DiracDec C++/Coq Implementation[1]

December 16, 2024

1 Consideration

- AC equivalence and alpha equivalence.
- Efficient AC equivalence checking by sorting on syntax.
- Sorting requires that alpha equivalent terms have the same syntax.
- de Bruijn expression satisfies this requirement, but the typing and substitution becomes very complicated in the type index scenario.
- only compute de Bruijn form in the equivalence checking and sorting phase.

2 TODO

- implemented trace output for the whole pipeline.
- Note that the context should also be maintained during rewriting matching.
- better parser and output.

3 Language Syntax

Definition 3.1 (syntax)**.** *The syntax for type indices are defined as*

$$\sigma ::= x \mid \sigma_1 \times \sigma_2.$$

The syntax for Dirac notation types is defined as

$$T ::= x \mid \mathsf{Basis}(\sigma) \mid \mathcal{S} \mid \mathcal{K}(\sigma) \mid \mathcal{B}(\sigma) \mid \mathcal{O}(\sigma_1, \sigma_2) \mid T_1 \to T_2 \mid \forall x.T \mid \mathsf{Set}(\sigma).$$

The syntax for Dirac notation terms is defined as

$$\begin{aligned} e ::= {} & x \mid (e_1, e_2) \mid \lambda x : T.e \mid \mu x.e \mid e_1\ e_2 \\ & \mid 0 \mid 1 \mid \mathsf{ADDS}(e_1 \cdots e_n) \mid e_1 \times \cdots \times e_n \mid e^* \mid \delta_{e_1,e_2} \mid \mathsf{DOT}(e_1\ e_2) \\ & \mid \mathbf{0}_{\mathcal{K}}(\sigma) \mid \mathbf{0}_{\mathcal{B}}(\sigma) \mid \mathbf{0}_{\mathcal{O}}(\sigma_1, \sigma_2) \mid \mathbf{1}_{\mathcal{O}}(\sigma) \\ & \mid |e\rangle \mid \langle t| \mid e^\dagger \mid e_1.e_2 \mid \mathsf{ADD}(e_1 \cdots e_n) \mid e_1 \otimes e_2 \\ & \mid \mathsf{MULK}(e_1\ e_2) \mid \mathsf{MULB}(e_1\ e_2) \mid \mathsf{OUTER}(e_1\ e_2) \mid \mathsf{MULO}(e_1\ e_2) \\ & \mid \mathbf{U}(e) \mid e_1 \star e_2 \mid \sum_{e_1} e_2. \end{aligned}$$

Here i is a natural number and $\$i$ represents the i-th bound variable in de Bruijn notation. Compared to [?], this syntax for Dirac notations merges the symbols with overlapped properties, such as the addition and scaling symbols for ket, bra and operator. Here ADDS and ADD are two different AC symbols representing the scalar addition and the linear algebra addition respectively. They will be denoted as $a_1 + \cdots + a_n$ and $X_1 + \cdots + X_n$. There are five kinds of linear algebra multiplications among ket, bra and operator, whose properties are similar but still diverge to some extent. For example, the rules $(O_1 \cdot O_2) \cdot K \ \triangleright\ O_1 \cdot (O_2 \cdot K)$ and $B \cdot (O_1 \cdot O_2) \ \triangleright\ (B \cdot O_1) \cdot O_2$ indicate that the sorting of multiplication sequences depends on the subterm types. To avoid frequent but unnecessary type checkings, we encode the typing information by using five different symbols, namely DOT, MULK, MULB, OUTER and MULO. They are denoted as $B \cdot K$, $K_1 \cdot K_2$, $B_1 \cdot B_2$, $K \cdot B$ and $O_1 \cdot O_2$, respectively.

Usually, the sum body is specified by an abstraction. Therefore we use notation $\sum_s X$ to denote $\sum_{x \in s} \lambda x : T.X$ as well.

4 Typing System

The type checking of Dirac notations involves maintaining a well-formed environment and context $E[\Gamma]$, which specifies the definitions and typing assumtpions for variables. The environment and the conetxt are defined as follows.

Definition 4.1 (environment and context)**.**

$$E ::= [] \mid E; x : \mathsf{Index} \mid E; T : \mathsf{Type} \mid E; x : T \mid E; x := t : T.$$
$$\Gamma ::= [] \mid \Gamma; x : \mathsf{Index} \mid \Gamma; x : T.$$

Note that in the following rules, $x : \mathsf{Index}$ and $x : \mathsf{Type}$ are not considered as $x : T$ judgements.

W-Empty $\quad \dfrac{}{\mathcal{WF}([])[]}$

W-AssumE-Index $\quad \dfrac{\mathcal{WF}(E)[] \qquad x \notin E}{\mathcal{WF}(E; x : \mathsf{Index})[]}$ **W-AssumE-Type** $\quad \dfrac{\mathcal{WF}(E)[] \qquad x \notin E}{\mathcal{WF}(E; x : \mathsf{Type})[]}$

W-AssumE-Term $\quad \dfrac{E[] \vdash T : \mathsf{Type} \qquad x \notin E}{\mathcal{WF}(E; x : T)[]}$ **W-Def-Term** $\quad \dfrac{E[] \vdash t : T \qquad x \notin E}{\mathcal{WF}(E; x := t : T)[]}$

W-AssumC-Index $\quad \dfrac{\mathcal{WF}(E)[\Gamma]}{\mathcal{WF}(E)[\Gamma; x : \mathsf{Index}]}$ **W-AssumC-Term** $\quad \dfrac{E[\Gamma] \vdash T : \mathsf{Type}}{\mathcal{WF}(E)[\Gamma; x : T]}$

Figure 1: Rules for a well-formed environment and context.

Index-Var $\quad \dfrac{\mathcal{WF}(E)[\Gamma] \qquad x : \mathsf{Index} \in E[\Gamma]}{E[\Gamma] \vdash x : \mathsf{Index}}$ **Index-Prod** $\quad \dfrac{E[\Gamma] \vdash \sigma : \mathsf{Index} \qquad E[\Gamma] \vdash \tau : \mathsf{Index}}{E[\Gamma] \vdash \sigma \times \tau : \mathsf{Index}}$

Figure 2: Rules for type index.

$$\textbf{Type-Arrow}\quad \frac{E[\Gamma] \vdash T : \mathsf{Type} \qquad E[\Gamma] \vdash U : \mathsf{Type}}{E[\Gamma] \vdash T \to U : \mathsf{Type}} \qquad \textbf{Type-Index}\quad \frac{E[\Gamma; x : \mathsf{Index}] \vdash U : \mathsf{Type}}{E[\Gamma] \vdash \forall x.U : \mathsf{Type}}$$

$$\textbf{Type-Basis}\quad \frac{E[\Gamma] \vdash \sigma : \mathsf{Index}}{E[\Gamma] \vdash \mathsf{Basis}(\sigma) : \mathsf{Type}}$$

$$\textbf{Type-Ket}\quad \frac{E[\Gamma] \vdash \sigma : \mathsf{Index}}{E[\Gamma] \vdash \mathcal{K}(\sigma) : \mathsf{Type}} \qquad \textbf{Type-Bra}\quad \frac{E[\Gamma] \vdash \sigma : \mathsf{Index}}{E[\Gamma] \vdash \mathcal{B}(\sigma) : \mathsf{Type}}$$

$$\textbf{Type-Opt}\quad \frac{E[\Gamma] \vdash \sigma : \mathsf{Index} \qquad E[\Gamma] \vdash \tau : \mathsf{Index}}{E[\Gamma] \vdash \mathcal{O}(\sigma, \tau) : \mathsf{Type}} \qquad \textbf{Type-Scalar}\quad \frac{\mathcal{WF}(E)[\Gamma]}{E[\Gamma] \vdash \mathcal{S} : \mathsf{Type}}$$

$$\textbf{Type-Set}\quad \frac{E[\Gamma] \vdash \sigma : \mathsf{Index}}{E[\Gamma] \vdash \mathsf{Set}(\sigma) : \mathsf{Type}}$$

Figure 3: Rules for types.

$$\textbf{Term-Var}\quad \frac{\mathcal{WF}(E)[\Gamma] \qquad (x : T) \in E[\Gamma] \text{ or } (x := t : T) \in E \text{ for some } t}{E[\Gamma] \vdash x : T}$$

$$\textbf{Lam}\quad \frac{E[\Gamma; x : T] \vdash t : U}{E[\Gamma] \vdash (\lambda x : T.t) : T \to U} \qquad \textbf{Index}\quad \frac{E[\Gamma; x : \mathsf{Index}] \vdash t : U}{E[\Gamma] \vdash (\mu x.t) : \forall x.U}$$

$$\textbf{App-Arrow}\quad \frac{E[\Gamma] \vdash t : U \to T \qquad E[\Gamma] \vdash u : U}{E[\Gamma] \vdash (t\ u) : T} \qquad \textbf{App-Index}\quad \frac{E[\Gamma] \vdash t : \forall x.U \qquad E[\Gamma] \vdash u : \mathsf{Index}}{E[\Gamma] \vdash (t\ u) : U\{x/u\}}$$

Figure 4: Rules for variable and function typings. Here $U[u]$ means instantiate the first bound variable with u in U.

$$\textbf{Pair-Base}\quad \frac{E[\Gamma] \vdash s : \mathsf{Basis}(\sigma) \qquad E[\Gamma] \vdash t : \mathsf{Basis}(\tau)}{E[\Gamma] \vdash (s, t) : \mathsf{Basis}(\sigma \times \tau)}$$

Figure 5: Typing rules for Basis.

$$\textbf{Sca-0}\quad \frac{\mathcal{WF}(E)[\Gamma]}{E[\Gamma] \vdash 0 : \mathcal{S}} \qquad \textbf{Sca-1}\quad \frac{\mathcal{WF}(E)[\Gamma]}{E[\Gamma] \vdash 1 : \mathcal{S}}$$

$$\textbf{Sca-Delta}\quad \frac{E[\Gamma] \vdash s : \mathsf{Basis}(\sigma) \qquad E[\Gamma] \vdash t : \mathsf{Basis}(\sigma)}{E[\Gamma] \vdash \delta_{s,t} : \mathcal{S}}$$

$$\textbf{Sca-Add}\quad \frac{E[\Gamma] \vdash a_i : \mathcal{S} \text{ for all } i}{E[\Gamma] \vdash a_1 + \cdots + a_n : \mathcal{S}} \qquad \textbf{Sca-Mul}\quad \frac{E[\Gamma] \vdash a_i : \mathcal{S} \text{ for all } i}{E[\Gamma] \vdash a_1 \times \cdots \times a_n : \mathcal{S}}$$

$$\textbf{Sca-Conj}\quad \frac{E[\Gamma] \vdash a : \mathcal{S}}{E[\Gamma] \vdash a^* : \mathcal{S}} \qquad \textbf{Sca-Dot}\quad \frac{E[\Gamma] \vdash B : \mathcal{B}(\sigma) \qquad E[\Gamma] \vdash K : \mathcal{K}(\sigma)}{E[\Gamma] \vdash B \cdot K : \mathcal{S}}$$

Figure 6: Scalar typing rules.

$$\textbf{Ket-0}\quad \frac{E[\Gamma] \vdash \sigma : \mathsf{Index}}{E[\Gamma] \vdash \mathbf{0}_{\mathcal{K}}(\sigma) : \mathcal{K}(\sigma)} \qquad \textbf{Ket-Base}\quad \frac{E[\Gamma] \vdash t : \mathsf{Basis}(\sigma)}{E[\Gamma] \vdash |t\rangle : \mathcal{K}(\sigma)}$$

$$\textbf{Ket-Adj}\quad \frac{E[\Gamma] \vdash B : \mathcal{B}(\sigma)}{E[\Gamma] \vdash B^\dagger : \mathcal{K}(\sigma)} \qquad \textbf{Ket-Scr}\quad \frac{E[\Gamma] \vdash a : \mathcal{S} \qquad E[\Gamma] \vdash K : \mathcal{K}(\sigma)}{E[\Gamma] \vdash a.K : \mathcal{K}(\sigma)}$$

$$\textbf{Ket-Add}\quad \frac{E[\Gamma] \vdash K_i : \mathcal{K}(\sigma) \text{ for all } i}{E[\Gamma] \vdash K_1 + \cdots + K_n : \mathcal{K}(\sigma)} \qquad \textbf{Ket-MulK}\quad \frac{E[\Gamma] \vdash O : \mathcal{O}(\sigma,\tau) \qquad E[\Gamma] \vdash K : \mathcal{K}(\tau)}{E[\Gamma] \vdash O \cdot K : \mathcal{K}(\sigma)}$$

$$\textbf{Ket-Tsr}\quad \frac{E[\Gamma] \vdash K_1 : \mathcal{K}(\sigma) \qquad E[\Gamma] \vdash K_2 : \mathcal{K}(\tau)}{E[\Gamma] \vdash K_1 \otimes K_2 : \mathcal{K}(\sigma \times \tau)}$$

Figure 7: Ket typing rules.

$$\textbf{Bra-0}\quad \frac{E[\Gamma] \vdash \sigma : \mathsf{Index}}{E[\Gamma] \vdash \mathbf{0}_{\mathcal{B}}(\sigma) : \mathcal{B}(\sigma)} \qquad \textbf{Bra-Base}\quad \frac{E[\Gamma] \vdash t : \mathsf{Basis}(\sigma)}{E[\Gamma] \vdash \langle t| : \mathcal{B}(\sigma)}$$

$$\textbf{Bra-Adj}\quad \frac{E[\Gamma] \vdash K : \mathcal{K}(\sigma)}{E[\Gamma] \vdash K^\dagger : \mathcal{B}(\sigma)} \qquad \textbf{Bra-Scr}\quad \frac{E[\Gamma] \vdash a : \mathcal{S} \qquad E[\Gamma] \vdash B : \mathcal{B}(\sigma)}{E[\Gamma] \vdash a.B : \mathcal{B}(\sigma)}$$

$$\textbf{Bra-Add}\quad \frac{E[\Gamma] \vdash B_i : \mathcal{B}(\sigma) \text{ for all } i}{E[\Gamma] \vdash B_1 + \cdots + B_n : \mathcal{B}(\sigma)} \qquad \textbf{Bra-MulB}\quad \frac{E[\Gamma] \vdash B : \mathcal{K}(\sigma) \qquad E[\Gamma] \vdash O : \mathcal{O}(\sigma,\tau)}{E[\Gamma] \vdash B \cdot O : \mathcal{B}(\tau)}$$

$$\textbf{Bra-Tsr}\quad \frac{E[\Gamma] \vdash B_1 : \mathcal{B}(\sigma) \qquad E[\Gamma] \vdash B_2 : \mathcal{B}(\tau)}{E[\Gamma] \vdash B_1 \otimes B_2 : \mathcal{B}(\sigma \times \tau)}$$

Figure 8: Bra typing rules.

$$\textbf{Opt-0}\quad \frac{E[\Gamma] \vdash \sigma : \mathsf{Index} \qquad E[\Gamma] \vdash \tau : \mathsf{Index}}{E[\Gamma] \vdash \mathbf{0}_{\mathcal{O}}(\sigma, \tau) : \mathcal{O}(\sigma, \tau)} \qquad \textbf{Opt-1}\quad \frac{E[\Gamma] \vdash \sigma : \mathsf{Index}}{E[\Gamma] \vdash \mathbf{1}_{\mathcal{O}}(\sigma) : \mathcal{O}(\sigma, \sigma)}$$

$$\textbf{Opt-Adj}\quad \frac{E[\Gamma] \vdash O : \mathcal{O}(\sigma, \tau)}{E[\Gamma] \vdash O^{\dagger} : \mathcal{O}(\tau, \sigma)} \qquad \textbf{Opt-Scr}\quad \frac{E[\Gamma] \vdash a : \mathcal{S} \qquad E[\Gamma] \vdash O : \mathcal{O}(\sigma, \tau)}{E[\Gamma] \vdash a.O : \mathcal{O}(\sigma, \tau)}$$

$$\textbf{Opt-Add}\quad \frac{E[\Gamma] \vdash O_i : \mathcal{O}(\sigma, \tau) \text{ for all } i}{E[\Gamma] \vdash O_1 + \cdots + O_n : \mathcal{O}(\sigma, \tau)} \qquad \textbf{Opt-Outer}\quad \frac{E[\Gamma] \vdash K : \mathcal{K}(\sigma) \qquad E[\Gamma] \vdash B : \mathcal{B}(\tau)}{E[\Gamma] \vdash K \cdot B : \mathcal{O}(\sigma, \tau)}$$

$$\textbf{Opt-MulO}\quad \frac{E[\Gamma] \vdash O_1 : \mathcal{O}(\sigma, \tau) \qquad E[\Gamma] \vdash O_2 : \mathcal{O}(\tau, \rho)}{E[\Gamma] \vdash O_1 \cdot O_2 : \mathcal{O}(\sigma, \rho)}$$

$$\textbf{Opt-Tsr}\quad \frac{E[\Gamma] \vdash O_1 : \mathcal{O}(\sigma_1, \tau_1) \qquad E[\Gamma] \vdash O_2 : \mathcal{O}(\sigma_2, \tau_2)}{E[\Gamma] \vdash O_1 \otimes O_2 : \mathcal{O}(\sigma_1 \times \sigma_2, \tau_1 \times \tau_2)}$$

Figure 9: Operator typing rules.

$$\textbf{Set-U}\quad \frac{E[\Gamma] \vdash \sigma : \mathsf{Index}}{E[\Gamma] \vdash \mathbf{U}(\sigma) : \mathsf{Set}(\sigma)} \qquad \textbf{Set-Prod}\quad \frac{E[\Gamma] \vdash A : \mathsf{Set}(\sigma) \qquad E[\Gamma] \vdash B : \mathsf{Set}(\tau)}{E[\Gamma] \vdash A \star B : \mathsf{Set}(\sigma \times \tau)}$$

Figure 10: Set typing rules.

$$\textbf{Sum-Scalar}\quad \frac{E[\Gamma] \vdash s : \mathsf{Set}(\sigma) \qquad E[\Gamma] \vdash f : \mathsf{Basis}(\sigma) \to \mathcal{S}}{E[\Gamma] \vdash \sum_s f : \mathcal{S}}$$

$$\textbf{Sum-Ket}\quad \frac{E[\Gamma] \vdash s : \mathsf{Set}(\sigma) \qquad E[\Gamma] \vdash f : \mathsf{Basis}(\sigma) \to \mathcal{K}(\tau)}{E[\Gamma] \vdash \sum_s f : \mathcal{K}(\tau)}$$

$$\textbf{Sum-Bra}\quad \frac{E[\Gamma] \vdash s : \mathsf{Set}(\sigma) \qquad E[\Gamma] \vdash f : \mathsf{Basis}(\sigma) \to \mathcal{B}(\tau)}{E[\Gamma] \vdash \sum_s f : \mathcal{B}(\tau)}$$

$$\textbf{Sum-Opt}\quad \frac{E[\Gamma] \vdash s : \mathsf{Set}(\sigma) \qquad E[\Gamma] \vdash f : \mathsf{Basis}(\sigma) \to \mathcal{O}(\tau, \rho)}{E[\Gamma] \vdash \sum_s f : \mathcal{O}(\tau, \rho)}$$

Figure 11: Sum typing rules.

5 Conversions and Reductions

$$\frac{E[\Gamma] \vdash K : \mathcal{K}(\sigma)}{E[\Gamma] \vdash K \ \triangleright\ \sum_{i\in\mathbf{U}(\sigma)}(\langle i| \cdot K).\,|i\rangle}$$

$$\frac{E[\Gamma] \vdash B : \mathcal{B}(\sigma)}{E[\Gamma] \vdash B \ \triangleright\ \sum_{i\in\mathbf{U}(\sigma)}(B \cdot |i\rangle).\,\langle i|}$$

$$\frac{E[\Gamma] \vdash O : \mathcal{O}(\sigma,\tau)}{E[\Gamma] \vdash O \ \triangleright\ \sum_{i\in\mathbf{U}(\sigma)}\sum_{j\in\mathbf{U}(\tau)}(\langle i| \cdot O \cdot |j\rangle).(|i\rangle \cdot \langle j|)}$$

Beta-Arrow $\dfrac{}{E[\Gamma] \vdash ((\lambda x : T.t)\ u) \ \triangleright\ t\{x/u\}}$ **Beta-Index** $\dfrac{}{E[\Gamma] \vdash ((\mu x.t)\ u) \ \triangleright\ t\{x/u\}}$

Delta $\dfrac{\mathcal{WF}(E)[\Gamma] \qquad (c := t : T) \in E}{E[\Gamma] \vdash c \ \triangleright\ t}$

Eta-Arrow $\dfrac{}{E[\Gamma] \vdash \lambda x : T.(t\ x) \ \triangleright\ t}$ **Eta-Index** $\dfrac{}{E[\Gamma] \vdash \mu x.(t\ x) \ \triangleright\ t}$

Figure 12: Conversions and reductions for the typed lambda calculus.

Table 1: The special flattening rule.

Rule	**Description**
R-FLATTEN	$a_1 + \cdots + (b_1 + \cdots + b_m) + \cdots + a_n \ \triangleright\ a_1 + \cdots + b_1 + \cdots + b_m + \cdots + a_n$
	$a_1 \times \cdots \times (b_1 \times \cdots \times b_m) \times \cdots \times a_n \ \triangleright\ a_1 \times \cdots \times b_1 \times \cdots \times b_m \times \cdots \times a_n$
	$X_1 + \cdots + (X'_1 + \cdots + X'_m) + \cdots + X_n \ \triangleright\ X_1 + \cdots + X'_1 + \cdots + X'_m + \cdots + X_n$
	A special rule to flatten all AC symbols within one call.

Table 2: Rules for scalar addition and multiplication.

Rule	**Description**
R-ADDSID	$+(a) \ \triangleright\ a$
	Reduce the term if the AC symbol + only has one argument.
R-MULSID	$\times(a) \ \triangleright\ a$
	Similar to (R-ADDSID).
R-ADDS0	$a_1 + \cdots + 0 + \cdots + a_n \ \triangleright\ a_1 + \cdots + a_n$
	This rule removes all 0 occurances and keeps the order of remaining subterms.
R-MULS0	$a_1 \times \cdots \times 0 \times \cdots \times a_n \ \triangleright\ 0$
R-MULS1	$a_1 \times \cdots \times 1 \times \cdots \times a_n \ \triangleright\ a_1 \times \cdots \times a_n$
	Similar to (R-ADDS0).
	Continued on the next page

Rule	Description
R-MULS2	$b_1 \times \cdots \times (a_1 + \cdots + a_n) \times \cdots \times b_m$ $\triangleright (b_1 \times \cdots \times a_1 \times \cdots \times b_m) + \cdots + (b_1 \times \cdots \times a_n \times \cdots \times b_m)$ This rule matches the first scalar addition subterm in the list.

Table 3: Rules for other scalar symbols.

Rule	Description
R-CONJ0	$0^* \triangleright 0$
R-CONJ1	$1^* \triangleright 1$
R-CONJ2	$(a_1 + \cdots + a_n)^* \triangleright a_1^* + \cdots + a_n^*$
R-CONJ3	$(a_1 \times \cdots \times a_n)^* \triangleright a_1^* \times \cdots \times a_n^*$
R-CONJ4	$(a^*)^* \triangleright a$
R-CONJ5	$\delta_{s,t}^* \triangleright \delta_{s,t}$
R-CONJ6	$(B \cdot K)^* \triangleright K^\dagger \cdot B^\dagger$
R-DOT0	$\mathbf{0}_\mathcal{B}(\sigma) \cdot K \triangleright 0$
R-DOT1	$B \cdot \mathbf{0}_\mathcal{K}(\sigma) \triangleright 0$
R-DOT2	$(a.B) \cdot K \triangleright a \times (B \cdot K)$
R-DOT3	$B \cdot (a.K) \triangleright a \times (B \cdot K)$
R-DOT4	$(B_1 + \cdots + B_n) \cdot K \triangleright B_1 \cdot K + \cdots + B_n \cdot K$
R-DOT5	$B \cdot (K_1 + \cdots + K_n) \triangleright B \cdot K_1 + \cdots + B \cdot K_n$
R-DOT6	$\langle s\| \cdot \|t\rangle \triangleright \delta_{s,t}$
R-DOT7	$(B_1 \otimes B_2) \cdot \|(s,t)\rangle \triangleright (B_1 \cdot \|s\rangle) \times (B_2 \cdot \|t\rangle)$
R-DOT8	$\langle (s,t)\| \cdot (K_1 \otimes K_2) \triangleright (\langle s\| \cdot K_1) \times (\langle t\| \cdot K_2)$
R-DOT9	$(B_1 \otimes B_2) \cdot (K_1 \otimes K_2) \triangleright (B_1 \cdot K_1) \times (B_2 \cdot K_2)$
R-DOT10	$(B \cdot O) \cdot K \triangleright B \cdot (O \cdot K)$
R-DOT11	$\langle (s,t)\| \cdot ((O_1 \otimes O_2) \cdot K) \triangleright ((\langle s\| \cdot O_1) \otimes (\langle t\| \cdot O_2)) \cdot K$
R-DOT12	$(B_1 \otimes B_2) \cdot ((O_1 \otimes O_2) \cdot K) \triangleright ((B_1 \cdot O_1) \otimes (B_2 \cdot O_2)) \cdot K$
R-DELTA0	$\delta_{a,a} \triangleright 1$
R-DELTA1	$\delta_{(a,b),(c,d)} \triangleright \delta_{a,c} \times \delta_{b,d}$

Table 4: Rules for linear algebra scaling.

Rule	Description
R-SCR0	$1.X \triangleright X$
R-SCR1	$a.(b.X) \triangleright (a \times b).X$
R-SCR2	$a.(X_1 + \cdots + X_n) \triangleright a.X_1 + \cdots + a.X_n$
R-SCRK0	$K : \mathcal{K}(\sigma) \Rightarrow 0.K \triangleright \mathbf{0}_\mathcal{K}(\sigma)$
R-SCRK1	$a.\mathbf{0}_\mathcal{K}(\sigma) \triangleright \mathbf{0}_\mathcal{K}(\sigma)$
R-SCRB0	$B : \mathcal{B}(\sigma) \Rightarrow 0.B \triangleright \mathbf{0}_\mathcal{B}(\sigma)$
R-SCRB1	$a.\mathbf{0}_\mathcal{B}(\sigma) \triangleright \mathbf{0}_\mathcal{B}(\sigma)$
R-SCRO0	$O : \mathcal{O}(\sigma, \tau) \Rightarrow 0.O \triangleright \mathbf{0}_\mathcal{O}(\sigma, \tau)$
R-SCRO1	$a.\mathbf{0}_\mathcal{O}(\sigma, \tau) \triangleright \mathbf{0}_\mathcal{O}(\sigma, \tau)$

Table 5: Rules for linear algebra addition.

Rule	Description
R-ADDID	$+(X) \ \triangleright\ X$ Reduce the term if the AC symbol + only has one argument.
R-ADD0	$Y_1 + \cdots + X + \cdots + X + \cdots + Y_n \ \triangleright\ Y_1 + \cdots + Y_n + \cdots + (1+1).X$ This rule matches the first X in the list satisfying the pattern. The result $(1+1).X$ will be placed at the end of the list.
R-ADD1	$Y_1 + \cdots + X + \cdots + a.X + \cdots + Y_n \ \triangleright\ Y_1 + \cdots + Y_n + (1+a).X$ Similar to (R-ADD0).
R-ADD2	$Y_1 + \cdots + a.X + \cdots + X + \cdots + Y_n \ \triangleright\ Y_1 + \cdots + Y_n + (a+1).X$ Similar to (R-ADD0).
R-ADD3	$Y_1 + \cdots + a.X + \cdots + b.X + \cdots + Y_n \ \triangleright\ Y_1 + \cdots + Y_n + (a+b).X$ Similar to (R-ADD0).
R-ADDK0	$K_1 + \cdots + \mathbf{0}_{\mathcal{K}}(\sigma) + \cdots + K_n \ \triangleright K_1 + \cdots + K_n$ Similar to (R-ADDS0).
R-ADDB0	$B_1 + \cdots + \mathbf{0}_{\mathcal{B}}(\sigma) + \cdots + B_n \ \triangleright\ B_1 + \cdots + B_n$ Similar to (R-ADDS0).
R-ADDO0	$O_1 + \cdots + \mathbf{0}_{\mathcal{O}}(\sigma, \tau) + \cdots + O_n \ \triangleright\ O_1 + \cdots + O_n$ Similar to (R-ADDS0).

Table 6: Rules for adjoint.

Rule	Description
R-ADJ0	$(X^\dagger)^\dagger \ \triangleright\ X$
R-ADJ1	$(a.X)^\dagger \ \triangleright\ (a^*).(X^\dagger)$
R-ADJ2	$(X_1 + \cdots + X_n)^\dagger \ \triangleright\ X_1^\dagger + \cdots + X_n^\dagger$
R-ADJ3	$(X \otimes Y)^\dagger \ \triangleright\ X^\dagger \otimes Y^\dagger$
R-ADJK0	$\mathbf{0}_{\mathcal{B}}(\sigma)^\dagger \ \triangleright\ \mathbf{0}_{\mathcal{K}}(\sigma)$
R-ADJK1	$\langle t\|^\dagger \ \triangleright\ \|t\rangle$
R-ADJK2	$(B \cdot O)^\dagger \ \triangleright\ O^\dagger \cdot B^\dagger$
R-ADJB0	$\mathbf{0}_{\mathcal{K}}(\sigma)^\dagger \ \triangleright\ \mathbf{0}_{\mathcal{B}}(\sigma)$
R-ADJB1	$\|t\rangle^\dagger \ \triangleright\ \langle t\|$
R-ADJB2	$(O \cdot K)^\dagger \ \triangleright\ K^\dagger \cdot O^\dagger$
R-ADJO0	$\mathbf{0}_{\mathcal{O}}(\sigma, \tau)^\dagger \ \triangleright\ \mathbf{0}_{\mathcal{O}}(\tau, \sigma)$
R-ADJO1	$\mathbf{1}_{\mathcal{O}}(\sigma)^\dagger \ \triangleright\ \mathbf{1}_{\mathcal{O}}(\sigma)$
R-ADJO2	$(K \cdot B)^\dagger \ \triangleright\ B^\dagger \cdot K^\dagger$
R-ADJO3	$(O_1 \cdot O_2)^\dagger \ \triangleright\ O_2^\dagger \cdot O_1^\dagger$

Table 7: Rules for tensor product.

Rule	Description
R-TSR0	$(a.X_1) \otimes X_2 \ \triangleright\ a.(X_1 \otimes X_2)$
R-TSR1	$X_1 \otimes (a.X_2) \ \triangleright\ a.(X_1 \otimes X_2)$
R-TSR2	$(X_1 + \cdots + X_n) \otimes X' \triangleright X_1 \otimes X' + \cdots + X_n \otimes X'$

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Rule	Description
R-TSR3	$X' \otimes (X_1 + \cdots + X_n) \triangleright X' \otimes X_1 + \cdots + X' \otimes X_n$
R-TSRK0	$K : \mathcal{K}(\tau) \Rightarrow \mathbf{0}_{\mathcal{K}}(\sigma) \otimes K \ \triangleright\ \mathbf{0}_{\mathcal{K}}(\sigma \times \tau)$
R-TSRK1	$K : \mathcal{K}(\tau) \Rightarrow K \otimes \mathbf{0}_{\mathcal{K}}(\sigma) \ \triangleright\ \mathbf{0}_{\mathcal{K}}(\tau \times \sigma)$
R-TSRK2	$\|s\rangle \otimes \|t\rangle \ \triangleright\ \|(s,t)\rangle$
R-TSRB0	$B : \mathcal{B}(\tau) \Rightarrow \mathbf{0}_{\mathcal{B}}(\sigma) \otimes B \ \triangleright\ \mathbf{0}_{\mathcal{B}}(\sigma \times \tau)$
R-TSRB1	$B : \mathcal{B}(\tau) \Rightarrow B \otimes \mathbf{0}_{\mathcal{B}}(\sigma) \ \triangleright\ \mathbf{0}_{\mathcal{B}}(\tau \times \sigma)$
R-TSRB2	$\langle s\| \otimes \langle t\| \ \triangleright\ \langle (s,t)\|$
R-TSRO0	$O : \mathcal{O}(\sigma,\tau) \Rightarrow O \otimes \mathbf{0}_{\mathcal{O}}(\sigma',\tau') \ \triangleright\ \mathbf{0}_{\mathcal{O}}(\sigma \times \sigma', \tau \times \tau')$
R-TSRO1	$O : \mathcal{O}(\sigma,\tau) \Rightarrow \mathbf{0}_{\mathcal{O}}(\sigma',\tau') \otimes O \ \triangleright\ \mathbf{0}_{\mathcal{O}}(\sigma' \times \sigma, \tau' \times \tau)$
R-TSRO2	$\mathbf{1}_{\mathcal{O}}(\sigma) \otimes \mathbf{1}_{\mathcal{O}}(\tau) \ \triangleright\ \mathbf{1}_{\mathcal{O}}(\sigma \times \tau)$
R-TSRO3	$(K_1 \cdot B_1) \otimes (K_2 \cdot B_2) \ \triangleright\ (K_1 \otimes K_2) \cdot (B_1 \otimes B_2)$

Table 8: Rule for $O \cdot K$.

Rule	Description
R-MULK0	$\mathbf{0}_{\mathcal{O}}(\sigma,\tau) \cdot K \ \triangleright\ \mathbf{0}_{\mathcal{K}}(\sigma)$
R-MULK1	$O : \mathcal{O}(\sigma,\tau) \Rightarrow O \cdot \mathbf{0}_{\mathcal{K}}(\tau) \ \triangleright\ \mathbf{0}_{\mathcal{K}}(\sigma)$
R-MULK2	$\mathbf{1}_{\mathcal{O}}(\sigma) \cdot K \ \triangleright K$
R-MULK3	$(a.O) \cdot K \ \triangleright\ a.(O \cdot K)$
R-MULK4	$O \cdot (a.K) \ \triangleright\ a.(O \cdot K)$
R-MULK5	$(O_1 + \cdots + O_n) \cdot K \ \triangleright\ O_1 \cdot K + \cdots + O_n \cdot K$
R-MULK6	$O \cdot (K_1 + \cdots + K_n) \ \triangleright\ O \cdot K_1 + \cdots + O \cdot K_n$
R-MULK7	$(K_1 \cdot B) \cdot K_2 \ \triangleright\ (B \cdot K_2).K_1$
R-MULK8	$(O_1 \cdot O_2) \cdot K \ \triangleright\ O_1 \cdot (O_2 \cdot K)$
R-MULK9	$(O_1 \otimes O_2) \cdot ((O_1' \otimes O_2') \cdot K) \ \triangleright\ ((O_1 \cdot O_1') \otimes (O_2 \cdot O_2')) \cdot K$
R-MULK10	$(O_1 \otimes O_2) \cdot \|(s,t)\rangle \ \triangleright\ (O_1 \cdot \|s\rangle) \otimes (O_2 \cdot \|t\rangle)$
R-MULK11	$(O_1 \otimes O_2) \cdot (K_1 \otimes K_2) \ \triangleright\ (O_1 \cdot K_1) \otimes (O_2 \cdot K_2)$

Table 9: Rule for $B \cdot O$.

Rule	Description
R-MULB0	$B \cdot \mathbf{0}_{\mathcal{O}}(\sigma,\tau) \ \triangleright\ \mathbf{0}_{\mathcal{B}}(\tau)$
R-MULB1	$O : \mathcal{O}(\sigma,\tau) \Rightarrow \mathbf{0}_{\mathcal{B}}(\sigma) \cdot O \ \triangleright\ \mathbf{0}_{\mathcal{B}}(\tau)$
R-MULB2	$B \cdot \mathbf{1}_{\mathcal{O}}(\sigma) \ \triangleright B$
R-MULB3	$(a.B) \cdot O \ \triangleright\ a.(B \cdot O)$
R-MULB4	$B \cdot (a.O) \ \triangleright\ a.(B \cdot O)$
R-MULB5	$(B_1 + \cdots + B_n) \cdot O \ \triangleright\ B_1 \cdot O + \cdots + B_n \cdot O$
R-MULB6	$B \cdot (O_1 + \cdots + O_n) \ \triangleright\ B \cdot O_1 + \cdots + B \cdot O_n$
R-MULB7	$B_1 \cdot (K \cdot B_2) \ \triangleright\ (B_1 \cdot K).B_2$
R-MULB8	$B \cdot (O_1 \cdot O_2) \ \triangleright\ (B \cdot O_1) \cdot O_2$
R-MULB9	$(B \cdot (O_1' \otimes O_2')) \cdot (O_1 \otimes O_2) \ \triangleright\ B \cdot ((O_1' \otimes O_2') \cdot (O_1 \otimes O_2))$
R-MULB10	$\langle (s,t)\| \cdot (O_1 \otimes O_2) \ \triangleright\ (\langle s\| \cdot O_1) \otimes (\langle t\| \cdot O_2)$
R-MULB11	$(B_1 \otimes B_2) \cdot (O_1 \otimes O_2) \ \triangleright\ (B_1 \cdot O_1) \otimes (B_2 \cdot O_2)$

Table 10: Rules for $K \cdot B$.

Rule	Description
R-OUTER0	$B : \mathcal{B}(\tau) \Rightarrow \mathbf{0}_{\mathcal{K}}(\sigma) \cdot B \ \triangleright\ \mathbf{0}_{\mathcal{O}}(\sigma, \tau)$
R-OUTER1	$K : \mathcal{K}(\sigma) \Rightarrow K \cdot \mathbf{0}_{\mathcal{B}}(\tau) \ \triangleright\ \mathbf{0}_{\mathcal{O}}(\sigma, \tau)$
R-OUTER2	$(a.K) \cdot B \ \triangleright\ a.(K \cdot B)$
R-OUTER3	$K \cdot (a.B) \ \triangleright\ a.(K \cdot B)$
R-OUTER4	$(K_1 + \cdots + K_n) \cdot B \ \triangleright\ K_1 \cdot B + \cdots + K_n \cdot B$
R-OUTER5	$K \cdot (B_1 + \cdots + B_n) \ \triangleright\ K \cdot B_1 + \cdots + K \cdot B_n$

Table 11: Rules for $O_1 \cdot O_2$.

Rule	Description
R-MULO0	$O : \mathcal{O}(\tau, \rho) \Rightarrow \mathbf{0}_{\mathcal{O}}(\sigma, \tau) \cdot O \ \triangleright\ \mathbf{0}_{\mathcal{O}}(\sigma, \rho)$
R-MULO1	$O : \mathcal{O}(\sigma, \tau) \Rightarrow O \cdot \mathbf{0}_{\mathcal{O}}(\tau, \rho) \ \triangleright\ \mathbf{0}_{\mathcal{O}}(\sigma, \rho)$
R-MULO2	$\mathbf{1}_{\mathcal{O}}(\sigma) \cdot O \ \triangleright\ O$
R-MULO3	$O \cdot \mathbf{1}_{\mathcal{O}}(\sigma) \ \triangleright\ O$
R-MULO4	$(K \cdot B) \cdot O \ \triangleright\ K \cdot (B \cdot O)$
R-MULO5	$O \cdot (K \cdot B) \ \triangleright\ (O \cdot K) \cdot B$
R-MULO6	$(a.O_1) \cdot O_2 \ \triangleright\ a.(O_1 \cdot O_2)$
R-MULO7	$O_1 \cdot (a.O_2) \ \triangleright\ a.(O_1 \cdot O_2)$
R-MULO8	$(O_1 + \cdots + O_n) \cdot O' \ \triangleright\ O_1 \cdot O' + \cdots + O_n \cdot O'$
R-MULO9	$O' \cdot (O_1 + \cdots + O_n) \ \triangleright\ O' \cdot O_1 + \cdots + O' \cdot O_n$
R-MULO10	$(O_1 \cdot O_2) \cdot O_3 \ \triangleright\ O_1 \cdot (O_2 \cdot O_3)$
R-MULO11	$(O_1 \otimes O_2) \cdot (O_1' \otimes O_2') \ \triangleright\ (O_1 \cdot O_1') \otimes (O_2 \cdot O_2')$
R-MULO12	$(O_1 \otimes O_2) \cdot ((O_1' \otimes O_2') \cdot O_3) \ \triangleright\ ((O_1 \cdot O_1') \otimes (O_2 \cdot O_2')) \cdot O_3$

Table 12: Rules for sets.

Rule	Description
R-SET0	$\mathbf{U}(\sigma) \star \mathbf{U}(\tau) \ \triangleright\ \mathbf{U}(\sigma \times \tau)$

Table 13: Rules for sum operators.

Rule	Description
R-SUM-CONST0	$\sum_{x \in s} 0 \ \triangleright\ 0$
R-SUM-CONST1	$\sum_{x \in s} \mathbf{0}_{\mathcal{K}}(\sigma) \ \triangleright\ \mathbf{0}_{\mathcal{K}}(\sigma)$
R-SUM-CONST2	$\sum_{x \in s} \mathbf{0}_{\mathcal{B}}(\sigma) \ \triangleright\ \mathbf{0}_{\mathcal{B}}(\sigma)$
R-SUM-CONST3	$\sum_{x \in s} \mathbf{0}_{\mathcal{O}}(\sigma, \tau) \ \triangleright\ \mathbf{0}_{\mathcal{O}}(\sigma, \tau)$
R-SUM-CONST4	$\mathbf{1}_{\mathcal{O}}(\sigma) \ \triangleright\ \sum_{i \in \mathbf{U}(\sigma)} \lvert i \rangle \cdot \langle i \rvert$

Table 14: Rules for eliminating $\delta_{s,t}$. Note that these rules will match the δ operator modulo the commutativity of its arguments.

Rule	Description
R-SUM-ELIM0	i free in $t \vdash \sum_{i\in\mathbf{U}(\sigma)} \sum_{k_1\in s_1} \cdots \sum_{k_n\in s_n} \delta_{i,t} \ \triangleright\ \sum_{k_1\in s_1} \cdots \sum_{k_n\in s_n} 1$
R-SUM-ELIM1	i free in $t \vdash \sum_{i\in\mathbf{U}(\sigma)} \sum_{k_1\in s_1} \cdots \sum_{k_n\in s_n} (a_1 \times \cdots \times \delta_{i,t} \times \cdots \times a_n)$ $\triangleright\ \sum_{k_1\in s_1} \cdots \sum_{k_n\in s_n} a_1\{i/t\} \times \cdots \times a_n\{i/t\}$
R-SUM-ELIM2	i free in $t \vdash \sum_{i\in\mathbf{U}(\sigma)} \sum_{k_1\in s_1} \cdots \sum_{k_n\in s_n} (\delta_{i,t}.A) \ \triangleright\ \sum_{k_1\in s_1} \cdots \sum_{k_n\in s_n} A\{i/t\}$
R-SUM-ELIM3	i free in $t \vdash \sum_{i\in\mathbf{U}(\sigma)} \sum_{k_1\in s_1} \cdots \sum_{k_n\in s_n} (a_1 \times \cdots \times \delta_{i,t} \times \cdots \times a_n).A$ $\triangleright\ \sum_{k_1\in s_1} \cdots \sum_{k_n\in s_n} (a_1\{i/t\} \times \cdots \times a_n\{i/t\}).A\{i/t\}$
R-SUM-ELIM4	$\sum_{i\in M} \sum_{j\in M} \sum_{k_1\in s_1} \cdots \sum_{k_n\in s_n} \delta_{i,j} \ \triangleright\ \sum_{j\in M} \sum_{k_1\in s_1} \cdots \sum_{k_n\in s_n} 1$
R-SUM-ELIM5	$\sum_{i\in M} \sum_{j\in M} \sum_{k_1\in s_1} \cdots \sum_{k_n\in s_n} (a_1 \times \cdots \times \delta_{i,j} \times \cdots \times a_n)$ $\triangleright\ \sum_{j\in M} \sum_{k_1\in s_1} \cdots \sum_{k_n\in s_n} (a_1\{j/i\} \times \cdots \times a_n\{j/i\})$
R-SUM-ELIM6	$\sum_{i\in M} \sum_{j\in M} \sum_{k_1\in s_1} \cdots \sum_{k_n\in s_n} (\delta_{i,j}.A) \ \triangleright\ \sum_{j\in M} \sum_{k_1\in s_1} \cdots \sum_{k_n\in s_n} A\{j/i\}$
R-SUM-ELIM7	$\sum_{i\in M} \sum_{j\in M} \sum_{k_1\in s_1} \cdots \sum_{k_n\in s_n} (a_1 \times \cdots \times \delta_{i,j} \times \cdots \times a_n).A$ $\triangleright\ \sum_{j\in M} \sum_{k_1\in s_1} \cdots \sum_{k_n\in s_n} (a_1\{j/i\} \times \cdots \times a_n\{j/i\}).A\{j/i\}$

Table 15: Rules for pushing terms into sum operators

Rule	Description
R-SUM-PUSH0	$b_1 \times \cdots \times (\sum_{i\in M} a) \times \cdots \times b_n \ \triangleright\ \sum_{i\in M}(b_1 \times \cdots \times a \times \cdots \times b_n)$
R-SUM-PUSH1	$(\sum_{i\in M} a)^* \ \triangleright\ \sum_{i\in M} a^*$
R-SUM-PUSH2	$(\sum_{i\in M} X)^\dagger \ \triangleright\ \sum_{i\in M} X^\dagger$
R-SUM-PUSH3	$a.(\sum_{i\in M} X) \ \triangleright\ \sum_{i\in M}(a.X)$
R-SUM-PUSH4	$(\sum_{i\in M} a).X \ \triangleright\ \sum_{i\in M}(a.X)$
R-SUM-PUSH5	$(\sum_{i\in M} B) \cdot K \ \triangleright\ \sum_{i\in M}(B \cdot K)$
R-SUM-PUSH6	$(\sum_{i\in M} O) \cdot K \ \triangleright\ \sum_{i\in M}(O \cdot K)$
R-SUM-PUSH7	$(\sum_{i\in M} B) \cdot O \ \triangleright\ \sum_{i\in M}(B \cdot O)$
R-SUM-PUSH8	$(\sum_{i\in M} K) \cdot B \ \triangleright\ \sum_{i\in M}(K \cdot B)$
R-SUM-PUSH9	$(\sum_{i\in M} O_1) \cdot O_2 \ \triangleright\ \sum_{i\in M}(O_1 \cdot O_2)$
R-SUM-PUSH10	$B \cdot (\sum_{i\in M} K) \ \triangleright\ \sum_{i\in M}(B \cdot K)$
R-SUM-PUSH11	$O \cdot (\sum_{i\in M} K) \ \triangleright\ \sum_{i\in M}(O \cdot K)$
R-SUM-PUSH12	$B \cdot (\sum_{i\in M} O) \ \triangleright\ \sum_{i\in M}(B \cdot O)$
R-SUM-PUSH13	$K \cdot (\sum_{i\in M} B) \ \triangleright\ \sum_{i\in M}(K \cdot B)$
R-SUM-PUSH14	$O_1 \cdot (\sum_{i\in M} O_2) \ \triangleright\ \sum_{i\in M}(O_1 \cdot O_2)$
R-SUM-PUSH15	$(\sum_{i\in M} X_1) \otimes X_2 \ \triangleright\ \sum_{i\in M}(X_1 \otimes X_2)$
R-SUM-PUSH16	$X_1 \otimes (\sum_{i\in M} X_2) \ \triangleright\ \sum_{i\in M}(X_1 \otimes X_2)$

[YX] : Note: because we apply type checking on variables, and stick to unique bound variables, these pushing in operations are always sound.

Table 16: Rules for addition and index in sum

Rule	Description
R-SUM-ADDS0	$\sum_{i\in M}(a_1 + \cdots + a_n) \ \triangleright\ (\sum_{i\in M} a_1) + \cdots + (\sum_{i\in M} a_n)$
R-SUM-ADD0	$\sum_{i\in M}(X_1 + \cdots + X_n) \ \triangleright\ (\sum_{i\in M} X_1) + \cdots + (\sum_{i\in M} X_n)$

Continued on the next page

Rule	Description
R-SUM-ADD1	$Y_1 + \cdots + \sum_{i \in M}(a.X) + \cdots + \sum_{i \in M} X + Y_n \ \triangleright\ Y_1 + \cdots + Y_n + \sum_{i \in M}(a+1).X$
R-SUM-ADD2	$Y_1 + \cdots + \sum_{i \in M} X + \cdots + \sum_{i \in M}(a.X) + Y_n \ \triangleright\ Y_1 + \cdots + Y_n + \sum_{i \in M}(1+a).X$
R-SUM-ADD3	$Y_1 + \cdots + \sum_{i \in M}(a.X) + \cdots + \sum_{i \in M}(b.X) + Y_n \ \triangleright\ Y_1 + \cdots + Y_n + \sum_{i \in M}(a+b).X$
R-SUM-INDEX0	$\sum_{i \in \mathbf{U}(\sigma \times \tau)} A \ \triangleright\ \sum_{j \in \mathbf{U}(\sigma)} \sum_{k \in \mathbf{U}(\tau)} A\{i/(j,k)\}$
R-SUM-INDEX1	$\sum_{i \in M_1 \star M_2} A \ \triangleright\ \sum_{j \in M_1} \sum_{k \in M_2} A\{i/(j,k)\}$
R-SUM-SWAP	$M_1 < M_2 \vdash \sum_{i \in M_2} \sum_{j \in M_1} X \ \triangleright\ \sum_{j \in M_1} \sum_{i \in M_2} X$

[YX] : The rules R-SUM-ADD1 to R-SUM-ADD3 needs identical sum terms, which requires the rewriting to be after the alpha normalization. We don't implement them for now.

6 `Diracoq` language

$$
\begin{aligned}
\mathit{cmd} ::= {} & \texttt{Def}(\texttt{ID}\ \mathit{term}) \\
& \mid \texttt{Def}(\texttt{ID}\ \mathit{term}\ \mathit{type}) \\
& \mid \texttt{Var}(\texttt{ID}\ \mathit{term}) \\
& \mid \texttt{Check}(\mathit{term}) \\
& \mid \texttt{Show}(\texttt{ID}) \\
& \mid \texttt{ShowAll} \\
& \mid \texttt{Normalize}(\mathit{term}) \mid \texttt{Normalize}(\mathit{term}\ \mathsf{Trace}) \\
& \mid \texttt{CheckEq}(\mathit{term}\ \mathit{term}) \\
\mathit{type} ::= {} & \texttt{Type} \mid \texttt{Arrow}(\mathit{type}\ \mathit{type}) \\
& \mid \texttt{Base} \\
\mathit{term} ::= {} & \texttt{Type} \mid \texttt{fun}(\texttt{ID}\ \mathit{type}\ \mathit{term}) \mid \texttt{apply}(\mathit{term}\ \mathit{term}) \mid \texttt{ID}
\end{aligned}
$$

7 Rewriting Control and Intermediate Language

The associativity is already handled by the (R-FLATTEN) rule. In order to decide two terms A and B are equivalent under commutativity, we need to proof that A can be transformed into B with a structured permutation, which is described by the *permutation tree*.

Definition 7.1 (permutation tree)**.** *The syntax for permutation trees are inductively defined below:*

$$P ::= \mathsf{E} \mid [(i : P)^+].$$

Here i represents positive numbers.

We always only consider *well-formed* permutation trees. That is, if $P \equiv [i_1 : P_1\ i_2 : P_2\ \cdots\ i_n : P_n]$, then $\{i_1, ...i_n\}$ forms the set of integers from 0 to $n-1$.

We can transform a term A with a suitable permutation tree. The transformation is defined as

$$
\begin{aligned}
\texttt{apply}(A, P) := {} & \texttt{match}\ P\ \texttt{with} \\
& \quad \mid \mathsf{E} \Rightarrow A \\
& \quad \mid [i_1 : P_1\ \cdots\ i_n : P_n] \Rightarrow \texttt{A.head(apply(A.args[}i_1\texttt{], }P_{i_1}\texttt{)}\ \cdots\ \texttt{apply(A.args[}i_n\texttt{], }P_{i_n}\texttt{))} \\
& \texttt{end}
\end{aligned}
$$

8 Labelled Dirac Notation

Definition 8.1 (quantum registers)**.**

$$R ::= x \mid (R, R) \mid \mathsf{fst}\ R \mid \mathsf{snd}\ R$$

We define the following relations for quantum registers:

- *R **belongs to** Q, written as R in Q,*
- *R **is disjoint with** Q, written as $R \| Q$.*

Remark: We still have a speical algorithm deciding the relations.

Definition 8.2 (register set)**.**

$$S ::= \emptyset \mid \{R\} \mid S \cup S \mid S \setminus S$$

Remark: $S_1 \cap S_2 \equiv S_1 \cup S_2 \setminus (S_1 \setminus S_2) \setminus (S_2 \setminus S_1)$

REG

$$\mathsf{fst}\ (R_1, R_2) \rhd R_1 \qquad \mathsf{snd}\ (R_1, R_2) \rhd R_2 \qquad (\mathsf{fst}\ R, \mathsf{snd}\ R) \rhd R$$

RSET

$$S \cup \emptyset \rhd S \qquad S \cup S \rhd S \qquad \{\mathsf{fst}\ R\} \cup \{\mathsf{snd}\ R\} \rhd R$$
$$S \setminus \emptyset \rhd S \qquad \emptyset \setminus S \rhd \emptyset \qquad S \setminus S \rhd \emptyset$$
$$(S_1 \cup S_2) \setminus X \rhd (S_1 \setminus X) \cup (S_2 \setminus X) \qquad S_1 \setminus (S_2 \cup S_3) \rhd (S_1 \setminus S_2) \setminus S_3$$

$$\frac{R_1 \text{ in } R_2}{\{R_1\} \cup \{R_2\} \rhd \{R_2\}} \qquad \frac{R_1 \text{ in } R_2}{\{R_1\} \setminus \{R_2\} \rhd \emptyset}$$

$$\frac{R_1 \text{ in } R_2}{\{R_2\} \setminus \{R_1\} \rhd (\{\mathsf{fst}\ R_2\} \setminus \{R_1\}) \cup (\{\mathsf{snd}\ R_2\} \setminus \{R_1\})} \qquad \frac{R_1 \| R_2}{\{R_1\} \setminus \{R_2\} \rhd \{R_1\}}$$

Definition 8.3 (labelled core language)**.** *The **labelled core langauge** includes all symbols in the core language of Dirac notation, as well as symbols for the three new sorts.*

$$
\begin{array}{ll}
 & S ::= \ \tilde{B} \cdot \tilde{K} \\
\tilde{\mathcal{K}} \textit{(labelled ket)} & \tilde{K} ::= \ K_R \mid \tilde{B}^\dagger \mid S.\tilde{K} \mid \tilde{K} + \tilde{K} \mid \tilde{O} \cdot \tilde{K} \mid \tilde{K} \otimes \tilde{K} \\
\tilde{\mathcal{B}} \textit{(labelled bra)} & \tilde{B} ::= \ B_R \mid \tilde{K}^\dagger \mid S.\tilde{B} \mid \tilde{B} + \tilde{B} \mid \tilde{B} \cdot \tilde{O} \mid \tilde{B} \otimes \tilde{B} \\
\tilde{\mathcal{O}} \textit{(labelled operator)} & \tilde{O} ::= \ O_{R;R} \mid \tilde{K} \otimes \tilde{B} \mid \tilde{O}^\dagger \mid S.\tilde{O} \mid \tilde{O} + \tilde{O} \mid \tilde{O} \cdot \tilde{O} \mid \tilde{O} \otimes \tilde{O}
\end{array}
$$

In other words, we don't allow variables for labelled core language for now.

LABEL-CORE

We generally copied the symbols $(\dagger, S.\tilde{X}, +, \cdot, \otimes)$ from the core language. Therefore we also need a copy of the corresponding rewriting rules.

TSR-DECOMP

$$|(s,t)\rangle_{(Q,R)} \triangleright |s\rangle_Q \otimes |t\rangle_R \qquad \langle(s,t)|_{(Q,R)} \triangleright \langle s|_Q \otimes \langle s|_R$$
$$\mathbf{0}_{\mathcal{K}(Q,R)} \triangleright \mathbf{0}_{\mathcal{K}Q} \otimes \mathbf{0}_{\mathcal{K}R} \qquad \mathbf{0}_{\mathcal{B}(Q,R)} \triangleright \mathbf{0}_{\mathcal{B}Q} \otimes \mathbf{0}_{\mathcal{B}R}$$
$$\mathbf{0}_{\mathcal{O}(Q,R);(S,T)} \triangleright \mathbf{0}_{\mathcal{O}(Q,S)} \otimes \mathbf{0}_{\mathcal{O}(R,T)}$$
$$\mathbf{0}_{\mathcal{O}(Q_1,Q_2);R} \triangleright \mathbf{0}_{\mathcal{O}Q_1;\mathsf{fst}\ R} \otimes \mathbf{0}_{\mathcal{O}Q_2;\mathsf{fst}\ R} \qquad \mathbf{0}_{\mathcal{O}Q;(R_1,R_2)} \triangleright \mathbf{0}_{\mathcal{O}\mathsf{fst}\ Q;R_1} \otimes \mathbf{0}_{\mathcal{O}\mathsf{snd}\ Q;R_2}$$
$$\mathbf{1}_{\mathcal{O}(Q,R);(Q,R)} \triangleright \mathbf{1}_{\mathcal{O}Q;Q} \otimes \mathbf{1}_{\mathcal{O}R;R}$$
$$\mathbf{1}_{\mathcal{O}(Q_1,Q_2);R} \triangleright \mathbf{1}_{\mathcal{O}Q_1;\mathsf{fst}\ R} \otimes \mathbf{1}_{\mathcal{O}Q_2;\mathsf{fst}\ R} \qquad \mathbf{1}_{\mathcal{O}Q;(R_1,R_2)} \triangleright \mathbf{1}_{\mathcal{O}\mathsf{fst}\ Q;R_1} \otimes \mathbf{1}_{\mathcal{O}\mathsf{snd}\ Q;R_2}$$

$$(K_1 \otimes K_2)_{(Q,R)} \triangleright K_{1Q} \otimes K_{2R} \qquad (B_1 \otimes B_2)_{(Q,R)} \triangleright B_{1Q} \otimes B_{2R}$$
$$(O_1 \otimes O_2)_{(Q,R);(S,T)} \triangleright O_{1Q;S} \otimes O_{2R;T}$$
$$(O_1 \otimes O_2)_{(Q_1,Q_2);R} \triangleright O_{1Q_1;\mathsf{fst}\ R} \otimes O_{2Q_2;\mathsf{snd}\ R} \qquad (O_1 \otimes O_2)_{Q;(R_1,R_2)} \triangleright O_{1\mathsf{fst}\ Q;R_1} \otimes O_{2\mathsf{snd}\ Q;R_2}$$

TSR-COMP

$$K_{1\mathsf{fst}\ R} \otimes K_{1\mathsf{snd}\ R} \triangleright (K_1 \otimes K_2)_R \qquad B_{1\mathsf{fst}\ R} \otimes B_{2\mathsf{snd}\ R} \triangleright (B_1 \otimes B_2)_R$$
$$O_{1\mathsf{fst}\ Q;\mathsf{fst}\ R} \otimes O_{2\mathsf{snd}\ Q;\mathsf{snd}\ R} \triangleright (O_1 \otimes O_2)_{Q;R}$$

DOT-TSR

$$\frac{R \| S}{O_{1Q;R} \cdot O_{2S;T} \triangleright O_{1Q;R} \otimes O_{2S;T}}$$

LABEL-LIFT

$$(K_R)^\dagger \triangleright (K^\dagger)_R \qquad (B_R)^\dagger \triangleright (B^\dagger)_R \qquad (O_{Q;R})^\dagger \triangleright (O^\dagger)_{Q;R}$$
$$(K_R)^\top \triangleright (K^\top)_R \qquad (B_R)^\top \triangleright (B^\top)_R \qquad (O_{Q;R})^\top \triangleright (O^\top)_{Q;R}$$
$$(S.K)_R \triangleright S.(K_R) \qquad (S.B)_R \triangleright S.(B_R) \qquad (S.O)_{Q;R} \triangleright S.(O_{Q;R})$$
$$(K_1 + K_2)_R \triangleright K_{1R} + K_{2R} \qquad (B_1 + B_2)_R \triangleright B_{1R} + B_{2R} \qquad (O_1 + O_2)_{Q;R} \triangleright O_{1Q;R} + O_{2Q;R}$$

$$O_{1Q;R} \cdot O_{2R;S} \triangleright (O_1 \cdot O_2)_{Q;S} \qquad O_{Q;R} \cdot K_R \triangleright (O \cdot K)_Q \qquad B_Q \cdot O_{Q;R} \triangleright (B \cdot O)_R$$
$$B_R \cdot K_R \triangleright B \cdot K$$
$$(K \otimes B)_{Q;R} \triangleright K_Q \otimes B_R$$

OPT-EXT

I think the concept "cylinder extension" is only limited to endomorphisms. Besides, one quantum register should a sub-register of the other one, which is defined as follows:

Definition 8.4 (sub-register)**.**

$$\mathsf{fst}\ R \preceq R \qquad \mathsf{snd}\ R \preceq R \qquad Q \preceq (Q,R) \qquad R \preceq (Q,R) \qquad \frac{Q \preceq R \qquad R \preceq S}{Q \preceq S}$$

And we can further calculate the "position" of sub-register, which will be utilized during cylinder extension: assume Q is a sub-register of R, then the position of Q in R is a string defined as follows:

$$\begin{aligned}
&pos(\mathsf{fst}\ R, R) = 0\\
&pos(\mathsf{snd}\ R, R) = 1\\
&pos(Q, (Q,R)) = 0\\
&pos(R, (Q,R)) = 1\\
&pos(Q, S) = pos(R, S)\ pos(Q, R)
\end{aligned}$$

Remark: For a well-formed quantum register, the sub-register position is well-defined.

Definition 8.5 (cylinder extension)**.**

$$ext(O, \epsilon) \equiv O \qquad ext(O, p :: 0) \equiv O \otimes \mathbf{1}_{\mathcal{O}} \qquad ext(O, p :: 1) \equiv \mathbf{1}_{\mathcal{O}} \otimes O$$

CYLINDER-EXT

$$\frac{Q \text{ is a subterm of } R \text{ at } p}{O_{Q;Q} \cdot K_R \triangleright (\mathrm{ext}(O,p) \cdot K)_R} \qquad \frac{Q \text{ is a subterm of } R \text{ at } p}{O_{1Q;Q} \cdot O_{2R;T} \triangleright (\mathrm{ext}(O_1,p) \cdot O_2)_{R;T}}$$

$$\frac{R \text{ is a subterm of } Q \text{ at } p}{B_Q \cdot O_{R;R} \triangleright (B \cdot \mathrm{ext}(O,p))_Q} \qquad \frac{R \text{ is a subterm of } Q \text{ at } p}{O_{1T;Q} \cdot O_{2R;T} \triangleright (O_1 \cdot \mathrm{ext}(O_2,p))_{T;Q}}$$

8.1 Labelled Extended Language

Definition 8.6 (labelled extended language)**.** *The* ***labelled extended language*** *consists of the symbols in labelled core langauge and unlabelled extended language, and add the new symbols of transpose and big-op for labelled bra, ket and operators, which is described in the following.*

$$\tilde{K} ::= \tilde{B}^\top \mid \sum_{i \in M} \tilde{K} \qquad \tilde{B} ::= \tilde{K}^\top \mid \sum_{i \in M} \tilde{B} \qquad \tilde{O} ::= \tilde{O}^\top \mid \sum_{i \in M} \tilde{O}$$

LABEL-SUM

$$(\sum_{i \in M} K)_R \triangleright \sum_{i \in M} (K_R) \qquad (\sum_{i \in M} B)_R \triangleright \sum_{i \in M} (B_R) \qquad (\sum_{i \in M} O)_{Q;R} \triangleright \sum_{i \in M} (O_{Q;R})$$

LABEL-TEMP

[YX] : These are ad-hoc rules for the examples for now. They are still not organized and require further investigations.

$$\frac{R \text{ in } Q}{O_{P;R} \cdot (K_Q \otimes \tilde{K}') \triangleright (O_{P;R} \cdot K_Q) \otimes \tilde{K}'} \qquad \frac{R \text{ in } Q}{O_{P;R} \cdot (K_Q \otimes \tilde{B}) \triangleright (O_{P;R} \cdot K_Q) \otimes \tilde{B}}$$

$$\frac{R \text{ in } Q}{O_{1P;R} \cdot (O_{2Q;T} \otimes \tilde{O}_3) \triangleright (O_{1P;R} \cdot O_{2Q;T}) \otimes \tilde{O}_3}$$

$$\frac{Q \text{ in } R}{(B_R \otimes \tilde{B}') \cdot O_{Q;P} \triangleright (B_R \cdot O_{Q;P}) \otimes \tilde{B}'} \qquad \frac{Q \text{ in } R}{(O_{1T;R} \otimes \tilde{O}') \cdot O_{2Q;P} \triangleright (O_{1T;R} \cdot O_{2Q;P}) \otimes \tilde{O}'}$$

$$B_R \cdot (K_R \otimes \tilde{B}') \triangleright (B_R \cdot K_R).\tilde{B}' \qquad (\tilde{K}' \otimes B_R) \cdot K_R \triangleright (B_R \cdot K_R).\tilde{K}'$$

References

[1] Walliam Shakespeard. Halmet. 2666.