Constructing (Super)Space Using Cosets

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Introduction

Lagrangians in which Supersymmetry is manifest require superfields, dependent on 'unnatural' *Grassmann* parameters in addition to the usual spacetime. This begs the question: on what sort of space can such a function be well defined? We are used to producing the Poincaré group by analysis on Minkowski space itself (via the metric) but our starting point in supersymmetry is merely an extended algebra.

In 1872, in his famous Erlangen Programme, Felix Klein proposed that "*geometric properties are characterized by their remaining invariant under the transformations of the principal group*".

Figure 1: Felix Klein

Here 'principal group' refers to the group of isometries of the (Riemannian) metric. It is through this *Kleinian concept* that we endeavour to construct superspace from the algebra of its isometries.

Definitions and Theorems

Let G be a Lie group and M a smooth manifold:

- The **(left) action** of G on M, $G \curvearrowright M$, is a map $\mathcal{L}: G \times M \to M$, $(g,p) \mapsto g \cdot p \equiv \mathcal{L}_g(p)$. The action is **transitive** if $\forall p, q \in M \exists g \in G : p = g \cdot q$. The **isotropy subgroup** of $p \in M$ is $G_p \equiv \{g \in G | g \cdot p = p\}$. For $H \subset G$ acting on G from the **right**, we define the **left coset** to be $gH := \{g \cdot h | h \in H\}$.
- A map $F : M \to N$ on manifolds M, N is **equivariant** if $\mathcal{L}^N_g(F(p)) = F(\mathcal{L}^M_g(p))$
- A **homogeneous space** is a smooth manifold M endowed with a transitive smooth action by a Lie group.

Theorem 1: For G a Lie group and $H \subset G$ a closed subgroup, the left coset space G/H is a topological manifold with unique smooth structure, and is a homogeneous space under the left action $G \curvearrowright G/H$ given by $g_1 \cdot (g_2 H) = (g_1 \cdot g_2)H$.

The Poincaré group is the semi-direct product $ISO(1,3) \equiv \mathcal{R}^{1,3} \rtimes SO^+(1,3)$ with group product:

$$(\Lambda_2, y)(\Lambda_1, x) = (\Lambda_2\Lambda_1, \Lambda_2 x + y), \text{ Identity: } (\mathbb{1}, 0) \quad (1)$$

Minkowski Space

To motivate the case of superspace, we will first construct Minkowski space from the Poincaré group.

- The right action of $SO^+(1,3)$ on $ISO(1,3)$ fixes the translation component:
$$(\Lambda, x)(L, 0) = (\Lambda L, x) = (\mathbb{1}, x)(\Lambda L, 0), \ \Lambda, L \in SO^+(1,3) \quad (2)$$
i.e $SO^+(1,3)$ is the isotropy subgroup for a given pure translation $(\mathbb{1}, x) \in \mathcal{R}^{1,3}$.
- Thus we take the set of left cosets, $(\Lambda, x)SO^+(1,3) = \{(\Lambda, x)(L, 0) | L \in SO^+(1,3)\}$, of $SO^+(1,3)$ in $ISO(1,3)$:
$$ISO(1,3)/SO^+(1,3) = \{(\Lambda_1, x_1)SO^+(1,3), (\Lambda_2, x_2)SO^+(1,3), ...\} \quad (3)$$
Note: $SO^+(1,3)$ a closed Lie subgroup of $ISO(1,3)$ so Theorem 1 gives us our result. However, we will continue so as to make the machinery of this theorem explicit.
- Using (1) and (2), and since ΛL spans $SO^+(1,3)$ provided L does, we can choose $L = \Lambda^{-1}$ wlog:
$$(\Lambda, x)SO^+(1,3) = \{(\Lambda L, x) | L \in SO^+(1,3)\} = \{(\mathbb{1}, x) | L \in SO^+(1,3)\} = (\mathbb{1}, x)SO^+(1,3) \quad (4)$$
$$\implies ISO(1,3)/SO^+(1,3) = \{(\mathbb{1}, x_1)SO^+(1,3), (\mathbb{1}, x_2)SO^+(1,3), ...\} \quad (5)$$
- Now we can define an equivariant map $F : ISO(1,3)/SO^+(1,3) \to \mathcal{M}$ by $(\Lambda, x)SO^+(1,3) \mapsto x$. The equivariance means F induces the transitive action of $\mathcal{R}^{1,3}$ on $\mathcal{M}$,
$$(\mathbb{1}, y) \cdot x = (\mathbb{1}, y)F[(\mathbb{1}, x)SO^+(1,3)] = F[(\mathbb{1}, x)(\Lambda, y)SO^+(1,3)] = F[(\mathbb{1}, x+y)SO^+(1,3)] = x + y \quad (6)$$
and the general Poincaré transformation we expect by $ISO(1,3) \curvearrowright \mathcal{M}$:
$$(\Lambda, y)x = (\Lambda, y)F[(\mathbb{1}, x)SO^+(1,3)] = F[(\Lambda, y)(\mathbb{1}, x)SO^+(1,3)] = F[(\Lambda, \Lambda x + y)SO^+(1,3)] = \Lambda x + y \quad (7)$$
Thus we have constructed Minkowski space as a homogeneous space with Poincaré isometries.

Superspace

To apply this procedure to superspace we want to start with a group but all we have is a *graded* Lie algebra i.e. it contains anticommutators as well as commutators. Assuming central charges are zero, and ignoring internal symmetries, we obtain the following non-zero relations involving the supercharges Q that extend our Poincaré algebra to the $(N = 1)$ superPoincaré algebra:

$$[M_{\mu\nu}, Q^I_\alpha] = i(\sigma_{\mu\nu})_\alpha{}^\beta Q^I_\beta, \quad [M_{\mu\nu}, \bar{Q}^{I\dot{\alpha}}] = i(\bar{\sigma}_{\mu\nu})^{\dot{\alpha}}{}_{\dot{\beta}} \bar{Q}^{I\dot{\beta}}, \quad \{Q^I_\alpha, \bar{Q}^J_{\dot{\beta}}\} = 2\sigma^\mu_{\alpha\dot{\beta}} P_\mu \delta^{IJ} \quad (8)$$

We can exponentiate a normal Lie algebra to produce its Lie group but this does not give well defined group structure when we have anticommutators involved (multiplication of group elements given by the *Baker-Campbell-Hausdorff Formula* depends only on commutators). To solve this we convert them to commutators by contracting the fermionic generators with Grassmann coordinates $\theta^\alpha, \bar{\theta}^{\dot{\beta}}$:

$$(4) \implies \theta^\alpha \{Q_\alpha, \bar{Q}_{\dot{\beta}}\}\bar{\theta}^{\dot{\beta}} = 2\theta\sigma^\mu\bar{\theta}P_\mu \implies [\theta Q, \bar{\theta}\bar{Q}] = 2\theta\sigma^\mu\bar{\theta}P_\mu \text{ and similarly, } [\theta Q, \theta Q] = [\bar{\theta}\bar{Q}, \bar{\theta}\bar{Q}] = 0$$

The algebra we obtain turns out to be that of the group:

$$\overline{OSp(4|1)}, \quad OSp(4|1) \equiv Sp(4, \mathbb{C}) \times O(1, \mathbb{C}) \quad (9)$$

where the bar denotes the *Inönü-Wigner Contraction*, $Sp(n, \mathbb{C})$ the symplectic group and $O(n, \mathbb{C})$ the complex orthogonal group. A general group element is given by: $\exp(ixP + i\theta Q + i\bar{\theta}\bar{Q} + \frac{1}{2}i\omega M)$. Now we consider a 'supertranslation', $\exp(ixP + i\theta Q + i\bar{\theta}\bar{Q})$, i.e. a translation w.r.t both normal and Grassmann coordinates. Like in the Minkowski case, we find these are fixed by $SO^+(1,3)$ so again produce our space via the left coset: $\frac{\overline{OSp(4|1)}}{SO^+(1,3)}$. Since $SO^+(1,3)$ is a closed Lie subgroup of $\overline{OSp(4|1)}$, Theorem 1 gives us our result. We can again define an equivariant map $F : \frac{\overline{OSp(4|1)}}{SO^+(1,3)} \to \mathcal{S}$, $\exp(ixP + i\theta Q + i\bar{\theta}\bar{Q}) \mapsto (x^\mu, \theta, \bar{\theta})$ which induces the correct general supersymmetric transformation on $\mathcal{S}$, again analogous to the Minkowski space procedure.

Conclusion

We conclude with a couple of remarks. Firstly, why construct and understand space in this complicated manner? Below are a number of applications where this approach is illuminating:

- **Supersymmetric Field Theory**: As mentioned earlier, for such field theories our lagrangians must depend on fields defined over a special 'superspace'. We have now given this space rigorous definition.
- **Spontaneous Symmetry Breaking**: In SSB a global symmetry group G is broken down to some subgroup $H \subset G$ and for each broken generator, produces a *Goldstone Boson*. These bosons live in the coset space G/H.
- **Gravity and Supergravity**: Some of the more complicated structures in theories of gravity are elegantly categorised by this analysis of homogeneous spaces. E.g. Anti de Sitter space in n-spatial dimensions is $AdS_n = \frac{O(2,n-1)}{O(1,n-1)}$.

Finally, we note that despite introducing isometries in relation to a (pseudo-)Riemannian metric of a space (it is the behaviour of the metric that defines the geometry) I made no mention of how one might introduce a metric canonically on the spaces we constructed. The answer lies in the *Killing Form* (10) of the corresponding Lie algebra. This is a well defined inner product on a Lie algebra and tells us something fundamental about the structure of a given Lie algebra and in turn the Lie group. From this inner product we can induce a metric on our coset manifold. The details of this can be found in [1], pg 87.

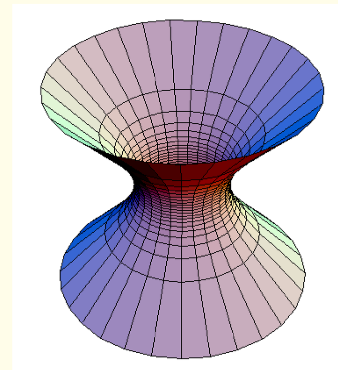

Figure 2: Anti de Sitter Space

$$\kappa(X, Y) = \text{tr}(\text{ad}_X \circ \text{ad}_Y), \ X, Y \in \mathfrak{g} \quad (10)$$

References

[1] P. G. Fré. *Advances in Geometry and Lie algebras from Supergravity*. Springer, 1st edition, 2018.

[2] M. Bertollini. Lectures on supersymmetry. pages 57–64, 2020.