# Mathcamp Day 2: Integrals

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# Outline

# Section 1: Introduction

## Introduction: Some ideas

As a political science graduate student...

- Why do we care about math?

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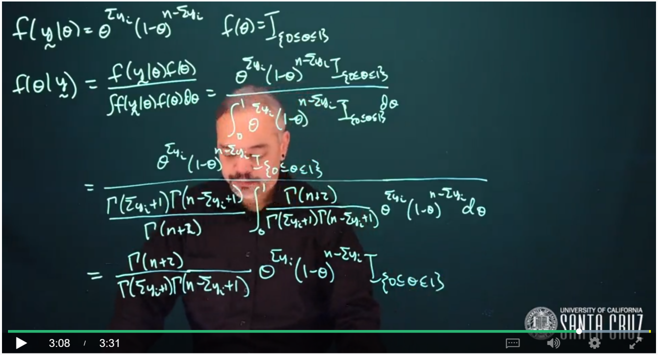


# Section 1: Introduction

- Why do we care about learning integral?
- What is the learning objective in today's math camp?
- How do you participate in the morning sessions?

# Introduction: Link to Day 1

A brief overview of functions and derivatives

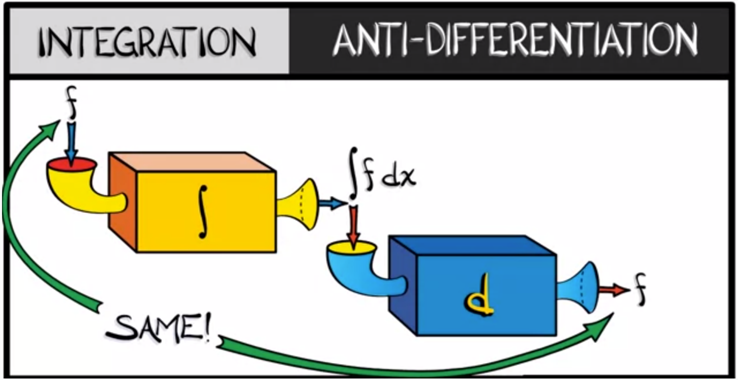



**Figure 1:** Integration and differentiation

For further references see Calculus: Single Variable Part 3 - Integration

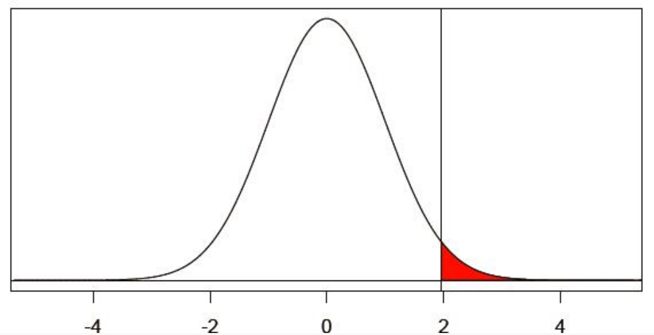
# Section 2: Concept of Integral

Area under curve:

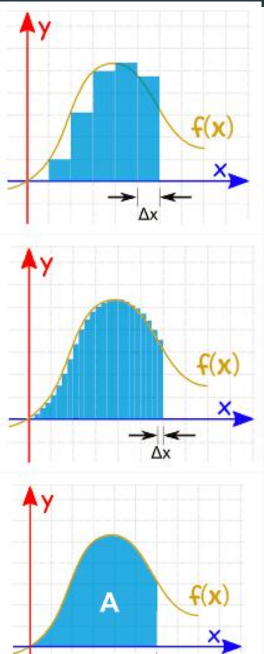
Let's take the standard normal curve as an example.

What if we want to know the area below the curve when x $\geq$ 1.96?

We'll come back to this example later.

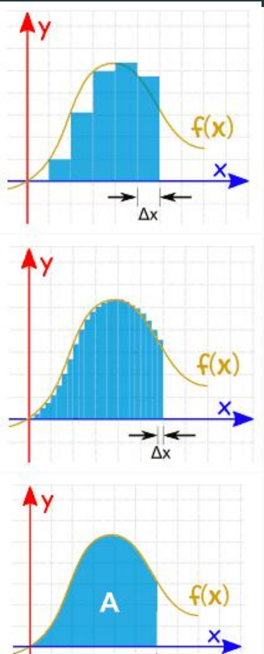


# Concept of Integrals




i. We could calculate the function at a few points and add up slices of width $\Delta x$.

[a]Reference: Math is fun: Introduction to Integration




i. We could calculate the function at a few points and add up slices of width $\Delta x$.

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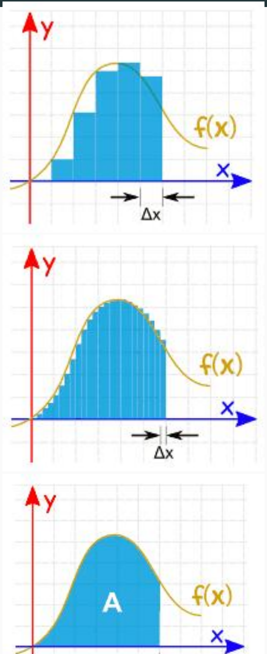


i. We could calculate the function at a few points and add up slices of width $\Delta$x.
ii. We can make $\Delta$x a lot smaller and add up many small slices.
iii. As the slices approach zero in width, the answer approaches the true answer. [a]

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## Concept of Integral

The way to find the area **under a curve** is to take the integral.

*What is an integral?*

- The area under the curve f(x) for some range of x = (a, b) is defined as the definite integral for f from a to b.

$$\int_a^b f(x)dx$$

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- Forms: indefinite integrals and definite integrals. A definite integral has a boundary, e.g. [a, b].

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- Forms: indefinite integrals and definite integrals. A definite integral has a boundary, e.g. [a, b].
- Useful illustration in Coursera (7 minutes)

## Concept of Integral

The integral is the inverse function of the derivative:

$$F(x) = \int_a^b f(x)dx$$

$$f(x) = \frac{dF(x)}{dx}$$

Then, for any x,

$$F(x) = \int_a^b f(x)dx = \int_a^b \frac{dF(x)}{dx}dx = F(x)$$

$$\frac{dF(x)}{dx} = \frac{d}{dx}\int_a^b f(x)dx = f(x)$$

In other words, F(x) is the anti-derivative of f(x).

# Concept of Integral

Too abstract?

Some examples...

**What's an integral of 2x?**

- We know that the derivative of $x^2$ is ......
- An integral of 2x is ......
- We will write $\int 2x dx = x^2 + C$

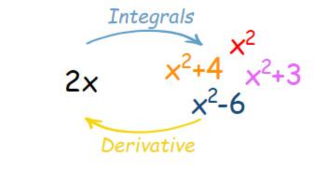
**Power rule:**

- $\int x^n dx = \frac{x^{n+1}}{n+1} + C$

# Concept of (Indefinite) Integrals

## Why plus C?

It is the "Constant of Integration". It is there because of all the functions whose derivative is 2x:




- The derivative of $x^2 + 4$ is 2x, and the derivative of $x^2 + 99$ is also 2x, and so on! Because the derivative of a constant is **zero**.
- So when we reverse the operation (to find the integral) we only know 2x, but there could have been a constant of any value.

# Concept of (Indefinite) Integrals

The anti-derivative, also called the indefinite integral, is not unique.

- There are multiple anti-derivatives for each function (for each C).
- This shifts the curve up or down the y-axis
- With information, such as a point the function passes through, you can solve for c.

# Concept of (Indefinite) Integrals: Some Classic Anti-derivative Formulas

antiderivative = indefinite integral

$$\begin{aligned}
\int 1 dx &= x + c \\
\int k dx &= kx + c \\
\int x^n dx &= \frac{x^{n+1}}{n+1} + c \\
\int \frac{1}{x} dx &= \log x + c \\
\int e^x dx &= e^x + c \\
\int a^x dx &= \frac{a^x}{\log a} + c
\end{aligned}$$

# Concept of (Indefinite) Integrals

Pause......Questions?

Recall Power Rule: $\int x^n dx = \frac{x^{n+1}}{n+1} + C$

Practices:

i $\int 8x dx = ......$

ii $\int 6x^2 dx = ......$

iii $\int x^3 dx = ......$

iv $\int e^x dx = ......$

# Section 3: Fundamental Theorem of Calculus and Definite Integral

# Fundamental Theorem of Calculus and Definite Integral

A **deep** connection between derivatives and integrals makes integration much easier.

The definite integral, again, is the defined by a specific range:

**Fundamental Theorem of Calculus**

Suppose $F$ is differentiable on $[a, b]$ and that its derivative, $f$, is integrable. Then,

$$\int_a^b f(x)dx = F(b) - F(a)$$

- This formula helps you to find area under the curve f (x) for some range of x from a to b.
- Simply find an antiderivative $F$, evaluate at $b$, evaluate at $a$, and take $F(b) - F(a)$.

# Fundamental Theorem of Calculus and Definite Integral: Uniform Distribution

Example I:

Uniform Distribution Suppose $f : \Re \rightarrow \Re$, with

$$
\begin{aligned}
f(x) &= 1 \text{ if } x \in [0,1] \\
f(x) &= 0 \text{ otherwise}
\end{aligned}
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$$\int_0^{1/2} f(x)dx = \int_0^{1/2} 1dx$$

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We will call $f(x) = 1$ the uniform distribution.

## Fundamental Theorem of Calculus and Definite Integral: Area Under a Line

Example II:

Area Under a Line

Suppose $f : \Re \to \Re$, with

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&= \frac{x^2}{2}\Big|_2^t \\
&= \frac{t^2}{2} - \frac{2^2}{2} \\
&= \frac{t^2}{2} - \frac{4}{2} = \frac{t^2}{2} - 2
\end{aligned}$$

## Fundamental Theorem of Calculus and Definite Integral: Integration Facts

If $f_1, f_2 : [a, b] \to \Re$ and $f_1, f_2$ are integrable on [a,b], then

- Consider the interval $[a, b]$ and $c \in [a, b]$. Then,

$$\begin{aligned}
\int_c^c f_1^{'}(x)dx &= f_1(c) - f_1(c) = 0 \\
\int_a^b f_1^{'}(x)dx &= \int_a^c f_1^{'}(x)dx + \int_c^b f_1^{'}(x)dx \\
&= (f_1(c) - f_1(a) + (f_1(b) - f_1(c)) \\
&= f_1(b) - f_1(a)
\end{aligned}$$

# Section 4: Integrals and Distributions

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- What is probability mass function (PMF)? What is probability density function (PDF)? What is the difference between the two?

# Integrals and Distributions

- What is distribution, and why should we care?
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- What is probability mass function (PMF)? What is probability density function (PDF)? What is the difference between the two?
    1. Intuitively, PDF is the probability that a continuous X drawn in a region that is indefinitely small at the limit of a shrinking process.
    2. Formally, $Pr(X\in$ [a, b]$) = \int_a^b f(x)dx$. The probability of being in some interval [a, b] is the integral of the PDF over that region.

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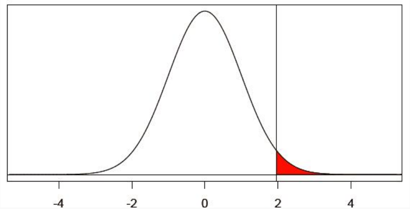
# Integrals and Distributions

- What is cumulative distribution function (CDF) for both discrete and continuous variables?
    1. Intuitively, if we pick a number, and we want to know the probability that a random draw from a population that produces a value less than that specified number, we sum the individual probabilities of each value below the specified value.
    2. For discrete variable is
       $$Pr(Y \leq y) = \sum_{i \leq y} p(i)$$
       .
    3. For continuous variable, we replace the sum with an integral to get the equation for the CDF:
       $$Pr(X \leq x) = F(x) = \int_{-\infty}^{x} f(t)dt$$
       .
    4. $0 \leq$ CDF $\leq 1$.

# Integrals and Distributions

Back to the normal distribution that started this off:

What if we want to know the area below the curve when x $\geq$ 1.96?




To find the area, we need to integrate the function:

$$f(x) = \frac{1}{2\pi} \exp^{-\frac{x^2}{2}}$$

# Integrals and Distributions

To find the area, we need to integrate the PDF function of the normal distribution:

$$f(x) = \frac{1}{2\pi} e^{-\frac{x^2}{2}}$$

Pause...

What's the next step to calculate the area under the curve when x $\geq 1.96$?

# Integrals and Distributions

To find the area, we need to integrate the PDF function of the normal distribution:

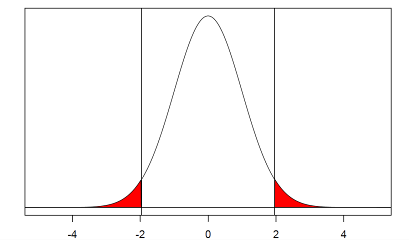
$$f(x) = \frac{1}{2\pi} e^{-\frac{x^2}{2}}$$

Pause...

What's the next step to calculate the area under the curve when x $\geq$ 1.96?

# Integrals and Distributions

The area under the curve when x $\geq$ 1.96 is 0.025. The function is symmetric, so the area when x $\leq$ - 1.96 is also 0.025. Together it is 0.05.




**P-values are integrals under the normal curve.** This is where the famous P-value coming from.

One of the most important **assumption** is the continuous variable X follows a normal distribution.

# Section 5: References

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Moore, Will H. A Mathematics Course for Political and Social Research. Princeton, NJ: Princeton University Press, 2013. Ch. 3, 10, 11.

Math is Fun: Introduction to Integration

Calculus: Single Variable Part 3 - Integration