# **Image -** TD2

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**Exercice 1**

$\omega_N = e^{\frac{2i\pi}{N}}$. Pour $p \in \mathbb{Z}$, $\omega_N^p = 1 \iff N|p$.

$$\sum_{k=0}^{N-1} \omega_N^k = \frac{\omega_N^N - 1}{\omega_N - 1} = 0$$

Soit $l \neq 0 \mod N$,

$$\sum_{k=0}^{N-1} \omega_N^{kl} = \frac{\omega_N^{Nl} - 1}{\omega_N^l - 1} = 0$$

Et,

$$\sum_{k=k_0}^{k_0+N-1} \omega_N^{lk} = \omega^{lk_0} \sum_{k=0}^{N-1} \omega_N^{kl} = 0$$

**Exercice 3**

$$\begin{aligned} c_n(u) &= \frac{1}{a} \int_0^a u(x) e^{-\frac{2i\pi nx}{a}} dx \\ &= \frac{1}{a} \frac{a}{N} \sum_{k=0}^{N-1} u\left(\frac{ka}{N}\right) e^{-\frac{2i\pi n}{N}} = \tilde{u}_n \end{aligned}$$

**Exercice 2**

$u \in \mathbb{C}^N$, $\tilde{u} = DFT(u)$.

$$U = \begin{pmatrix} U_0 \\ \vdots \\ U_{N-1} \end{pmatrix} \qquad \tilde{U} = \begin{pmatrix} \tilde{U}_{-\frac{N}{2}} \\ \vdots \\ \tilde{U}_{\frac{N}{2}-1} \end{pmatrix}$$

$$\begin{pmatrix} 1 & \omega_N^{-\frac{-N}{2}} & \dots & \\ 1 & \omega_N^{\left(\frac{-N}{2}+1\right)} & \dots & \\ & & & \\ 1 & \omega_N^{-\left(\frac{N}{2}-1\right)} & \dots & \omega_N^{-(N-1)\left(\frac{N}{2}-1\right)} \end{pmatrix} \begin{pmatrix} U_0 \\ \vdots \\ U_{N-1} \end{pmatrix} = \begin{pmatrix} \tilde{U}_{-\frac{N}{2}} \\ \vdots \\ \tilde{U}_{\frac{N}{2}-1} \end{pmatrix}$$

$$\frac{\sqrt{N}}{N}\begin{pmatrix}1 & \omega_N^{-\left(-\frac{N}{2}+i\right)} & \dots & \omega_N^{-(N-1)\left(-\frac{N}{2}+i\right)}\end{pmatrix}$$

$$\sqrt{\sum_{k=0}^{N-1}\left(\frac{\sqrt{N}}{N}\right)^2|\omega_N^{-k\left(-\frac{N}{2}+i\right)}|^2}=\frac{1}{\sqrt{N}}\sqrt{\sum_{k=0}^{N-1}|\omega_N^{-k\left(-\frac{N}{2}+i\right)}|^2}=\frac{1}{\sqrt{N}}\times\sqrt{N}=1$$

$$\frac{1}{N}\sum_{k=0}^{N-1}\omega_N^{-k\left(-\frac{N}{2}+i-\left(-\frac{N}{2}+j\right)\right)}=\frac{1}{N}\sum_{k=0}^{N-1}\omega_N^{k(j-i)}\quad j,i\in\{0,\dots,N-1\}$$

puisque $i\neq j$, $j-i\neq 0 \mod N$ donc $\sum_{k=0}^{N-1}\omega_N^{k(j-i)}=0$.

**Exercice 4**

$$\tilde{u}_{m,n}=\frac{1}{N^2}\sum_{k,l=1}^{N-1}u_{k,l}\omega_N^{-km}\omega_N^{-ln}$$

$$\begin{aligned}\tilde{u}_{m,n}&=\frac{1}{N}\sum_{l=0}^{N-1}\left(\frac{1}{N}\sum_{k=0}^{N-1}u_{k,l}\omega_N^{-km}\right)\omega_N^{-ln}\\&=\frac{1}{N}\sum_{l=0}^{N-1}DFT_1(u)\omega_N^{-ln}\\&=DFT_2(DFT_1(u))\end{aligned}$$

$N$ opérations, $DFT(u)$, $N^2$ opérations pour $DFT_2(DFT_1(u))$

**Exercice 6**

$$c_{m,n}=\frac{1}{a^2}\int_{[0,a]^2}u(x,y)e^{-\frac{2i\pi}{}}$$

On suppose juste $u(x+a,y)=u(x,y+a)=u(x,y)$.

$$\begin{aligned}T(c_{m,n})=&\sum_{0\leqslant k,l\leqslant N-1}\frac{1}{4N^2}\Bigg(f\left(\frac{ka}{N},\frac{la}{N}\right)+f\left(\frac{(k+1)a}{N},\frac{la}{N}\right)\\&+f\left(\frac{ka}{N},\frac{(l+1)a}{N}\right)+f\left(\frac{(k+1)a}{N},\frac{(l+1)a}{N}\right)\Bigg)\\&=\frac{1}{N^2}\left(f(0,0)+\sum_{k=1}^{N-1}f\left(\frac{ka}{N},0\right)+\sum_{l=1}^{N-1}f\left(0,\frac{la}{N}\right)+\sum_{1\leqslant k,l,N-1}f\left(\frac{ka}{N},\frac{la}{N}\right)\right)\end{aligned}$$

**Exercice 9**

On introduit $v$ dont la DFT est :

$$(\tilde{v}_{m,n})_{0\leqslant m,n\leqslant N-1} = \begin{cases} \tilde{u}_{m,n} & \text{si } -\frac{N}{2} \leqslant m,n \leqslant \frac{N}{2}-1 \\ 0 & \text{sinon} \end{cases}$$

$$\begin{aligned} v_{2k,2l} &= \sum_{-N\leqslant m,n\leqslant N-1} \tilde{v}_{m,n}\omega_{2N}^{2km+2ln} \\ &= \sum_{-\frac{N}{2}\leqslant m,n\leqslant \frac{N}{2}-1} \tilde{u}_{m,n}\omega_{N}^{km+ln} \\ &= u_{k,l} \end{aligned}$$

**Exercice 7**

$$u(x,y) = \sum_{n=-\infty}^{+\infty}\sum_{m=-\infty}^{+\infty} c_{n,m}e^{\frac{2i\pi nx}{a}}e^{\frac{2i\pi my}{a}}$$

$u\colon [0,a]^2 \longrightarrow \mathbb{C}$.

$$\begin{aligned} u\left(\frac{ka}{N},\frac{la}{N}\right) &= \sum_n\sum_m c_{n,m}e^{\frac{2i\pi mk}{N}}e^{\frac{2i\pi ml}{N}} \\ &= \sum_{-\frac{N}{2}\leqslant n',m'\leqslant \frac{N}{2}-1}\left(\sum_q\sum_{q'} c_{n'+qN,m'+q'N}\right)e^{\frac{2i\pi n'k}{N}}e^{\frac{2i\pi m'l}{N}} \end{aligned}$$

Si on pose :

$$P(x,y) = \sum_{-\frac{N}{2}\leqslant n',m'\leqslant \frac{N}{2}-1}\left(\sum_q\sum_{q'} c_{n'+qN,m'+q'N}\right)e^{\frac{2i\pi n'x}{a}}e^{\frac{2i\pi m'y}{a}}$$

On a bien $P\left(\frac{ka}{N},\frac{la}{N}\right) = u_{k,l}$.

**Exercice 8**

$(u_{k,l})_{0\leqslant k,l\leqslant N-1}$. On a un unique polynôme trigonométrique coïncidant avec les $(u_{k,l})$ :

$$P(x,y) = \sum m,n = -\frac{N}{2}^{\frac{N}{2}-1}\tilde{u}_{m,n}e^{\frac{2i\pi mx}{a}}e^{\frac{2i\pi ny}{a}}$$

Soit $(v_{k,l}) = u_{2k,2l})_{0\leqslant k,l\leqslant \frac{N}{2}-1}$

$$c_{\bar{m},\bar{n}}(P) = \frac{1}{a^2}\int_{[0,a]^2} P(x,y)e^{-\frac{2i\pi\bar{n}x}{a}}e^{-\frac{2i\pi\bar{m}y}{a}}dxdy = u_{\bar{m},\bar{n}}$$

Par le théorème 2.10,

$$v_{k,l} = \sum_{q,r=-\infty}^{+\infty} c_{k+q\frac{N}{2},l+r\frac{N}{2}}(P) = \sum_{q,r=-1}^{1} u_{k+q\frac{N}{2},l+r\frac{N}{2}}(P)$$

**Exercice 10**
$T(N) = O(N \log_2(N))$.

$$T(N) = 2T\left(\frac{N}{2}\right) + 2N$$

$$T(N) = 2^k T\left(\frac{N}{2^k}\right) + 2kN$$

Pour $N = 2^n$,

$$T(N) = \frac{N}{2}T(2) + 2\log_2(N)N = \frac{N}{2} + 2\log_2(N)N$$

D'où, $T(N) = O(N \log_2(N))$.

**Exercice 11**
$\alpha \in \mathbb{R}^2$, $v$ image translatée.

$$\tilde{v}_{m,n} = \tilde{u}_{m,n} = e^{-\frac{2i\pi m\alpha_1}{a}} e^{-\frac{2i\pi n\alpha_2}{a}}$$

$$T_\alpha P(x,y) = T_{\alpha_1}(T_{\alpha_2} P(x,y))$$