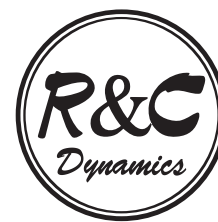**H. CENDRA, V. DÍAZ**

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THE LAGRANGE–D'ALEMBERT–POINCARÉ EQUATIONS AND INTEGRABILITY FOR THE ROLLING DISK

Classical nonholonomic systems are described by the Lagrange–d'Alembert principle. The presence of symmetry leads, upon the choice of an arbitrary principal connection, to a reduced variational principle and to the Lagrange–d'Alembert–Poincaré reduced equations. The case of rolling bodies has a long history and it has been the purpose of many works in recent times, in part because of its applications to robotics. In this paper we study the classical example of the rolling disk. We consider a natural abelian group of symmetry and a natural connection for this example and obtain the corresponding Lagrange–d'Alembert–Poincaré equations written in terms of natural reduced variables. One interesting feature of this reduced equations is that they can be easily transformed into a single ordinary equation of second order, which is a Heun's equation.

1. Introduction

Reduction theory for mechanical systems with symmetry has its roots in the classical works on mechanics of Euler, Jacobi, Lagrange, Hamilton, Routh, Poincaré and others. Excellent classical expositions on which the modern ones are based can be found for instance in [17], [18], [19], [13], [41], [2], [20], [50]. The modern theory of reduction is exposed in the excellent and well known book [1]. A thorough understanding of the several types of symmetries appearing in given examples, sometimes in an obvious way and sometimes in a very subtle way, provides a deep insight, in part because reduction theory gives conservation laws and a description in terms of fewer variables. Much research effort has gone into the development of the symplectic and Poisson view of reduction theory, but recently, the Lagrangian view with an emphasis on the reduction of variational principles, has attracted considerable attention, see for instance [36], [14], [15] and references therein.

The universal formalism created by Euler and Lagrange is not applicable to surprisingly simple systems, like those having rolling constraints. The systematic treatment of nonholonomic systems and the reduction of d'Alembert principle in the presence of symmetry is relatively recent, see for instance [31], [8] and references therein. The task of providing an intrinsic geometric formulation of the reduction theory for nonholonomic systems from the point of view of Lagrangian reduction using a given connection has been established in [16]. In this work the reduced Lagrange–d'Alembert equations, and in particular its vertical part called *the momentum equation*, are written intrinsically using covariant derivatives. The resulting equations in terms of the given connection are called the *Lagrange–d'Alembert–Poincaré equations.* We have mentioned only a few references on nonholonomic systems directly related to the present work. However, one should be aware that nonholonomic systems have been approached by several authors in the last few decades using different techniques. Among them we mention for instance [40], [28], [34], [6], [31], [5], [7], [23], [25], [32], [33], [11], [22], [46], [27], [37], [48], [9], [10], [21].

The main purpose of the present paper is to study the example of the rolling disk using reduction by the symmetry. Our approach is variational rather than Hamiltonian and leads to Lagrange–d'Alembert–Poincaré equations in terms of angular variables which are different, and so our equations are also different, from the ones used in [9] and also in [23]. This two papers realize a deep study of the rolling disk, including important aspects of the dynamics. In the present paper we choose an abelian group of symmetry which leaves invariant both the Lagrangian and the constraint, namely, the group $SO(2) \times \mathbb{R}^2$. Even if this is not the biggest obvious group of symmetry for the rolling disk it is enough to consider this group to obtain useful equations. In particular, we show in full detail that the Lagrange–d'Alembert–Poincaré equations corresponding to this group naturally lead to a single second order scalar equation involving meaningful parameters of the system. This equation, which is a Heun's equation, can be solved, in principle, by using Frobenius expansions. Heun's equation have been studied for more than a century and a complete classification of the different types and their solutions is still not finished, see [26], [45], [47]; and references therein. Another important aspect of our equations is that, being Lagrange–d'Alembert–Poincaré equations, they can be naturally Legendre-transformed to give equations in terms of Leibnitz brackets, see [44]. This is being planed for a future work.

A few comments on the effects of friction in more realistic models of the rolling disk is in order. The study of the effect of the several sources of dissipation on the motion of the science toy known as *Euler disk* has been the purpose of many works in the last few years, both theoretical and experimental ones. For instance, [38], [39] study the effect of air viscous dissipation as being the main reason of the finite-time singularity which causes the disk to stop. Subsequent works claim that air drag is not the primary dissipation mechanism. See for instance [49], [37], [48], [24], [29], [43], [12]. After all this work, the real cause of the dissipation effects seems to be still a controversial subject. It is possible that a good description of dissipation effects could be given in terms of Leibnitz brackets.

This paper is organized as follows. In section 2 we apply the reduction techniques of [16] to the example of the rolling disk and we obtain a detailed description of the main equations. In Appendix A we recall the basic facts about Lagrange–d'Alembert–Poincaré equations.

2. The Rolling Disk

Kinematics of the Rigid Body. The configuration space for the rigid body is the group $SO(3)$, see for instance [1], [3], [4]. The motion of the rigid body is given by a curve $A(t)$ on $SO(3)$. The *space angular velocity* $\hat{\omega}$ and the *body angular velocity* $\hat{\Omega}$ are elements of the Lie algebra $\mathfrak{so}(3)$ and they are defined by the conditions $\dot{A} = A\hat{\Omega} = \hat{\omega}A$.

We recall that there is a natural identification $\hat{}: \mathbb{R}^3 \to \mathfrak{so}(3)$ given by

$$\hat{x} = \begin{pmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{pmatrix},$$

where $x = x_1\,\mathbf{e}_1 + x_2\,\mathbf{e}_2 + x_3\,\mathbf{e}_3$.

We have the formulas

$$\widehat{x \times y} = [\hat{x}, \hat{y}], \quad x \cdot y = -\frac{1}{2}\mathrm{tr}\,\hat{x}\hat{y} \quad \text{and} \quad \hat{x}\,y = x \times y.$$

Besides, if A is any element of $SO(3)$ and x is any element of $\mathbb{R}^3$ we have

$$\widehat{A\,x} = A\hat{x}A^{-1}.$$

For any motion $A(t)$, define $z(t) = A(t)\,\mathbf{e}_3$. Then

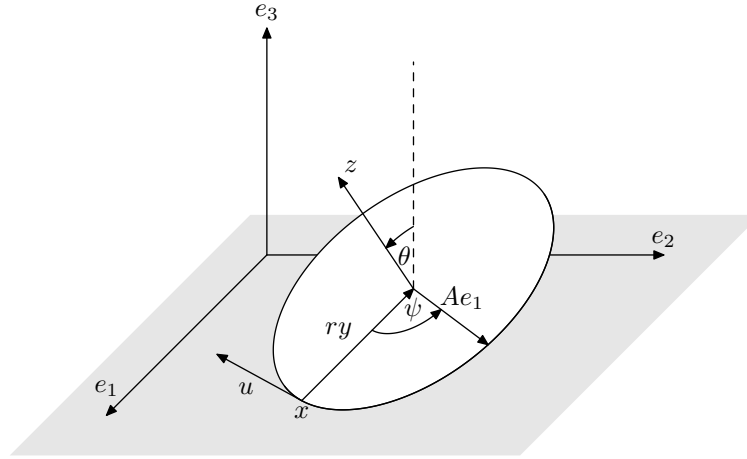$$\dot{z} = \dot{A}\,\mathbf{e}_3 = \hat{\omega}z = \omega \times z.$$

Fig. 1. Rolling Disk

We have that $\langle \omega, z \rangle = \langle \Omega, \mathbf{e}_3 \rangle = \Omega_3$, and that $A(\Omega_1 \hat{\mathbf{e}}_1 + \Omega_2 \hat{\mathbf{e}}_2) A^{-1} = \widehat{(z \times \dot{z})}$. Therefore the space velocity ω can be written $\omega = A\Omega$ and then $\omega = \Omega_3 z + z \times \dot{z}$. This gives a decomposition of ω as a sum of its component parallel to z plus its component normal to z.

Kinematics of the Rolling Disk. We are going to apply the reduction theory described in the Appendix A to the rolling disk. We will model the rolling disk as a rigid body with only one point of contact with the floor subjected to the usual nonsliding condition. To describe mathematically the system we choose a fixed reference frame $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ such that the axis $\mathbf{e}_3$ is directed upwards and the axis $\mathbf{e}_1$ and $\mathbf{e}_2$ lie on the floor (see Fig. 1). For each $A \in SO(3)$ the orthonormal frame $(A\mathbf{e}_1, A\mathbf{e}_2, A\mathbf{e}_3)$ is rigidly attached to the disk in such a way that $z = A\mathbf{e}_3$ is perpendicular to the plane of the disk. The point of contact of the disk with the floor is $x = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 = (x_1, x_2, 0)$.

Since we are interested only in the motion of the disk satisfying the condition $-1 < \langle A(t)\mathbf{e}_3, \mathbf{e}_3 \rangle < 1$ we can choose the configuration space for the rolling disk to be

$$Q = (0, \pi) \times S^1 \times S^1 \times \mathbb{R}^2.$$

Then a configuration of the disk is given by a point $(\theta, \varphi, \psi, x) \in (0, \pi) \times S^1 \times S^1 \times \mathbb{R}^2$ where the meaning of the variables is the following. The variable θ represents the angle from the axis $\mathbf{e}_3$ to the vector $z = A\mathbf{e}_3$. The unit vector y is directed from the point of contact x with the plane to the geometric center of the disk. The vector u has the direction of the horizontal line parallel to the plane of the disk, so it is tangent to the disk, and it also has the direction of the motion of x as a point of the floor. More precisely, it is defined by $u = z \times y$ and has the expression $u = (-\cos\varphi, -\sin\varphi, 0)$. So the angle φ is determined by the unit vector u. The variable ψ represents the angle from the vector $-y$ to the vector $A\mathbf{e}_1$ where the positive sense for measuring the angle ψ on the plane of the disk is given by the counterclockwise sense, as viewed from z.

For any element

$$A = \begin{pmatrix} -\cos\theta\cos\psi\sin\varphi - \cos\varphi\sin\psi & \cos\theta\sin\psi\sin\varphi - \cos\varphi\cos\psi & \sin\theta\sin\varphi \\ \cos\theta\cos\psi\cos\varphi - \sin\varphi\sin\psi & -\cos\theta\sin\psi\cos\varphi - \cos\psi\sin\varphi & -\sin\theta\cos\varphi \\ \sin\theta\cos\psi & -\sin\theta\sin\psi & \cos\theta \end{pmatrix},$$

$A \in SO(3)$, we can calculate $z = A\mathbf{e}_3$.

We are going to assume without loss of generality that the initial position of the disk, lying on the floor, is the case in which $\theta = 0$, $\varphi = \pi/2$, $\psi = \pi$ and $x = (r, 0, 0)$, where r is the radius of the disk.

After straightforward calculations we have that the space angular velocity is given by

$$\omega = (\dot{\theta}\cos\varphi + \dot{\psi}\sin\theta\sin\varphi, \dot{\theta}\sin\varphi - \dot{\psi}\cos\varphi\sin\theta, \dot{\varphi} + \dot{\psi}\cos\theta)$$

and the body angular velocity is given by

$$\Omega = (-\dot{\theta}\sin\psi + \dot{\varphi}\cos\psi\sin\theta, -\dot{\theta}\cos\psi - \dot{\varphi}\sin\theta\sin\psi, \dot{\varphi}\cos\theta + \dot{\psi}).$$

The nonholonomic constraint is given by the distribution

$$\mathcal{D}_{(\theta,\varphi,\psi,x)} = \left\{(\theta,\varphi,\psi,x,\dot{\theta},\dot{\varphi},\dot{\psi},\dot{x})|\dot{x} = \dot{\psi}ru\right\}.$$

The symmetry group is $SO(2) \times \mathbb{R}^2$. The group $SO(2)$ is identified with the set of elements of $SO(3)$ of the type

$$\begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and we identify the factor $\mathbb{R}^2$ of the group $SO(2) \times \mathbb{R}^2$ with the subspace of $\mathbb{R}^3$ defined by $x_3 = 0$.

The group $SO(2) \times \mathbb{R}^2$ acts on Q on the right by

$$(\theta,\varphi,\psi,x)(\alpha,a) = (\theta,\varphi,\psi+\alpha,x+a)$$

where the sum $\psi + \alpha$ is defined modulo 2π. With this action Q becomes a right $SO(2) \times \mathbb{R}^2$–principal bundle. The map $\pi\colon Q \to C$, where $C = (0,\pi) \times S^1$, given by $\pi(\theta,\varphi,\psi,x) = (\theta,\varphi)$ is a submersion. We have an identification $Q/\big(SO(2) \times \mathbb{R}^2\big) \equiv C$ given by

$$[(\theta,\varphi,\psi,x)]_{SO(2)\times\mathbb{R}^2} \equiv (\theta,\varphi)$$

The vertical distribution $\mathcal{V}$ is given by

$$\mathcal{V}_{(\theta,\varphi,\psi,x)} = \left\{(\theta,\varphi,\psi,x,\dot{\theta},\dot{\varphi},\dot{\psi},\dot{x})|\dot{\theta} = 0, \dot{\varphi} = 0\right\}.$$

The vector bundle $\mathcal{S} = \mathcal{D} \cap \mathcal{V}$ is given by

$$\mathcal{S}_{(\theta,\varphi,\psi,x)} = \{(\theta,\varphi,\psi,x,0,0,\xi,\xi ru)\}.$$

Since $\dim\mathcal{D}_{(\theta,\varphi,\psi,x)} = 3, \dim\mathcal{V}_{(\theta,\varphi,\psi,x)} = 3$ and $\dim\mathcal{S}_{(\theta,\varphi,\psi,x)} = 1$, we have

$$\mathcal{D}_{(\theta,\varphi,\psi,x)} + \mathcal{V}_{(\theta,\varphi,\psi,x)} = T_{(\theta,\varphi,\psi,x)}\mathcal{Q}$$

that is, the dimension assumption (see Appendix A) is satisfied.

We choose the horizontal spaces

$$\mathcal{H}_{(\theta,\varphi,\psi,x)} = \{(\theta,\varphi,\psi,x,\alpha,\beta,0,0)\}$$

satisfying

$$\mathcal{H}_{(\theta,\varphi,\psi,x)} \oplus \mathcal{S}_{(\theta,\varphi,\psi,x)} = \mathcal{D}_{(\theta,\varphi,\psi,x)}.$$

It is also clear that the distribution $\mathcal{H}$ is $SO(2) \times \mathbb{R}^2$-invariant. The connection 1-form $\mathcal{A}$ whose horizontal spaces are $\mathcal{H}_{(\theta,\varphi,\psi,x)}$ is

$$\mathcal{A}(\theta,\varphi,\psi,x,\dot{\theta},\dot{\varphi},\dot{\psi},\dot{x}) = (\dot{\psi},\dot{x}).$$

The adjoint bundle $\widetilde{\mathfrak{so}(2) \times \mathbb{R}^2}$ is a trivial bundle and we have an identification

$$\widetilde{\mathfrak{so}(2) \times \mathbb{R}^2} \equiv C \times (\mathfrak{so}(2) \times \mathbb{R}^2)$$

given by
$$[(\theta, \varphi, \psi, x, 0, 0, \xi, a)]_{SO(2) \times \mathbb{R}^2} \equiv (\theta, \varphi, \xi, a).$$

The vector bundle isomorphism
$$\alpha_{\mathcal{A}} : TQ/G \to T(Q/G) \oplus \tilde{\mathfrak{g}},$$

is described as follows. Since $G \equiv SO(2) \times \mathbb{R}^2$ is abelian and $Q/\left(SO(2) \times \mathbb{R}^2\right) \equiv C$ we obtain $\tilde{\mathfrak{g}} \equiv C \times \left(\mathfrak{so}(2) \times \mathbb{R}^2\right)$ and
$$TC \oplus \tilde{\mathfrak{g}} \equiv C \times \mathbb{R}^2 \oplus C \times \left(\mathfrak{so}(2) \times \mathbb{R}^2\right) \equiv C \times \mathbb{R}^2 \oplus \left(\mathfrak{so}(2) \times \mathbb{R}^2\right).$$

Then
$$\alpha_{\mathcal{A}} \left([(\theta, \varphi, \psi, x, \dot{\theta}, \dot{\varphi}, \dot{\psi}, \dot{x})]_{SO(2) \times \mathbb{R}^2}\right) = (\theta, \varphi, \dot{\theta}, \dot{\varphi}) \oplus (\theta, \varphi, \dot{\psi}, \dot{x}).$$

The subbundle $\tilde{\mathfrak{s}} \subset \widetilde{\mathfrak{so}(2) \times \mathbb{R}^2}$ is given by
$$\tilde{\mathfrak{s}} = \{(\theta, \varphi, 0, 0) \oplus (\theta, \varphi, \xi, \xi r u)\}.$$

Now we shall describe the structure of the bundle $\widetilde{\mathfrak{so}(2) \times \mathbb{R}^2}$. First of all, the Lie algebra structure on each fiber of $\widetilde{\mathfrak{so}(2) \times \mathbb{R}^2}$ is abelian because the Lie algebra $\mathfrak{so}(2) \times \mathbb{R}^2$ is abelian. Let $(\theta, \varphi, \xi, a)$ be a curve on $\mathfrak{so}(2) \times \mathbb{R}^2$. Using the formula for the covariant derivative and using the fact that the group $SO(2) \times \mathbb{R}^2$ is abelian, we see that the covariant derivative of this curve is given by
$$\frac{D(\theta, \varphi, \xi, a)}{Dt} = (\theta, \varphi, \dot{\xi}, \dot{a}).$$

Since the distribution of horizontal spaces is integrable, the $\tilde{\mathfrak{g}}$-valued 2-form $\tilde{\mathcal{B}}$ is equal to zero.

Dynamics of the Rolling Disk. We assume that the center of mass of the disk coincides with the geometric center, and the two moments of inertia with respect to the axis $A\mathbf{e}_1$ and $A\mathbf{e}_2$, say I_1 and I_2, are equal while the moment of inertia I_3 with respect to the axis $z = A\mathbf{e}_3$ is not necessarily equal to $I_1 = I_2$.

Let $w = x + ry$ be the center of mass. Then the Lagrangian of the system $L \colon TQ \to \mathbb{R}$ is given by the kinetic minus the potential energy
$$L = 1/2 \, I_1 \dot{z}^2 + 1/2 \, I_3 \Omega_3^2 + 1/2 \, M\dot{w}^2 - Mgr \sin\theta,$$

and may be written as
$$L(\theta, \varphi, \psi, x, \dot{\theta}, \dot{\varphi}, \dot{\psi}, \dot{x}) = \frac{1}{2}\left[-2Mgr \sin\theta + I_1(\dot{\theta}^2 + \dot{\varphi}^2 \sin^2\theta) + M(\dot{x}^2 + r^2\dot{\theta}^2 + \dot{\varphi}^2 r^2 \cos^2\theta + \right.$$
$$\left. +2r\dot{x}_1(\dot{\theta}\sin\theta \sin\varphi - \dot{\varphi}\cos\theta \cos\varphi) - 2r\dot{x}_2(\dot{\theta}\cos\varphi \sin\theta + \dot{\varphi}\cos\theta \sin\varphi)) + I_3(\dot{\varphi}\cos\theta + \dot{\psi})^2\right],$$

where g is the acceleration of gravity and M is the mass of the disk. This Lagrangian is invariant under the right action of the abelian group $SO(2) \times \mathbb{R}^2$, which also leaves the constraint $\mathcal{D}$ invariant.

Using the isomorphism
$$\alpha_{\mathcal{A}} \left([(\theta, \varphi, \psi, x, \dot{\theta}, \dot{\varphi}, \dot{\psi}, \dot{x})]_{SO(2) \times \mathbb{R}^2}\right) = (\theta, \varphi, \dot{\theta}, \dot{\varphi}) \oplus (\theta, \varphi, \bar{v}),$$

where $\bar{v} = (v_0, v_1) = (\dot{\psi}, \dot{x})$, the reduced Lagrangian $\ell(\theta, \varphi, \dot{\theta}, \dot{\varphi}, \bar{v})$ is given by
$$\ell(\theta, \varphi, \dot{\theta}, \dot{\varphi}, \bar{v}) = -Mgr \sin\theta + \frac{1}{2}I_1(\dot{\theta}^2 + \dot{\varphi}^2 \sin^2\theta) + \frac{1}{2}Mv_1^2 + \frac{1}{2}Mr^2\dot{\theta}^2 + \frac{1}{2}Mr^2\dot{\varphi}^2 \cos^2\theta +$$
$$+Mrv_{11}(\dot{\theta}\sin\theta \sin\varphi - \dot{\varphi}\cos\theta \cos\varphi) - Mrv_{12}(\dot{\theta}\cos\varphi \sin\theta + \dot{\varphi}\cos\theta \sin\varphi) + \frac{1}{2}I_3(\dot{\varphi}\cos\theta + v_0)^2.$$

Now we shall write the Lagrange–d'Alembert–Poincaré equations. Since the group is abelian the vertical Lagrange–d'Alembert–Poincaré equation becomes

$$\frac{D}{Dt}\frac{\partial\ell}{\partial\overline{v}}\bigg|_{\tilde{\mathfrak{s}}} = 0.$$

We have

$$\frac{\partial\ell}{\partial\overline{v}} = \left(\frac{\partial\ell}{\partial v_0}, \frac{\partial\ell}{\partial v_1}\right) = \Big(I_3 v_0 + I_3\dot\varphi\cos\theta, Mv_1 - Mr\dot\theta\sin\theta\cos\varphi\ \mathbf{e}_2 + Mr\dot\theta\sin\theta\sin\varphi\ \mathbf{e}_1 -$$
$$-Mr\dot\varphi\cos\theta\cos\varphi\ \mathbf{e}_1 - Mr\dot\varphi\cos\theta\sin\varphi\mathbf{e}_2\Big).$$

Then

$$\frac{D}{Dt}\frac{\partial\ell}{\partial\overline{v}} = \frac{d}{dt}\frac{\partial\ell}{\partial\overline{v}} =$$

$$= \Big(I_3(\dot v_0 + \ddot\varphi\cos\theta - \dot\theta\dot\varphi\sin\theta), M\dot v_1 + Mr(2\dot\theta\dot\varphi\sin\theta\sin\varphi - \cos\theta\cos\varphi(\dot\theta^2 + \dot\varphi^2) - \ddot\varphi\cos\theta\sin\varphi -$$
$$-\ddot\theta\sin\theta\cos\varphi)\mathbf{e}_2 + Mr(2\dot\theta\dot\varphi\sin\theta\cos\varphi + \cos\theta\sin\varphi(\dot\theta^2 + \dot\varphi^2) + \ddot\theta\sin\theta\sin\varphi - \ddot\varphi\cos\theta\cos\varphi)\mathbf{e}_1\Big).$$

But since $(\theta, \varphi, \overline{v}) \in \tilde{\mathfrak{s}}$ we have

$$v_1 = v_0 ru.$$

A generator of $\tilde{\mathfrak{s}}$ is

$$(\theta, \varphi, 1, ru),$$

then

$$\frac{d}{dt}\frac{\partial\ell}{\partial\overline{v}}(\theta, \varphi, 1, ru) = I_3(\dot v_0 + \ddot\varphi\cos\theta - \dot\theta\dot\varphi\sin\theta) + Mr^2(-2\dot\theta\dot\varphi\sin\theta + \ddot\varphi\cos\theta) + Mr\langle\dot v_1, u\rangle.$$

Therefore the vertical *Lagrange–d'Alembert–Poincaré equations* becomes

$$I_3(\dot v_0 + \ddot\varphi\cos\theta - \dot\theta\dot\varphi\sin\theta) + Mr\langle\dot v_1, u\rangle + Mr^2(\ddot\varphi\cos\theta - 2\dot\theta\dot\varphi\sin\theta) = 0,$$
$$v_0 ru = v_1.$$

Now let us calculate the horizontal Lagrange–d'Alembert–Poincaré equations which are given by

$$\left(\frac{\partial^C\ell}{\partial\theta} - \frac{D}{Dt}\frac{\partial\ell}{\partial\dot\theta}\right).\delta\theta = \frac{\partial\ell}{\partial\overline{v}}\left(\tilde{\mathcal{B}}(\dot\theta, \delta\theta)\right) = 0,$$
$$\left(\frac{\partial^C\ell}{\partial\varphi} - \frac{D}{Dt}\frac{\partial\ell}{\partial\dot\varphi}\right).\delta\varphi = \frac{\partial\ell}{\partial\overline{v}}\left(\tilde{\mathcal{B}}(\dot\varphi, \delta\varphi)\right) = 0.$$

We have

$$\frac{\partial^C\ell}{\partial\theta}.\delta\theta = \Big\langle(I_1\dot\varphi^2\sin\theta\cos\theta - Mgr\cos\theta - Mr^2\dot\varphi^2\cos\theta\sin\theta + Mrv_{11}(\dot\theta\cos\theta\sin\varphi + \dot\varphi\sin\theta\cos\varphi) -$$
$$-Mrv_{12}(\dot\theta\cos\theta\cos\varphi - \dot\varphi\sin\theta\sin\varphi) - I_3\dot\varphi\sin\theta(v_0 + \dot\varphi\cos\theta), \delta\theta\Big\rangle,$$

and

$$\frac{\partial^C\ell}{\partial\varphi}.\delta\varphi = \Big\langle Mrv_{11}(\dot\theta\sin\theta\cos\varphi + \dot\varphi\cos\theta\sin\varphi) + Mrv_{12}(\dot\theta\sin\theta\sin\varphi - \dot\varphi\cos\theta\cos\varphi), \delta\varphi\Big\rangle.$$

On the other hand, we have

$$\frac{D}{Dt}\frac{\partial \ell}{\partial \dot\theta} = \frac{d}{dt}\frac{\partial \ell}{\partial \dot\theta} = I_1\ddot\theta + Mr^2\ddot\theta + Mr\dot{v_{11}}\sin\theta\sin\varphi + Mr\dot\theta v_{11}\cos\theta\sin\varphi +$$

$$+Mr\dot\varphi v_{11}\sin\theta\cos\varphi - Mr\dot{v_{12}}\sin\theta\cos\varphi + Mr\dot\varphi v_{12}\sin\theta\sin\varphi - Mr\dot\theta v_{12}\cos\theta\cos\varphi,$$

and

$$\frac{D}{Dt}\frac{\partial \ell}{\partial \dot\varphi} = \frac{d}{dt}\frac{\partial \ell}{\partial \dot\varphi} = 2I_1\dot\theta\dot\varphi\sin\theta\cos\theta + I_1\ddot\varphi\sin^2\theta - 2Mr^2\dot\theta\dot\varphi\sin\theta\cos\theta - Mr\dot{v_{11}}\cos\theta\cos\varphi +$$

$$+Mr\dot\theta v_{11}\sin\theta\cos\varphi + Mr\dot\varphi v_{11}\cos\theta\sin\varphi - Mr\dot\varphi v_{12}\cos\theta\sin\varphi + Mr^2\ddot\varphi\cos^2\theta - Mr\dot\varphi v_{12}\cos\theta\cos\varphi +$$

$$+Mr\dot\theta v_{12}\sin\theta\sin\varphi + I_3\cos\theta(\dot{v_0} + \ddot\varphi\cos\theta - \dot\theta\dot\varphi\sin\theta) - I_3\dot\theta\sin\theta(v_0 + \dot\varphi\cos\theta).$$

Then the horizontal *Lagrange–d'Alembert–Poincaré equations* are

$$2(Mr^2 + I_3 - I_1)\dot\theta\dot\varphi\sin\theta\cos\theta - (I_1\sin^2\theta + (I_3 + Mr^2)\cos^2\theta)\ddot\varphi +$$
$$+Mr\cos\theta(\dot{v_{11}}\cos\varphi + \dot{v_{12}}\sin\varphi) + I_3(\dot\theta v_0\sin\theta - \dot{v_0}\cos\theta) = 0,$$
$$(I_1 - I_3 - Mr^2)\dot\varphi^2\sin\theta\cos\theta - I_3 v_0\dot\varphi\sin\theta - (I_1 + Mr^2)\ddot\theta +$$
$$+Mr\sin\theta(\dot{v_{12}}\cos\varphi - \dot{v_{11}}\sin\varphi) - Mgr\cos\theta = 0.$$

We have obtained the following system of reduced equations for the rolling disk

$$I_3(\dot{v_0} + \ddot\varphi\cos\theta - \dot\theta\dot\varphi\sin\theta) + Mr\langle \dot{v_1}, u\rangle + Mr^2(\ddot\varphi\cos\theta - 2\dot\theta\dot\varphi\sin\theta) = 0, \qquad (2.1)$$

$$v_0 r u = v_1, \qquad (2.2)$$

$$2(Mr^2 + I_3 - I_1)\dot\theta\dot\varphi\sin\theta\cos\theta - (I_1\sin^2\theta + (I_3 + Mr^2)\cos^2\theta)\ddot\varphi +$$
$$+Mr\cos\theta(\dot{v_{11}}\cos\varphi + \dot{v_{12}}\sin\varphi) + I_3(\dot\theta v_0\sin\theta - \dot{v_0}\cos\theta) = 0, \qquad (2.3)$$

$$(I_1 - I_3 - Mr^2)\dot\varphi^2\sin\theta\cos\theta - I_3 v_0\dot\varphi\sin\theta - (I_1 + Mr^2)\ddot\theta +$$
$$+Mr\sin\theta(\dot{v_{12}}\cos\varphi - \dot{v_{11}}\sin\varphi) - Mgr\cos\theta = 0, \qquad (2.4)$$

where the two first equations are the vertical Lagrange–d'Alembert–Poincaré equations and the two last equations are the horizontal Lagrange–d'Alembert–Poincaré equations.

This system immediately gives rise to the system

$$\begin{pmatrix} 0 & (I_3 + Mr^2)\cos\theta & I_3 & -Mr\cos\varphi & -Mr\sin\varphi \\ 0 & -(I_1\sin^2\theta + (I_3 + Mr^2)\cos^2\theta) & -I_3\cos\theta & Mr\cos\theta\cos\varphi & Mr\cos\theta\sin\varphi \\ -(I_1 + Mr^2) & 0 & 0 & -Mr\sin\theta\sin\varphi & Mr\sin\theta\cos\varphi \\ 0 & 0 & r\cos\varphi & 1 & 0 \\ 0 & 0 & r\sin\varphi & 0 & 1 \end{pmatrix}\begin{pmatrix} \dot a \\ \dot b \\ \dot{v_0} \\ \dot{v_{11}} \\ \dot{v_{12}} \end{pmatrix} =$$

$$= \begin{pmatrix} (I_3 + 2Mr^2)ab\sin\theta \\ 2(I_1 - I_3 - Mr^2)ab\sin\theta\cos\theta - I_3 a v_0\sin\theta \\ (Mr^2 + I_3 - I_1)b^2\sin\theta\cos\theta + I_3 b v_0\sin\theta + Mgr\cos\theta \\ v_0 b r\sin\varphi \\ -v_0 b r\cos\varphi \end{pmatrix},$$

where $a = \dot\theta$ and $b = \dot\varphi$. The last two equations are the derivative with respect to time of the constraint equation (2.2).

Solving for $(\dot{a}, \dot{b}, \dot{v}_0, \dot{v}_{11}, \dot{v}_{12})$, we have the following system of equations for the rolling disk

$$\dot{a} = \frac{-Mgr\cos\theta - b((I_3 + Mr^2)v_0 + (I_3 - I_1 + Mr^2)b\cos\theta)\sin\theta}{I_1 + Mr^2}, \tag{2.5}$$

$$\dot{b} = \frac{a((I_3 - 2I_1)b\cot\theta + I_3 v_0 \csc\theta)}{I_1}, \tag{2.6}$$

$$\dot{v}_0 = \frac{-a(I_3 v_0 + (I_3 - 2I_1)b\cos\theta)\cot\theta}{I_1} + \frac{(I_3 + 2Mr^2)ab\sin\theta}{I_3 + Mr^2}, \tag{2.7}$$

$$\begin{aligned}
\dot{v}_{11} = {} & \frac{1}{2I_1(I_3 + Mr^2)}(ar\cos\varphi(2I_3(I_3 + Mr^2)v_0\cot\theta + (I_3^2 - 3I_1 I_3 - 4I_1 Mr^2 + \\
& + I_3 Mr^2 + I_3(I_3 - I_1 + Mr^2)\cos(2\theta))b\csc\theta) + I_1 r(2I_3 + Mr^2 + \\
& + Mr^2\cos^2\varphi)bv_0\sin\varphi + I_1 Mr^3 bv_0\sin^3\varphi),
\end{aligned} \tag{2.8}$$

$$\begin{aligned}
\dot{v}_{12} = {} & -\frac{1}{2I_1(I_3 + Mr^2)}(r(I_1 Mr^2 bv_0\cos^3\varphi - a(2I_3(I_3 + Mr^2)v_0\cos\theta + \\
& + b(-3I_1 I_3 + I_3^2 - 4I_1 Mr^2 + I_3 Mr^2 + I_3(I_3 - I_1 + Mr^2)\cos(2\theta)))\csc\theta\sin\varphi + \\
& + I_1 bv_0\cos\varphi(2I_3 + Mr^2 + Mr^2\sin^2\varphi))).
\end{aligned} \tag{2.9}$$

Checking Balance of Momentum. The balance of momentum in the body axes is given by the formula $I\dot{\Omega} - I\Omega \times \Omega = T$, where $I = diag(I_1, I_2, I_3)$ is the tensor of inertia of the disk and $T = A^{-1}(M\ddot{w} \times ry + Mg\mathbf{e}_3 \times ry)$ is the torque of the external forces about the center of mass. By replacing

$$A = \begin{pmatrix} -\cos\theta\cos\psi\sin\varphi - \cos\varphi\sin\psi & \cos\theta\sin\psi\sin\varphi - \cos\varphi\cos\psi & \sin\theta\sin\varphi \\ \cos\theta\cos\psi\cos\varphi - \sin\varphi\sin\psi & -\cos\theta\sin\psi\cos\varphi - \cos\psi\sin\varphi & -\sin\theta\cos\varphi \\ \sin\theta\cos\psi & -\sin\theta\sin\psi & \cos\theta \end{pmatrix},$$

$$\Omega = (-\dot{\theta}\sin\psi + \dot{\varphi}\cos\psi\sin\theta, -\dot{\theta}\cos\psi - \dot{\varphi}\sin\theta\sin\psi, \dot{\varphi}\cos\theta + \dot{\psi}),$$
$$w = x + ry \qquad \text{and} \qquad y = (-\cos\theta\sin\varphi, \cos\theta\cos\varphi, \sin\theta)$$

in the formula $I\dot{\Omega} - I\Omega \times \Omega = T$, we obtain the following system of equations:

$$\begin{aligned}
& (I_1 - I_3)b^2\cos\theta\sin\theta\sin\psi + (2I_1 - I_3)ab\cos\theta\cos\psi - I_3 av_0\cos\psi - \\
& -I_1\dot{a}\sin\psi - I_3 bv_0\sin\theta\sin\psi + I_1\dot{b}\cos\psi\sin\theta = Mgr\sin\psi\cos\theta + \\
& + Mr^2 b^2\sin\psi\sin\theta\cos\theta + Mrv_{11}\sin\psi\sin\theta\sin\varphi - \\
& - Mrv_{12}\sin\psi\sin\theta\cos\varphi + Mr^2\dot{a}\sin\psi,
\end{aligned} \tag{2.10}$$

$$\begin{aligned}
& (I_1 - I_3)b^2\cos\theta\cos\psi\sin\theta + (I_3 - 2I_1)ab\cos\theta\sin\psi + I_3 av_0\sin\psi - \\
& -I_1\dot{a}\cos\psi - I_3 bv_0\sin\theta\cos\psi - I_1\dot{b}\sin\theta\sin\psi = Mgr\cos\theta\cos\psi + \\
& + Mr^2 b^2\cos\psi\sin\theta\cos\theta + Mrv_{11}\cos\psi\sin\theta\sin\varphi - \\
& - Mrv_{12}\cos\psi\sin\theta\cos\varphi + Mr^2\dot{a}\cos\psi,
\end{aligned} \tag{2.11}$$

$$\begin{aligned}
& -I_3 ab\sin\theta + I_3\dot{b}\cos\theta + I_3\dot{v}_0 = 2Mr^2 ab\sin\theta + Mrv_{11}\cos\varphi + \\
& + Mrv_{12}\sin\varphi - Mr^2\dot{b}\cos\theta.
\end{aligned} \tag{2.12}$$

We observe that the last equation (2.12) is the vertical equation (2.1) of the reduced system.

The system (2.10), (2.11), (2.12) is consistent with the formulas (2.5) to (2.9). In other words, the system of equations (2.10) to (2.12) is satisfied by the expressions (2.5) to (2.9).

Solving the equations. By introducing the derivative with respect to time of the rolling constraint (2.2) in the equations (2.1), (2.3) and (2.4), we obtain the following system of equations equivalent to the system (2.1) to (2.4):

$$v_0 r u = v_1, \tag{2.13}$$

$$(I_3 + Mr^2)\dot{v}_0 + (I_3 + Mr^2)\dot{b}\cos\theta - (I_3 + 2Mr^2)ab\sin\theta = 0, \tag{2.14}$$

$$2(Mr^2 + I_3 - I_1)ab\sin\theta\cos\theta - (I_1\sin^2\theta + (I_3 + Mr^2)\cos^2\theta)\dot{b} -$$
$$-(I_3 + Mr^2)\dot{v}_0\cos\theta + I_3 a\, v_0\sin\theta = 0, \tag{2.15}$$

$$(I_1 - I_3 - Mr^2)b^2\sin\theta\cos\theta - (I_1 + Mr^2)\dot{a} -$$
$$-(I_3 + Mr^2)bv_0\sin\theta - Mgr\cos\theta = 0. \tag{2.16}$$

Before we continue, a comment is in order about the comparison of our work with that of Pars in [42].

The three last equations (2.14), (2.15), (2.16) form a system equivalent to the one that has been obtained in [42], which is the following:

$$(A + Ma^2)\ddot{\theta} = A\dot{\varphi}^2\cos\theta\sin\theta - (C + Ma^2)\omega_3\dot{\varphi}\sin\theta - Mga\cos\theta, \tag{2.17}$$

$$(C + Ma^2)\dot{\omega}_3 = Ma^2\dot{\theta}\dot{\varphi}\sin\theta, \tag{2.18}$$

$$\frac{d}{dt}(A\dot{\varphi}\sin^2\theta) = C\omega_3\dot{\theta}\sin\theta. \tag{2.19}$$

In fact, since $\omega_3 = \dot{\psi} + \dot{\varphi}\cos\theta$, $A = I_1$, $C = I_3$ and $a = r$, the first equation of Pars' system is the equation (2.16). The equation (2.18) is the same as (2.14) and the Pars' third equation (2.19) is the equation (2.15) plus $\cos\theta$-times the equation (2.14) of our system.

The system formed by the equations (2.14) to (2.16) can be written in matrix form as follows

$$\begin{pmatrix} 0 & (I_3 + Mr^2)\cos\theta & I_3 + Mr^2 \\ 0 & I_1\sin^2\theta + (I_3 + Mr^2)\cos^2\theta & (I_3 + Mr^2)\cos\theta \\ I_1 + Mr^2 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} \dot{a} \\ \dot{b} \\ \dot{v}_0 \end{pmatrix} =$$
$$= \begin{pmatrix} (I_3 + 2Mr^2)ab\sin\theta \\ 2(Mr^2 + I_3 - I_1)ab\sin\theta\cos\theta + I_3 v_0 a\,\sin\theta \\ (I_1 - Mr^2 - I_3)b^2\sin\theta\cos\theta - (I_3 + Mr^2)bv_0\sin\theta - Mgr\cos\theta \end{pmatrix}. \tag{2.20}$$

From (2), we obtain $\dot{a}$, $\dot{b}$ and $\dot{v}_0$ in explicit form as follows:

$$\dot{a} = \frac{-Mgr\cos\theta - ((I_3 + Mr^2)v_0 + b(I_3 - I_1 + Mr^2)\cos\theta)b\sin\theta}{I_1 + Mr^2}, \tag{2.21}$$

$$\dot{b} = \frac{a((I_3 - 2I_1)b\cot\theta + I_3 v_0\csc\theta)}{I_1}, \tag{2.22}$$

$$\dot{v}_0 = \frac{-a(I_3 v_0 + (I_3 - 2I_1)b\cos\theta)\cot\theta}{I_1} + \frac{(I_3 + 2Mr^2)ab\sin\theta}{I_3 + Mr^2}. \tag{2.23}$$

To the system (2.21) to (2.23) we must add the algebraic equation $v_1 = v_0 r u$, and we obtain a system equivalent to the system (2.5) to (2.9).

By dividing the equations (2.14) and (2.15) by $a = \dot{\theta}$, we obtain the following system of two linear differential equations in two variables b and v_0:

$$(I_3 + Mr^2)\frac{dv_0}{d\theta} + (I_3 + Mr^2)\cos\theta\,\frac{db}{d\theta} - (I_3 + 2Mr^2)b\sin\theta = 0, \tag{2.24}$$

$$(I_3 + Mr^2)\cos\theta\,\frac{dv_0}{d\theta} + (I_1\sin^2\theta + (I_3 + Mr^2)\cos^2\theta)\frac{db}{d\theta} -$$
$$-2(Mr^2 + I_3 - I_1)b\sin\theta\cos\theta - I_3 v_0\sin\theta = 0. \tag{2.25}$$

This system can be transformed in only one second order differential equation with nonconstant coefficients as we will explain next. The system $(2.24), (2.25)$ can be solved for $db/d\theta$, $dv_0/d\theta$ as follows:

$$\frac{db}{d\theta} = \frac{I_3}{I_1} \csc \theta \; v_0 + \frac{I_3 - 2I_1}{I_1} \cot \theta \; b, \tag{2.26}$$

$$\frac{dv_0}{d\theta} = -\frac{I_3}{I_1} \cot \theta \; v_0 + \left(\frac{I_3 + 2Mr^2}{I_3 + Mr^2} \sin \theta - \frac{I_3 - 2I_1}{I_1} \cos \theta \cot \theta \right) b \tag{2.27}$$

By defining the dimensionless constants $\alpha = I_1/Mr^2$ and $\beta = I_3/Mr^2$, and thinking of b and v_0 as being functions dependent on θ, we obtain from $(2.26), (2.27)$ the following second order differential equation for $b = b(\theta)$:

$$b'' = -3 \cot \theta \; b' + \frac{2\alpha + \beta + 2\alpha\beta}{\alpha + \alpha\beta} \; b. \tag{2.28}$$

We observe that this equation can be written in its self-adjoint form as

$$(\sin^3 \theta \; b')' - \frac{2\alpha + \beta + 2\alpha\beta}{\alpha + \alpha\beta} \; \sin^3 \theta \; b = 0. \tag{2.29}$$

After the substitution $u = \tan(\theta/2)$ in (2.28), we have the same equation written as

$$b'' + \frac{3 - u^2}{u(1 + u^2)} \; b' - \frac{4\gamma}{(1 + u^2)^2} \; b = 0 \tag{2.30}$$

where $\gamma = \dfrac{2\alpha + \beta + 2\alpha\beta}{\alpha + \alpha\beta}$.

This equation is a Heun's equation with the four singularities given by $0, \infty, \pm i$. Notice that since $u = \tan(\theta/2)$ none of these singularities are in the interval $0 < \theta < \pi$. Heun's equation is the most general linear Fuchsian equation of second order with four regular singularities. This equations have been first studied by Heun in [26], and they are still not completely understood. See [45], [47] for an account of results on Heun's equations.

Of course we can directly apply the Frobenius method and obtain some information about the solutions. Set $b(u) = u^m \displaystyle\sum_{n=0}^{\infty} a_n u^n$, then we obtain the equation

$$(u + 2u^3 + u^5) \left(\sum_{n=0}^{\infty} (n + m - 1)(n + m)a_n u^{n+m-2} \right) + (3 + 2u - u^4) \left(\sum_{n=0}^{\infty} (n + m)a_n u^{n+m-1} \right) -$$

$$-4\gamma u \left(\sum_{n=0}^{\infty} a_n u^{n+m} \right) = 0,$$

or,

$$\sum_{n=0}^{\infty} (n+m+2)(n+m)a_n u^{n+m-1} + \sum_{n=0}^{\infty} \left(2(n + m)^2 - 4\gamma \right) a_n u^{n+m+1} + \sum_{n=0}^{\infty} (n+m-2)(n+m)a_n u^{n+m+3} = 0.$$

Then the indicial equation for the second order differential equation (2.30) is

$$m(m + 2) = 0.$$

For the case $m = 0$, the recursion formula to calculate the coefficients a_n is the following:

$$
\begin{cases}
a_{2k-1} & = 0, \ k \in \mathbb{N}; \\
a_2 & = 1/2\gamma a_0; \\
a_{2k} & = \dfrac{(\gamma - 2(k-1)^2)a_{2k-2} - (k-2)(k-3)a_{2k-4}}{k(k+1)}, \ \ k \in \mathbb{N}, \ k > 1.
\end{cases}
\tag{2.31}
$$

For the case $m = -2$, the coefficients are

$$
\begin{cases}
a_0 & = 0; \\
a_{2k-1} & = 0, \ k \in \mathbb{N}; \\
a_4 & = 1/2\gamma a_2; \\
a_{2k} & = \dfrac{(\gamma - 2(k-2)^2)a_{2k-2} - (k-4)(k-3)a_{2k-4}}{k(k-1)}, \ \ k \in \mathbb{N}, \ k > 2.
\end{cases}
\tag{2.32}
$$

In the case $m = 0$ the solution is

$$
b_1(u) = a_0 \left(1 + \frac{1}{2}\gamma \, u^2 + \frac{\gamma(\gamma - 2)}{12}u^4 + \frac{\gamma(\gamma - 2)(\gamma - 8)}{12^2} \, u^6 + ... \right),
$$

and in the case $m = -2$ we obtain the solution

$$
b_2(u) = a_2 u^2 \left(1 + \frac{\gamma}{2}u^2 + \frac{\gamma(\gamma - 2)}{12}u^4 + \frac{\gamma(\gamma - 2)(\gamma - 8)}{12^2}u^6 + \right.
$$
$$
\left. + \left(\frac{\gamma(\gamma - 2)(\gamma - 8)(\gamma - 18)}{12^2.20} - \frac{\gamma(\gamma - 2)}{12.20} \right) u^8 + ... \right).
$$

In an analogous way, we can obtain a second order differential equation for v_0 from (2.26) and (2.27) as follows. By deriving with respect to θ the equation (2.27), and substituting b and $v_0' = dv_0/d\theta$ given by the same equation (2.27) and $db/d\theta$ given by the equation (2.26) in the expression of $d^2v_0/d\theta^2$, we have:

$$
\left((2\alpha - \beta)(\beta + 1)\cos^2\theta \sin\theta + \alpha(\beta + 2)\sin^3\theta \right) v_0'' + \left(\beta(1 - \alpha + \beta)\cos\theta \sin^2\theta - \right.
$$
$$
-3(\beta + 1)(\beta - 2\alpha)\cos\theta \right) v_0' - \left(\frac{\beta}{\alpha}(2\alpha(\beta + 3) - \beta)\sin\theta + \frac{\beta^2(1 - \alpha + \beta)}{\alpha(\beta + 1)}\sin^3\theta \right) v_0 = 0.
$$

This second order equation can be transformed in a second order equation with polynomial coefficients in terms of the variable $u = \tan(\theta/2)$ and then solved for v_0 using Frobenius expansions, however we do not know how to classify it. An alternative procedure would be to introduce the general solution to the Heun's equation (2.28), which are of the type $b(\theta) = C_1 \, b_1(\theta) + C_2 \, b_2(\theta)$ with C_1, C_2 constants and $b_1(\theta)$, $b_2(\theta)$ linearly independent solutions, into equation (2.27). This way we would obtain a linear first order equation for $v_0(\theta)$, which can be solved by quadratures involving a constant C_3. So we finally obtain an analytic formula $f(\theta, v_0, C_1, C_2, C_3) = 0$, that must be satisfied by the solutions $(\theta(t), a(t), b(t), v_0(t))$ to the equations (2.21), (2.22) and (2.23).

Appendix A. The Lagrange–d'Alembert–Poincaré Equations

The Nonholonomic Connection. Let $\pi \colon Q \to Q/G$ be a principal bundle with structure group G. Assume that there is a given G-invariant metric on Q. In many important physical examples there is a natural way of choosing an invariant metric, representing, for instance, the inertia tensor of

the system, see [8]. Now assume that $\mathcal{D}$ is a given invariant distribution on Q. In physical examples this distribution often represents a nonholonomic constraint. We are going to assume the following **dimension assumption**, see [16],

$$TQ = \mathcal{D} + \mathcal{V},$$

where $\mathcal{V}$ is the vertical distribution. Let $\mathcal{S} = \mathcal{D} \cap \mathcal{V}$. We can then define the principal connection form $\mathcal{A}: TQ \to \mathfrak{g}$ such that the horizontal distribution $\mathrm{Hor}^{\mathcal{A}}TQ$ satisfies the condition that, for each q, the space $\mathrm{Hor}^{\mathcal{A}}T_qQ$ coincides with the orthogonal complement $\mathcal{H}_q$ of the space $\mathcal{S}_q$ in $\mathcal{D}_q$. This connection is called the **nonholonomic connection**. For each $q \in Q$, let us denote $\mathcal{U}_q$ the orthogonal complement of $\mathcal{S}_q$ in $\mathcal{V}_q$. Then it is easy to see that $\mathcal{U}$ is a smooth distribution and we have the Whitney sum decomposition

$$TQ = \mathcal{H} \oplus \mathcal{S} \oplus \mathcal{U}.$$

We obviously have

$$\mathcal{D} = \mathcal{H} \oplus \mathcal{S}$$

and

$$\mathcal{V} = \mathcal{S} \oplus \mathcal{U}.$$

Under the invariance assumption, all three distributions $\mathcal{H}$, $\mathcal{S}$, and $\mathcal{U}$ are G-invariant, so we can write,

$$TQ/G = \mathcal{H}/G \oplus \mathcal{S}/G \oplus \mathcal{U}/G.$$

The Geometry of the Reduced Bundles. Recall from [15] that there is a vector bundle isomorphism

$$\alpha_{\mathcal{A}}: TQ/G \to T(Q/G) \oplus \tilde{\mathfrak{g}},$$

where $\tilde{\mathfrak{g}}$ is the adjoint bundle of the principal bundle Q, defined as follows

$$\alpha_{\mathcal{A}}([q, \dot{q}]_G) = T\pi(q, \dot{q}) \oplus [q, \mathcal{A}(q, \dot{q})]_G,$$

where $(q, \dot{q}) \in TQ$ and the index G denotes equivalence classes under the action of G. Notice that the bundle $T(Q/G) \oplus \tilde{\mathfrak{g}}$ does not depend on the connection $\mathcal{A}$, however, the vector bundle isomorphism $\alpha_{\mathcal{A}}$ does depend on $\mathcal{A}$. It is easy to see that

$$\alpha_{\mathcal{A}}(\mathcal{H}/G) = T(Q/G),$$

and

$$\alpha_{\mathcal{A}}(\mathcal{V}/G) = \tilde{\mathfrak{g}}.$$

Define the subbundles $\tilde{\mathfrak{s}}$ and $\tilde{\mathfrak{u}}$ of $\tilde{\mathfrak{g}}$ by

$$\tilde{\mathfrak{s}} = \alpha_{\mathcal{A}}(\mathcal{S}/G)$$

and

$$\tilde{\mathfrak{u}} = \alpha_{\mathcal{A}}(\mathcal{U}/G)$$

respectively. Clearly, we have

$$\tilde{\mathfrak{g}} = \tilde{\mathfrak{s}} \oplus \tilde{\mathfrak{u}}.$$

Recall from [15], [16] that the bundle $\tilde{\mathfrak{g}}$ has a structure defined as follows

a. A Lie algebra structure on each fiber of $\tilde{\mathfrak{g}}$ defined by

$$[[q, \xi_1]_G, [q, \xi_2]_G] = [q, [\xi_1, \xi_2]]_G$$

b. A covariant derivative of curves on $\tilde{\mathfrak{g}}$, given by

$$\frac{D}{Dt}\left[q(t), \xi(t)\right]_G = \left[q(t), \left[-\mathcal{A}(q(t), \dot{q}(t)), \xi(t)\right] + \dot{\xi}(t)\right]_G.$$

The corresponding connection in $\tilde{\mathfrak{g}}$ is denoted $\tilde{\nabla}^{\mathcal{A}}$.

c. A $\tilde{\mathfrak{g}}$-valued 2-form $\tilde{\mathcal{B}}$ on the base Q/G defined by

$$\tilde{\mathcal{B}}\left((x, \dot{x}_1), (x, \dot{x}_2)\right) = \left[q, \mathcal{B}\left((x, \dot{x}_1)_q^h, (x, \dot{x}_2)_q^h\right)\right]_G$$

where q satisfies $[q]_G = x$, $(x, \dot{x}_i)_q^h$ is the horizontal lift of $(x, \dot{x}_i)$ at the point q, for $i = 1, 2$, and $\mathcal{B}$ is the curvature of $\mathcal{A}$.

Now let $L\colon TQ \to \mathbb{R}$ be a given invariant Lagrangian. Then it naturally induces a reduced Lagrangian $\ell\colon TQ/G \to \mathbb{R}$. Via the identification given by the vector bundle isomorhism $\alpha_{\mathcal{A}}$ we will often think of ℓ as being a map $\ell\colon T(Q/G) \oplus \tilde{\mathfrak{g}} \to \mathbb{R}$, or, with the usual notation in terms of variables, $\ell(x, \dot{x}, \bar{v})$. We should remark that x, $\dot{x}$ and $\bar{v}$ are not independent variables, unless $T(Q/G)$ and $\tilde{\mathfrak{g}}$ are trivial bundles, see [15]. Finally, given any torsionless connection ∇ on Q/G we have a naturally defined connection $\nabla \oplus \tilde{\nabla}^{\mathcal{A}}$ on $T(Q/G) \oplus \tilde{\mathfrak{g}}$. It is with respect to this connection that the covariant derivatives appearing in the following theorem, like

$$\frac{\partial^C \ell}{\partial x}(x, \dot{x}, \bar{v}) \quad \text{and} \quad \frac{D}{Dt}\frac{\partial \ell}{\partial \dot{x}}(x, \dot{x}, \bar{v})$$

should be understood. See for instance [30] or [15], [16] for details.

The following theorem (see [16]) is the main result that we need to study the rolling disk.

Theorem 1. *Let $q(t)$ be a curve in Q such that $(q(t), \dot{q}(t)) \in \mathcal{D}_{q(t)}$ for all t and let $(x(t), \dot{x}(t), \bar{v}(t)) = \alpha_{\mathcal{A}}\left([q(t), \dot{q}(t)]_G\right)$ be the corresponding curve in $T(Q/G) \oplus \tilde{\mathfrak{s}}$. The following conditions are equivalent.*

(i) *The **Lagrange–d'Alembert principle** holds:*

$$\delta \int_{t_0}^{t_1} L(q, \dot{q})dt = 0$$

for variations δq of the curve q such that $\delta q(t_i) = 0$, for $i = 0, 1$, and $\delta q(t) \in \mathcal{D}_{q(t)}$, for all t.

(ii) *The **reduced Lagrange–d'Alembert principle** holds: The curve $x(t) \oplus \bar{v}(t)$ satisfies*

$$\delta \int_{t_0}^{t_1} \ell\left(x(t), \dot{x}(t), \bar{v}(t)\right) dt = 0,$$

for variations $\delta x \oplus \delta^{\mathcal{A}}\bar{v}$ of the curve $x(t) \oplus \bar{v}(t)$, where $\delta^{\mathcal{A}}\bar{v}$ has the form

$$\delta^{\mathcal{A}}\bar{v} = \frac{D\bar{\eta}}{Dt} + [\bar{v}, \bar{\eta}] + \tilde{\mathcal{B}}(\delta x, \dot{x}),$$

with the boundary conditions $\delta x(t_i) = 0$ and $\bar{\eta}(t_i) = 0$, for $i = 0, 1$, and where $\bar{\eta}(t) \in \tilde{\mathfrak{s}}_{x(t)}$.

(iii) *The following **vertical Lagrange–d'Alembert-Poincaré equations**, corresponding to vertical variations, hold:*

$$\left.\frac{D}{Dt}\frac{\partial \ell}{\partial \bar{v}}(x, \dot{x}, \bar{v})\right|_{\tilde{\mathfrak{s}}} = \left.\mathrm{ad}_{\bar{v}}^{*}\frac{\partial \ell}{\partial \bar{v}}(x, \dot{x}, \bar{v})\right|_{\tilde{\mathfrak{s}}},$$

and the **horizontal Lagrange–d'Alembert-Poincaré equations**, *corresponding to horizontal variations,*

$$\frac{\partial^C \ell}{\partial x}(x, \dot{x}, \bar{v}) - \frac{D}{Dt}\frac{\partial \ell}{\partial \dot{x}}(x, \dot{x}, \bar{v}) = \left\langle \frac{\partial \ell}{\partial \bar{v}}(x, \dot{x}, \bar{v}), \mathbf{i}_{\dot{x}}\tilde{\mathcal{B}}(x) \right\rangle.$$

hold.

In part (ii) of this theorem, if $\bar{v} = [q, v]_G$ with $v = \mathcal{A}(q, \dot{q})$ then $\bar{\eta}$ can be always written $\bar{\eta} = [q, \eta]_G$, and the condition $\bar{\eta}(t_i) = 0$ for $i = 0, 1$, is equivalent to the condition $\eta(t_i) = 0$ for $i = 0, 1$. Also, if $x(t) = [q(t)]_G$ and $\bar{v} = [q, v]_G$ where $v = \mathcal{A}(q, \dot{q})$, then variations $\delta x \oplus \delta^{\mathcal{A}}\bar{v}$ such that

$$\delta^{\mathcal{A}}\bar{v} = \frac{D\bar{\eta}}{Dt} + [\bar{v}, \bar{\eta}] \equiv \frac{D[q, \eta]_G}{Dt} + [q, [v, \eta]]_G$$

with $\bar{\eta}(t_i) = 0$ (or, equivalently, $\eta(t_i) = 0$) for $i = 0, 1$, and $\bar{\eta}(t) \in \tilde{\mathfrak{s}}_{x(t)}$, correspond exactly to vertical variations δq of the curve q such that $\delta q(t_i) = 0$ for $i = 0, 1$, and $\delta q(t) \in \mathcal{S}_{q(t)}$, while variations $\delta x \oplus \delta^{\mathcal{A}}\bar{v}$ such that

$$\delta^{\mathcal{A}}\bar{v} = \tilde{\mathcal{B}}(\delta x, \dot{x})$$

with $\delta x(t_i) = 0$ for $i = 0, 1$, correspond exactly to horizontal variations δq of the curve q such that $\delta q(t_i) = 0$.

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