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Does the Euler Disk slip during its motion?

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The motion of a disk that is spun on a smooth flat surface slowly damps out due to friction. To help identify the nature of the friction, we test experimentally whether the disk slips during its motion. We find that, at least during the early stages, the disk rolls without slipping, thus ruling out sliding friction as the cause of the damping. Together with the results of the experiments of van der Engh *et al.* that rule out air friction, our results establish that rolling friction is mainly responsible for the damping in the early stages of the motion. Student projects are suggested that could establish whether our conclusion of rolling without slipping holds for the later stages of the motion.

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I. INTRODUCTION

Everyone has observed the interesting motion of a coin that has been spun on a table. In the simplest case, as the coin rolls, the center of mass remains fixed horizontally and slowly sinks vertically. In the final stage, with the coin nearly horizontal, the angular velocity of the contact point increases dramatically, while the rotation of the face of the coin simultaneously slows. As the coin shudders to a halt, a rapidly increasing whirring sound is heard.

The motion of a coin or disk has often been discussed theoretically for the case of no dissipation in the limit of an infinitely thin disk,[1–4] and occasionally for disks of finite thickness.[5]

Interest in the motion of a spinning disk was rejuvenated by the toy, "Euler Disk," [6] appropriately named after Euler[7] who pioneered the study of rotational motion of rigid bodies. The Euler disk is larger (diameter ~ 7.5 cm) and heavier (mass ~ 440 g) than a coin, and rolls on a smooth, slightly concave base, thus substantially prolonging the motion (up to 2 minutes or more, depending on the initial conditions). Recent theoretical work[8–12] has included the dissipation due to various mechanisms that are discussed below, and has been concerned with the nature of the finite time singularity that arises when the disk finally shudders to a halt. The nature of this singularity depends on the nature of the friction that damps the motion.

A key unresolved issue is the cause of the friction that slowly damps the motion. The three most plausible proposed mechanisms are air friction,[8] rolling friction,[9] and sliding friction.[10] Air friction would appear to be ruled out as the main cause of the damping by the work of Ref. 10; experiments with hollowed-out disks (rings), and with disks in evacuated chambers (pressure <1 mtorr$\sim 1.3\times 10^{-6}$ atm) lead to very little change in the damping rate.

The sound emitted during the motion does not allow us to distinguish between rolling and sliding friction. To distinguish between them, we carry out measurements on the Euler disk to see whether or not slipping occurs during the motion. We find that slipping does not in fact occur, at least in the early stages. Together with the results of Ref. 10, our results suggest that, at least in the early stages, rolling friction is mainly responsible for the damping, thus supporting the hypothesis of McDonald and McDonald.[9]

II. THEORY

Consider a uniform solid disk of mass M, radius a, and half-thickness b, which rolls on a flat horizontal surface. As is standard for describing rigid body motions, we employ space-fixed axes XYZ, with Z vertical, and body-fixed axes 123, with 3 along the symmetry axis (Fig. 1). We also employ the standard Euler angles[3] ϕ, θ, ψ specifying the orientation of 123 with respect to XYZ, and corresponding angular velocities $\dot{\phi}$, $\dot{\theta}$, $\dot{\psi}$, where the dot represents d/dt with t the time. We see from Fig. 1 that $\dot{\phi}$ (denoted by Ω in Ref. 8) is the precession rate about Z, $\dot{\psi}$ is the spin rate about the 3 axis, and $\dot{\theta}$ is the nutation rate.

We consider first the case of pure rolling (no slip) with steady precession at constant tilt angle θ, that is, $\dot{\phi}=$constant, $\dot{\psi}=$constant, and $\dot{\theta}=0$. If the disk is rolling into the paper in Fig. 1, then $\dot{\psi}>0$ and $\dot{\phi}<0$. (It is useful to remember that $\dot{\psi}$ and $\dot{\phi}$ have opposite signs.) For the experiments we consider, the center of mass G is at rest, and the contact point C rolls around a circle of radius ρ with its center at the intersection of the Z axis and the horizontal plane, where (see Fig. 1)

$$\rho = a\cos\theta - b\sin\theta. \tag{1}$$

For the case of no slipping, the velocity of the contact point C (into the paper) is $v=a\dot{\psi}$. Because we also have $v=-\rho\dot{\phi}$, as seen by looking down from above, we have

$$a\dot{\psi} = -\rho\dot{\phi}. \tag{2}$$

If we combine Eqs. (1) and (2), we obtain the no-slipping condition as

$$\dot{\psi} + \dot{\phi}\left(\cos\theta - \frac{b}{a}\sin\theta\right) = 0. \tag{3}$$

Equation (3) is one of the two we test in our experiment by measuring (at the same time) $\dot{\psi}$, $\dot{\phi}$, and θ. Due to the damping, $\dot{\psi}$, $\dot{\phi}$, and θ all change slowly with time, and we test Eq. (3) for each value of θ.

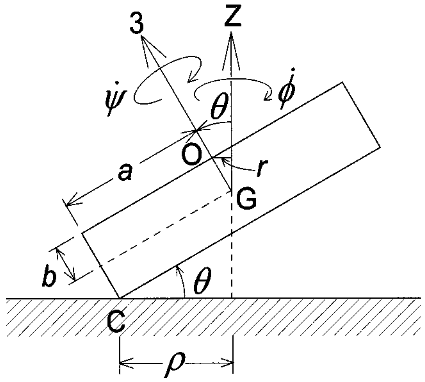

Fig. 1. The Euler Disk. XYZ are space-fixed axes, with Z vertical and X,Y (not shown) horizontal. 123 are body-fixed axes, with 3 along the symmetry axis and 1,2 not shown. The contact point C rolls on a circle of radius ρ about a vertical axis passing through the fixed center of mass G. The point O is the center of the top of the disk, which moves in a horizontal circle of radius r. The positive directions of $\dot\psi$ and $\dot\phi$ are shown. (When the disk rolls into the paper, $\dot\phi$ has the direction opposite to that shown.)

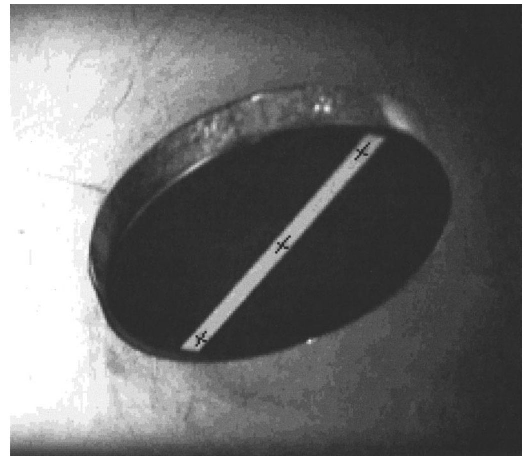

Fig. 2. A frame from the resulting video showing the disk rolling on the mirror. The white paper bar has the three crosshairs used for the measurements.

A second relation depending on the no-slip condition can also be checked experimentally. For the case of G at rest, the uniform precession rate $\dot\phi$ at fixed θ can be derived from the equations of motion,[5]

$$\dot\phi^2=\frac{(g/a)(\cos\theta-(b/a)\sin\theta)}{(\frac{1}{4}+\frac{1}{3}(b/a)^2)\sin\theta\cos\theta-\frac{1}{2}(b/a)\sin^2\theta}, \qquad (4)$$

where g is the gravitational acceleration. For very thin disks $(b\rightarrow 0)$, Eq. (4) reduces to $\dot\phi^2=4g/a\sin\theta$, the expression given in Refs. 8 and 9. [This limit, and the corresponding one from Eq. (3), cannot be used for the Euler disk because $b/a\sim 0.17$ is not negligible.] In reality, due to damping, $|\dot\phi|$ slowly increases as θ slowly decreases (adiabatic motion),[8] and we check whether Eq. (4) holds for each value of θ.

III. EXPERIMENT

The disk supplied by the manufacturer[6] was used. It has a diameter of 7.5 cm and a thickness of 1.3 cm ($a=3.75$ cm and $b=0.65$ cm); the mass is 0.44 kg. The original base made of hard plastic was not used, as the rolling motion of the disk induced considerable distortion in the surface, and there was clearly an energy interchange between the base and the disk.[13] A Pyrex telescope mirror 25 cm in diameter and 4 cm thick with a radius of curvature of 150 cm was used instead. The rolling took place on the over-coated aluminum layer of the mirror and no distortions were observed.

The motion of the disk was recorded with a video camera operating at a rate of 30 frames/s with an exposure time of 1/3500 s. The horizontally mounted camera observed the motion from directly above using a high-quality front-surface mirror. To achieve uniform illumination for all orientations of the disk, the apparatus was surrounded with white paper curtains to form a ''light furnace'' which was illuminated with two 750 watt floodlights.

The top surface of the disk was painted flat black to eliminate glare and a white paper bar 6.5×0.5 cm was pasted along a diameter. Crosshairs were drawn on the bar in the center (point O in Fig. 1) and near the ends separated by 6.0 cm. The arrangement is shown in Fig. 2, which is one of the frames from the final video chosen for analysis. This sequence has 4190 frames for a total time of about 140 s. For each of the frames the horizontal X,Y coordinates of the three points on the bar were recorded in pixels, which were converted into centimeters (130 pixels = 6 cm). These measurements permit the determination of the various angular frequencies as follows.

(a) $\dot\phi$ is the precession rate of the symmetry axis about the vertical. It is the rotation rate of the center of the top face (point O in Fig. 1), and also the rotation rate of the contact point C. It is determined by analyzing the periodicity of the X (or Y) coordinate of the center point O.

(b) $\dot\psi$ is the rotation rate of the bar about the 3 axis. It is obtained by analyzing the time dependence of the apparent length of the bar as seen from above. The period between two apparent minima is half the period of $\dot\psi$.

(c) $\overline{\Omega}$ is the mean angular velocity of the bar as seen from above. It is obtained by analyzing the time dependence of the X (or Y) coordinate of an end point of the bar.

The relation between these quantities is

$$\overline{\Omega}=\dot\phi+\dot\psi. \qquad (5)$$

$\dot\phi$ and $\dot\psi$ are always of opposite sign and their magnitudes increase with time; $|\overline{\Omega}|$ decreases with time and comes to zero when the disk stops. Because of the relation (5), only two of $\overline{\Omega}$, $\dot\phi$, and $\dot\psi$ need be measured; we measure all three as a check.

The measurement of these three frequencies was made by performing a Fourier analysis on each 1 s interval of data over which the angular velocities can be considered constant; the relation between them is shown in Fig. 3(a). As expected, $\dot\phi$ and $\dot\psi$ have opposite signs [the measured values are shown as dots in Figs. 3(b) and 3(c)] and become equal in magnitude as time increases and the tilt of the disk becomes flatter. The measured and the calculated values of $\overline{\Omega}$ from Eq. (5) are virtually indistinguishable in Fig. 3(a), showing that the frequencies can be measured with considerable precision.

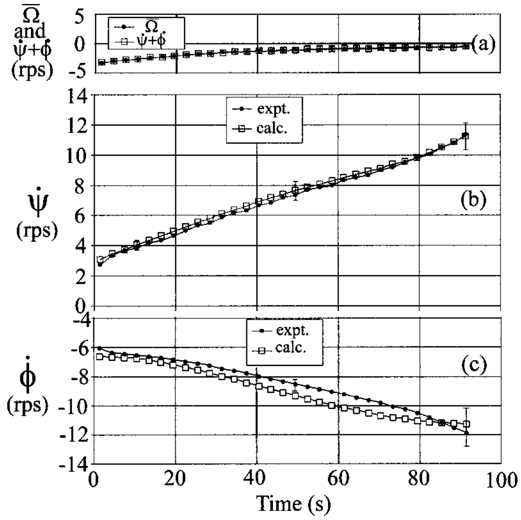

Fig. 3. (a) Test of the relation (5), $\bar{\Omega}=\dot{\phi}+\dot{\psi}$. (b) Test of the no-slip condition (3) for the spin rate $\dot{\psi}$. (c) Test of the no-slip value (4) for the precession rate $\dot{\phi}$. The angular velocities are all given in revolutions per second (rps).

The data set for $\dot{\phi}$ and $\dot{\psi}$ for the particular run we analyzed can only be carried to 90 s of the 140 s of the motion as the frame rate of 30 Hz has a Nyquist limit at 15 Hz; that is, after 90 s the disk was spinning so rapidly that it made half a revolution between frames, so that the true rotational rates cannot be measured. The $\bar{\Omega}$ data can be carried out to the end, but is not shown beyond 90 s.[14]

The final experimental variable is the angle of inclination θ. This quantity was determined from the variation of the apparent length of the bar as seen from above. The angle is determined from $\cos\theta = L_{\min}/L_{\max}$, where $L_{\min}$ and $L_{\max}$ ($=130$ pixels) are the apparent minimum and maximum lengths of the bar measured between the two outer crosses (see Fig. 2). This is the noisiest data in the experiment and the experimental error in the analysis that follows predominantly originates in the determination of the angle. The value of $\cos\theta$ was again determined by averaging the data over 1 s intervals. The resulting values of $\cos\theta$ versus t were fitted with a smooth curve (a sixth order polynomial), and the angle and its functions were determined as needed from this fit. All the data were taken from one run. The angle θ could also be determined from measurement of the radius r of the circle traced out by the center point O using the relation $\sin\theta = r/b$ (see Fig. 1), but because r is always small, this determination would be less accurate than the method used here. Different runs would give essentially the same results after the initial transient (which lasts about 60 s in our launch) is gone. The transient is due to an imperfect launch; that is, initially there is some nutation and G is initially not exactly above the lowest point of the concave surface.

IV. ANALYSIS

The no-slip condition given by Eq. (3) is now tested; the experimental and calculated values of $\dot{\psi}$ are shown in Fig. 3(b). These two curves are essentially coincident and well within the uncertainty introduced by the measurement of $\cos\theta$. This error is shown in Fig. 3(b) by the (unsymmetrical) error bars at calculated points near 10, 50, and 90 s. Figure 3(b) appears to give strong evidence that the disk rolls without slipping.

Finally the precession is shown in Fig. 3(c) where the calculated and measured values of $\dot{\phi}$ are plotted. The calculated value is obtained using the no-slip value in Eq. (4) giving $\dot{\phi}$ in terms of θ, and using θ as a function of time from the smooth curve of $\cos\theta$ versus t found earlier. This calculated curve is more sensitive to the uncertainty in θ than the corresponding curve in Fig. 3(b), and error bars are shown in Fig. 3(c) at calculated points near 50 and 90 s. The error near 10 s is of the order of the size of the plot symbol. These results again confirm the no-slipping condition.

V. STUDENT PROJECTS

All of the analysis in the experiment was carried out using a spreadsheet program and its graphing facilities. The analysis is an excellent vehicle for instruction in these techniques. There are opportunities for further investigation.

(1) Is it possible to alter the nature of the rolling surface (for example, with lubricants or disk size or shape) to change the general conclusion of rolling without slipping?

(2) The recording of X, Y coordinates on 4000 frames is rather onerous. Can this process be further automated?

(3) The availability of cameras with a higher frame rate than 30 per second would permit investigation of the motion nearer to the end. Are there departures from the results obtained here? Better quality motion-picture cameras have this ‘‘slow motion’’ facility.

(4) An analysis of the sound spectrum has been briefly examined (see Ref. 11). The sound recording does not suffer from Nyquist limitations, so simultaneous recording of the motion as performed in this work and an analysis of the sound might make it possible to examine the high frequency, final seconds of the motion.

(5) A third possible way to reach larger times would be to extend an optical method. McDonald and McDonald[9] obtained $\dot{\phi}(t)$ for large t by reflecting a beam of light off the top surface of the spinning disk and measuring the frequency of the reflected beam by a photodetector. They can record values of $\dot{\phi}$ up to several hundred Hz by this method. If one could record $\theta(t)$ simultaneously, then the no-slip value (4) of $\dot{\phi}(\theta)$ could be checked for larger times.

VI. CONCLUSION

By measuring the angular velocities of the Euler Disk, we found that the no-slip condition is satisfied for the early part of the motion, that is, for $\theta \gtrsim 10°$. This result appears to rule out sliding friction as the cause of the damping, at least in the early stages of the motion. Because air friction has been ruled out by the work of Ref. 10, our results show that rolling friction is mainly responsible for the damping in the early stages of the motion. Three methods were suggested that could establish whether our conclusion of rolling without slipping continues to hold for the later stages of the motion.

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[14]The Nyquist limit for $\dot{\psi}$ is actually 15/2 Hz [see point (b) of Sec. III for the factor of 2], but can be extended to 15 Hz by examining each Fourier transform spectrum and knowing that the frequency must increase monotonically.