

From the Earth to the Moon

Lucien Huber, Aman Berdyev
lucien.huber@epfl.ch , aman.berdyev@epfl.ch

October 3, 2018

1 Introduction

In this exercise we were tasked with simulating the path of a bullet/rocket from the famous book *De la Terre à la Lune* by Jules Verne. The projectile leaves the earth with an initial velocity v_0 and goes in a straight line towards the Moon.

2 Analytical Computations

2.1 Differential equations

To tackle this problem the axes and origin (at the center of the earth) are positioned and then the different forces acting upon the projectile are considered:

$$m\vec{a} = \sum \vec{F}_{ext} = \vec{F}_{moon-bullet} + \vec{F}_{earth-bullet} + \vec{F}_t \quad (1)$$

Projecting on the z axis gives:

$$m\ddot{z} = \frac{-zGm_{Earth}m}{|z|^3} + \frac{(z_{Moon} - z)Gm_{Moon}m}{|z_{Moon} - z|^3} - \frac{\dot{z}^3 \rho SC_x}{2|\dot{z}|} \quad (2)$$

Thus the two differential equations obtained are:

$$\begin{cases} \frac{dz(t)}{dt} = v(t) \\ \frac{dv(t)}{dt} = \frac{-z(t)Gm_{Earth}}{|z(t)|^3} + \frac{(z_{Moon} - z(t))Gm_{Moon}}{|z_{Moon} - z(t)|^3} - \frac{v(t)^3 \rho SC_x}{2|v(t)|} \end{cases} \quad (3)$$

2.2 Equilibrium state

This section will address the existence of an equilibrium state on the z axis, such that the sum of the gravitational fields of the Earth and the Moon is null. One such equilibrium state $\vec{z}_E = (0, 0, z_E)$ is expected between the two bodies ($z_{Earth} < z_E < z_{Moon}$).

The equilibrium state satisfies

$$\vec{g}_{Earth}(\vec{z}_E) + \vec{g}_{Moon}(\vec{z}_E) = 0 \quad (4)$$

where \vec{g}_{Earth} and \vec{g}_{Moon} are the gravitational fields of the Earth and the Moon, respectively.

The projection on the z axis gives

$$\frac{-z_E G m_{Earth}}{|z_E|^3} + \frac{(z_{Moon} - z_E) G m_{Moon}}{|z_{Moon} - z_E|^3} = 0 \quad (5)$$

The condition $z_{Earth} = 0 < z_E < z_{Moon}$ gives

$$-\frac{G m_{Earth}}{z_E^2} + \frac{G m_{Moon}}{(z_{Moon} - z_E)^2} = 0 \quad (6)$$

$$(m_{Earth} - m_{Moon})z_E^2 - (2m_{Earth}z_{Moon})z_E + m_{Earth}z_{Moon}^2 = 0 \quad (7)$$

The two solutions are of the form

$$z_{E1,2} = z_{Moon} \left(\frac{m_{Earth} \pm \sqrt{m_{Earth}m_{Moon}}}{m_{Earth} - m_{Moon}} \right) \quad (8)$$

Only the solution that satisfies $z_E < z_{Moon}$ is kept

$$z_E = z_{Moon} \left(\frac{m_{Earth} - \sqrt{m_{Earth}m_{Moon}}}{m_{Earth} - m_{Moon}} \right) \quad (9)$$

Thus, $(0, 0, z_E)$ is the only equilibrium state on the z axis situated between the Earth and the Moon. With the reference values given in the file `configuration.in` $z_E = 3.4603 \times 10^8$ m.

2.3 Initial velocity to reach z_E

The numerical simulations will not faithfully replicate the voyage described by Jules Verne. Instead, the focus is set on launching the bullet from the surface to the equilibrium state described above. The initial velocity v_0 will

be computed analytically, so that the bullet reaches z_E at zero speed under the hypothesis that the air drag can be ignored ($\rho_0 = 0$).

The forces of gravity of both the Earth and the Moon are conservative, with the potentials

$$U_{Earth}(z) = -\frac{Gm_{Earth}m}{|z|} \quad (10)$$

$$U_{Moon}(z) = -\frac{Gm_{Moon}m}{|z - z_{Moon}|} \quad (11)$$

Therefore, the mechanical energy of the bullet is conserved:

$$\frac{mv_0^2}{2} + U_{Earth}(z_0) + U_{Moon}(z_0) = U_{Earth}(z_E) + U_{Moon}(z_E) \quad (12)$$

$$\frac{mv_0^2}{2} - \frac{Gm_{Earth}m}{z_0} - \frac{Gm_{Moon}m}{z_{Moon} - z_0} = -\frac{Gm_{Earth}m}{z_E} - \frac{Gm_{Moon}m}{z_{Moon} - z_E} \quad (13)$$

Solving for v_0 gives:

$$v_0 = \sqrt{2G \left(m_{Earth} \left(\frac{1}{z_0} - \frac{1}{z_E} \right) + m_{Moon} \left(\frac{1}{z_E - z_{Moon}} - \frac{1}{z_0 - z_{Moon}} \right) \right)} \quad (14)$$

With the reference values this gives: $v_0 = 1.1066 \times 10^4 \text{ m s}^{-1}$

3 Numerical simulations

Two types of simulations will be conducted. The first simulation takes place over a span of 24 hours and does not take the air drag into account. The second one looks more closely at the movement of the bullet through the atmosphere during 10 seconds. Both of them use the system (3) to determine the acceleration and then compute the velocity and position by iterating the Euler method:

$$a(t) = \sum F_z(z(t), v(t))/m \quad (15)$$

$$v(t + \Delta t) = v(t) + a(t) \cdot \Delta t \quad (16)$$

$$z(t + \Delta t) = z(t) + v(t) \cdot \Delta t \quad (17)$$

3.1 Simulation without air drag ($\rho_0 = 0$)

The simulation conducted was done over 86400s. It was repeated 6 times with 1000, 2000, 4000, 8000, 16000, 32000 iterations. The graphs (Fig. 1, 2)

plot the final position and final velocity with respect to Δt . By observing both graphs the linear convergence of Euler's method becomes clear as the points plotted form a straight line.

3.2 Simulation with air drag ($\rho_0 = 1.3 \text{ kg m}^{-3}$)

The simulation conducted was done over 10s. It was repeated 6 times with 200, 400, 800, 1600, 3200, 6400 iterations. The graphs (Fig. 3, 4) plot the final position and final velocity with respect to Δt . Both graphs confirm the linear convergence of Euler's method.

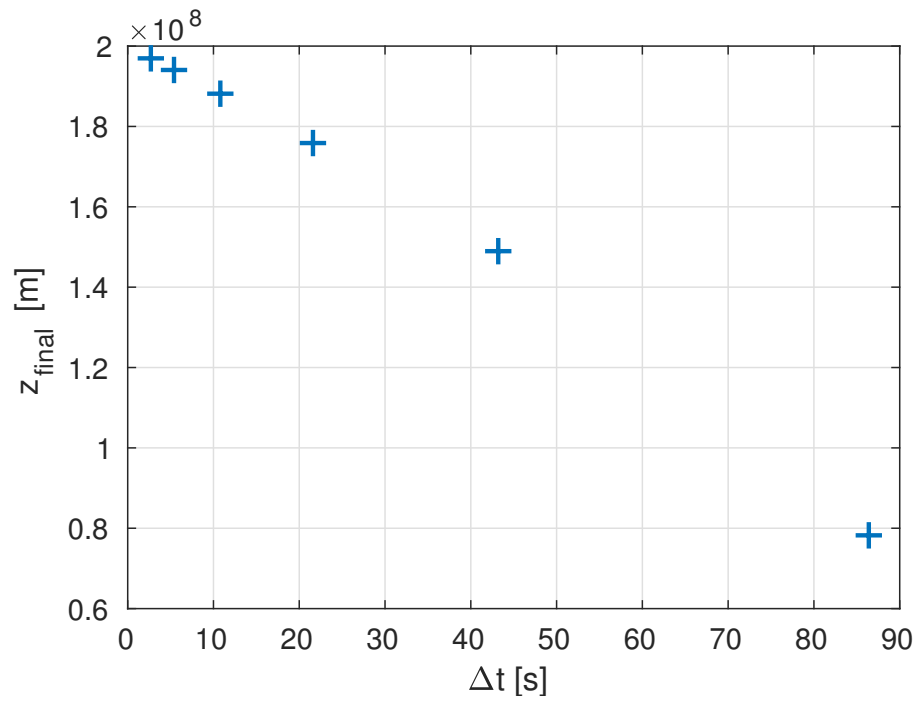


Figure 1: Position of the bullet relative to the time step length with no drag

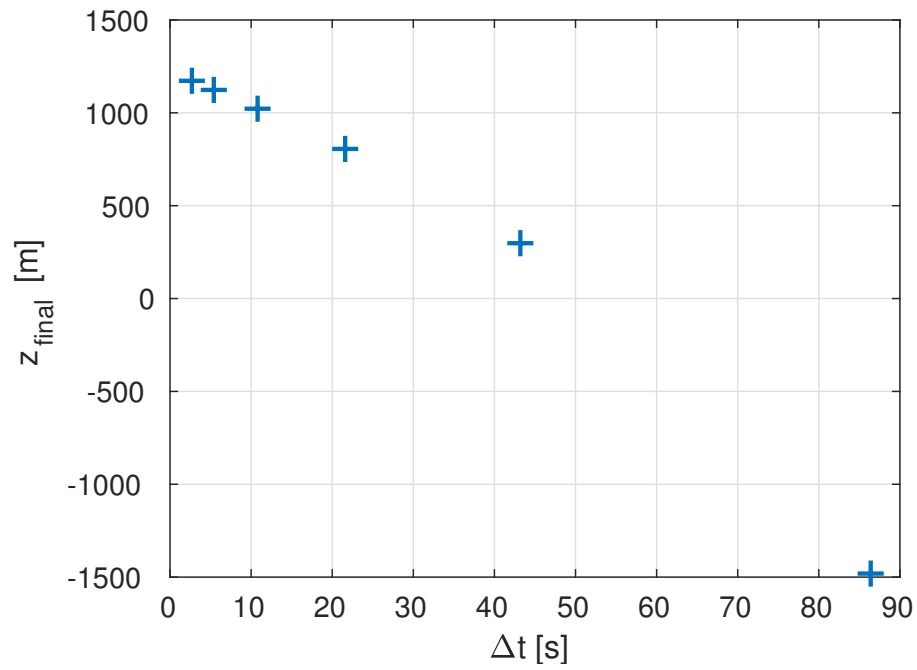


Figure 2: Velocity of the bullet relative to the time step length with no drag

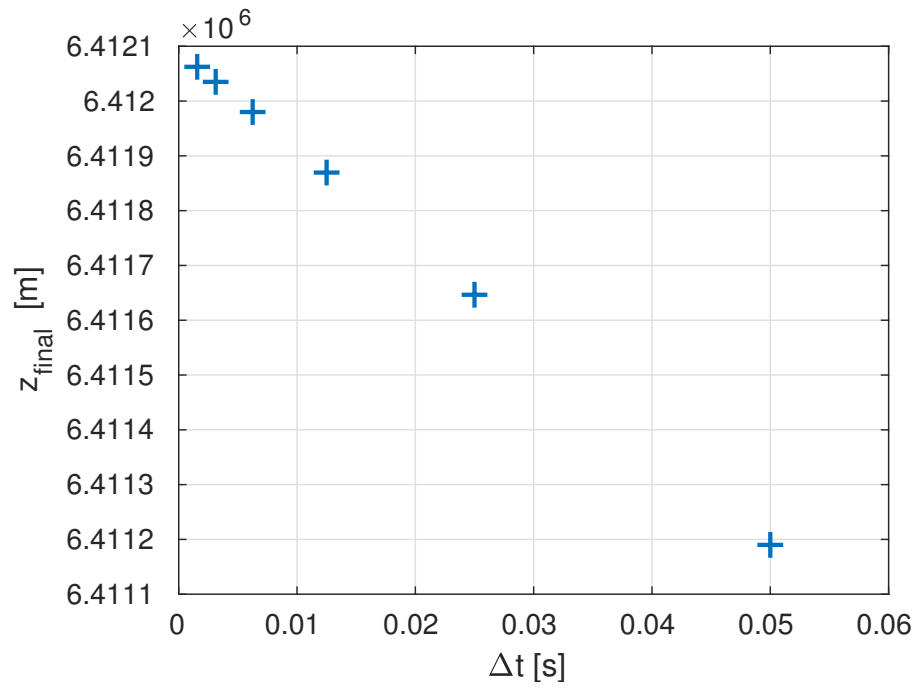


Figure 3: Position of the bullet relative to the time step length with drag

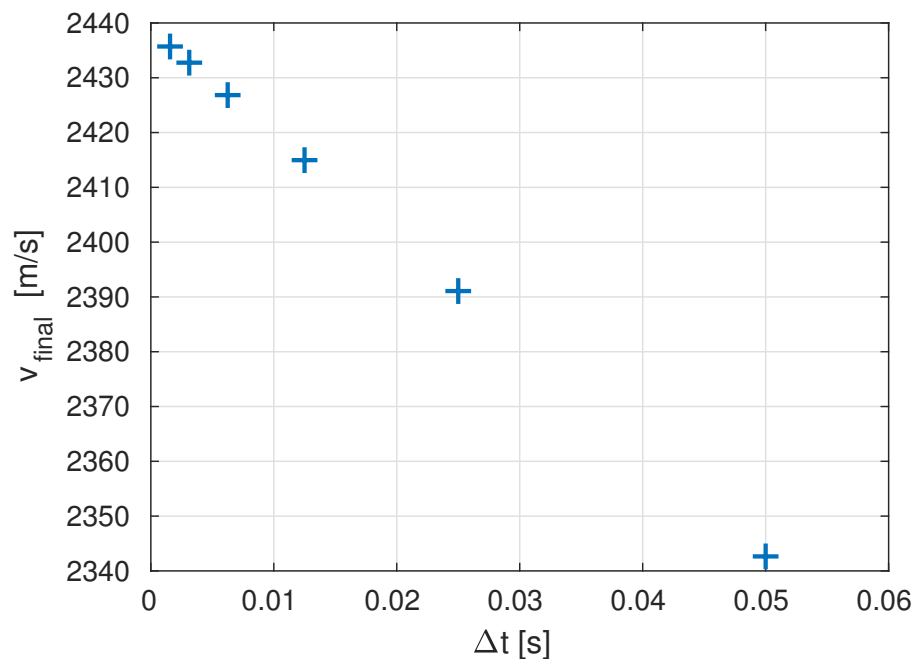


Figure 4: Velocity of the bullet relative to the time step length with drag