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# Laborator 2

Ex 6 $\begin{cases} -u''(x)+u(x)=f(x), & x\in(0,1) \\ u(0)=u(1)=0 \end{cases}$

cu discretizarea lui $I=[0,1]$

$x_i = ih, \quad i=\overline{0,n+1}, \quad h:=\frac{1}{n+1}$

a) $u_i := u(x_i), \quad i=\overline{0,n+1}$

$f_i := f(x_i), \quad i=\overline{0,n+1}$

$u''(x) \approx \frac{1}{h^2}[u(x-h)-2u(x)+u(x+h)] \ \forall x\in(0,1)$

$A_h u_h = f_h, \quad A_h \in \mathcal{M}_n(\mathbb{R}), \quad u_h=(u_1,\dots,u_n)^T \in \mathbb{R}^n$

$f_h=(f_1,\dots,f_n)^T \in \mathbb{R}^n$

$-u''(x)+u(x)=f(x) \Leftrightarrow -c(x)u''(x)+b(x)u'(x)+a(x)u(x)=f(x)$

$\Rightarrow \begin{cases} c(x)=1 \\ b(x)=0 \\ a(x)=1 \end{cases}$

$-c(x)u''(x)+a(x)u(x) = -\frac{1}{h^2}u_{i-1}+\frac{2}{h^2}u_i-\frac{1}{h^2}u_{i+1}+1\cdot u_i$

$\Rightarrow A_h=(a_{ij})_{i,j=\overline{1,n}} = \begin{cases} \frac{2}{h^2}+a(x_i)=\frac{2}{h^2}+1, & i=j, \ i=\overline{1,n} \\ -\frac{1}{h^2}, & j=i+1 \text{ sau } j=i-1, \ i=\overline{2,n-1} \\ 0, & \text{altfel} \end{cases}$

$$A_h = \begin{pmatrix} \frac{2}{h^2}+1 & -\frac{1}{h^2} & 0 & 0 & \cdots & 0 \\ -\frac{1}{h^2} & \frac{2}{h^2}+1 & -\frac{1}{h^2} & 0 & \cdots & 0 \\ 0 & -\frac{1}{h^2} & \frac{2}{h^2}+1 & -\frac{1}{h^2} & \cdots & 0 \\ \vdots & \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & -\frac{1}{h^2} & \cdots & \frac{2}{h^2}+1 \end{pmatrix}$$

Sistemul de ecuații liniare asociat FDM este:

$$\begin{pmatrix} \frac{2}{h^2}+1 & -\frac{1}{h^2} & 0 & \cdots & 0 \\ -\frac{1}{h^2} & \frac{2}{h^2}+1 & -\frac{1}{h^2} & \cdots & 0 \\ 0 & -\frac{1}{h^2} & \frac{2}{h^2}+1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{2}{h^2}+1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_m \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ \vdots \\ f_m \end{pmatrix}$$

b) $A_h \in \mathcal{M}_m(\mathbb{R})$ SPD? $A_h$ simetrică (evident). $A_h$ PD:

Fie $v = (v_1, \ldots, v_m)^T \neq \underline{0}_m$

$v^T A_h v \overset{?}{>} 0$

$$(v_1, v_2, v_3, \ldots, v_m) \begin{pmatrix} \frac{2}{h^2}+1 & -\frac{1}{h^2} & 0 & \cdots & 0 \\ -\frac{1}{h^2} & \frac{2}{h^2}+1 & -\frac{1}{h^2} & \cdots & 0 \\ \vdots & \vdots & \vdots & & \\ 0 & 0 & 0 & \cdots & \frac{2}{h^2}+1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{pmatrix} =$$

$$\left(v_1\left(\frac{2}{h^2}+1\right)-v_2\cdot\frac{1}{h^2}, -v_1\frac{1}{h^2}+v_2\left(\frac{2}{h^2}+1\right)-v_3\frac{1}{h^2}, \dots, -\frac{1}{h^2}v_{m-1}+v_m\left(\frac{2}{h^2}+1\right)\right)\cdot$$

$$\cdot(v_1, v_2, \dots, v_m)^T = v_1^2\left(\frac{2}{h^2}+1\right)-v_1v_2\frac{1}{h^2}-v_1v_2\frac{1}{h^2}+v_2^2\left(\frac{2}{h^2}+1\right)+\dots+$$

$$+ -\frac{1}{h^2}v_{m-1}\cdot v_m + v_m^2\left(\frac{2}{h^2}+1\right) = \sum_{i=1}^{m} v_i^2\left(\frac{2}{h^2}+1\right)-\sum_{i=1}^{m-1} v_i\, v_{i+1}\cdot\frac{1}{h^2}\cdot 2 =$$

$$=\left(\frac{2}{h^2}+1\right)\sum_{i=1}^{m} v_i^2+\frac{1}{h^2}\sum_{i=1}^{m-1}-2v_iv_{i+1}$$

$$-2v_iv_{i+1}=(v_i-v_{i+1})^2-(v_i^2+v_{i+1}^2)$$

$$\left(\frac{2}{h^2}+1\right)\sum_{i=1}^{m}v_i^2+\frac{1}{h^2}\sum_{i=1}^{m-1}(v_i-v_{i+1})^2-\frac{1}{h^2}\sum_{i=1}^{m-1}(v_i^2+v_{i+1}^2)=$$

$$=\left(\frac{2}{h^2}\sum_{i=1}^{m}v_i^2-\frac{1}{h^2}\sum_{i=1}^{m-1}v_i^2\right)+\left(\sum_{i=1}^{m}v_i^2-\sum_{i=1}^{m-1}\frac{1}{h^2}v_{i+1}^2\right)+\frac{1}{h^2}\sum_{i=1}^{m-1}(v_i-v_{i+1})^2=$$

$$=\left(\frac{2}{h^2}v_m^2+\frac{1}{h^2}\sum_{i=1}^{m-1}v_i^2\right)+\left(\frac{h^2-1}{h^2}\sum_{i=1}^{m-1}v_i^2+v_m^2\right)+\frac{1}{h^2}\sum_{i=1}^{m-1}(v_i-v_{i+1})^2=$$

$$=\frac{2+h^2}{h^2}v_m^2+\sum_{i=1}^{m-1}v_i^2+\frac{1}{h^2}\sum_{i=1}^{m-1}(v_i-v_{i+1})^2=$$

$$=\frac{2}{h^2}v_m^2+\sum_{i=1}^{m}v_i^2+\frac{1}{h^2}\sum_{i=1}^{m-1}(v_i-v_{i+1})^2>0 \quad \forall\, v\neq \underline{0}_m \Leftrightarrow$$

$\Leftrightarrow A_h$ PD.

□