Probabilités

Correction TD3

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Remarque.
Typo dans l'exercice 7 : $\frac{1}{n-1}\sum$

Exercice 5.

1. $TV(P,Q) = P(A^*) - Q(A^*)$ Soit A un évènement.

$$\begin{aligned}
|P(A) - Q(A)| &= \left|\int_A p(x) - q(x) d\lambda x\right| \\
&= \left|\int_{A\cup A^*} \underbrace{p(x) - q(x)}_{\geq 0} d\lambda x + \int_{A\cup (A^*)^C} \underbrace{p(x) - q(x)}_{\leq 0} d\lambda x\right| \\
&\leq \max\left(\int_{A\cup A^*} p(x) - q(x) d\lambda x, \int_{A\cup (A^*)^C} q(x) - p(x) d\lambda x\right) \\
&\leq \max(P(A^*) - Q(A^*), Q((A^*)^C) - P((A^*)^C) \\
&\leq P(A^*) - Q(A^*)
\end{aligned}$$

$$\begin{aligned}
\frac{1}{2}\int |p-q| d\lambda &= \frac{1}{2}\int_{A^*} (p-q) d\lambda + \frac{1}{2}\int_{(A^*)^C} (q-p) d\lambda \\
&= \frac{1}{2}[P(A^*) - Q(A^*) + Q((A^*)^C) - P((A^*)^C)] \\
&= P(A^*) - Q(A^*)
\end{aligned}$$

2. $B^* = \{Y \geq 1\} = \{q(x) \geq p(x)\} = (A^*)^C$

   Donc $E_Q(Z) - E_P(Z) \underset{\text{def}}{=} Q((A^*)^C) - P((A^*)^C) = P(A^*) - Q(A^*) = TV(P,Q)$

3. Z est bornée par 0 et 1. Donc $Z - E(Z)$ est $\frac{1}{2}$-sous-gaussienne (d'après Hoeffding)
   Ainsi :

$$M_{Z_E(Z)}(\alpha) \leq e^{\alpha^2/8} \qquad \Lambda_{Z-E(Z)}(\alpha) \leq \frac{\alpha^2}{8}$$

4. $\mathcal{D}(P||Q) = \sup_{Z'|E(e^{Z'})} \left\{ E_G(Z') - \log(E_P(e^{Z'})) \right\}$

   Prenons $Z' = \alpha(Z - E(Z))$

   On a $\mathcal{D}(P||Q) \geq \alpha(E_Q(Z) - E_P(Z)) - \underbrace{\log(E_P(e^{\alpha(Z-E(Z))}))}_{=\Lambda_{Z-E(Z)}(\alpha)}$

   Donc :

$$\begin{aligned} TV(P,Q) &\leq \frac{1}{\alpha}(\mathcal{D}(Q||P) + \Lambda_{Z-E(Z)}(\alpha)) \\ &\leq \frac{\alpha^2/8 + \mathcal{D}(Q||P)}{\alpha} \end{aligned}$$

5. On minimise : on trouve pour $\alpha = \sqrt{8\mathcal{D}(Q||P)}$

$$\Rightarrow TV(P,Q)^2 \leq \frac{4\mathcal{D}(Q||P)^2}{8\mathcal{D}(Q||P)} \leq \frac{\mathcal{D}(Q||P)}{2}$$

Exercice 6.

Soit $\lambda \in [0,1]$, Q_1 et Q_2 deux mesures de probabilité absolument continues par rapport à μ et P aussi.

$$\begin{aligned} \lambda\mathcal{D}(P||Q_1) + (1-\lambda)\mathcal{D}(P||Q_2) &= \lambda \int \ln\left(\frac{p(x)}{q_1(x)}\right) p(x)d\mu(x) + (1-\lambda)\int \ln\left(\frac{p(x)}{q_2(x)}\right) p(x)d\mu(x) \\ &= \int [\ln(p(x)) - (\lambda \ln(q_1(x)) + (1-\lambda)\ln(q_2(x)))]p(x)d\mu(x) \\ &\geq \int \left(\frac{p(x)}{\lambda q_1(x) + (1-\lambda)q_2(x)}\right) p(x)d\mu(x) \\ &\geq \mathcal{D}(P||\lambda Q_1 + (1-\lambda)Q_2) \end{aligned}$$

Exercice 7.

Montrons que $H(X|Y,Z) \leq H(X|Y)$ (1) et que $H(X_1,...,X_i) = \sum_{j=1}^{i} H(X_j|X_1,...,X_{j-1})$ (2) (chain rule).
De (1) on peut déduire $H(X|Y_1,...,Y_n) \leq H(X|Y_1,...,Y_i) \quad 1 \leq i \leq n$ (avec $Y = (Y_1,...,Y_i)$ et $Z = (Y_{i+1},...,Y_n)$).

$$\begin{aligned} H(X|Y) &= H(X,Y) - H(Y) \\ &= -\sum_{x,y} p(x,y)\ln p(x,y) + \sum_{y} p(y)\ln p(y) \\ &= -\sum_{x,y} p(x,y)\ln \underbrace{\frac{p(x,y)}{p(y)}}_{p(x|y)} \end{aligned}$$

$$I(X,Y) = H(X) + H(Y) - H(X,Y) \underset{\text{cours}}{\geq} 0$$

$$\begin{aligned}
H(X|Y) &= H(X,Y) - H(Y) \\
&= H(X) - (\underbrace{H(X) + H(Y) - H(X,Y)}_{I(X,Y)\geq 0}) \\
&\leq H(X)
\end{aligned}$$

On veut montrer que :

$$(n-1)H(X_1, ..., X_n) \leq \sum_{i=1}^{n} H(X_1, ..., X_{i-1}, X_{i+1}, ..., X_n)$$

$$\begin{aligned}
H(X_1, ..., X_n) &= H(X_i|X_1, ..., X_{i-1}, X_{i+1}, ..., X_n) + H(X_1, ..., X_{i-1}, X_{i+1}, ..., X_n) \\
nH(X_1, ..., X_n) &= \underbrace{\sum_{i=1}^{n} H(X_i|X_1, ..., X_{i-1}, X_{i+1}, ..., X_n)}_{\text{mq } \leq H(X_1, ..., X_n)} + \sum_{i=1}^{n} H(X_1, ..., X_{i-1}, X_{i+1}, ..., X_n)
\end{aligned}$$

$$H(X_i|X_1, ..., X_{i-1}, X_{i+1}, ..., X_n) \leq H(X_i|X_1, ..., X_{i-1})$$

$$\begin{aligned}
H(X|Y,Z) &= -\sum_{x,y,z} p(x,y,z) \ln \frac{p(x,y,z)}{p(y,z)} \\
&= -\sum_{y} p(y) \sum_{x,y} p((x,y)|z) \ln \frac{p(y)p((x,y)|z)}{p(y)p(z|y)}
\end{aligned}$$

Notons $X'_y = X|Y = y$ et $Z'_y = Z|Y = y$.

$$p(x'_y) = p(x|Y = y) \geq 0 \text{ et } \sum p(x'_y) = 1$$

$$\begin{aligned}
H(X|Y,Z) &= \sum_{y} p(y) H(X'_y|Z'_y) \\
&\leq \sum_{y} p(y) H(X'_y)
\end{aligned}$$

$$\begin{aligned}
H(X|Y,Z) &\leq -\sum_{y} p(y) \sum_{x}' p(x') \ln p(x') \\
&\leq -\sum_{y} p(y) \sum_{x} p(x|y) \ln p(x|y) \\
&\leq -\sum_{x,y} p(x) \ln p(x|y) \\
&= H(X|Y)
\end{aligned}$$

$$\sum_{j=1}^{i} H(X_j|X_1, ..., X_{j-1}) = \sum_{j=1}^{i} H(X_j, H_1, ..., X_{j-1}) - H(X_1, ..., X_{j-1}) = H(X_1, ..., X_i)$$

Exercice 8.

Montrons que $g(t) = t\mathbb{E}[Z^{1/t}]$ est convexe. En déduire $h(t) = \exp(2g(t))$ convexe. En déduire la 1.