TECHNICAL REPORT

The Euler disk and its dynamic finite time singularity: Investigating a fascinating acoustical and mechanical phenomenon with simple means

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Abstract: Many have spun a coin or other disk on a table, observing — and listening! — that — while its motion comes to rest, the angular velocity of the contact point and the frequency of the emitted sound raise in a striking way. This dynamical system is known as the "Euler disk," and its physics is theoretically understood as a "finite time singularity" in form of an inverse power law. Various dissipation mechanisms lead to different exponents in the power law. The present investigation is about different configurations of the Euler disk using acoustical and mechanical data. We present results for the well-known Euler disk toy, well in accord with a model of viscous air dissipation. This is the first measurement using the acoustical signal for the dynamics of the Euler disk. Moreover, we present new cases where other dissipation mechanism must be dominant. Experiments are carried out with simple means and using a smartphone for recording and analysing data. This and interesting physics of the Euler disc make it a good candidate e.g. for undergraduate lab-work courses or research projects, which can spark students interest in acoustics, and more generally in applying physics to phenomena of everyday life.

1. THE PHENOMENON

Many have played with a coin, lid or other disk, spinning it on a table, and observing how its rotation comes to rest by friction (which is not surprising), with increasing angular velocity of the contact point and increasing frequency of the emitted sound, in the final stage both raising dramatically (which is more surprising).

Moffatt (2000) [1] was the first to show how this can be described by a "finite time singularity" in form of an inverse power law, arising from viscous damping of the air layer between the disc and the surface. Subsequently, other dissipation mechanisms were discussed, leading to different exponents in the power law describing the singularity [2–4].

While the sound emission, in form of "ringing" [2] or "chirp" (near the end of the process, [5]) belongs to the striking features of the Euler disc motion, to best of our knowledge no study has used it to carry out an acoustical frequency analysis. In the present work we report measurements both of acoustic and mechanical data, taking advantage of the internal sensors of mobile phones (micro-

phone and gyroscope), and show that they achieve a satisfactory degree of precision. We apply these methods to three different disks as shown in Fig. 1 (a lid, a grill, and a toy) and compare it to the finite-time singularity model. Satisfactory agreement with theory is found, and consequences for the nature of damping process are discussed. The easy availability of both the measurement device and of the measured systems make the approach a good candidate for undergraduate research projects on acoustics and dynamics. On this basis, the present example is meant a contribution to the collection of innovative educational experiments and activities on acoustics published in "Acoustical Science and Technology" [6,7] and other journals [8].

2. THEORY

2.1. Basic Description

The system consists of the disk of radius R and mass m rotating on a smooth surface (Fig. 2). Due to dissipative effects, the initial rotation around a (nearly) vertical axis (angular velocity ω), passing through the contact point and the centre the disk, comes to rest, this axis approaches the horizontal (inclination angle α), and the angular velocity of the contact point (Ω) rapidly increases rapidly towards the end of the process.
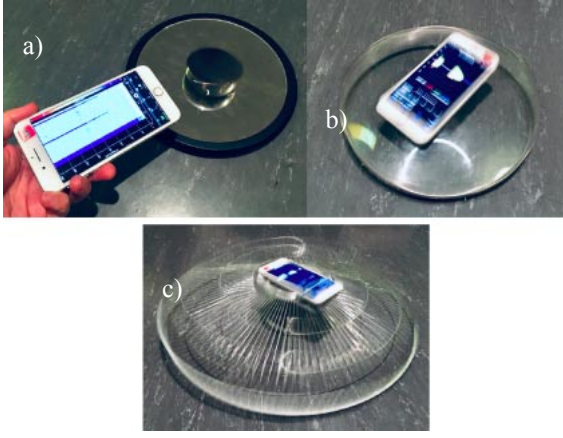
Fig. 1 Three "Euler disk" systems investigated in this work: a) the Euler disk toy, b) a common cooking pot lid and c) a fan grill. While a) oscillations are measured with a microphone b) and c) oscillations are measured with a gyroscope.
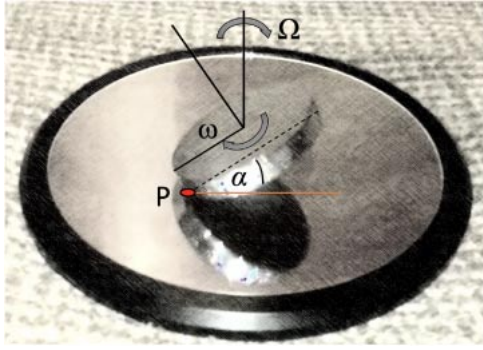

Fig. 2 Schematic representation of Euler's disk where P is the contact point between the disk and the surface, and α the inclination angle α, ω the angular velocity, and Ω the precession velocity of the disk.

Basically, the motion of the disk can be understood as top with tilted axis on which the gravitational force exerts a torque, causing precession. The angular momentum is given by $\boldsymbol{L} = I\omega\boldsymbol{e}_r$, ($I = 1/4\ mR^2$ is the moment of inertia of a flat disk around the axis of rotation, $\boldsymbol{e}_r$ a unit vector along that axis), and the gravitational torque by $T = mgR$. The precession frequency is $\Omega = T/L = T/\omega I$ [9]; with I and T as given, and $\omega = \sin\alpha\,\Omega$ [1], one arrives at

$$\Omega^2 = 4g/(R \cdot \sin\alpha) \approx 4g/(R \cdot \alpha) \text{ (for small } \alpha) \quad (1)$$

This relates Ω to α, and the temporal development of the latter can be obtained by the following energy consideration.

2.2. Energy Dissipation and Temporal Asymptote

The total energy of the system is the sum of its potential energy $mgR\sin\alpha$ and kinetic energy $1/2\ I\omega^2 = 1/2\ mgR\sin\alpha$ (with $\omega = \sin\alpha\,\Omega$ and Eq. (0)), thus

$$E = \frac{3}{2}mgR\sin\alpha \approx \frac{3}{2}mgR \cdot \alpha \text{ (for a small } \alpha) \quad (2)$$

The change rate of energy, dE/dt must be equal to the dissipated power P. Moffatt [1] has proposed that dissipation is due to a thin layer of air (viscous dissipation) between the disk and the supporting surface, which for small angles is described by $P = \pi\mu gR^2\alpha^{-2}$. From $dE/dt = P$ one then obtains (for late times, i.e. small α) a differential equation $d\alpha/dt = \text{const} \cdot \alpha^{-2}$ with solution $\alpha(t) \propto (t_f - t)^{1/3}$, where t_f is the moment where the motion stops and Ω diverges. Later, an improved model of viscous damping was proposed with $P \propto \alpha^{-5/4}$ and solution $\alpha(t) \propto (t_f - t)^{4/9}$ [10].

Other approaches to the spinning disk phenomenon were described in the literature, taking account of different dissipation mechanisms to explain the finite-time singularity such as sliding friction [11], rolling friction [4], pivoting friction [4], and a combination of mechanisms [4,10]. The contribution by [4] analyses a broad range of other damping mechanisms with dissipation rates proportional to various powers of α, and also leading also to power laws of the form $\alpha \propto (t_f - t)^{n_\alpha}$ with positive n_α. According to this result, α decreases, as it should. Then we can then conclude from 0 $\Omega(t) = (t_f - t)^{n_\Omega}$ with $n_\Omega = -1/2\ n_\alpha$, which diverges at t_f; i.e. the precession frequency indeed shows divergence at a finite time, if the α shows a power-law decrease in time.

Based on the above research, we will use

$$\Omega \propto (t_f - t)^{-2/9} \quad (3)$$

for the case of the disk [9] and, as proposed by Caps *et al.* [12],

$$\Omega \propto (t_f - t)^{-n_\Omega} \quad (4)$$

with a dynamical exponent n_Ω for the general case.

Summarising, the fact that Ω diverges when the disk comes to halt can be traced back to the precession relation $\Omega = T/\omega I$ [9]: the slower a top spins, the smaller its "rotational inertia," and the faster it precedes under the influence of an external torque; the fact that Ω diverges *at finite time* can be traced back to power laws for the dissipation rate P.

3. MEASUREMENT METHODS

Three "Euler disk" systems are investigated in this work: a) the Euler disk toy (Fig. 1), b) a cooking pot lid and c) a fan grill. The surface is the glass mirror support coming with the toy in case a), and a plastic vinyl floor in case b) and c).

For the Euler disk toy (system a), in order to focus on the phenomenon of interest, only measurements in the last phase approaching the singularity are presented and

discussed (5–10 s). It is also important to mention that the surface of the support is not perfectly flat but slightly concave. This is in order maintain the disk in the delimited area, also in case when the toy rests on a slightly inclined surface (as it often happens in a domestic setting), but it also affects energy dissipation as we will discuss later. The video link in the bibliography [13] shows the displacement of the disk on a slightly inclined surface.

For the other two systems b) and c), the smartphone is attached firmly to the lid and grill (strong double-sided adhesive tape for the lid and elastic bands for the grill). The specificity of the fan grill experiment is that it provides a simple approach to test for the power law and finite time singularity (Eq. (3)) when dissipation by air is much lower than in the other cases (in van den Engh *et al.* [11] a vacuum chamber was used).

For data collection we used an iPhone 7 plus (not all mobile devices are able to do these measurements, due to the lack of gyroscope sensor or bad resolution or insufficient interfacing between sensor and application).

The acoustic measurements from the microphone are directly transformed to a spectrogram (Fig. 3) by the application Spectrum View using a sample rate of 8 kHz, FFT of order 10 with 1,024 samples and Hamming windows function. The two-dimensional spectrum data are coded as amplitude, frequency and time and transferred as a csv file.
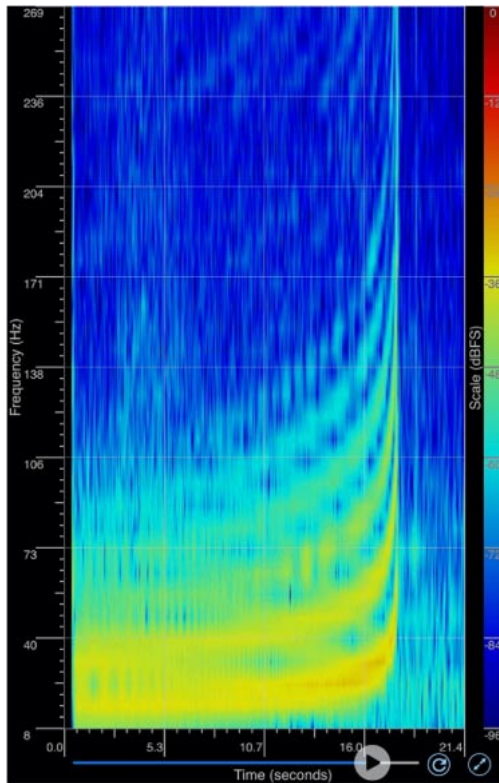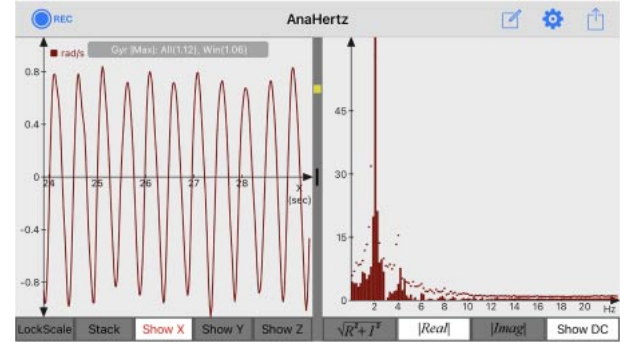

Fig. 4 iPhone 7 with a 2 Hz angular oscillation measured with the App AnaHerz.

The inbuilt gyroscope allows direct measurement of the angular variation of the of the lid and grill with a phone that is firmly attached to these. The mechanical measurements from the gyroscope are collected with the app AnaHertz with a sampling rate of 80 Hz in a time frame of 30 s and then transferred as a csv file to a computer for further analysis. In order to verify the reliability of the measurement we performed various tests about the operation of the gyroscope sensor. We provide one example in Fig. 4 where we use an angular oscillation of 2 Hz, with the app showing the temporal function and Fourier transform with a clear peak at 2 Hz.

Data are processed and filtered by Excel. Fits are carried out in two ways: First with the Excel solver tool (https://www.solver.com/excel-solver-change-options-grg-nonlinear-solving-method) using the Generalized Reduced Gradient (GRG) method for non-linear fits (the power law) which uses iterative methods to minimize differences between data and mathematical model [14]. Second, by a bi-logarithmic transformation which transforms a power law into a linear one with the dynamical exponent as slope,

$$\ln \Omega \propto n_\Omega \ln(t_f - t) \tag{5}$$

and then carrying out a linear fit with the least square method using a R routine [15]. Both methods give the consistent results.

Two complementary measurement approaches were used. For the lid and the fan grill the precession frequencies are lower than ≈ 15 Hz, i.e. below the frequencies a common smartphone microphone can detect (>20 Hz). For this reason, the gyroscope method is used in these two cases.

For the Euler disk toy, the situation is the opposite. The size is too small to attach a mobile device for a gyroscope measurement. On the other hand, the toy has been designed in a way that it produces higher frequencies in the later stages of the motion. A precession "sweep" between 70 to 220 Hz occurs in the last moments of motion (4 to 5 last seconds), allowing an acoustic treatment of the experiment.

Fig. 3 Euler disk toy spectrogram using the app Spectrum View on an iPhone 7.

Table 1 Description of the two different methods used. Gyroscope method for lower precession frequencies and acoustic method for higher and audible precession frequencies.

Disk	Gyroscopic method	Acoustic method
Lid	✓	admissible frequency range: $f > 70\,\text{Hz}$ actual frequency range: 3 to 15 Hz
Fan grill	✓	admissible frequency range: $f > 70\,\text{Hz}$ actual frequency range: 2 to 15 Hz
Toy disk	the toy is too small to measure with a mobile device Fig. 1	✓

Table 1 provides a summary of the two different methods used to analyse different experimental situations.

4.   EXPERIMENTAL RESULTS

4.1.   Acoustic Measurements for the Euler Disk Toy

The third experiment is made with the Euler disk toy. The temporal evolution of Ω and the bi-logarithmic representation is shown in Fig. 5. A power law is obtained, with $n_\Omega = 0.2 \pm 0.005$ (Eq. (6)), close to the theoretical result of Bildsten of $n_\Omega = 2/9$ [8].

$$\Omega \propto (t_\text{f} - t)^{-0.2} \tag{6}$$

4.2.   Gyroscope Measurements for Lid and Fan Grill

The temporal evolution of the precession frequency and its bi-logarithmic representation for the cooking pot lid are shown in Fig. 6, together with a model fit according to a power law (Eq. (7)). The two fit methods described in Sect. 3 consistently yield the same value $n_\Omega = 0.56 \pm 0.02$, i.e.

$$\Omega \propto (t_\text{f} - t)^{-0.56} \tag{7}$$

The temporal evolution of Ω and the bi-logarithmic representation for the fan grill system are shown in Fig. 7 together with a model fit according to a power law (Eq. (8)). The two fit methods described in Sect. 3 consistently yield the same value $n_\Omega = 0.59 \pm 0.01$, i.e.

$$\Omega \propto (t_\text{f} - t)^{-0.59} \tag{8}$$

4.3.   Additional Measurements for the Euler Disk Toy on Different Surfaces

The Euler disk toy with its specially manufactured support surface is designed to have very low rolling and sliding friction. In order to test for the value of the dynamical exponent when other dissipation mechanisms
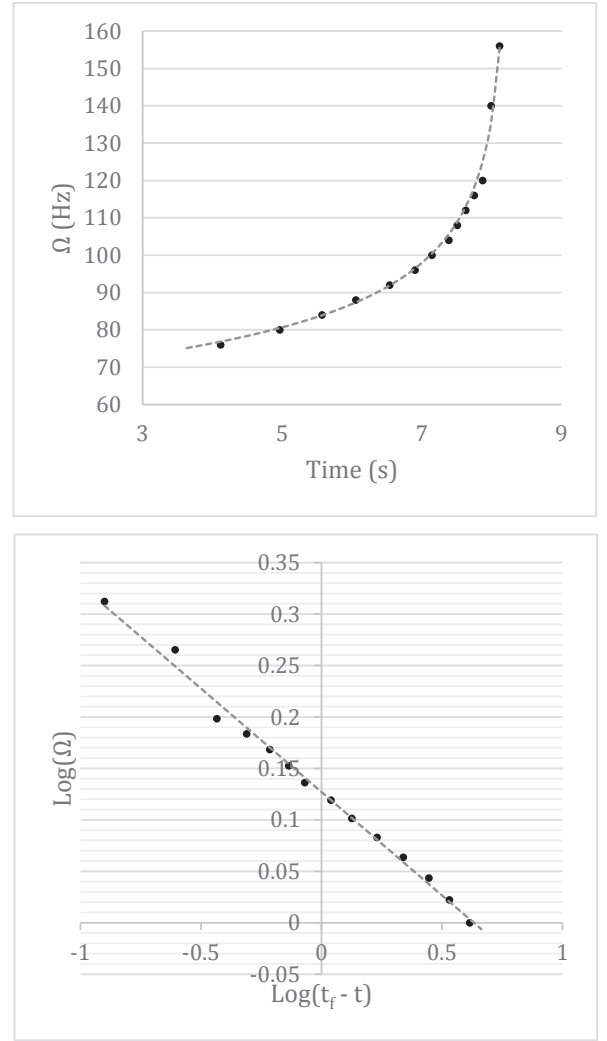

Fig. 5   Acoustic data (dots) and model (segmented line) of the temporal evolution and bi-Log representation of the precession frequency Ω for a Euler disk toy.

are present, we performed two other measurements using Euler disk toy on other surfaces than the one provided with the toy. In the first case we use a mirror, having surface properties close to that of the toy support, but being flat and not concave. In the second case the surface is a vinyl coated table.

The bi-Log representation of the precession frequency for both cases is shown in Fig. 8. Again, data are well compatible with a power law (Eqs. (7) and (8)), with $n_\Omega = 0.26 \pm 0.01$ for the mirror surface and $n_\Omega = 0.41 \pm 0.02$ for the vinyl coated table. These results of n_Ω are in between the values obtained for the disk toy on the provided support, and those for the lid and grill, as expected.

$$\Omega \propto (t_\text{f} - t)^{-0.26} \tag{9}$$

$$\Omega \propto (t_\text{f} - t)^{-0.41} \tag{10}$$
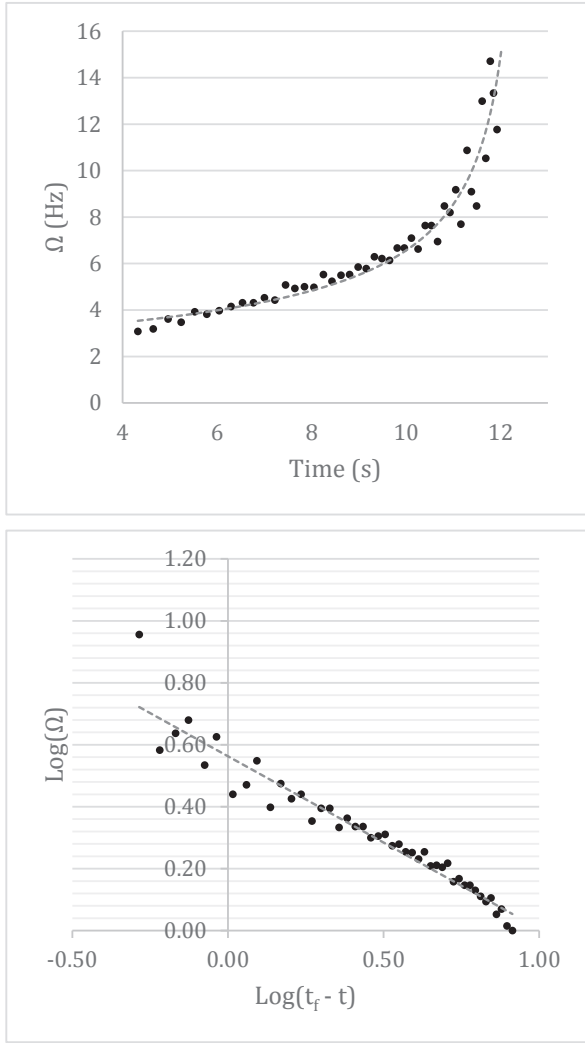
Fig. 6 Results for the lid system (gyroscopic data): precession frequency Ω (dots) and model (segmented line) fit for the temporal evolution and bi-logarithmic representation of the precession rate.
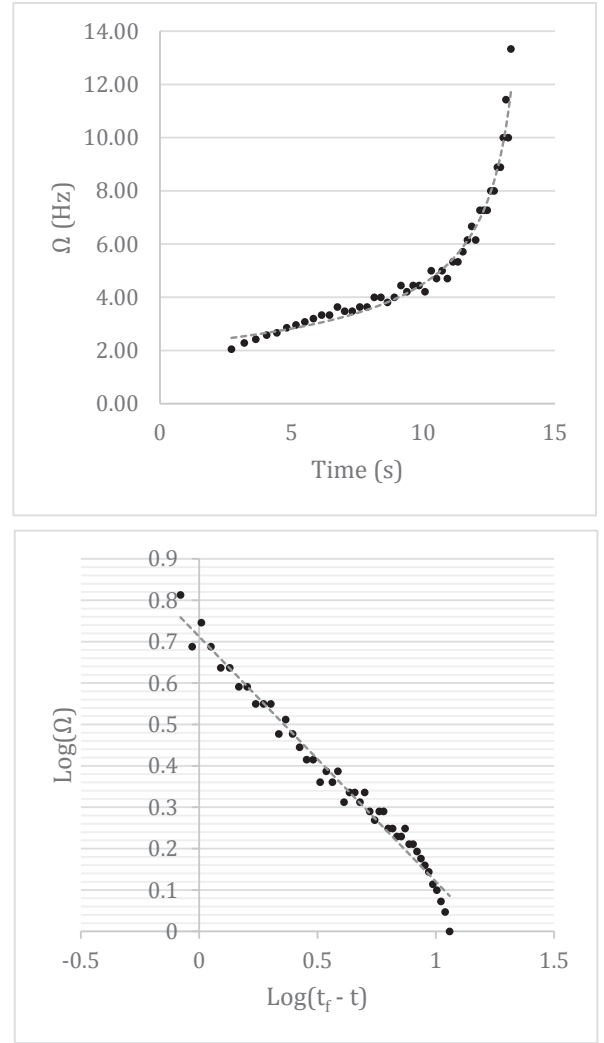

Fig. 7 Results for the fan grill system (gyroscopic data): precession frequency Ω (dots) and model (segmented line) fit for the temporal evolution and bi-logarithmic representation of the precession rate.

5.   DISCUSSION

For the toy on the specially manufactured surface the result $n_\Omega = 0.2$ (Eq. (0)) is quite close to the hypothetical result of Bildsten of $n_\Omega = 2/9$ [10]. For two other surfaces, or findings for the toy disk are in accord with previous experimental findings: For a flat glass surface, we find $n_\Omega = 0.26$ (Eq. (0)), very close to the result $n_\Omega = 1/4$ of [4] in the phase before the singularity, corresponding to contour friction as dissipation mechanism (resistance against the motion of the contact point over the edge of the disk). For a plastic surface (vinyl coated table) the result $n_\Omega = 0.41$ (Eq. (0)) is compatible with the range 0.35–0.4 found by [12] for two disks geometrically similar to the toy disk, also on plastic (type not specified). Taken together, our measurements confirm the power-law divergence of the precession frequency with time, and the values of dynamical exponents obtained in this work are in accord

with published values of other work (see overview in Table 2, [4,10,12]).

Comparing the values of n_Ω for the toy disk on the specially manufactured surface and on a glass surface, we can see an effect of the concave shape on the energy dissipation of the system because both surfaces have mirror finish but with different geometry. Therefore, considering that the concave surface was made for to keep the disk in the center of the surface, a side effect is that it leads to a smaller dynamical exponent, i.e. slower decrease of α and slower divergence of Ω. A detailed model for the interaction and dissipation mechanism of the disk with a concave surface is beyond the scope of the present paper.

For the lid and grill experiment, the n_Ω values obtained here confirm that dissipation mechanisms other than air viscosity are dominant for generic systems, in accord with the literature [4,12,16]. Note that (i) for the grill,
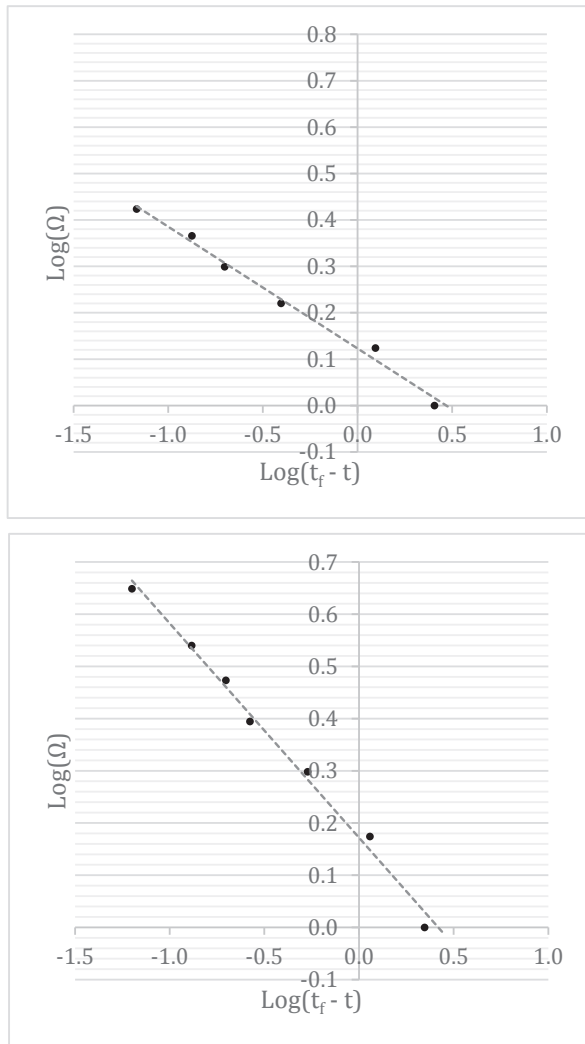
Fig. 8 Acoustic data of bi-Log representation of the precession frequency Ω (dots) and model (segmented line) for a Euler disk toy using a mirror surface (upper graph) and vinyl coated surface (bottom graph).

Table 2 Resume of the results of n_Ω from all experiments.

System	n_Ω power law fit (Excel)	n_Ω bi-Log linear fit (R)	Reference
Disk toy — glass mirror support	0.2	0.2 ± 0.005	$n_\Omega = 2/9$ [8]
Disk toy — glass surface	0.26	0.26 ± 0.01	$n_\Omega = 0.25$ [4]
Disk toy — vinyl coated table	0.41	0.41 ± 0.02	$n_\Omega = 0.34$, 0.39 [12]*

*for two disks with dimensions (diameter 75 mm, thickness 10 mm) similar to those of the toy, on a plastic surface (not specified).

dissipation by air viscosity is small by construction, and (ii) the value of the dynamical exponent for the lid is not very different from that of the grill ($n_\Omega = 0.56$ and $n_\Omega = 0.59$,

respectively), suggesting similar dissipation mechanisms. From qualitative observations (loudness), vibrations might play a role for the lid and the grill, but again a theoretical model for the dissipation mechanism is not intended by this paper.

6.   CONCLUSION AND PERSPECTIVES

In this work the Euler disk phenomenon was investigated in five different configurations, using two different experimental methods: First, an acoustic method with the microphone sensor of the smartphone, for a small disk with higher precession frequencies; three different settings allow to test for different regimes of the dominant dissipation mechanism.

Second, with the gyroscope sensor of a smartphone for larger and slower disks (lid and fan grill); the fan grill in particular allows to test a situation where another dissipation mechanism than viscous air damping is dominant. In all cases the results shown a satisfactory agreement with theory or extant measurements, e.g. the power law behaviour as such, or a value of $n_\Omega = 0.2 \pm 0.005$ of the dynamical exponent for the case of viscous air damping (i.e. within 10% coinciding with the theoretical value).

The contribution aims at a new experimental exploration of a curious and physically rich phenomenon, while using easily available equipment in the same time. It is intended for acoustics and physics lecturers interested in emphasizing the connection to everyday life, and potentially for undergraduate lab-work courses or research projects in the area of oscillations and sound. In this sense, the present example contributes to the collection of inspiring educational experiments and investigations published in this journal, in two special issues [6,7], and on topics like architectural acoustics [17], sound tract modelling [18], and others [19]. As a perspective for future work, we also think that this contribution provides a further example of how the availability and mobility of smartphones (or tablets) allows to investigate sound and oscillation phenomena in particular in everyday life settings. Other examples of this kind are bouncing balls [20], measurement of sound velocity [21], architectural acoustics [17,22], elevator oscillations [23], cracking knuckles [24], tunnel pressure waves [25], the Helmholtz resonator [26], the human vocal tract [27,28], and musical acoustics [29,30]. For further exploration by the interested reader, inspiring activity collections including acoustics and other areas of physics are available (e.g. Science on Stage [31]), and the journal "The Physics Teacher" runs a column about the use of smartphones as experimental tools since several years [32].

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