Diophantine Approximation

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Problem statement

Let $x \in \mathbb{R}$ be a real number, and let $\epsilon \in \mathbb{R}^+$ be a small positive constant. We want to find $p, q \in \mathbb{Z}$ such that

$$\left| x - \frac{p}{q} \right| < \epsilon$$

and p and q are relatively prime (thus $\frac{p}{q}$ is in lowest terms).

For example, if $x = \pi$ and $\epsilon = 10^{-5}$, then the rational number

$$\frac{355}{113} = 3.14159292035\ldots$$

would be a valid diophantine approximation for this ϵ.

Framing the problem

We only need to focus on the non-integer part of the number $x \geqslant 0$ to be approximated:

1. Let $x_f = x - \lfloor x \rfloor$ be the non-integer part of x
2. Let $\frac{p}{q}$ be the diophantine approximation of x_f.
3. Then, $\frac{p'}{q} = \frac{\lfloor x \rfloor}{1} + \frac{p}{q}$ is the diophantine approximation of x, where $p' = p + q\lfloor x \rfloor$.

Note that if $x < 0$ we can just apply the process above to $|x|$ and return $-\frac{p'}{q}$.

Hence, we can assume w.l.o.g. that $x \in [0, 1)$.

Assessing the solution

There are infinitely many rational numbers $\frac{p}{q}$ that approximate x within the given tolerance ϵ.

Best Diophantine Approximation

An approximation $\frac{p}{q}$ is said to be a best diophantine approximation if

$$\left|x - \frac{p}{q}\right| < \left|x - \frac{p'}{q'}\right|$$

for every rational number $\frac{p'}{q'}$ different from $\frac{p}{q}$ such that $0 < q' \leqslant q$.

We look for a best diophantine approximation within the given tolerance ϵ. In other words, we look for the rational number with the smallest denominator that provides a good enough approximation.

Binary Search

We can find the diophantine approximation by doing binary search on the unit interval:

1. Let $L = 0/1$ (i.e., 0) and let $R = 1/1$ (i.e., 1).
2. Pick some rational number M between L and R.
3. If $|x - M| < \epsilon$, return M and terminate.
4. Otherwise, update L (if $M < x$) or R (if $M > x$) and go back to 2

Using Bipartition

The bipartition approach picks the middle point $M = (L + R)/2$.

PRO The worst-case number of iterations to find the solution is minimal: $O(\log \frac{1}{\epsilon})$.

CON The denominator q is always a power of 2, and both p and q can grow very large, e.g., for $x = \pi$ and $\epsilon = 10^{-5}$, we get

$$\frac{3217}{1024} = 3.1416015625$$

Thus, it is not a best diophantine approximation.

Farey Sequences

Farey sequence of order n

The Farey sequence of order n is the sequence F_n of completely reduced fractions between 0 and 1 whose denominators are less than or equal to n, arranged in order of increasing size.

$$
\begin{aligned}
F_1 &= \langle \tfrac{0}{1}, \tfrac{1}{1} \rangle \\
F_2 &= \langle \tfrac{0}{1}, \tfrac{1}{2}, \tfrac{1}{1} \rangle \\
F_3 &= \langle \tfrac{0}{1}, \tfrac{1}{3}, \tfrac{1}{2}, \tfrac{2}{3}, \tfrac{1}{1} \rangle \\
F_4 &= \langle \tfrac{0}{1}, \tfrac{1}{4}, \tfrac{1}{3}, \tfrac{1}{2}, \tfrac{2}{3}, \tfrac{3}{4}, \tfrac{1}{1} \rangle \\
&\ldots
\end{aligned}
$$

Note that F_n includes F_{n-1} plus some new interleaved elements. Any new element is the mediant of two adjacent elements in F_{n-1}.

Farey Sequences

Mediant

Let $\frac{a}{c}$ and $\frac{b}{d}$ be two fractions. Their mediant is defined as

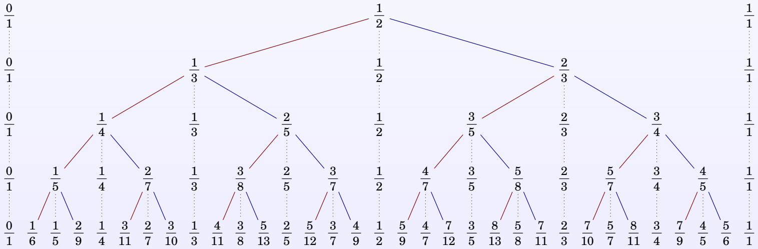
$$\frac{a+b}{c+d}$$

Note that:

- The mediant lies between the two fractions considered: $\frac{a}{c} < \frac{a+b}{c+d} < \frac{b}{d}$
- If $\frac{a}{c}$ and $\frac{b}{d}$ are adjacent in a Farey sequence, the mediant is the simplest fraction (i.e., the fraction with the smallest denominator) in the interval $(\frac{a}{c}, \frac{b}{d})$.

Stern-Brocot Tree

The Stern–Brocot tree is an infinite complete binary tree built using mediants, just like Farey Sequences.



Technically speaking, the Stern-Brocot tree is defined between $0/1$ and $1/0$. Hence the above figure illustrates the left branch of the tree, namely $\mathbb{Q} \cap (0, 1)$.

Stern-Brocot Tree

An inorder traversal of the first k levels of the Stern-Brocot tree contains the Farey sequence of order $k+1$ (assuming we add the boundary terms $\frac{0}{1}, \frac{1}{1}$).

The vertices of the Stern–Brocot tree correspond one-for-one to the positive rational numbers.

Hence, the Stern-Brocot tree forms an infinite binary search tree that contains every rational number exactly once.

We can perform a binary search of this tree to locate a particular rational number.

Using Mediants

The mediant approach picks $M = \frac{a+b}{c+d}$ as the mediant of $L = \frac{a}{c}$ and $R = \frac{b}{d}$. As shown before, this is the simplest fraction in (L, R).

PRO It will provide a best approximation of x.

CON The worst-case complexity is worse than bipartition: $O(1/\epsilon)$.

Goal of the Lab Session

What is the average complexity of this approach?

Complementary Bibliography

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