

Normal distribution

Assembler
Institute of Technology



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Master en Data Science
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ÍNDICE

- 1** Introduction to normal distribution
- 2** Calculating probabilities using Python
- 3** Modelling real life problems

Consider the lengths, in mm, of 50 leaves that have fallen from a coffee tree.


60	31	72	57	99	46	68	47	54	57
42	48	39	40	67	89	70	68	42	54
52	50	85	56	50	53	57	83	79	63
63	72	57	53	90	52	58	47	34	102
70	60	94	43	85	67	78	66	57	44

How could we find the probability that a leaf is L mm?



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 $62.5 \leq l < 63.5$

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How could we find the probability that a leaf is L mm?

How could we find the probability that the length L of a leaf lies in the interval $x_1 \leq L < x_2$?

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→ $62.5 \leq l < 63.5$

**Let's make a
histogram!**

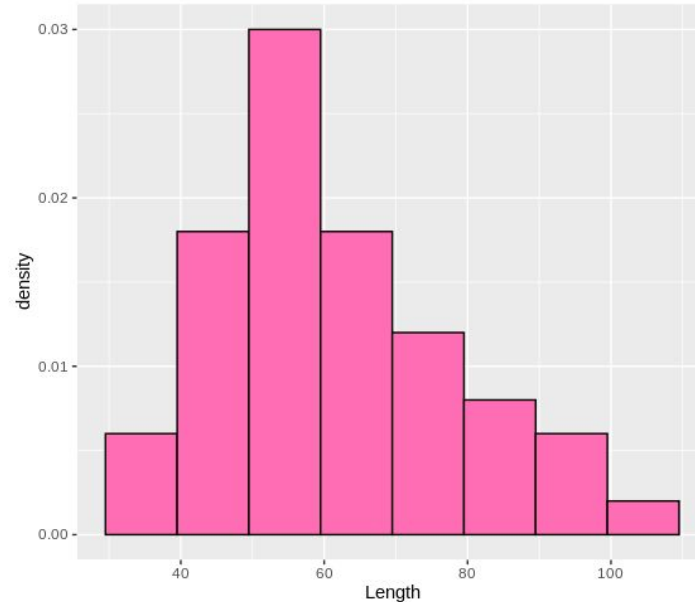
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→ $62.5 \leq l < 63.5$

How could we find the probability that the length L of a leaf lies in the interval $x_1 \leq L < x_2$?



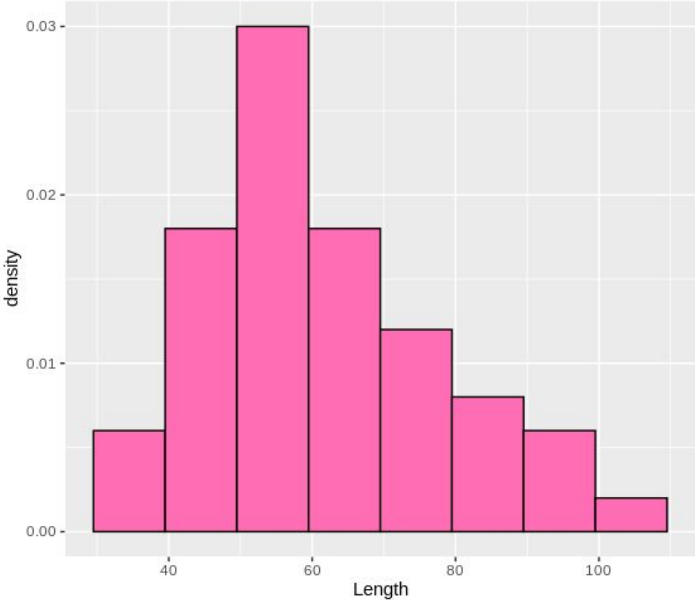
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How could we find the probability that the length L of a leaf lies in the interval $x_1 \leq L < x_2$?

$R\text{ Freq density}(x_i) = \frac{Rel\text{ Freq}}{Class\text{ width}}$



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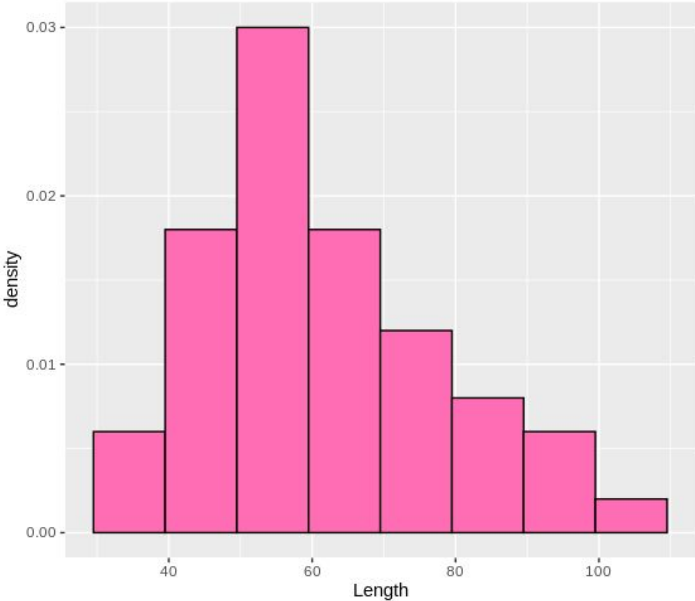
→ $62.5 \leq l < 63.5$

How could we find the probability that the length L of a leaf lies in the interval $x_1 \leq L < x_2$?

$$R\text{ Freq density}(x_i) = \frac{Rel\text{ Freq}}{Class\text{ width}}$$

Each area represents relative frequency.

$$P(x_i) \sim Relative\text{ Frequency}(x_i)$$



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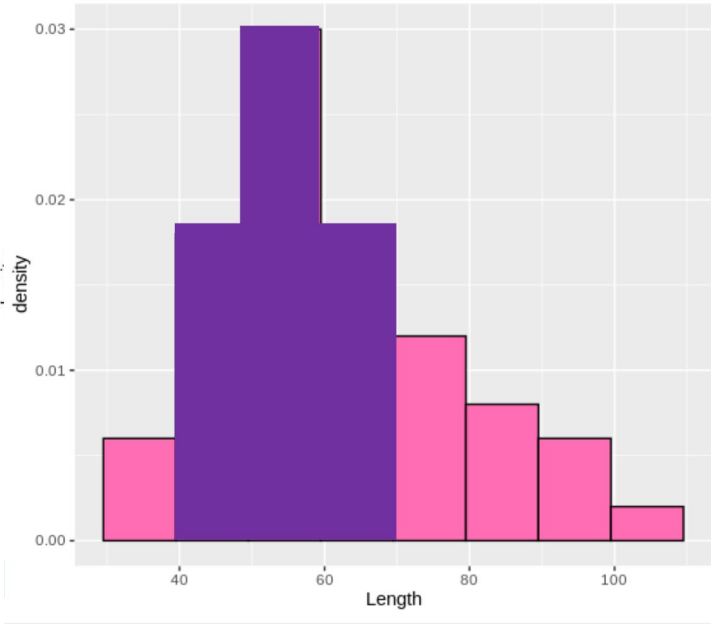
62.5 ≤ l < 63.5

How could we find the probability that the length L of a leaf lies in the interval $x_1 \leq L < x_2$?

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What is the probability that the length L lies in the interval $39.5 \leq L < 70.5$?

Consider the lengths, in mm, of 50 leaves that have fallen from a coffee tree.

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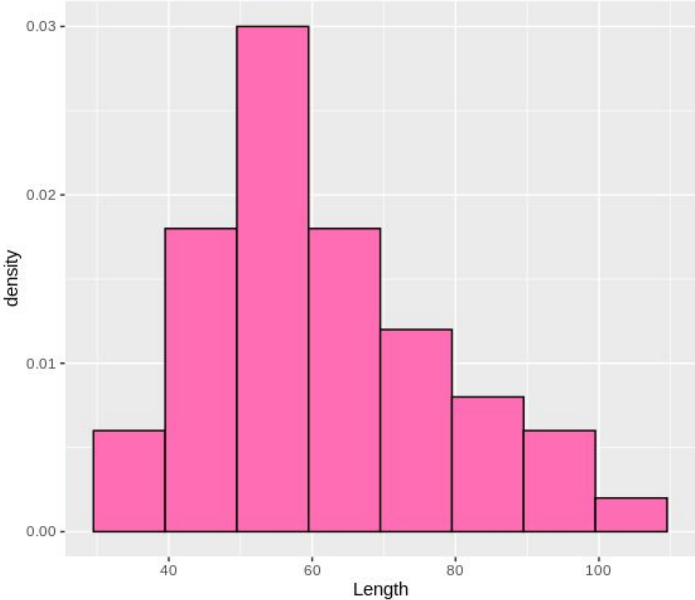
→ $62.5 \leq l < 63.5$

How could we find the probability that the length L of a leaf lies in the interval $x_1 \leq L < x_2$?

$$R\text{ Freq density}(x_i) = \frac{Rel\text{ Freq}}{Class\ width}$$

Each area represents relative frequency.

$$P(x_i) \sim Relative\ Frequency(x_i)$$



What is the probability that the length L is equal to 39.5?

Consider the lengths, in mm, of 50 leaves that have fallen from a coffee tree.

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42	48	39	40	67	89	70	68	42	54
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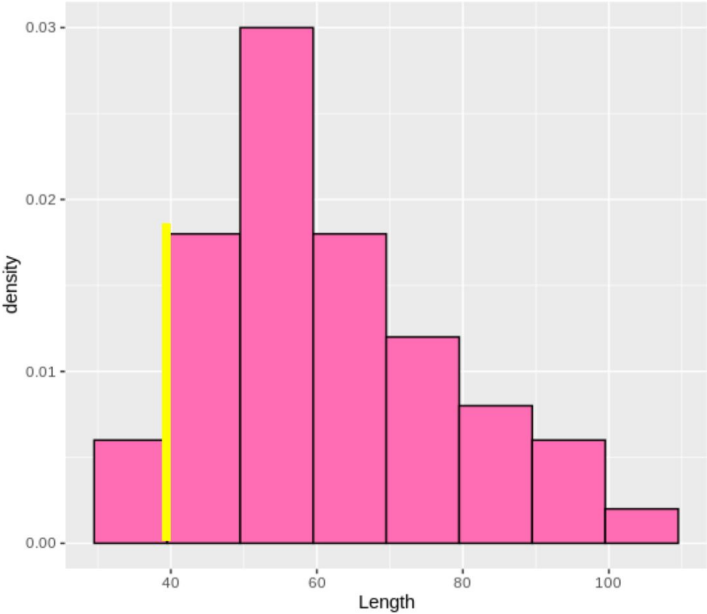
63 → 62.5 ≤ l < 63.5

How could we find the probability that the length L of a leaf lies in the interval $\bar{x}_1 \leq L < x_2$?

$$R\text{ Freq density}(x_i) = \frac{Rel\text{ Freq}}{Class\text{ width}}$$

Each area represents relative frequency.

$$P(x_i) \sim Relative\text{ Frequency}(x_i)$$



What is the probability that the length L is equal to 39.5?



Consider the lengths, in mm, of 50 leaves that have fallen from a coffee tree.

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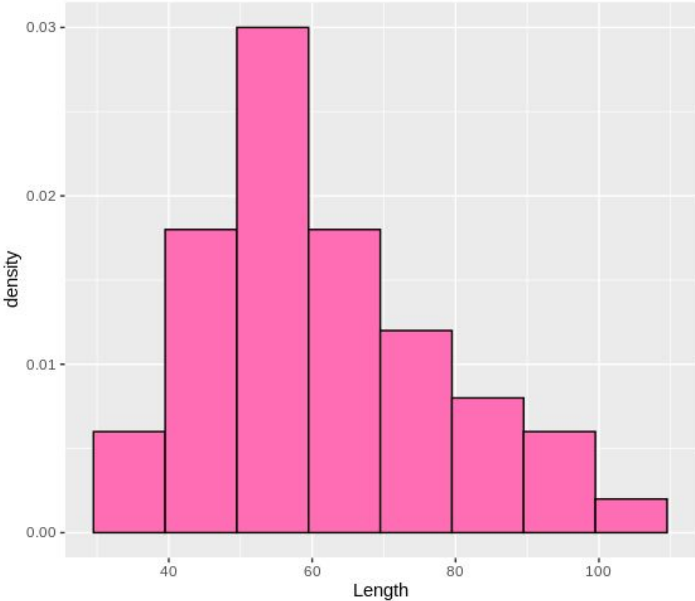
$\rightarrow 62.5 \leq l < 63.5$

How could we find the probability that the length L of a leaf lies in the interval $x_1 \leq L < x_2$?

$$R\text{ Freq density}(x_i) = \frac{Rel\text{ Freq}}{Class\text{ width}}$$

Each area represents relative frequency.

$$P(x_i) \sim Relative\text{ Frequency}(x_i)$$



$$P(39.5 \leq L < 70.5) = P(39.5 < L < 70.5) = P(39.5 < L \leq 70.5)$$

Consider the lengths, in mm, of 50 leaves that have fallen from a coffee tree.

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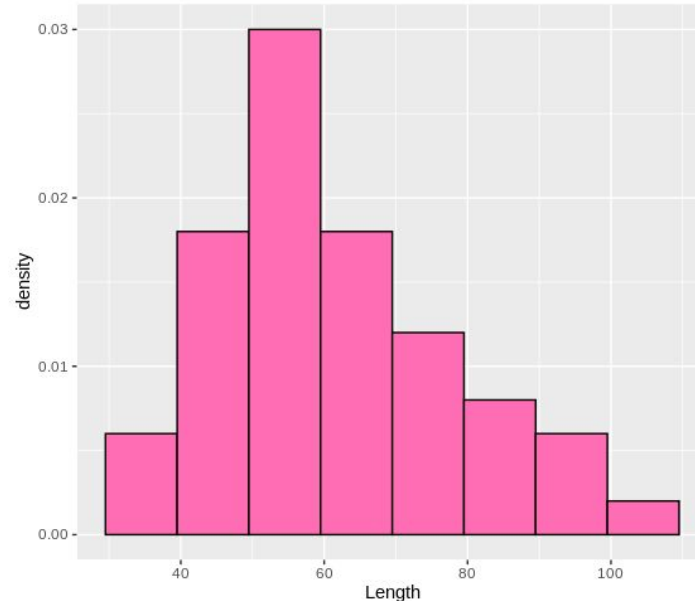
→ $62.5 \leq l < 63.5$

How could we find the probability that the length L of a leaf lies in the interval $x_1 \leq L < x_2$?

$$R \text{ Freq density}(x_i) = \frac{Rel \text{ Freq}}{Class \text{ width}}$$

Each area represents relative frequency.

$$P(x_i) \sim \text{Relative Frequency}(x_i)$$



$$P(x_1 \leq L < x_2) =$$

$$P(x_1 < L < x_2) =$$

$$P(x_1 < L \leq x_2)$$

$$P(x_i) = 0$$

Consider the lengths, in mm, of 50 leaves that have fallen from a coffee tree.

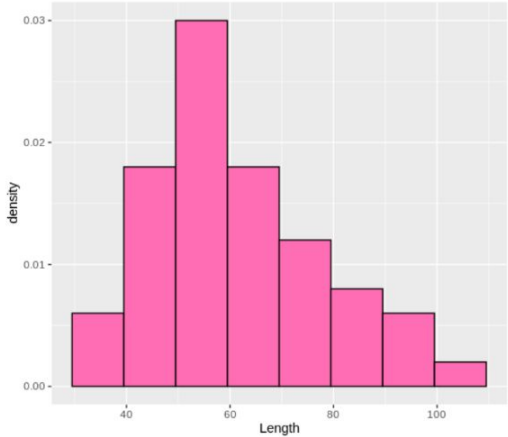
60	31	72	57	99	46	68	47	54	57
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What happens if we reduce the length of the class width ?

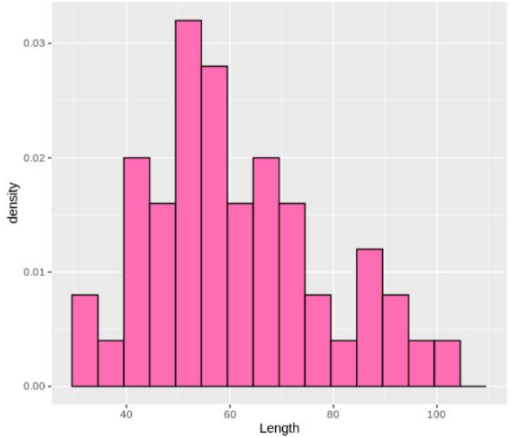
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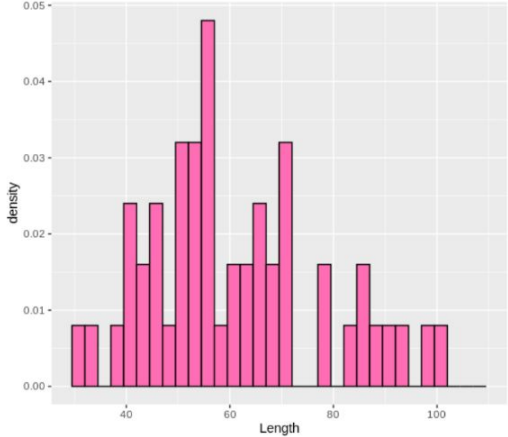
What happens if we reduce the length of the class width ?



Class width: 10

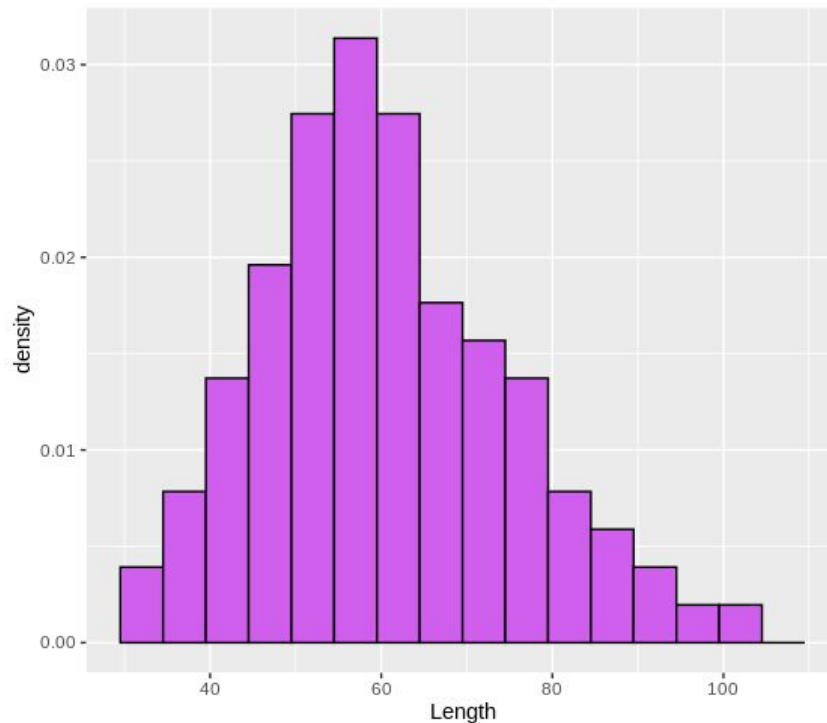


Class width: 5



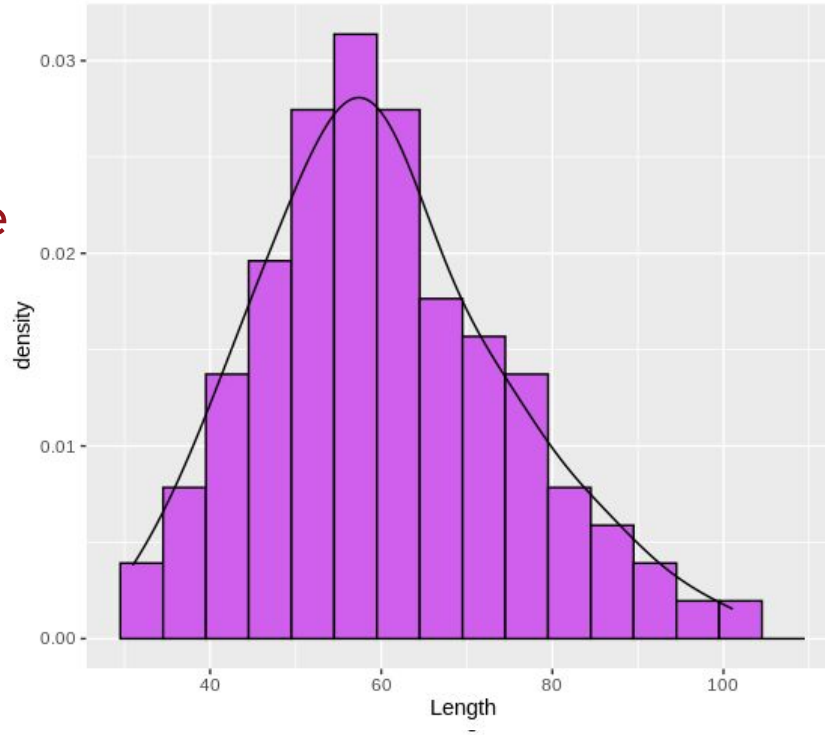
Class width: 2.5

We'll consider a better model with more data and smaller class widths. Imagine we have a dataset of size 100 and class widths of 5.



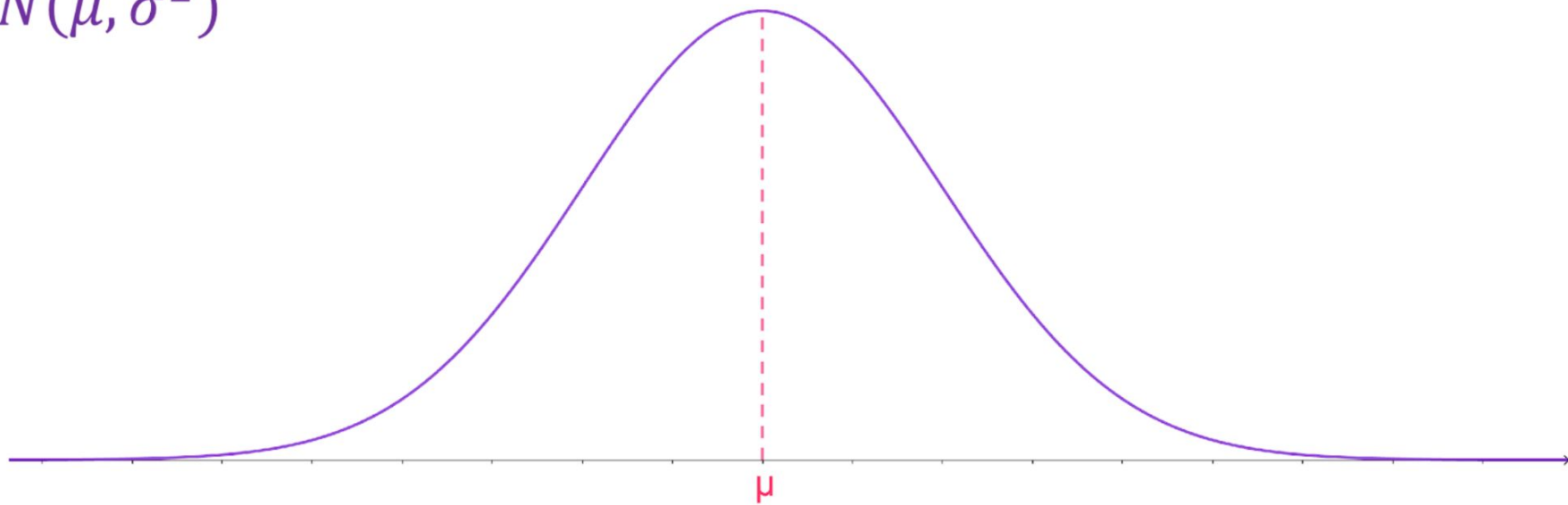
We'll consider a better model with more data and smaller class widths. Imagine we have a dataset of size 100 and class widths of 5.

The length of the
coffee leaves
follows a **normal
distribution**



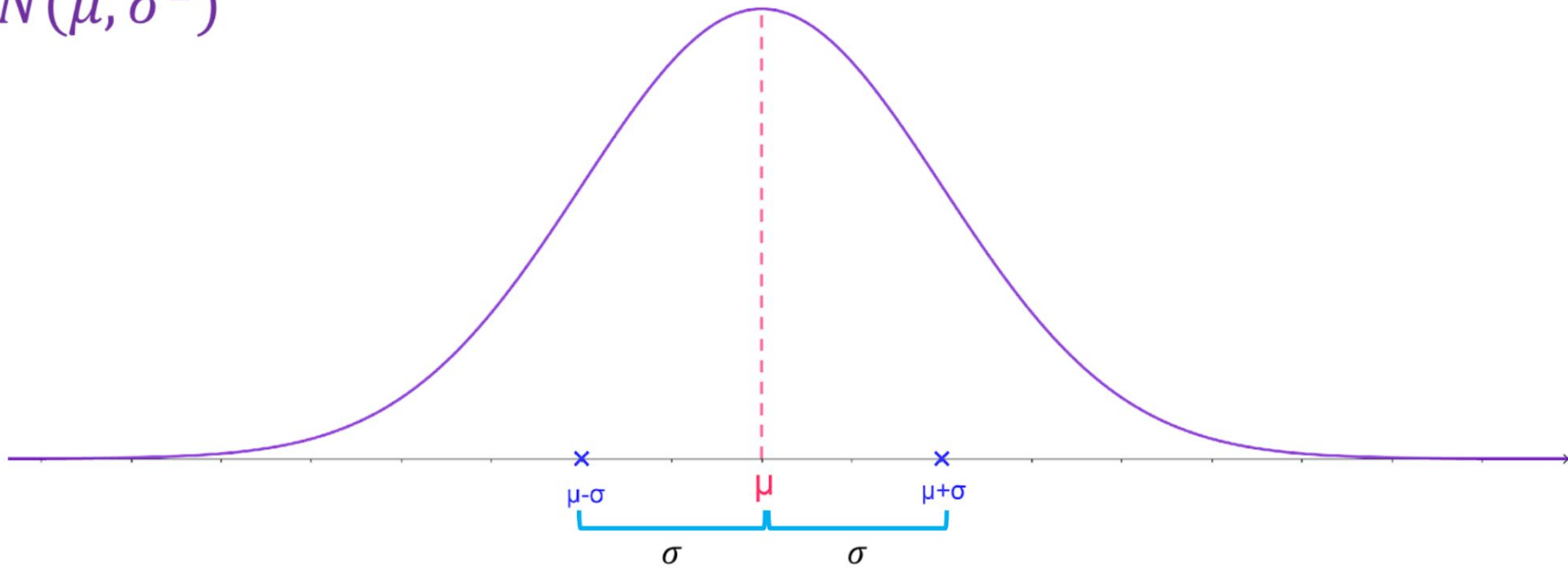
NORMAL DISTRIBUTION

$$N(\mu, \sigma^2)$$



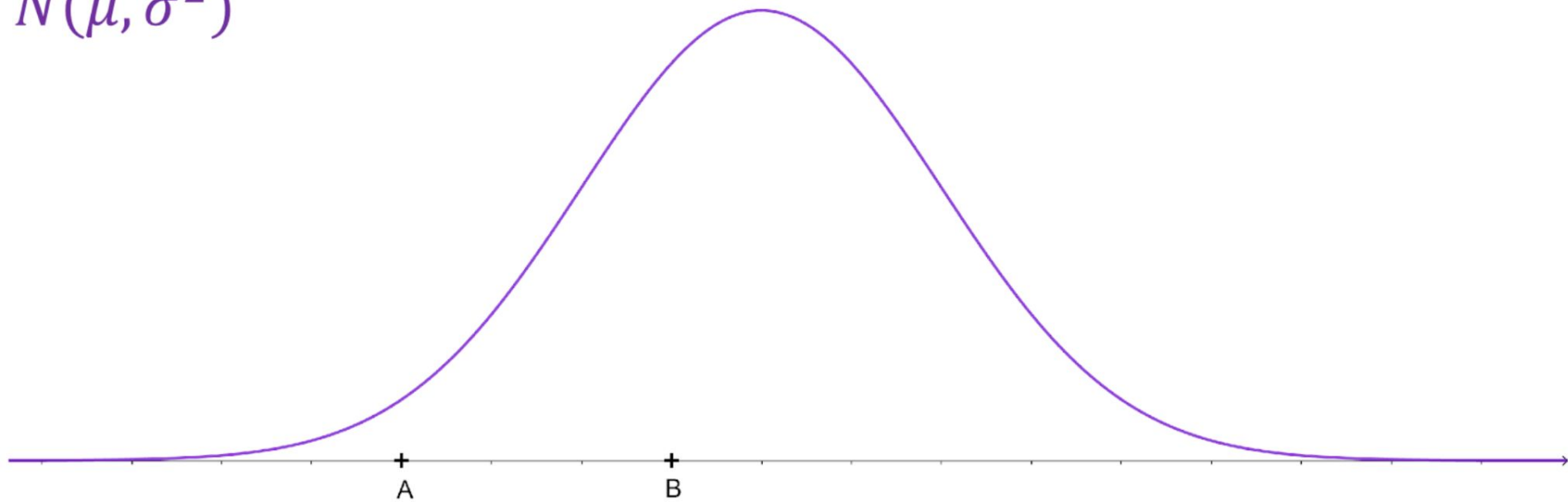
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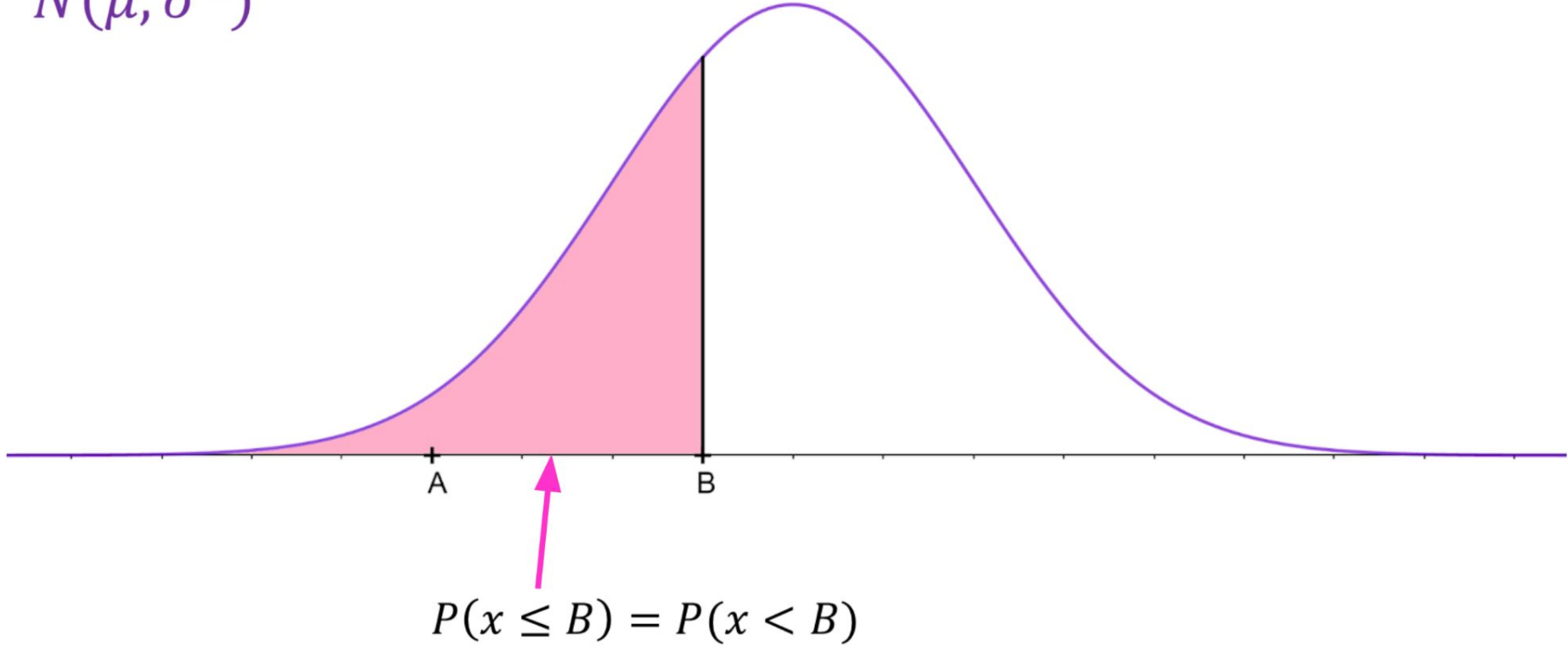
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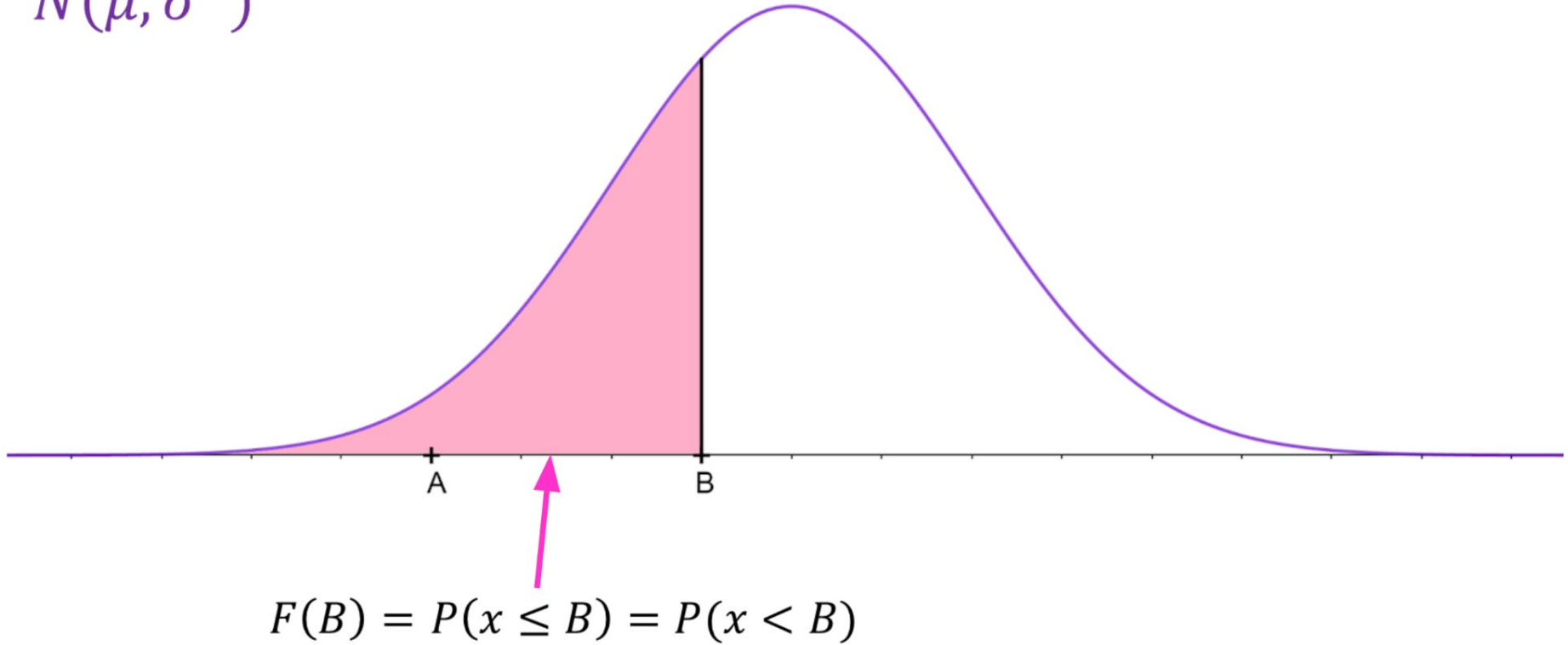
NORMAL DISTRIBUTION

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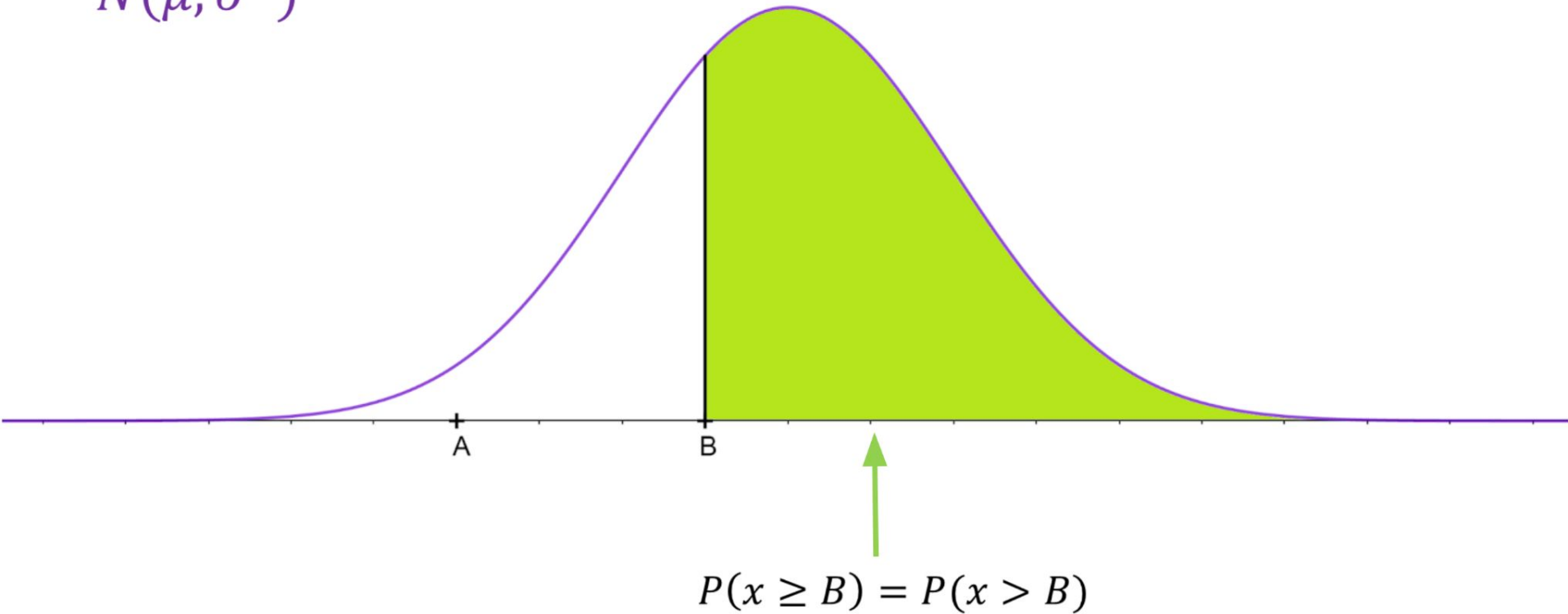
NORMAL DISTRIBUTION

$N(\mu, \sigma^2)$



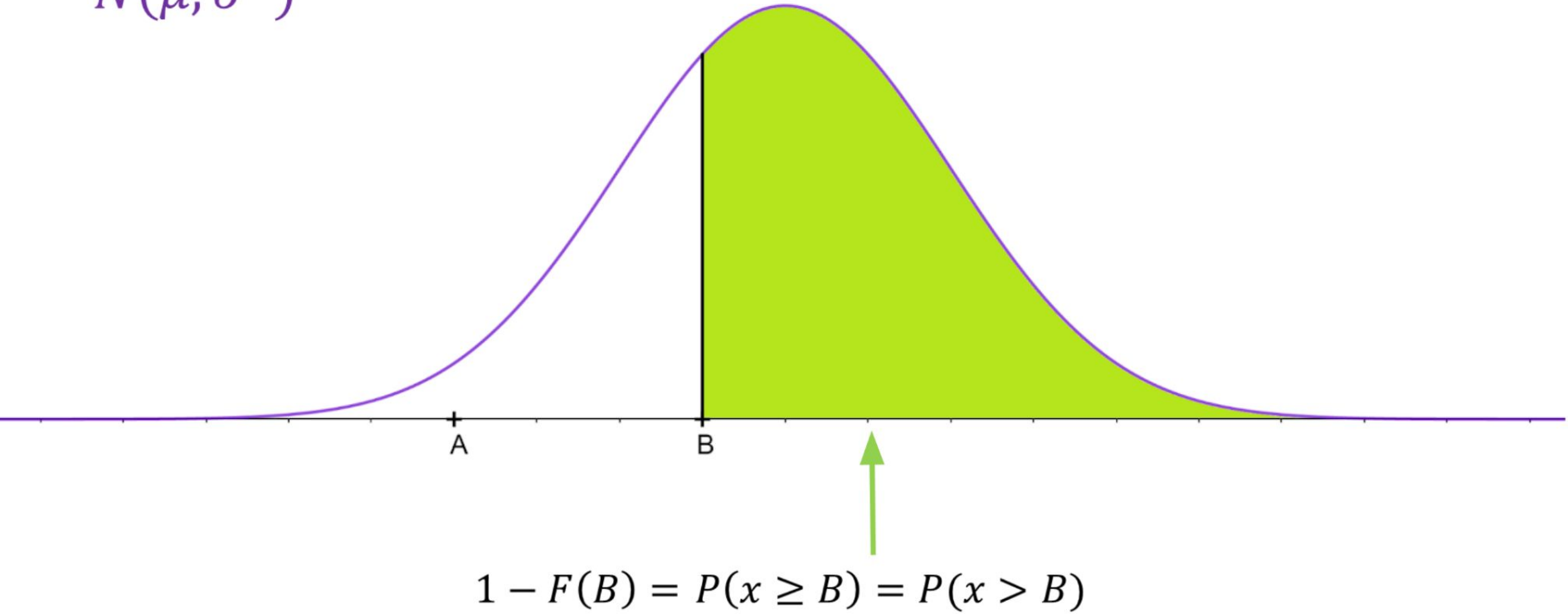
NORMAL DISTRIBUTION

$$N(\mu, \sigma^2)$$



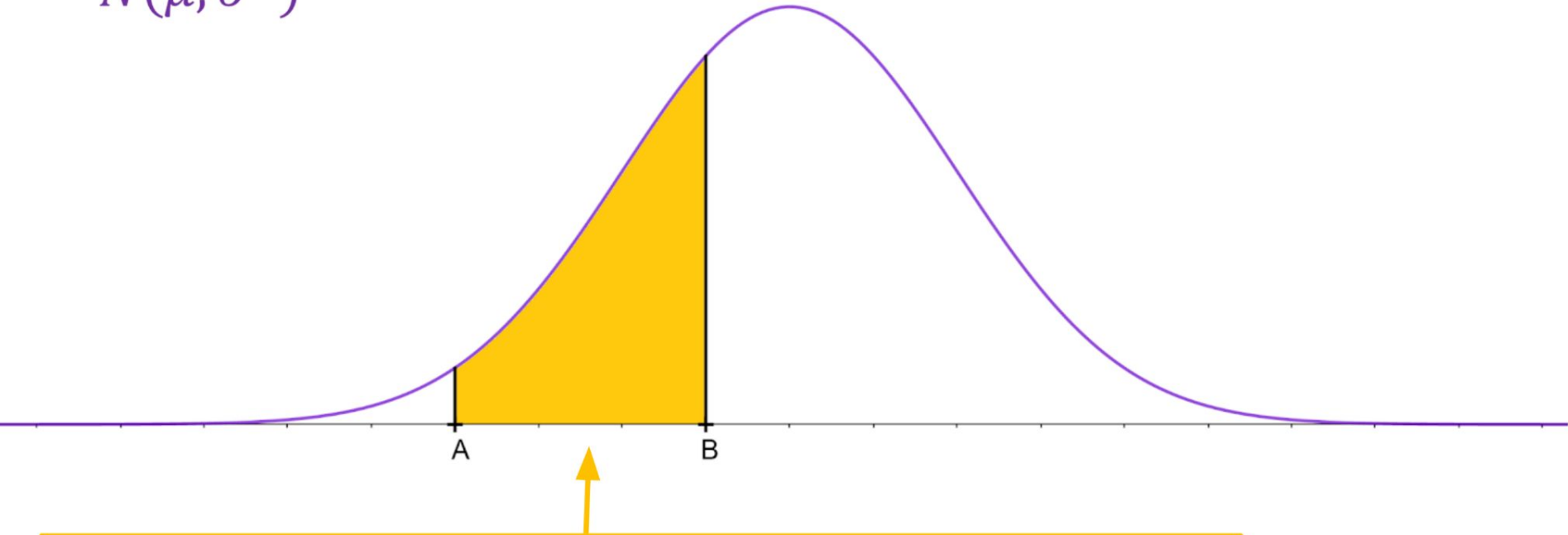
NORMAL DISTRIBUTION

$N(\mu, \sigma^2)$



NORMAL DISTRIBUTION

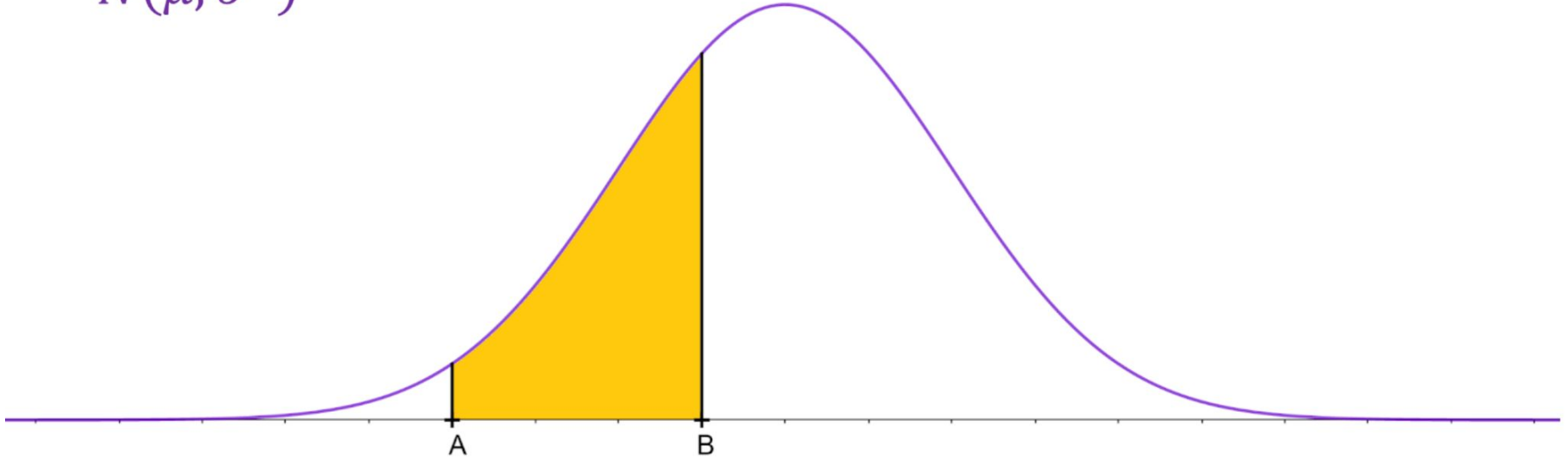
$N(\mu, \sigma^2)$



$$P(A < x < B) = P(A < x \leq B) = P(A \leq x < B) = P(A \leq x \leq B)$$

NORMAL DISTRIBUTION

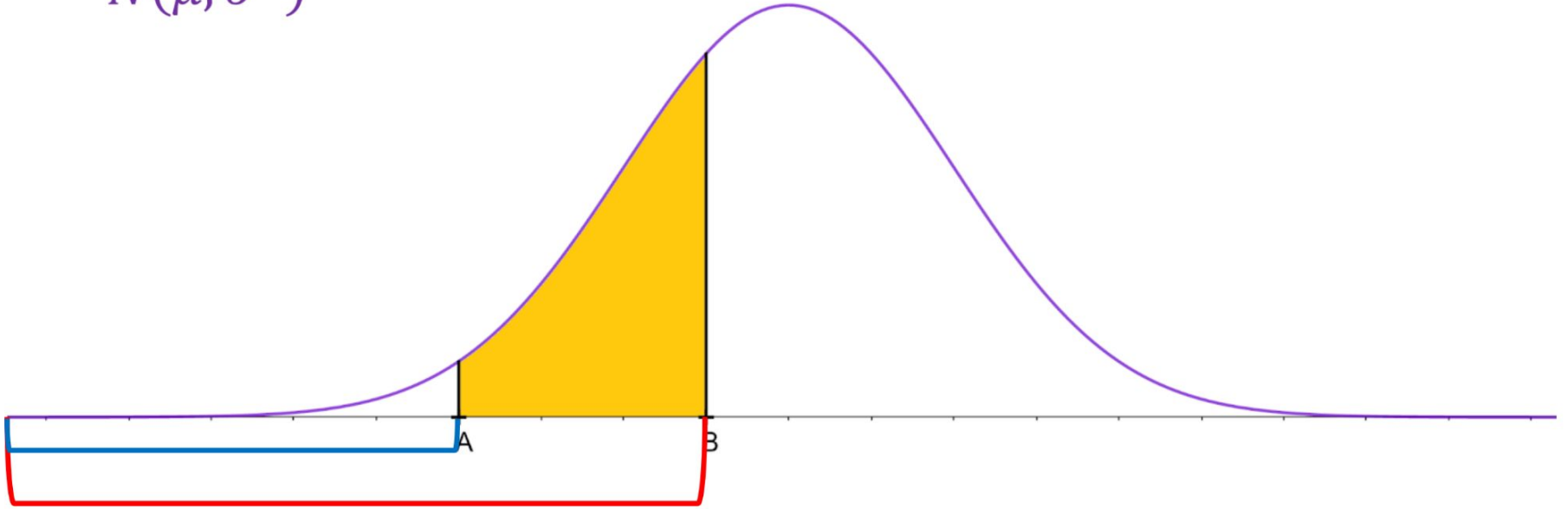
$$N(\mu, \sigma^2)$$



$$P(A \leq x \leq B) =$$

NORMAL DISTRIBUTION

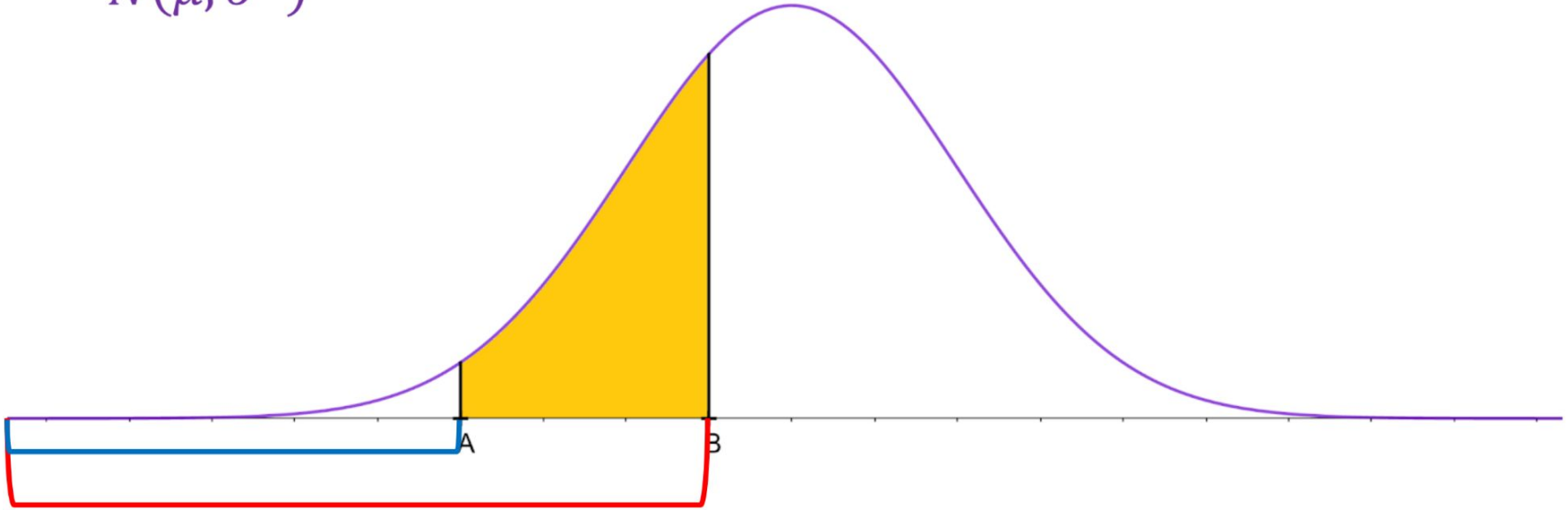
$N(\mu, \sigma^2)$



$$P(A \leq x \leq B) = P(x \leq B) - P(x \leq A)$$

NORMAL DISTRIBUTION

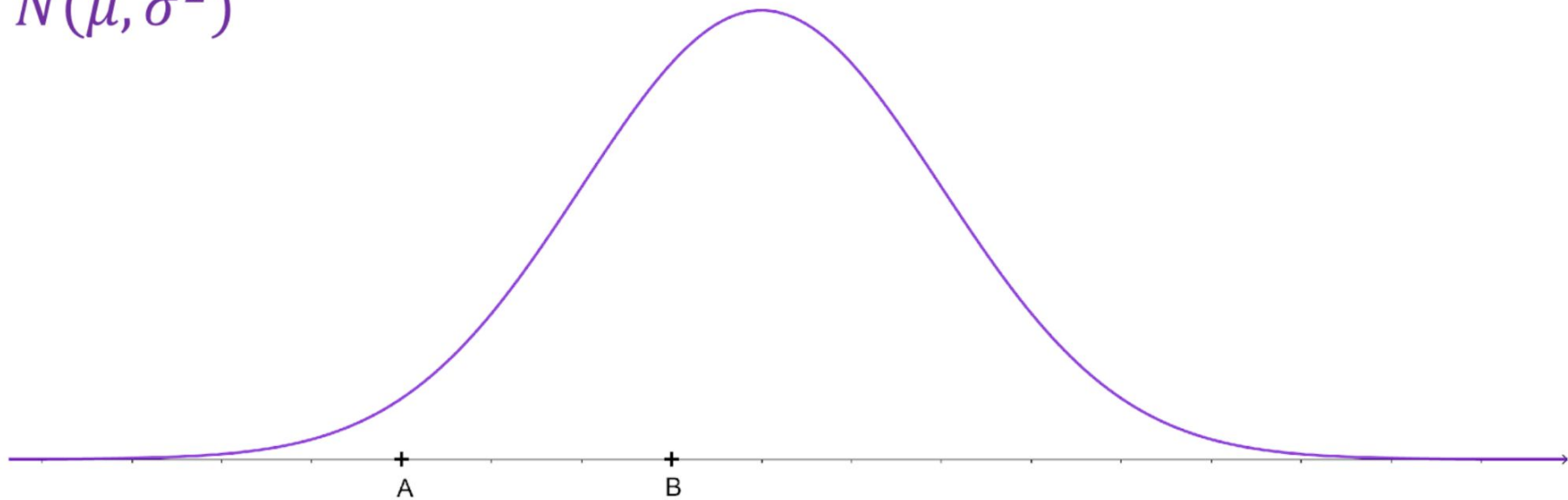
$N(\mu, \sigma^2)$



$$P(A \leq x \leq B) = P(x \leq B) - P(x \leq A) = F(B) - F(A)$$

NORMAL DISTRIBUTION

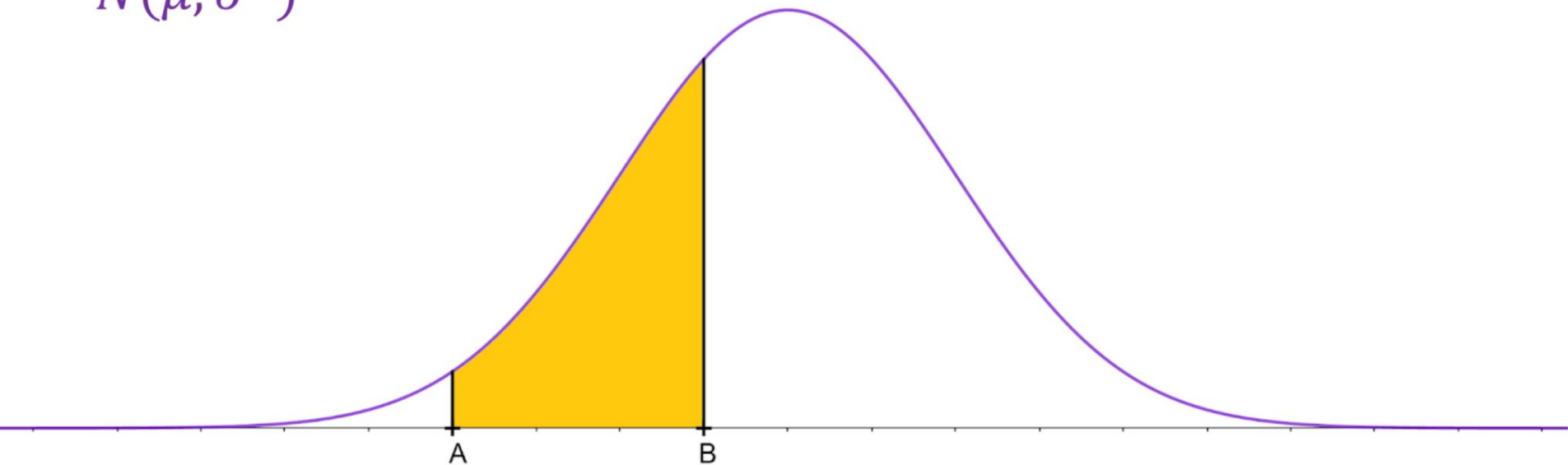
$$N(\mu, \sigma^2)$$



How do we calculate the areas? 🤔

NORMAL DISTRIBUTION

$$N(\mu, \sigma^2)$$

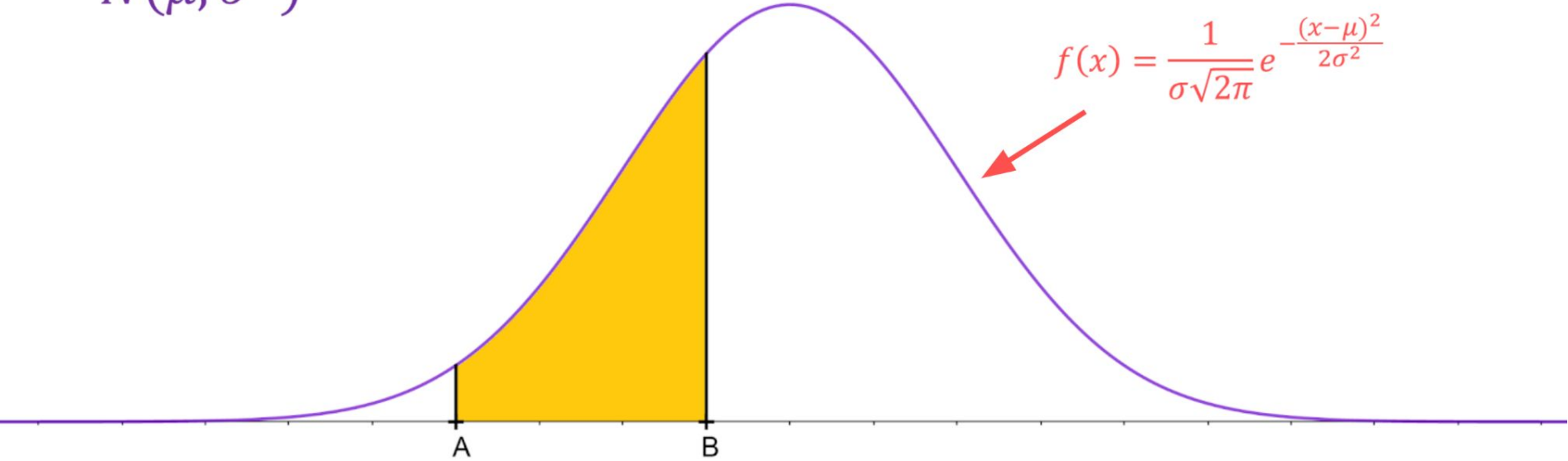


$$P(A \leq x \leq B) =$$

NORMAL DISTRIBUTION

$$N(\mu, \sigma^2)$$


$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

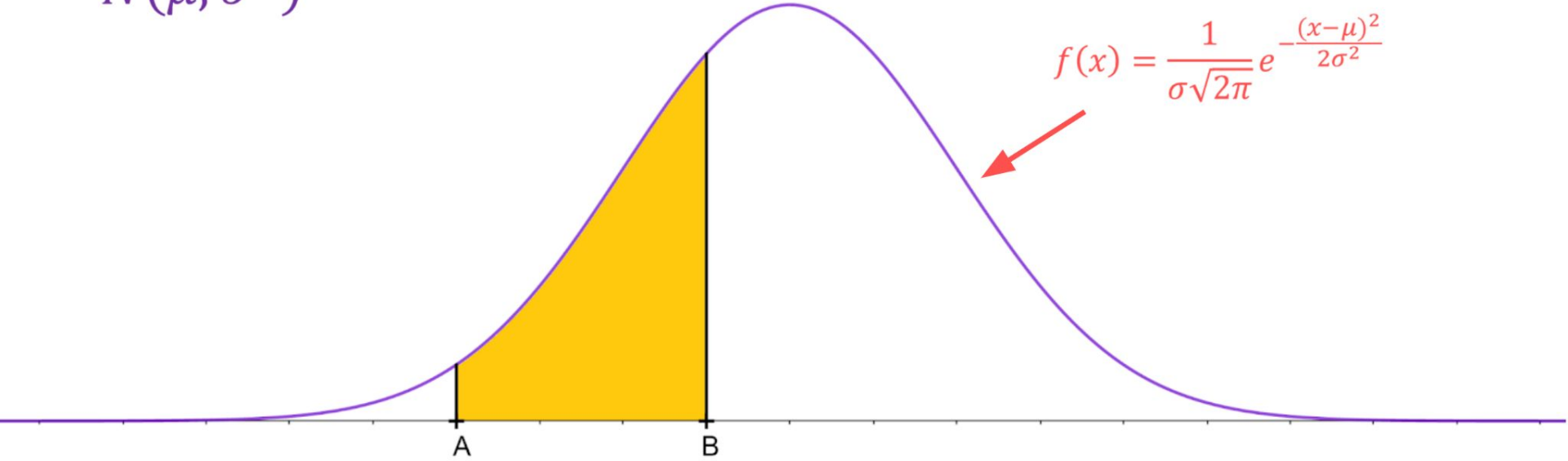


$$P(A \leq x \leq B) =$$

NORMAL DISTRIBUTION

$$N(\mu, \sigma^2)$$

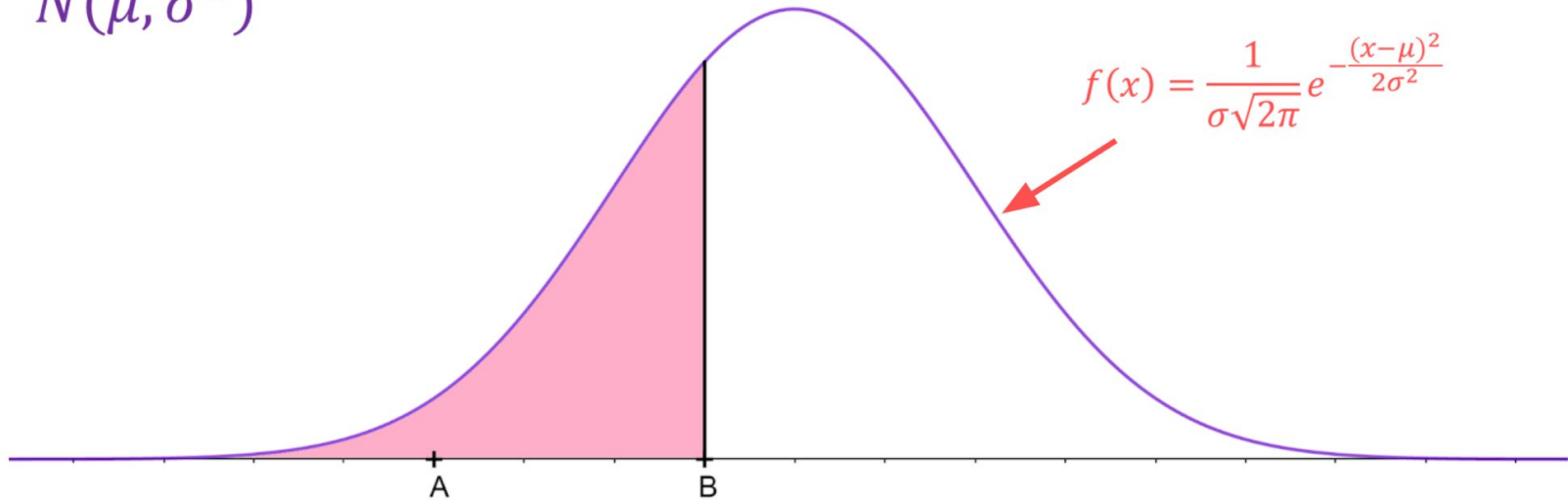
$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$




$$P(A \leq x \leq B) = \int_A^B \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} = \dots$$

NORMAL DISTRIBUTION

$$N(\mu, \sigma^2)$$

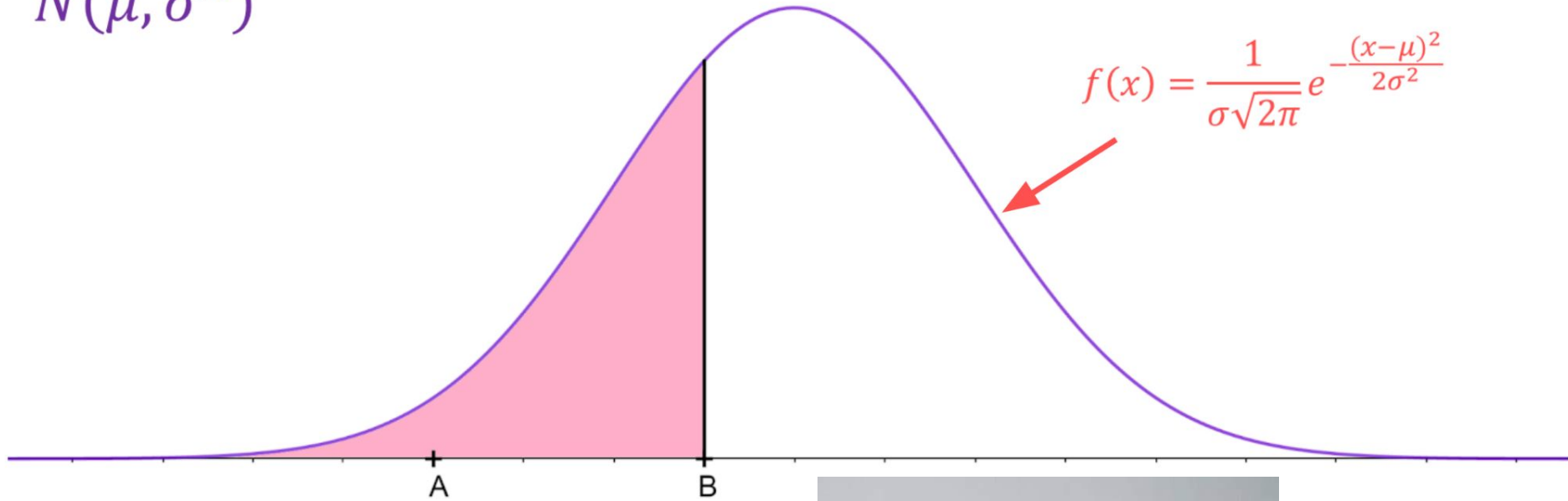


$$P(x \leq B) =$$

NORMAL DISTRIBUTION

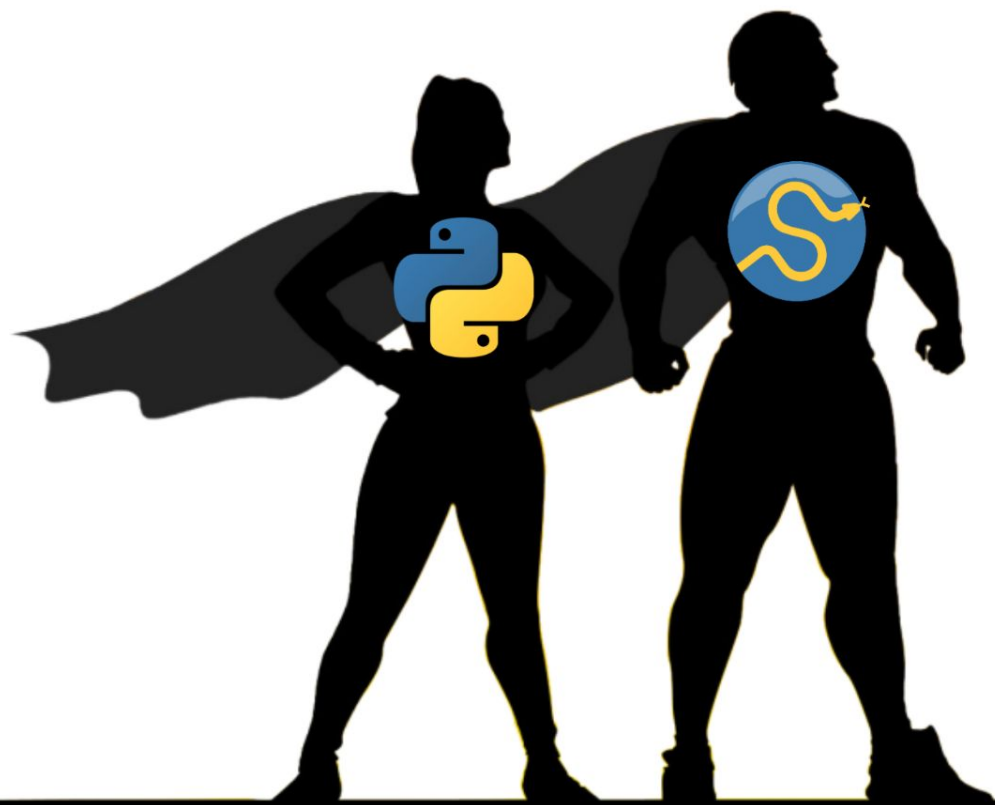
$$N(\mu, \sigma^2)$$

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$



$$P(x \leq B) = \int_{-\infty}^B \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

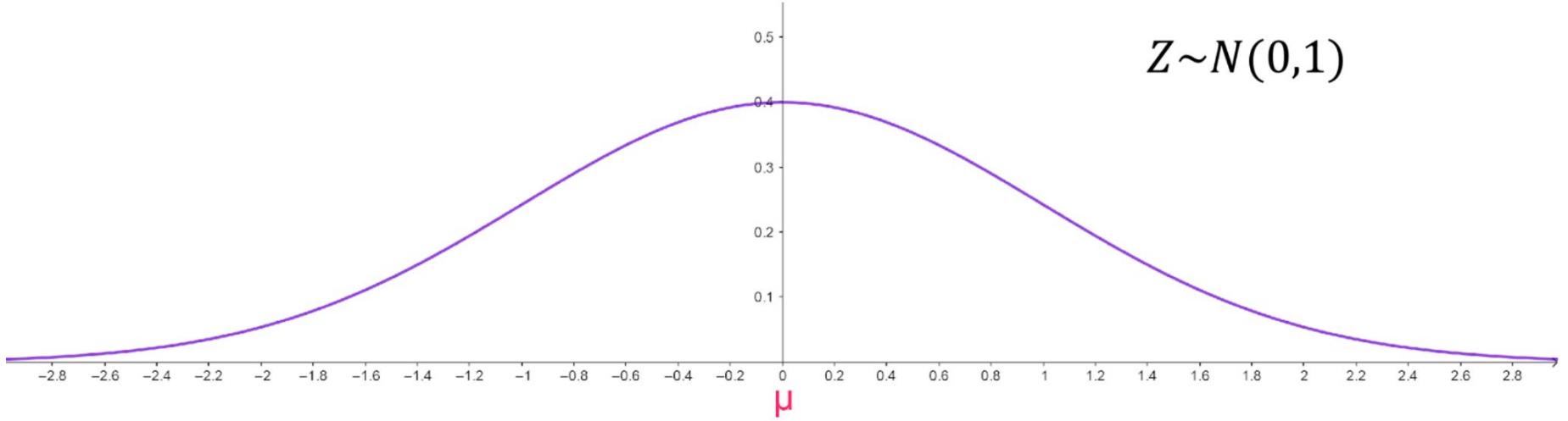




```
from scipy.stats import norm
```

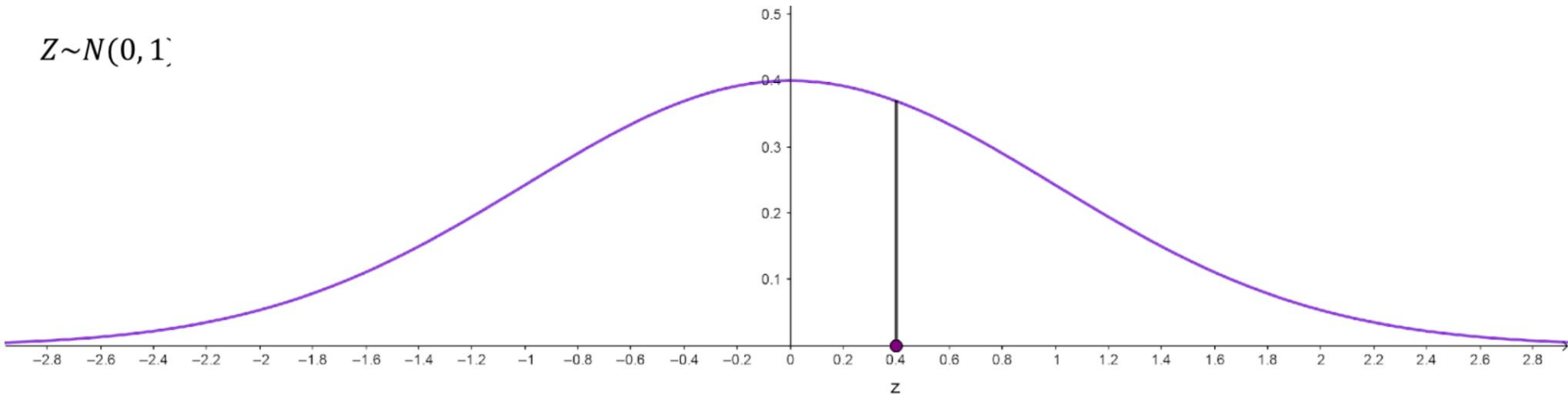
STANDARD NORMAL DISTRIBUTION

$$Z \sim N(0,1)$$



STANDARD NORMAL DISTRIBUTION

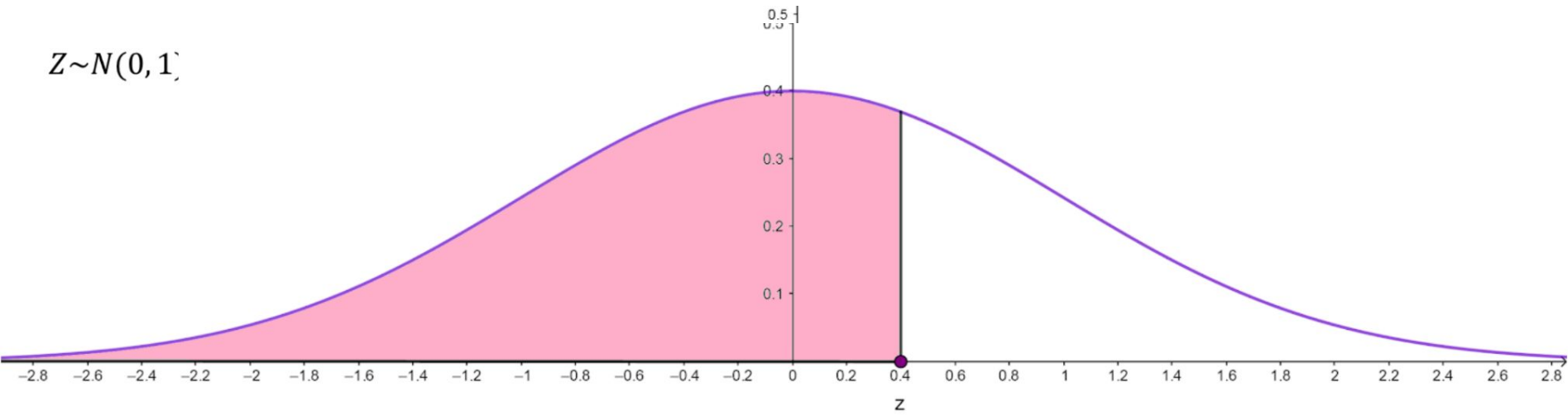
$$Z \sim N(0, 1)$$



$$P(Z \leq z)?$$

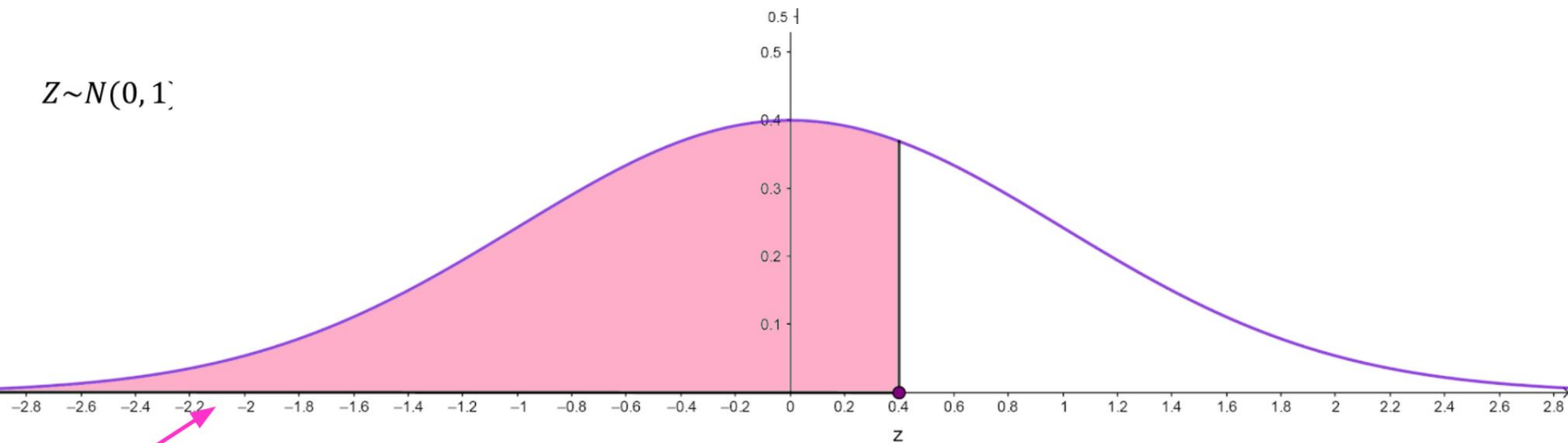
STANDARD NORMAL DISTRIBUTION

$$Z \sim N(0, 1)$$



$$P(Z \leq z)?$$

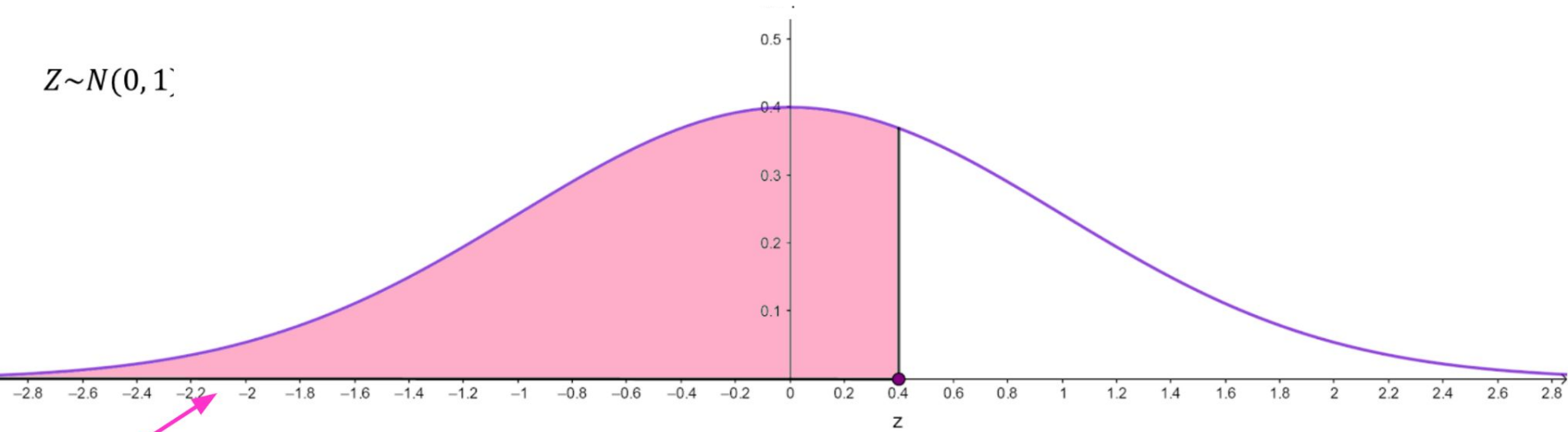
STANDARD NORMAL DISTRIBUTION



$$P(Z \leq z) = \Phi(z)$$

STANDARD NORMAL DISTRIBUTION

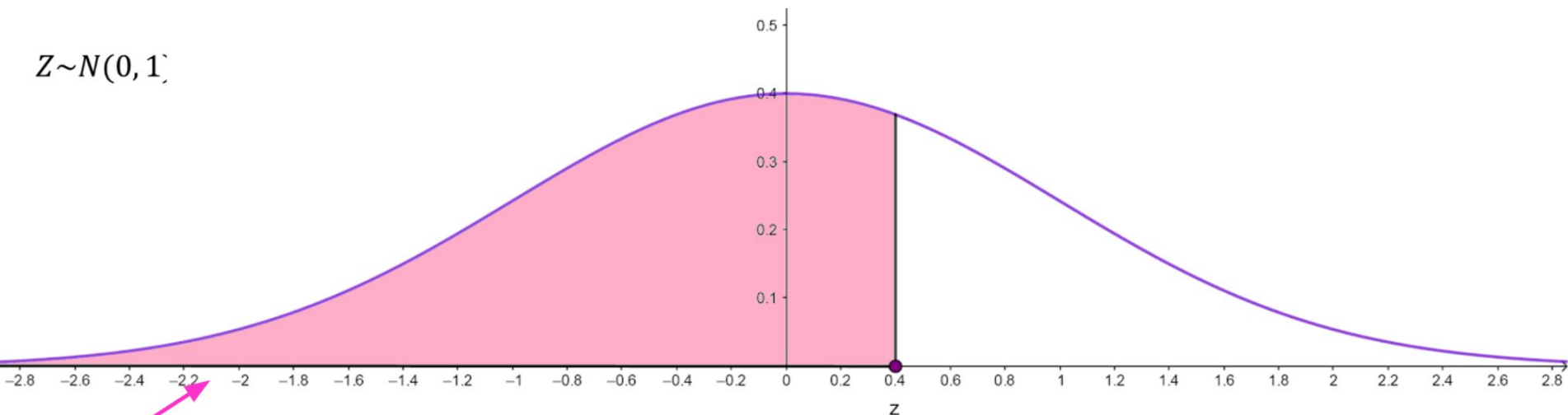
$$Z \sim N(0, 1)$$



$$P(Z \leq 0.4) = \Phi(0.4)$$

STANDARD NORMAL DISTRIBUTION

$$Z \sim N(0, 1)$$



$$P(Z \leq 0.4) = \Phi(0.4)$$

`cdf(x, loc=0, scale=1)`

Cumulative distribution function.

```
mean, var = 0, 1
```

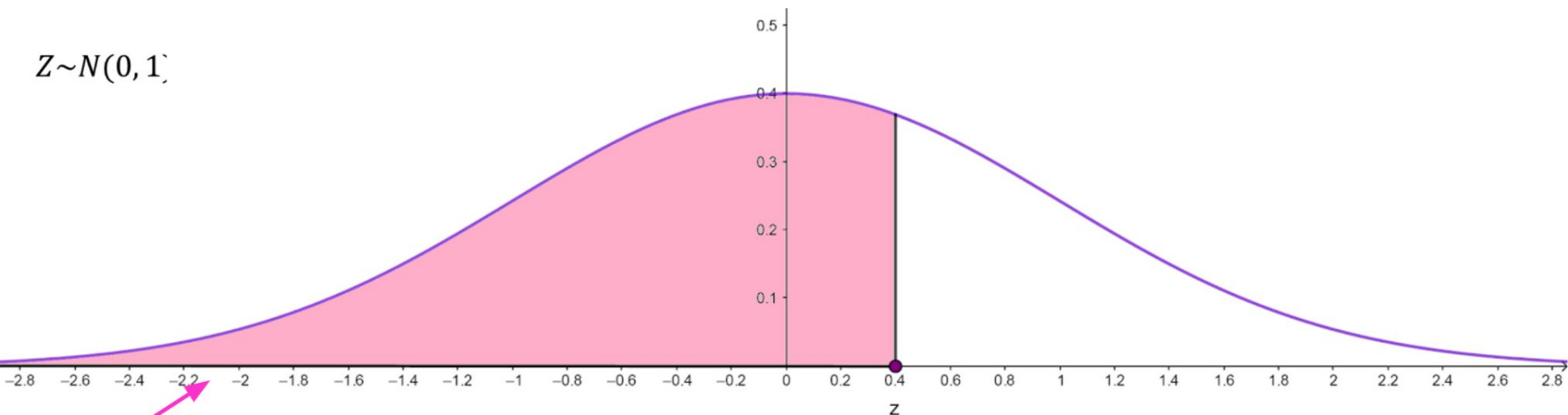
```
# P(Z<=0.4)
```

```
norm.cdf(0.4, loc=mean, scale=var)
```

```
0.6554217416103242
```

STANDARD NORMAL DISTRIBUTION

$$Z \sim N(0, 1)$$



$$P(Z \leq 0.4) = \Phi(0.4)$$

`cdf(x, loc=0, scale=1)`

Cumulative distribution function.

```
mean, var = 0, 1
```

```
# P(Z ≤ 0.4)
```

```
norm.cdf(0.4, loc=mean, scale=var)
```

```
0.6554217416103242
```

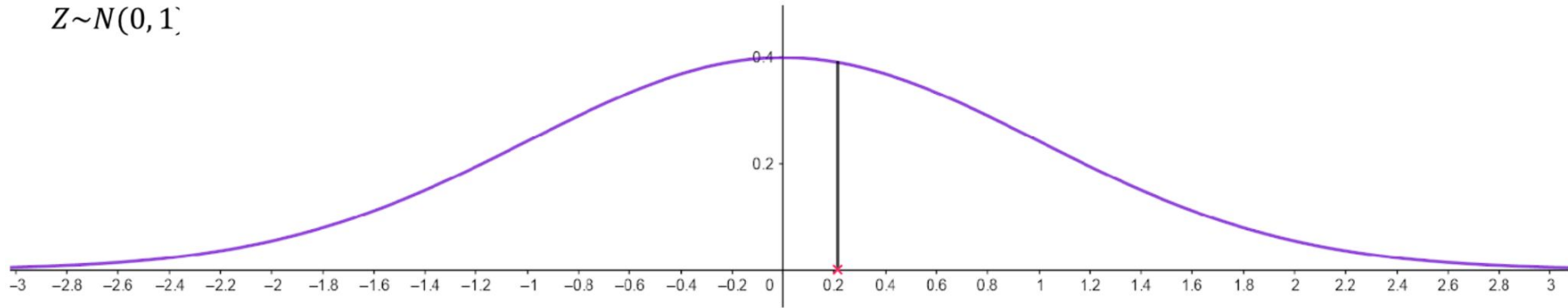
```
# P(Z ≤ 0.4)
```

```
norm.cdf(0.4)
```

```
0.6554217416103242
```

STANDARD NORMAL DISTRIBUTION

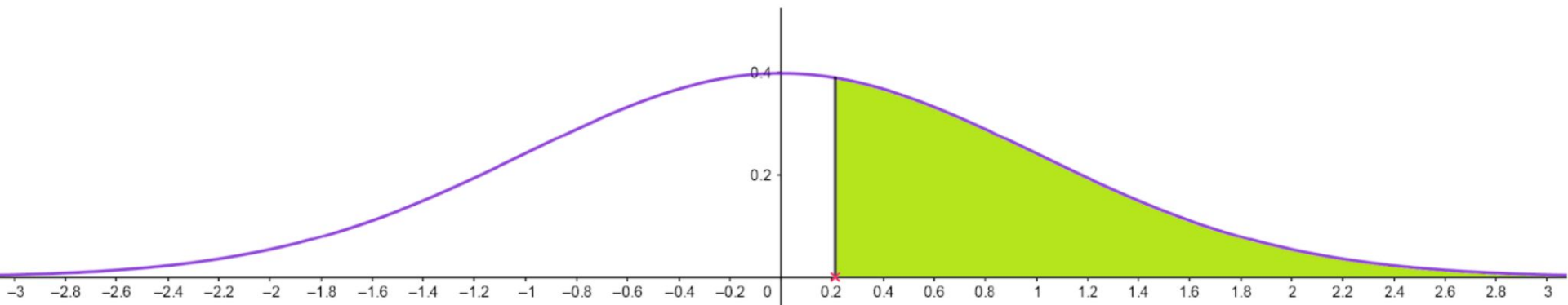
$$Z \sim N(0, 1)$$



$$P(Z \geq 0.213)$$

STANDARD NORMAL DISTRIBUTION

$$Z \sim N(0, 1)$$



$$P(Z \geq 0.213) = 1 - \Phi(0.213)$$

```
# P(Z>=0.213)  
1-norm.cdf(0.213)
```

```
0.415663481341247
```

STANDARD NORMAL DISTRIBUTION

$$Z \sim N(0, 1)$$

$$P(1.15 \leq Z \leq 1.35) = \Phi(1.35) - \Phi(1.15)$$

```
# P(1.15<=Z<=1.35)  
norm.cdf(1.35)-norm.cdf(1.15)
```

```
0.0365639441997484
```

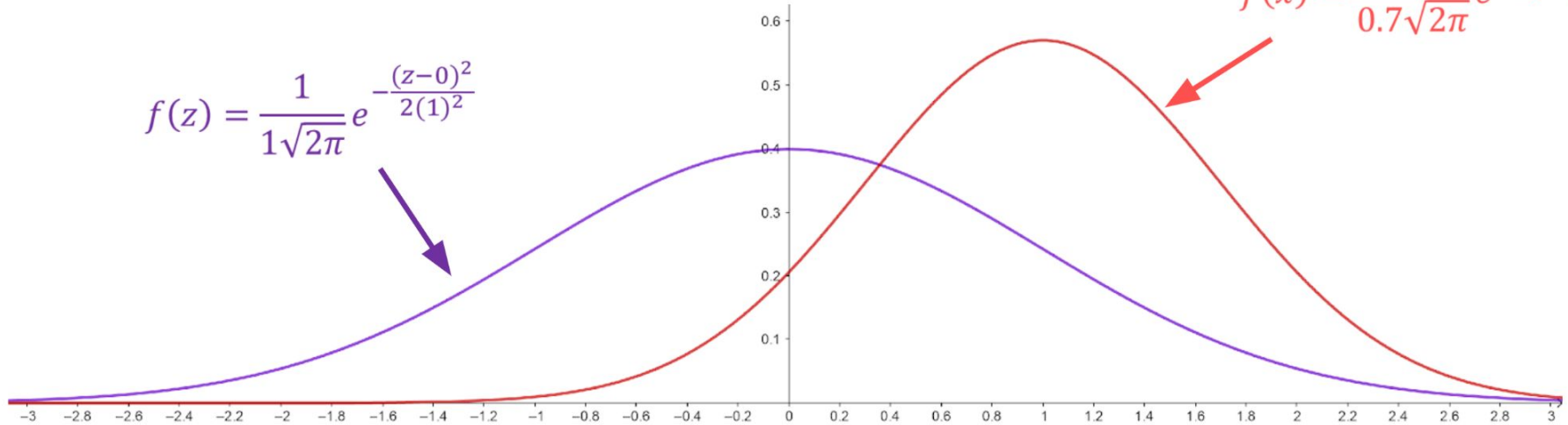
NORMAL DISTRIBUTION

$$Z \sim N(0,1)$$

$$X \sim N(1, 0.7^2)$$

$$f(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{(z-0)^2}{2(1)^2}}$$

$$f(x) = \frac{1}{0.7\sqrt{2\pi}} e^{-\frac{(x-1)^2}{2(0.7)^2}}$$



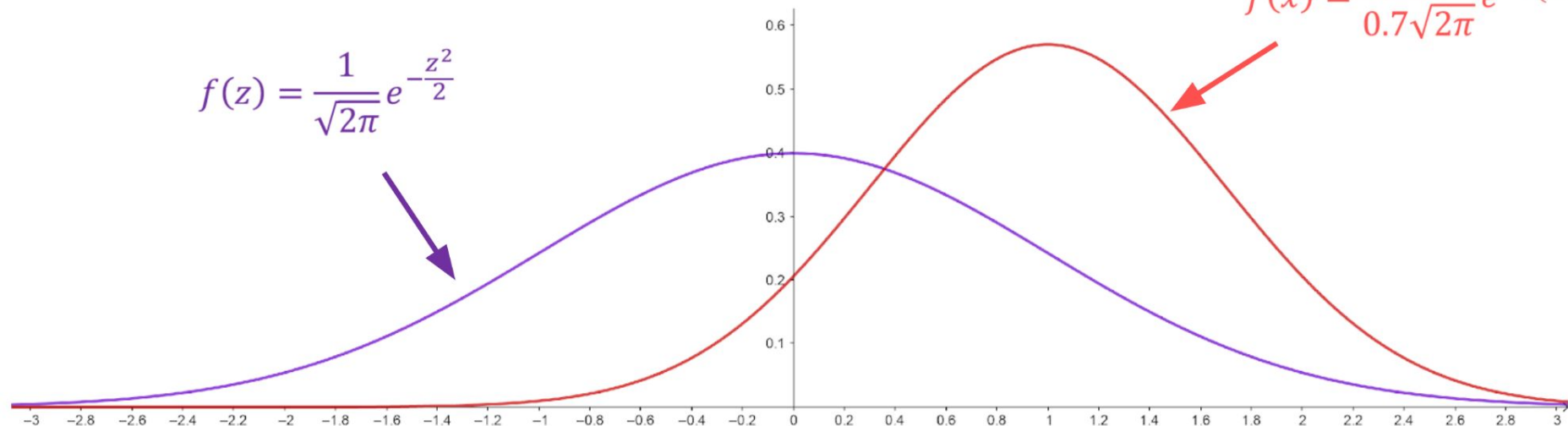
NORMAL DISTRIBUTION

$$Z \sim N(0, 1)$$

$$X \sim N(1, 0.7^2)$$

$$f(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$$

$$f(x) = \frac{1}{0.7\sqrt{2\pi}} e^{-\frac{(x-1)^2}{2(0.7)^2}}$$



NORMAL DISTRIBUTION

$$X \sim N(205, 400)$$

$$P(X \leq 230)$$

```
mean, var = 205, 400
```

```
# P(X<=230)  
norm.cdf(230, loc=mean, scale=sqrt(var))
```

```
0.8943502263331446
```

NORMAL DISTRIBUTION

Dado $X \sim N(6,4)$, hallar el valor de s tal que $P(X \leq s) = 0.6500$

```
mean, var = 6, 4
```

```
ppf(q, loc=0, scale=1)
```

Percent point function (inverse of cdf – percentiles).


```
[11] norm.ppf(0.65, loc=mean, scale=sqrt(var))
```

```
6.770640932815136
```

Two friends Sarah and Hannah often go to the post office together. They travel on Sarah's scooter. Sarah always drives Hannah to the post office and drops her off there. Sarah then drives around until she is ready to pick Hannah up some time later. Their experience has been that the time Hannah takes in the post office can be approximated by a normal distribution with mean 6 minutes and standard deviation 1.3 minutes. **How many minutes after having dropped Hannah off should Sarah return if she wants to be at least 95% certain that Hannah will not keep her waiting?**




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T = "tiempo que Hannah demora en el correo en un día aleatorio"  $T \sim N(6, 1.3^2)$



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$T = \text{"tiempo que Hannah demora en el correo en un día aleatorio"}$  $T \sim N(6, 1.3^2)$

$$P(T \leq t) = 95\%$$

$$P(T \leq t) = 0.95$$

```
mean, std = 6, 1.3
```

```
#  $P(T \leq t) = 0.95$   
norm.ppf(0.95, loc=mean, scale=std)
```

```
8.138309715036915
```

```
norm.cdf(8.2, loc=mean, scale=std)
```


```
0.9547063390392431
```

```
norm.cdf(8.1, loc=mean, scale=std)
```

```
0.946886285109998
```



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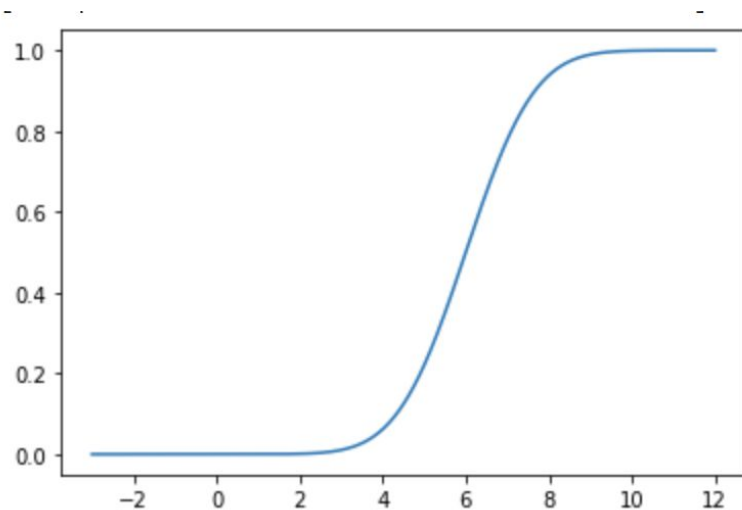
$$P(T \leq t) = 95\%$$

$$P(T \leq t) = 0.95$$

```
mean, std = 6, 1.3
```

```
# P(T≤t)=0.95  
norm.ppf(0.95, loc=mean, scale=std)
```

```
8.138309715036915
```



SUMMARY

Today we've learnt:

- Introduction to Normal distribution
 - Formula
 - Definition
 - Graph
- Calculating probabilities of a random variable that follows a normal distribution using Python
- Modelling a real life problem using normal distribution



