Part 3 - Question 1

$$\frac{dg}{dt} = -k_4 g - k_6 i + A(t) \quad - (i)$$

$$\frac{di}{dt} = k_3 g - k_1 i + B(t) \quad - (ii)$$

$$\frac{d^2 g}{dt^2} = -k_4 \frac{dg}{dt} - k_6 \frac{di}{dt} + \frac{d}{dt} A(t)$$

Let $A(t) = a.u(t)$ then from (i)

$$\frac{d^2 g}{dt^2} = -k_4 \frac{dg}{dt} - k_6 \frac{di}{dt} + a.\frac{du(t)}{dt}$$

Let $B(t) = 0$ and substitute $\frac{di}{dt}$ from (ii)

$$\frac{d^2 g}{dt^2} = -k_4 \frac{dg}{dt} - k_6 \left[k_3 g - k_1 i + 0 \right] + a.\frac{du(t)}{dt}$$

$$= -k_4 \frac{dg}{dt} - k_3 k_6 g + k_1 k_6 i + a \frac{du(t)}{dt}$$

substitute for $k_6 i$ from (i)

$$\frac{d^2 g}{dt^2} = -k_4 \frac{dg}{dt} - k_3 k_6 g + k_1 \left[-k_4 g + a.u(t) - \frac{dg}{dt} \right] + a.\frac{du(t)}{dt}$$

$$= -(k_4 + k_1) \frac{dg}{dt} - (k_3 k_6 + k_1 k_4) g - k_1 a u(t) + a \frac{du(t)}{dt}$$

$$\frac{d^2 g}{dt^2} + (k_1 + k_4) \frac{dg}{dt} + (k_1 k_4 + k_3 k_6) g = k_1 a + a \frac{du(t)}{dt}$$

Subtitute typical values

$k_1 = 0.8\ h^{-1}$ $k_3 = 0.2\ \text{l/h/g}$ $k_4 = 2\ h^{-1}$ $k_6 = 5\ \text{g/h/kc}$

$a = 1\ \text{g/l/h}$ $\frac{du(t)}{dt} = 0$ for $t > 0$

$$\frac{d^2g}{dt^2} + (0.8+2)\frac{dg}{dt} + (0.8\times 2 + 0.2\times 5)g = 0.8\times 1 + 1\times 0$$

$$\frac{d^2g}{dt^2} + 2.8\frac{dg}{dt} + 2.6g = 0.8$$

$$g(t) = g_c(t) + g_p(t)$$

complementary soln:

$$m^2 + 2.8m + 2.6 = 0 \Rightarrow m = -1.4 \pm 0.8i$$

$$g_c(t) = c_1 e^{(-1.4+0.8i)t} + c_2 e^{(-1.4-0.8i)t}$$

$$= e^{-1.4t}\left[A\cos(0.8t) + B\sin(0.8t)\right]$$

Particular soln:

$$g_p(t) = k$$

$$0 + 0 + 2.6k = 0.8 \Rightarrow k = 4/13$$

$$g(t) = e^{-1.4t}\left[A\cos(0.8t) + B\sin(0.8t)\right] + 4/13$$

using initial conditions

$$g(0) = 0 \Rightarrow A + 4/13 = 0 \Rightarrow A = -4/13$$

$$g'(0) = 1 \Rightarrow 0.8M - 1.4N = 1 \Rightarrow N = \frac{37}{52}$$

$$g(t) = e^{-1.4t}\left[-\tfrac{4}{13}\cos(0.8t) + \tfrac{37}{52}\sin(0.8t)\right] + \tfrac{4}{13}u(t)$$

Substituting g in (i)

$$\frac{dg}{dt} = -2g - 5i + 1$$

$$5i = -2g - \frac{dg}{dt} + 1$$

$$5i = -2 \cdot e^{-1.4t}\left[-\frac{4}{13}\cos(0.8t) + \frac{37}{52}\sin(0.8t) + \frac{4}{13}\right] - \frac{dg}{dt} + 1$$

$$\frac{dg}{dt} = e^{-1.4t}\left[\frac{4}{13}\times\frac{8}{10}\sin(0.8t) + \frac{37}{52}\times\frac{8}{10}\cos(0.8t)\right] - 1.4e^{-1.4t}\left[-\frac{4}{13}\cos(0.8t) + \frac{37}{52}\sin(0.8t) + \frac{4}{13}\right]$$

$$i(t) = e^{-1.4t}\left[-\frac{1}{13}\cos(0.8t) - \frac{7}{52}\sin(0.8t)\right] + \frac{1}{13}u(t)$$

Question 2

From Bolie's model

$$\frac{dG}{dt} = k_5 + A(t) - k_4 G - k_6 I + k_{10} G_n(t)$$

$$\frac{dI}{dt} = k_2 + k_3 G + B(t) - k_1 I$$

$$\frac{dG_n}{dt} = k_8 + C(t) + k_9 G - k_7 G_n$$

$G \rightarrow$ glucose

$G_n \rightarrow$ Glucogon

$I \rightarrow$ Insulin

At equillibrium

$$\frac{dG}{dt} = \frac{dI}{dt} = \frac{dG_n}{dt} = 0$$

$$\frac{dG}{dt} = 0 \Rightarrow \quad k_5 = k_4 G_0 + k_6 I_0 - k_{10} G_{n0}$$

$$\frac{dI}{dt} = 0 \Rightarrow \quad k_2 = k_1 I_0 - k_3 G_0$$

$$\frac{dG_n}{dt} = 0 \Rightarrow \quad k_8 = k_7 G_{n0} - k_4 G_0$$

$A(t) = a.u(t)$, $B(t) = 0$, $C(t) = 0$;

$$\frac{dg}{dt} = -k_4 g - k_6 i - k_{10} g_n + a.u(t) \qquad [g = G - G_0]$$

$$\frac{di}{dt} = k_3 g - k_1 i \qquad [i = I - I_0]$$

$$\frac{dg_n}{dt} = k_9 g - k_7 i \qquad [g_n = G_n - G_{n0}]$$

$$\begin{pmatrix} \frac{dg}{dt} \\ \frac{di}{dt} \\ \frac{dg_n}{dt} \end{pmatrix} = \begin{pmatrix} -k_4 & -k_6 & k_{10} \\ k_3 & -k_1 & 0 \\ k_9 & 0 & -k_8 \end{pmatrix} \begin{pmatrix} g \\ i \\ g_n \end{pmatrix} + \begin{pmatrix} a \cdot u(t) \\ 0 \\ 0 \end{pmatrix}$$