Logique

Chapitre 3 : Logique du premier ordre

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1 Theories

Remark.
s.t. means "such that"

Definition 1 (elementary arithmetics)

- $\forall x,\ 0 + x = x$
- $\forall x, \forall y,\ S(x) + y = S(x + y)$
- $\forall x,\ 0 \times x = x$
- $\forall x, \forall y,\ S(x) = S(y) \Rightarrow x = y$
- $\forall x, x \neq 0 \Rightarrow \exists y,\ x = S(y)$
- $\forall x, \forall y,\ S(x)y = (xy) + y$
- $\forall x,\ \neg(S(x) = 0)$

Definition 2 (group theory)

- $\forall x, \forall y, \forall z,\ (xy)z = x(yz)$
- $\exists e, (\forall x, xe = ex) \wedge (xx^{-1} = e) \wedge (x^{-1}x = e)$

Remark.
A theory equivalent to group theory :

$$\forall x \forall y : x^{-1} \times (xy) = y$$

2 Syntax

Let F be a set of symbols of *functions* $f \in F$, each one having arity $a(f) \in \mathbb{N}$. Let X be a set of *variables*.

Definition 3 (terms)

F,X given. The set of terms $T(F, X)$ is defined by :

- $x \in X$
- $f(t_1, ..., t_n)$ where the t_i are terms and $a(f) = n$

Example.
$F = \{z(0), +(2), \times(3), s(1)\}$

3 Formulas of the first-order predicate calculus

Let $\mathcal{P}$ be a set of relation symbols s.t. $P \in \mathcal{P}$ has arity $a(P) \in \mathbb{N}$.

Definition 4 ($CP_1(F, \mathcal{P})$)

$CP_1(F, \mathcal{P})$:

- an atomic formula $P(t_1, ..., t_n)$, where $a(P) = n$ and $t_1, ..., t_n$ are terms
- if $\varphi, \psi \in CP_1(F, \mathcal{P})$, then : $\neg\varphi, \varphi \Rightarrow \psi, \varphi \wedge \psi, \varphi \vee \psi \in CP_1(F, \mathcal{P})$

3.1 F-algebra

Definition 5 (F-algebra)

- a given non empty set D_A
- for all $f \in F$ $f_A : D_A^{a(f)} \longrightarrow D_A$

Examples.

- $(\mathbb{N}, +_{\mathbb{N}}, \times_{\mathbb{N}}, 0_{\mathbb{N}}, S_N)$ is an F-algebra where $F = \{+(2), \times(2), 0(0), s(1)\}$
- A', with :
  - $D_{A'} = \Sigma^*$
  - $+_{A'}$ is the concatenation
  - $\times_{A'}$ is defined by
    $$\omega \times_{A'} \omega' = \omega[a \mapsto \omega'] \quad \text{for } a \in \Sigma$$
  - $S_{A'}(\omega) = \omega \cdot a$
- $T(F, X)$ is a F-algebra, where functions are trivially interpreted :
  $$f_{T(F,X)}(t_1, ..., t_n) = f(t_1, ..., t_n)$$
  As it happens, the domain is $T(F, X)$

3.2 Morphisms

Definition 6 (morphism of $A \longrightarrow A'$)

A, A' two F-algebras. A morphism of $A \longrightarrow A'$ is an application $h : D_A \longrightarrow D_{A'}$ s.t. for all $f \in F, e_1, ..., e_n \in D_A$ where $a(f) = n$:

$$h(f_A(e_1, ..., e_n)) = f_{A'}(h(e_1), ..., h(e_n))$$

Example.

$$h : \begin{cases} A_\Sigma \longrightarrow \mathbb{N} \\ \omega \mapsto |\omega|_a \end{cases}$$

- $h(z_A) = h(\varepsilon) = 0 = z_\mathbb{N}$
- $h(\omega +_A \omega') = h(\omega\omega') = |\omega\omega'| = |\omega|_a + |\omega'|_a = h(\omega) +_\mathbb{N} h(\omega')$
- $h(\omega \times_A \omega') = h(\omega[a \mapsto \omega']) = |\omega|_a|\omega'|_a$
- $h(s_A(\omega)) = h(\omega \cdot a) = |\omega|_a + 1 = S_\mathbb{N}(h(\omega))$

Theorem 1 (Birkhoff)

If $\sigma : X \longrightarrow A$, where A is a F-algebra, then there exists a unique morphism :
$\widehat{\sigma} : T(F, X) \longrightarrow A$ s.t. $\widehat{\sigma}(x) = \sigma(x)$ for all $x \in X$

Proof.
$\widehat{\sigma}(t)$ is constructed by structural induction on t, with :

- $\widehat{\sigma}(x) = \sigma(x)$
- $\widehat{\sigma}(f(t_1, ..., t_n)) = f_A(\widehat{\sigma}(t_1), ..., \widehat{\sigma}(t_n))$

□

Examples.
$x + s(y)$

$$\sigma = \begin{cases} X \longrightarrow A_\Sigma \\ x \longmapsto ab \\ y \longmapsto b \end{cases}$$

then $\widehat{\sigma}(x + s(y)) = abba$

$$\sigma' = \begin{cases} X \longrightarrow \mathbb{N} \\ x \longmapsto 1 \\ y \longmapsto 2 \end{cases}$$

then $\widehat{\sigma}(x + s(y)) = 4$

Remarks.

- $\hat{\sigma}(t)$ is denoted $[\![t]\!]\sigma, A$, or $t\sigma$
- $\sigma : X \longrightarrow A$ is called an *interpretation*
- $\sigma : X \longrightarrow T(F, X)$ is a *substitution*
- $\mathrm{Dom}(\sigma) = \{x \mid x\sigma \neq x\}$ if $\mathrm{Dom}(\sigma) = \{x_1, ..., x_n\}$, σ is denoted $\{x_1 \mapsto t_1, ..., x_n \mapsto t_n\}$