Project 3 - Image Deblurring

Linear Algebra and Learning from Data

1. Introduction

Tasks: You should deliver computer codes to

- implement the three methods described in this text to recover the original image (called deblurring),
- a report that answers the questions raised in this text.

Figure 1. Initial blurry image

2. Description of the methods

a. A naive method

Questions:

- How does this reconstructed image look like?
- For this image, give an estimated bound on

$$\frac{\|HV^{\mathrm{orig}} - V^{\mathrm{bour}}\|_F}{\|V^{\mathrm{blur}}\|_F}$$

and estimate the condition number of H (which e.g. can be done using condest in Matlab). Use these estimates to explain the picture.

Answer: The reconstructed image looks like a very noisy image. This noise is first introduced by the round off in the computing of V^{blur} and multiplicated by the coefficients of H.

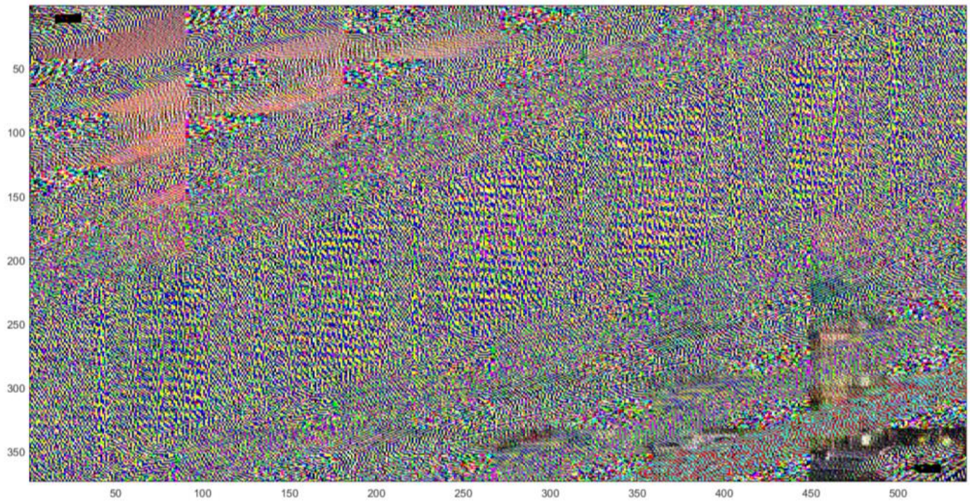

Figure 1. Reconstructed image with the naïve method

As mentioned before, we can consider that noise has been introduced by the round off so $V^{blur} \approx HV^{orig}$ can be written as $V^{blur} = HV^{orig} + \Delta V^{blur}$ with ΔV^{blur} the round off error.

Then we can define another error related to V^{orig} such that:

$$V^{orig} = H^{-1}V^{blur} + \Delta V^{orig}$$

By definition of ΔV^{orig} and ΔV^{blur} we have:

$$\Delta V^{orig} = H^{-1}\, \Delta V^{blur} = H^{-1}\, V^{blur} - V^{orig}$$

Then we can write the norm of the error related to the round off in V^{blur} as:

$$\left\|\Delta V^{blur}\right\|_F = \left\|HV^{orig} - V^{blur}\right\|_F = \left\|H\Delta V^{orig}\right\|_F \leq \|H\|_F \left\|\Delta V^{orig}\right\|_F$$

$$\left\|HV^{orig} - V^{blur}\right\|_F \leq \|H\|_F \left\|H^{-1}\Delta V^{blur}\right\|_F \leq \|H\|_F \|H^{-1}\|_F \left\|\Delta V^{blur}\right\|_F$$

Finally, we divide it by $\left\|V^{blur}\right\|_F$ to obtain the desired fraction.

$$\frac{\left\|HV^{orig} - V^{blur}\right\|_F}{\|V^{blur}\|_F} \leq \|H\|_F \|H^{-1}\|_F \frac{\left\|\Delta V^{blur}\right\|_F}{\|V^{blur}\|_F}$$

$\|H\|_F \|H^{-1}\|_F$ is the condition number of H and $\frac{\left\|\Delta V^{blur}\right\|_F}{\left\|V^{blur}\right\|_F}$ is the relative error in the value of V^{blur}.

Even though the relative error related to the round off can be neglected, the value of the condition number can induce significant noise in the reconstructed image. According to the value of the condition number of H, this phenomenon is responsible for the bad result obtained previously.

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Condition number of H : 119925901.7848
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Sources: D:/Work_Projects/_Strang/ila5/ILA_5th_edition/ila5.dvi (mit.edu)

b. Tikhonov regularization

Questions:

- We know that the Tikhonov regularization problem is equivalent to the least square problem in the following form

$$V^{\text{tik}} = \arg\min_V \left\| \begin{pmatrix} H \\ \beta I \end{pmatrix} V - \begin{pmatrix} V^{\text{blur}} \\ 0 \end{pmatrix} \right\|_F^2 .$$

  Give an analytic formula for the singular values of the regularized matrix

$$\begin{pmatrix} H \\ \beta I \end{pmatrix}$$

  in terms of β and of the singular values $\sigma_1, \ldots, \sigma_N$ for H. The largest singular value of H is just below one, using this fact, together with the condition number estimated earlier, give an estimate of the size of the smallest singular value for this matrix in the case of $\beta = 10^{-2}$.

- What would be the complexity of solving this ordinary least square system using dense QR factorization? Fortunately, we do not need to use dense factorization to solve this problem, the sparse QR factorization is recommended (In Matlab you can find information by help qr or just use the backslash \)

Answer: To compute the singular values of the regularized matrix $\begin{pmatrix} H \\ \beta I \end{pmatrix}$ we use its Gram matrix which is defined as:

$$G := \begin{pmatrix} H \\ \beta I \end{pmatrix}^{\mathrm{T}} \begin{pmatrix} H \\ \beta I \end{pmatrix} = \mathrm{H}^{\mathrm{T}}\mathrm{H} + \beta^2 \mathrm{I}$$

Consider the singular value decomposition of $H = U\Sigma V^T$. The singular values of $\begin{pmatrix} H \\ \beta I \end{pmatrix}$ are:

$$\sigma_i^2\left(\begin{pmatrix} H \\ \beta I \end{pmatrix}\right) = \sigma_i\left(\begin{pmatrix} H \\ \beta I \end{pmatrix}^{\mathrm{T}}\begin{pmatrix} H \\ \beta I \end{pmatrix}\right)$$

$$= \sigma_i\left(\mathrm{H}^{\mathrm{T}}\mathrm{H} + \beta^2\mathrm{I}\right)$$

$$= \sigma_i\left(\mathrm{V}\Sigma^{\mathrm{T}}\Sigma\mathrm{V}^{\mathrm{T}} + \beta^2\mathrm{I}\right)$$

$$= \sigma_i\left(\mathrm{V}(\Sigma^{\mathrm{T}}\Sigma + \beta^2\mathrm{I})\mathrm{V}^{\mathrm{T}}\right)$$

$$= \sigma_i^2(H) + \beta^2$$

$$\sigma_i\left(\begin{pmatrix} H \\ \beta I \end{pmatrix}\right) = \sqrt{\sigma_i^2(H) + \beta^2}$$

As the function is increasing, the i^{th} largest singular value of H gives the i^{th} largest singular value of $\begin{pmatrix} H \\ \beta I \end{pmatrix}$. Consider $\sigma_1 > \sigma_2 > \cdots > \sigma_r \geq 0$ the singular value of H.

$$\min_i \sigma_i\left(\begin{pmatrix} H \\ \beta I \end{pmatrix}\right) = \sqrt{\sigma_r^2 + \beta^2}$$

$$= \sqrt{\frac{\sigma_1^2}{\kappa^2(H)} + \beta^2}$$

If the largest singular value of H is just below one and $\beta = 10^{-2}$ together with the condition number $\kappa(H)$ estimated earlier, we can say that the smallest singular value for this matrix is equivalent to β^2:

$$\min_i \sigma_i\left(\begin{pmatrix} H \\ \beta I \end{pmatrix}\right) \approx \beta^2 = 10^{-4}$$

Figure 2.a With $\beta = 10^{-1}$

Figure 2.b With $\beta = 10^{-2}$

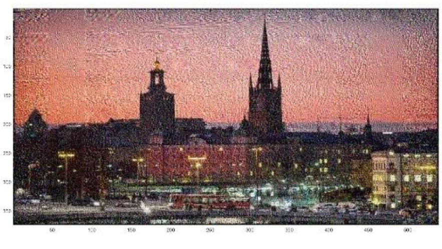

Figure 2.c With $\beta = 10^{-3}$

Figure 2. Reconstructed image with Tikhonov regularization

We obtain good results for $\beta = 10^{-2}$. With this parameter, the original image is recognizable even though some prints remain in the sky from the blurry image. When β is superior, the reconstructed image is still blurry. On the contrary, for a lower β such as $\beta = 10^{-3}$, the method introduces some random noise.

c. Landweber iteration

Questions:

- Show that the fixed-point iteration converges to the solution to the normal equations if

$$0 < \alpha < \frac{2}{\sigma_1^2},$$

where σ_1 is the largest singular value of H.
Calculating σ_1^2 can be expensive, but there is a cheap upper bound:

$$\sigma_1^2 \leq \|H\|_1 \|H\|_\infty.$$

In fact, in our case $\|H\|_1 = \|H\|_\infty = 1$ and σ_1^2 is bounded above by 1.
Prove these claims.

- Start with α = 2. Play with this parameter. Do you get better results for other values?

- How many iterations do you need to get a better image, almost as good as the one obtained by the Tikhonov regularization approach? Build a version that shows the images at each intermediate step of the iteration. At what step are you first able to see the image? You don't need to include your code, but ideally your writeup should include a picture of the reconstruction at the relevant step.

- Compare the run time and image quality for the Landweber iteration for this problem (with α = 2 and 100 respectively 1000 iterations) to the Tikhonov regularized computation (using e.g. in Matlabs sparse direct QR solver).

- Do you think there will be any improvement of speed/convergence if momentum is introduced, like the heavy ball, for solving the Tikhonov regularized method? Demonstrate it by implementation/simulation.

Answer: Suppose $0 < \alpha < \frac{2}{\sigma_1^2}$, as α is strictly positive, the sequence of X_k can be written as an arithmetic-geometric sequence:

$$X_{k+1} = (I - \alpha H^T H)X_k + \alpha H^T V^{blur}$$

Define another sequence U_k such that $U_0 = X_0$ and:

$$U_{k+1} = X_{k+1} - X_k$$

Then U_k is a geometric sequence with the common ratio $(I - \alpha H^T H)$.

$$U_k = (I - \alpha H^T H)^k X_0 \ \ for\ all\ k$$

Then,

$$X_{k+1} - X_k = \alpha H^T H X_k + \alpha H^T V^{blur} = (I - \alpha H^T H)^k X_0$$

$$\alpha H^T H X_k = (\alpha H^T H - I)^k X_0 + \alpha H^T V^{blur}$$

The matrix $(\alpha H^T H - I)^k$ is a diagonal of $(\alpha\sigma_1^2 - 1)^k$. As $\alpha < \frac{1}{\sigma_1^2}$, the matrix tends to the zero matrix when k tends to ∞. So $\alpha H^T H X_k$ converges toward $\alpha H^T V^{blur}$ which is constant and X_k when $k \to \infty$ is the solution of the following normal equation:

$$H^T H X = H^T V^{blur}$$

To prove that $\sigma_1^2 \leq \|H\|_1 \|H\|_\infty$, we must consider the usual norms as:

$$\|H\|_2 = largest\ singular\ value\ \sigma_1 of\ H$$

$$\|H\|_1 = largest\ l_1\ norm\ of\ the\ columns\ of\ H$$

$$\|H\|_\infty = largest\ l_1\ norm\ of\ the\ rows\ of\ H$$

Then $\|H\|_\infty = \|H^T\|_1$. Consider v the first singular vector of $H^T H$ and take the l_1 norm of it.

$$\left\|\sigma_1^2 v\right\|_1 = \|H^T H v\|_1 \leq \|H^T\|_1 \|Hv\|_1 \leq \|H\|_\infty \|H\|_1 \|v\|_1$$

Finally,

$$\|H\|_2 = \sigma_1^2 \leq \|H\|_\infty \|H\|_1$$

Sources: LINEAR ALGEBRA AND LEARNING FROM DATA GILBERT STRANG, p94

With $n = 100$ and for different α, we obtain the following results.

Figure 3.a With $\alpha = 0.5$

Figure 3.b With $\alpha = 1$

Figure 3.c With $\alpha = 1.5$

Figure 3.d With $\alpha = 2$

Figure 3. Reconstructed image with Landweber iteration for $n = 100$ and different values of α

We can clearly see a difference between the image with $\alpha = 0.5$ and the one with $\alpha = 2$. A high value for α seems to improve the results. However, the condition $\alpha < \frac{2}{\sigma_1^2}$ together with the proposition "The largest singular value of H is just below one" limit the value of α to 2.

For two near values of α, the difference in the result can be hard to see. We decide to keep $\alpha = 2$ as the best value for α parameter and try to see the effect of the parameter n.

Figure 4.a With $n = 10$

Figure 4.b With $n = 100$

Figure 4.c With $n = 1000$

Figure 4. Reconstructed image with Landweber iteration for $\alpha = 2$ *and different values of* n

The number of iterations n improves the results as it increases. For $n = 100$, the reconstructed image is quite good. The original image is recognizable, and the results are almost equivalent to the best result of Tikhonov method. However, for $n = 1000$ the computational time is considerable: $5{,}75\ min$ against $1{,}12\ min$ for Tikhonov method. In conclusion, Tikhonov regularization seems to be the best method in our case.