Lecture 31

Friday, April 5, 2024 11:34 AM

A decision problem (or a property P) is semi-decidable if $\exists$ a TM M where

M accepts $x \iff f(x) = 1$

$\iff P(x)$ holds

f is a binary function

[but if $f(x)=0$, M may either reject x or loop forever]

How many TMs can we design?

Each TM can be described by a finite-length string.

So, we have countably many TMs!

However there are uncountably many functions $f: \Sigma^* \to \{0,1\}$ (?)

powerset of natural numbers

So there are more "problems" than "solutions".

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* Any TM M can be described by a string. So, we can potentially give M as input to another TM M'.

* Designing a TM M is equivalent to writing a "code" for a function.

So we sometimes look at M as .

writing ...

So we sometimes look at M as a function in a programming language.

Halting problem: Given description of TM M and string x, decide if M will halt on x.

Halting problem is not decidable, (but is semi-decidable)

We can use the fact that the halting problem is not decidable to show that many other problems are not decidable: by reducing the halting problem to that problem.

Example: show the following is undecidable: given a TM M, decide if M accepts the null string.

Halting Problem Solver (M_1, x):
    if AcceptsNull (M_2)
        return true
    else
        return false

$M_2(y)$:

$\quad M_1(x)$

$\quad$ return true

M_1 halts on x $\iff$ M_2 accepts null

if $\exists$ Total TM that implements AcceptsNull, then so HaltingProblemSolver also always halts and returns the correct output.

Universal TM U:

$$U(M, x) = \begin{cases} \text{accept} & M \text{ accepts } x \\ \text{reject} & M \text{ rejects } x \\ \text{loop} & M \text{ loops on } x \end{cases}$$

We can implement U by just simulating M step-by-step. But U is not a total TM.

$$U'(M, x) = \begin{cases} \text{accept} & M \text{ accepts } x \\ \text{reject} & M \text{ rejects } x \\ \text{reject} & M \text{ loops on } x \end{cases}$$

Can we design such a total TM? we show that its impossible.

Proof:

For contradiction, assume we have such U'.

Based on U', we create another TM $\overline{U}$.

$\overline{U}(M)$:

if $U'(M, M) = \text{true}$

loop forever

else

accept

$$\overline{U}(M) \begin{cases} \text{accepts} & M \text{ rejects } M \text{ or } M \text{ loops on } M \\ \text{loops} & M \text{ accepts } M \end{cases}$$

$\overline{U}(\overline{U})$ what happens?

$\overline{U}$ accept $\overline{U} \Rightarrow$ $\overline{U}$ rejects $\overline{U}$ X

$\overline{U}$ loops on $\overline{U}$ X

$\overline{U}$ loops on $\overline{U} \Rightarrow \overline{U}$ accepts $\overline{U}$ X