 0.8 \Rightarrow k = 4/13$$

$$g(t) = e^{-1.4t}\left[A\cos(0.8t) + B\sin(0.8t)\right] + 4/13$$

using initial conditions

$$g(0) = 0 \Rightarrow A + 4/13 = 0 \Rightarrow A = -4/13$$

$$g'(0) = 1 \Rightarrow 0.8M - 1.4N = 1 \Rightarrow N = \frac{37}{52}$$

$$g(t) = e^{-1.4t}\left[-\tfrac{4}{13}\cos(0.8t) + \tfrac{37}{52}\sin(0.8t)\right] + \tfrac{4}{13}u(t)$$

Substituting g in (i)

$$\frac{dg}{dt} = -2g - 5i + 1$$

$$5i = -2g - \frac{dg}{dt} + 1$$

$$5i = -2 \cdot e^{-1.4t}\left[-\frac{4}{13}\cos(0.8t) + \frac{37}{52}\sin(0.8t) + 4/13\right] - \frac{dg}{dt} + 1$$

$$\frac{dg}{dt} = e^{-1.4t}\left[\frac{4}{13} \times \frac{8}{10}\sin(0.8t) + \frac{37}{52} \times \frac{8}{10}\cos(0.8t)\right] - 1.4e^{-1.4t}\left[-\frac{4}{13}\cos(0.8t) + \frac{37}{52}\sin(\cdot) + 4/13\right]$$

$$i(t) = e^{-1.4t}\left[-\frac{1}{13}\cos(0.8t) - \frac{7}{52}\sin(0.8t)\right] + \frac{1}{13}u(t)$$