Nonlinear Dyn
DOI 10.1007/s11071-014-1811-5

ORIGINAL PAPER

On the loss of contact of the Euler disk

Alexey V. Borisov · Ivan S. Mamaev · Yury L. Karavaev

Received: 25 October 2013 / Accepted: 13 November 2014

Abstract This paper is an experimental investigation of a round uniform disk rolling on a horizontal surface. Two methods for experimentally determining the loss of contact of the rolling disk from the horizontal surface before its stop are proposed. Results of experiments for disks having different masses and manufactured from different materials are presented. Causes of "microlosses of contact" detected in the processes of motion are discussed.

Keywords Euler's disk · Loss of contact · Experiment

1 Introduction

This paper is concerned with the experimental investigation of motion of a round uniform disk on a rough plane, especially at its final stage. For some reason, in recent years such a disk has been called Euler's disk [1], although in nonholonomic mechanics (i.e., when the disk moves on an absolutely rough plane), the problem of rolling of a disk was considered much later by Appel [2], Chaplygin [3], Korteweg [4] and others. An overview of the current status of the problem is given in [5]. Despite a large number of papers (including those devoted to nonholonomic problems) which prove integrability of this problem, they cannot explain some experimental effects: the increase in sound frequency during the motion of the disk and an abrupt stop, which are obviously caused by dissipative friction forces and possibly air viscosity.

We note that all models describing the process of rolling of a body on a surface are nonlinear and, as a rule, turn out to be fairly complicated. For example, in [5–7], it was shown that within the nonholonomic rolling model, the behavior of the body can be both regular and chaotic depending on its parameters and properties of the surface. Such systems exhibit the so-called hierarchy of dynamics depending on the conserved tensor invariants.

The current interest in the motion of the disk was largely aroused by the highly controversial paper of Moffatt [8], in which the termination of the motion of Euler's disk is attributed to the presence of viscous drag forces of the air. After experiments with Euler's disk in a vessel into which compressed air was pumped or in which vacuum was created, van den Engh et al. [9] showed sufficiently rigorously the negligibility of the viscous drag force of air. From their point of view, the motion pattern of Euler's disk is determined by the

A. V. Borisov · I. S. Mamaev
Laboratory of Nonlinear Analysis and the Design of New Types of Vehicles, Institute of Computer Science, Udmurt State University, Universitetskaya 1, Izhevsk 426034, Russia
e-mail: borisov@rcd.ru

I. S. Mamaev
e-mail: mamaev@rcd.ru

Y. L. Karavaev (✉)
Deparment of Mechatronic Systems, Kalashnikov Izhevsk State Technical University, Studencheskaya st. 7, Izhevsk 426069, Russia
e-mail: karavaev_yury@istu.ru

Published online: 23 November 2014

sliding friction force, and its impulsive behavior leads to the loss of contact of the disk with the surface in the process of motion and then to a stop.

The authors of subsequent works conducted investigations of the influence of rolling friction and sliding friction at various stages of the disk's motion [10–18]. For the power law of change in the inclination angle θ and the precession velocity $\dot{\alpha}$ of Euler's disk, the researchers determined the values of the index n depending on the dissipation mechanism under discussion:

$$\theta(t) \propto (t_f - t)^n, \ \dot{\alpha}(t) \propto (t_f - t)^{-\frac{1}{2}n}. \tag{1}$$

A detailed analysis of these studies is presented in Leine [19,20], in which different dissipation mechanisms are structured and compared. He concludes that dry friction prevails at the beginning of the stage of stationary rolling, and viscous friction predominates in the last one or two seconds before the disk comes to rest. These conclusions are in good agreement with the experimental results presented in [19,20]; however, as R.Leine himself states, other dissipation mechanisms may lead to analogous values of the exponents in the expression (1).

The authors of [21] consider the equations of motion taking into account the sliding of the disk and the air drag and neglecting the friction torque and deformation of the disk and the surface. They have also obtained conditions for a transition from sliding to rolling and performed simulations which showed that the reaction is positive, i.e., there is no loss of contact during the motion of the disk.

For convenience, all experiments, whose results are also summarized in [19,20], can be divided into two groups. The first group includes investigations of the evolution of the dynamical variables (inclination angle, precession velocity, angular velocity etc.) of Euler's disk by means of high-speed video cameras [15,19,20,22]. The parameters of the moving disk were determined by side view shooting [19,20] (the optical axis of the lens lies in the plane on which the disk moves) or top view shooting [22] (the optical axis of the lens is perpendicular to the plane of the surface) or by combined shooting to identify each parameter separately [15]. The work of McDonald and McDonald [14], who registered a light ray reflected from the disk's surface by means of a phototransistor, may also be included in this group. Research of the second group is concerned with sound vibrations generated during the motion of Euler's disk [17,21]. The authors of these papers conclude that the sound vibrations are nonlinear. Similar technical facilities were used by Shimomura and his colleagues [23,24] in an experimental study of the motion of a spheroid on a horizontal surface. Using acoustic and optical signals (recorded with high-speed cameras) and electric signals (the capacitance of the table–spheroid system was measured), the authors were able to register the losses of contact between the rotating spheroid and the horizontal surface. However, the rotation speed of the spheroid at which it loses contact with the surface is higher than 1,000 rpm. The experimental results are in good agreement with the simulation results for the case of a small coefficient of dry friction and a large initial angular momentum of the spheroid.

The problem of the loss of contact of a rolling disk was discussed in the theoretical works of Batista [25,26] and Ivanov [27]. For example, Ivanov analytically obtained conditions for the loss of contact of a rolling thin disk for different laws of friction [27]. In the case of a disk with nonzero thickness, these conditions do not hold. Batista obtained conditions for the loss of contact both for a rolling disk and a spheroid at the start of motion in the absence of friction [26]. Kessler and O'Reilly [12] defined the normal force (reaction) to be strictly positive, which rules out the possibility of the rolling disk losing contact during motion, although in conclusion, they advanced the hypothesis that the disk comes abruptly to rest due to the loss of contact between the disk and the surface in the process of vibrations at a small angle of inclination.

In spite of the above-mentioned studies, the question of the loss of contact of Euler's disk in the process of motion remains open. Therefore, to construct an appropriate model describing most accurately the dynamics of the disk, it is necessary to have a clear experimental conclusion about the losses of contact. Thus, the goal of this work is to perform experimental investigations of the moment when Euler's disk stops and to obtain an experimental confirmation of the loss of contact with a corresponding impact.

In this paper, we shall give no theoretical explanations of the phenomena considered, since this would require a detailed analysis and development of a suitable mathematical model.

To achieve this goal, two methods have been devised which allow the presence of contact to be investigated

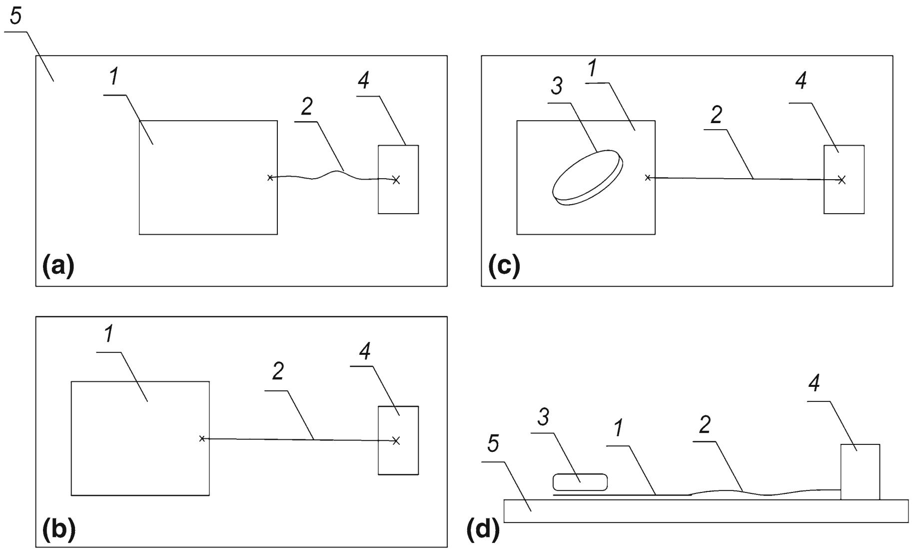

Fig. 1 Scheme of the experiment. **a** A sheet of paper in the initial position. **b** The sheet of paper taken aside to ensure the tension of the elastic is held in this position. **c** Euler's disk is set in motion on the sheet; at the end of motion, the sheet is released. **d** The sheet is withdrawn from under the disk under the action of the elastic's tension at the instant when it loses contact

both during the motion and at the moment of stop of the disk.

2 Simple experiment

The scheme of the first experiment (see Fig. 1) does not require sophisticated equipment, but allows one to confirm the loss of contact of the disk with the surface before a stop by using available materials. To conduct the experiment, it is necessary to attach one end of the elastic 2 to a sheet of paper 1. The other end of the elastic 2 is pressed with a heavy object 4 to the surface of the table 5 (the mass of the object 4 must be larger than that of Euler's disk 3). Thereafter, the sheet of paper is moved by hand for some distance on the table surface to ensure the tension of the elastic (Fig. 1b). The value of tension is chosen experimentally depending on the mass of Euler's disk. In a position where the elastic is tightened (the sheet of paper can be held by hand), the disk 3 is set in motion on the sheet 1. In the last seconds of the disk's motion, the sheet of paper should be withdrawn from under the disk. As the disk is moving on the sheet, the elastic remains tightened, and at the moment of stop, when the disk loses contact, the sheet of paper is withdrawn from under the disk by the elastic.

The disk on the surface of the sheet is set in motion by hand, and the parameters of its motion (initial velocity, initial angle of inclination) and the parameters of the disk itself (mass, roughness of the surface, material of the disk, radii of rounding) have no influence on the manifestation of this effect. The sheet can slightly twitch as Euler's disk is moving on it, but the "jerk" of the sheet from under the disk before its stop remains obvious and is more violent than the previous motions.

3 Advanced experiment

To carry out a more advanced experimental investigation of the disk's motion and to determine the factors that influence the loss of contact of the disk at the moment of its stop, an experimental setup has been developed (see Fig. 2). This setup allows one to confirm the loss of contact of the disk at the moment of its stop, to record the time of the loss of contact and to explore the final motions in more detail.

The disk 1 was set in motion on a sheet of stainless steel 2, 1.2 mm in thickness. For firm adherence of the sheet to the table 3, it was fastened by screw clamps 4. The table 3 was mounted by means of a digital level strictly horizontally, with an error of 0.050. The disk has a blind hole drilled and a thread M3 cut into the

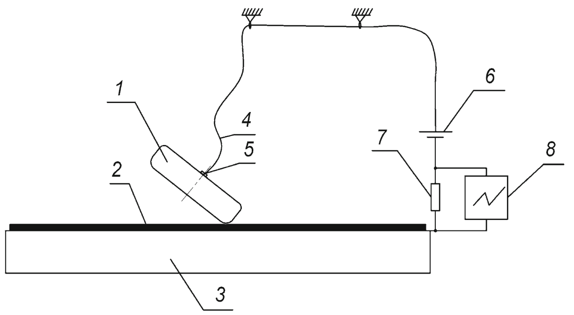

Fig. 2 Layout of the experimental setup for determining the time of the loss of contact of Euler's disk

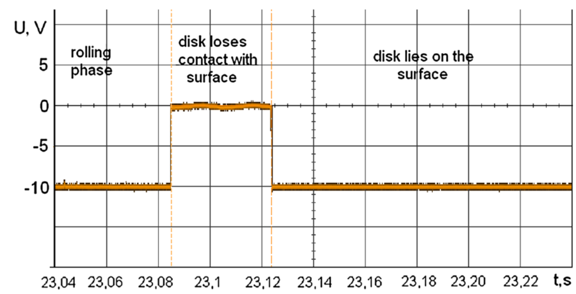

Fig. 3 A typical oscillogram showing the loss of contact of the disk (1 division of the abscissa scale is 20 ms, 1 division of the ordinate scale is 5 V)

Table 1 Experimental models

Designations of the disks	Diameter (mm)	Height (mm)	Mass (kg)	Material
Disk 1	100	20	0.435	Duralumin
Disk 2	100	40	0.868	Duralumin
Disk 3	100	20	1.215	Stainless steel

center of it so that a wire 5 (cross section $0.05\,\mathrm{mm}^2$) can be connected to the disk by means of a screw 6 (M3x10). The wire was chosen such that its mass and elasticity did not affect the motion pattern of the rotating disk. This influence was evaluated from the duration and motion pattern of the disk with the fastened wire and without it. The wire was connected through fixed supports (Fig. 2) to a DC power source 7 adjusted for a voltage of 10 V. In the case of loss of contact of the disk, an open-circuit failure occurred: Euler's disk 1—wire 5—power source 7—sheet of stainless steel 2—resistor 8 ($22\,\mathrm{k}\,\Omega$), which is connected into the circuit to limit current, since all circuit elements have a low ohmic resistance.

The presence or absence of electric current was registered on the resistor 8 by the oscillograph 9. The discretization frequency of the oscillograph 9 (manufactured by Agilent DSO-X 3024A) was 100 MHz. The signal was picked up by the oscillograph automatically via the trigger adjusted for the rear front of the signal with a voltage of 2 V. When the disk was located on the surface 2, electric current flew in the circuit, which was recorded by the oscillograph, and the voltage on the resistor was +10 V or −10 V depending on the polarity of connection of the devices. In the case of loss of contact of the disk with the surface, the electric circuit was broken, and the voltage on the resistor was 0 V. A slight torsion created in the wire after each experiment was eliminated by disconnecting it from the devices and bringing it to the initial state.

The minimal distance between the disk and the plane which can be detected by this method is theoretically determined by Paschen's law and does not exceed $10\,\mu\mathrm{s}$ [28]. Experimental investigations of the dependence of breakdown voltage on the distance between the wires in the air were carried out for much higher values of voltage (no <100 V). The results of investigation of the breakdown voltage for steel wires are presented in [29]. These results imply that the breakdown occurs at 150 V at a distance of $0.9\,\mu\mathrm{s}$. At lower voltage values, no breakthrough was observed, even at smaller distances.

At the same time, we recorded the sound that accompanied the rolling of the disk. A typical oscillogram showing the loss of contact of the disk is presented in Fig. 3.

For the experiments, three disks were chosen with characteristics presented in Table 1 (the radii of rounding of the disks were 2 mm, and the surfaces of the disks were ground).

Three experiments were carried out with each disk. Before each experiment, the surfaces of the sheet and the disk were wiped with a new cotton cloth moistened with distilled water to remove dust particles. Before the experiments, the sheet of stainless steel was treated with a weak solution of sulfuric acid to remove possible oil films and was rinsed with distilled water. The results of the experiments are presented in Table 2.

It is seen from Table 2 that the larger the mass of Euler's disk, the shorter the loss of contact with the surface before the stop. All disks were set in motion "by hand" under different initial conditions, but this did not influence the motion pattern and the duration of the loss of contact.

Table 2 Results of experiments

	Duration of the loss of contact of the disk at the moment of stop (ms)		
	Disk 1	Disk 2	Disk 3
Experiment 1	38.4	23.8	13.0
Experiment 2	38.6	21.6	13.0
Experiment 3	42	21.6	12.6
The average for the three experiments	39.67	23.33	12.87

To evaluate the influence of the rigidity of the surface and the coefficients of friction, the experiments were also conducted on a 310 mm × 310 mm × 30 mm duralumin plate. The duration of the loss of contact of the disk at the moment of stop for the disk 1 was on the average 102 ms for five experiments, which points to an influence of the factors considered on the motion pattern of the disk. Despite the fact that the sheet of stainless steel was pressed to the table by means of screw clamps, the sheet was deformed by the disk moving on it, resulting in a decrease in the duration of the loss of contact of the disk with the sheet.

4 Conclusions

The experimental results were unexpected and revealed a number of peculiarities in the motion of the disk:

1. The smoothness of the surface on which the disk moves and the quality of treatment of the surfaces contribute considerably to the motion pattern of the disk, although the duration of the loss of contact of the disk at the moment of stop remains the same.
2. The experiments also revealed that the disk loses contact not only at the moment of stop but also during the motion; however, the duration of the loss of contact is significantly smaller (up to 0.7 ms) (Fig. 4). The frequency and duration of "microlosses" of contact are hardly predictable and random. However, the contact point for the disks moves in the meantime over a distance of up to 3 mm.
3. The higher the quality of treatment of the surfaces, the less the duration and the number of the losses of contact during the motion. Hence, it may be concluded that the microasperities and roughnesses of the interacting surfaces are the main cause of the losses of contact of the disk during the motion.
4. The analysis of sound vibrations during the rolling motion of the disk using Fourier and wavelet transformations showed that the signal includes a large number of harmonics typical of shock pulses. As in [17,21], the peaks of the amplitude of oscillations on several intervals are in good agreement with speed fluctuations of the contact point, which allows one to classify them as first harmonics. The noisiness of the spectrum of the acoustic signal can be attributed to the presence of "microlosses" of contact.

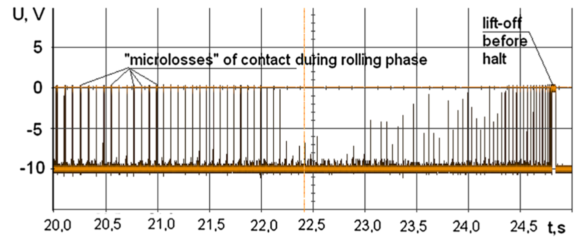

Fig. 4 Loss of contact of the disk during the motion (the last 5 s of motion)

Figure 5 shows a typical sound signal of the last 5 s of motion of Euler's disk. Figure 6 presents the result of wavelet transformation, which allows one to keep track of changes in the frequency of sound vibrations during the motion of the disk. The Gauss wavelet was chosen as a function with which the convolution took place, since it is best suited for use with signals similar to the acoustic signal [30]:

$$f(t) = e^{-t^2} \tag{2}$$

We consider the results of spectral transformations for typical intervals: pure rolling, microloss of contact, and the loss of contact of the disk with the surface before a stop. Figure 7 presents the results of fast Fourier transformation for 0.1 s intervals. The following typical intervals have been chosen for the signal shown in Fig. 5:

- an interval from 13.23 to 13.33 s, which corresponds to pure rolling;
- an interval from 14.53 to 14.63 s, which contains one "microloss of contact" with a duration of no more than 5 µs;

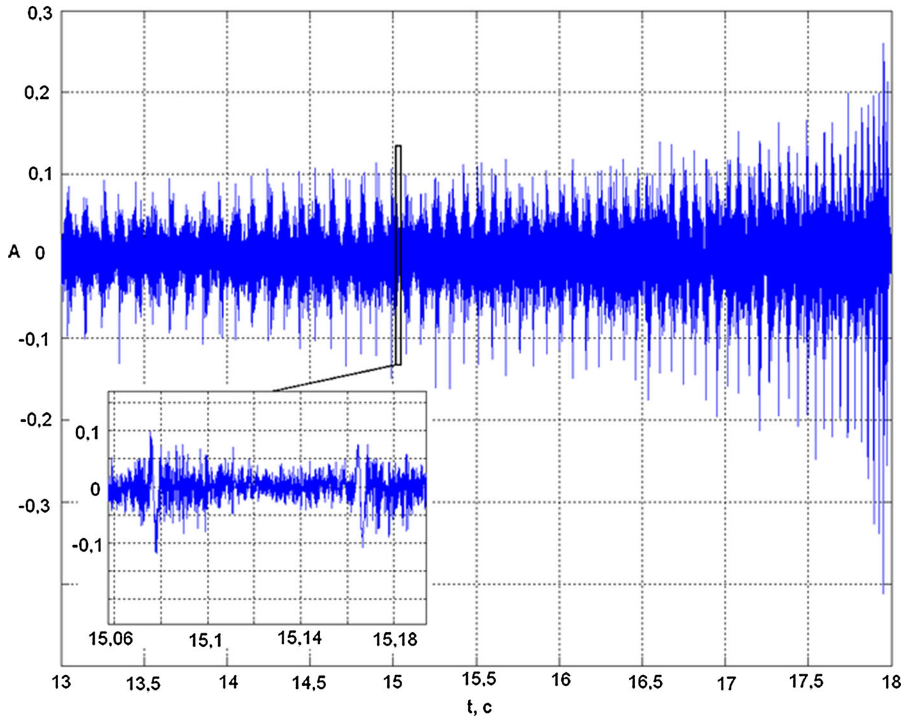

Fig. 5 The amplitude of typical sound vibrations during the rolling motion of the disk (the last 5 s of motion)

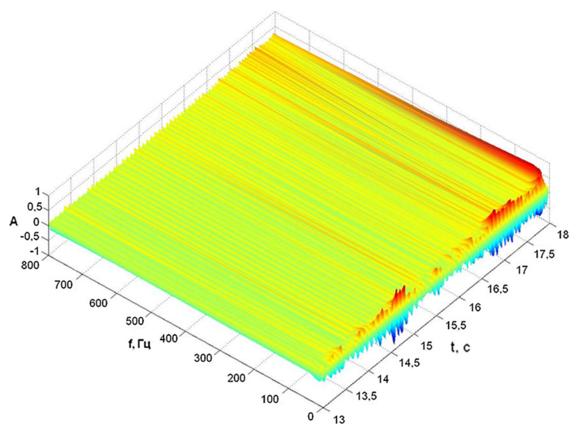

Fig. 6 The result of wavelet transformation (the last 5 s of motion)

– an interval from 17.9 to 18 s, which contains a loss of contact of the disk with the surface, before the stop.

On the spectrum of the first interval (Fig. 7a), one can see two carrier frequencies of about 10 and 30 Hz (probably the frequency related to the angular velocity of rotation of Euler's disk, since the closer the typical interval is located to the moment of stop, the higher are these values). The signals on other frequencies have a much smaller amplitude and can be classified as noise. In the process of motion, the carrier frequency ranges on the average from 5 to 15 Hz (depending on the initial conditions and parameters of the disk).

The spectrum of sound vibrations of the interval containing a typical loss of contact of the disk with the surface (Fig. 7c) has a large density and amplitude and is qualitatively similar to the spectrum of sound vibrations after an "ideal" shock pulse (for example, the fall of the disk onto the surface from some height). The spectrum of the interval with a "microloss of contact" (Fig. 7b) exhibits a smaller amplitude and density, but it is also closer to the spectrum of an "ideal" shock pulse.

Shock pulses are sources of vibrations in a wide range of frequencies. Each "microloss of contact" corresponds to a local blur of the frequency spectrum, but these vibrations subside rapidly. The loss of contact of the disk before its stop is characterized by a larger spectral density, which qualitatively resembles the spectrum of an "ideal" impact.

The "microlosses of contact" are most likely due to "microdeformations" of the surface and possibly of the disk itself. At the same time, there is no generally accepted theoretical explanation so far for the considerable loss of contact (immediately before the stop). In

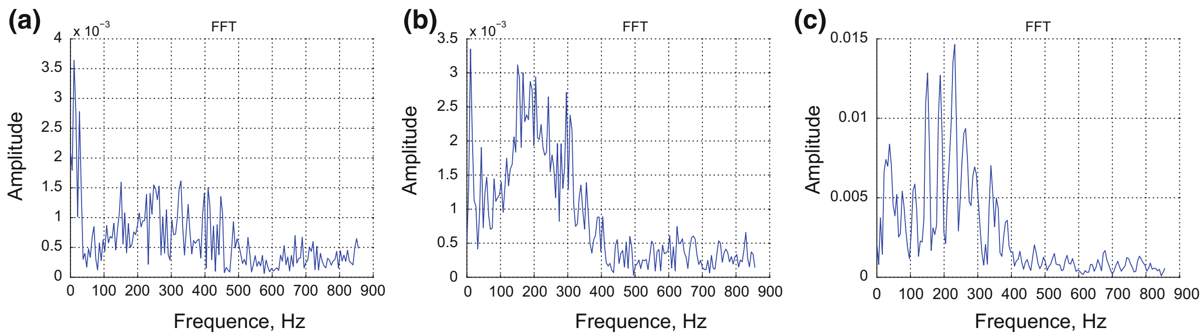

Fig. 7 The spectrum of sound vibrations of the intervals containing: **a** pure rolling (from 13.1 to 13.2 s), **b** "microlosses" (from 13.4 to 13.5 s), **c** loss of contact of the disk before a stop (from 17.9 to 18 s)

particular, it is not quite clear whether the loss of contact can be observed in models that do not take deformation into account.

The main goal of further research is to develop a theoretical model and to perform numerical simulations, which will allow a comparison of theoretical and experimental results.

Acknowledgments The authors thank A. Ruina, A.P. Ivanov, and D.V.Treshev for useful discussions, and S.R. Gallyamov and S.A. Trefilov for technical consultation.

References

1. Bendik, J.: The official Euler's disk website. http://www.eulerdisk.com (2000). Accessed 14 June 2013
2. Appel, P.: Sur l'integration des equations du mouvement d'un corps pesant de redolution roulant par une arete circulaire sur up plan horizontal; cas parculier du cerceau. Rendiconti del circolo matematico di Palermo. **14**, 1–6 (1900)
3. Chaplygin, S.A.: On motion of heavy rigid body of revolution on horizontal plane. Proc. Phys. Sci. Sect. Soc. Amat. Nat. Sci. **9**(1), 10–16 (1897)
4. Korteweg, D.: Extrait d'une lettre a M. Appel. Rendiconti del circolo matematico di Palermo. **14**, 7–8 (1900)
5. Borisov, A.V., Mamaev, I.S., Kilin, A.A.: Dynamic of rolling disk. Regul. Chaotic Dyn. **8**(2), 201–212 (2003)
6. Borisov, A.V., Mamaev, I.S.: The rolling motion of a rigid body on a plane and a sphere. Hierarchy of dynamics. Reg. Chaotic Dyn. **7**(2), 177–200 (2002)
7. Borisov, A.V., Mamaev, I.S., Bizyaev, I.A.: The hierarchy of dynamics of a rigid body rolling without slipping and spinning on a plane and a sphere. Reg. Chaotic Dyn. **18**(3), 277–328 (2013)
8. Moffatt, H.K.: Euler's disk and its finite-time singularity. Nature **404**, 833–834 (2000)
9. van den Engh, G., Nelson, P., Roach, J.: Numismatic gyrations. Nature **408**, 540 (2000)
10. Bildsten, L.: Viscous dissipation for Euler's disk. Phys. Rev. E. **66**(2), 056309 (2002)
11. Villanueva, R., Epstein, M.: Vibrations of Euler's disk. Phys. Rev. E. **71**(7), 066609 (2005)
12. Kessler, P., O'Reilly, O.M.: The Ringing of Euler's disk. Reg. Chaotic Dyn. **7**(1), 49–60 (2002)
13. O'Reilly, O.M.: The dynamics of rolling disks and sliding disks. Nonlinear Dyn. **10**, 287–305 (1996)
14. McDonald A.J., McDonald K.T.: The rolling motion of a disk on a horizontal plane. Preprint Archive, Los Alamos National Laboratory. arXiv: physics/008227(2000)
15. Le, Saux C., Leine, R.L., Glocker, C.: Dynamics of a rolling disk in the presence of dry friction. J. Nonlinear Sci. **15**, 27–61 (2005)
16. Caps, H., Dorbolo, S., Ponte, S., Croisier, H., Vandewalle, N.: Rolling and slipping motion of Euler's disk. Phys. Rev. E. **69**(6), 056610 (2004)
17. Stanislavsky, A.A., Weron, K.: Nonlinear oscillations in the rolling motion of Euler's disk. Physica D. **156**(10), 247259 (2001)
18. Easwar, K., Rouyer, F., Menon, N.: Speeding to a stop: the finite-time singularity of a spinning disk. Phys. Rev. E. **66**(3), 045102 (2002)
19. Leine R.L.: Measurements of the finite-time singularity of the Euler disk. In: 7th EUROMECH Solid Mechanics Conference, Lisbon, Portugal (2009)
20. Leine, R.L.: Experimental and theoretical investigation of the energy dissipation of a rolling disk during its final stage of motion. Arch. Appl. Mech. **79**, 1063–1082 (2009)
21. Saje, M., Zupan, D.: The rattling of Euler's disk. Multidiscip. Model. Mater. Struct. **2**(1), 49–66 (2006)
22. Petrie, D., Hunt, J.L., Gray, C.G.: Does the Euler disk slip during its motion? Am. J. Phys. **70**(10), 1025–1028 (2002)
23. Mitsui, T., Aihara, K., Terayama, C., Kobayashi, H., Shimomura, Y.: Can a spinning egg really jump? Proc. R. Soc. A. **462**, 2897–2905 (2006). doi:10.1098/rspa.2006.1718
24. Branicki, M., Shimomura, Y.: Dynamics of an axisymmetric body spinning on a horizontal surface. IV. Stability of steady spin states and the 'rising egg' phenomenon for convex axisymmetric bodies. Proc. R. Soc. A. **462**, 3253–3275 (2006)
25. Batista, M.: The nearly horizontally rolling of a thick disk on a rough plane. Regul. Chaotic Dyn. **13**(4), 344354 (2008)

26. Batista, M.: Self-induced jumping of a rigid body of revolution on a smooth horizontal surface. NonLinear Mech. **43**, 26–35 (2008)
27. Ivanov, A.P.: On detachment conditions in the problem on the motion of a rigid body on a rough plane. Reg. Chaotic Dyn. **13**(4), 355–368 (2008)
28. Go, D.B., Pohlman, D.A.: A mathematical model of the modified Paschen's curve for breakdown in microscale gaps. J. Appl. Phys. **107**, 103303 (2010)
29. Albert J., Levit W., Levit L.: Electrical breakdown and ESD phenomena for devices with nanometer-to-micron gaps. In: Proc. SPIE 4980, Reliability, Testing, and Characterization of MEMS/MOEMS II, 87. (2003). doi:10.1117/12.478191
30. Torrence, C., Compo, G. P.: A practical guide to wavelet analysis. Bull. Am. Meteorol. Soc. **79**(1), 61–78 (1998)