Lecture 17

Friday, February 16, 2024 11:39 AM

$A = \{ a^{2^n} : n \geq 0 \}$

* Demon has picked K.

* $x = \varepsilon$, $y = a^{2^K}$, $z = \varepsilon$ $\left\{ \begin{array}{l} xyz \in A \checkmark \\ |y| = 2^K \geq K \end{array} \right.$

* Demon chooses u, v, w
  where $uvw = y$, $v \neq \varepsilon$.

* pick i ?
  [goal: $x u v^i w z = u v^i w \notin A$]

$$y = \overbrace{\underbrace{aaa \cdots}_{u} \mid \underbrace{\cdots aa \cdots}_{v \neq \varepsilon} \mid \underbrace{\cdots \cdots a}_{w}}^{2^K}$$

* what if we pick $i = 0$?

  Does not work, since the demon can pick u, v, w such that

  $v = a^{2^{K-1}}$

  $y = \underbrace{a}_{u}\underbrace{aa}_{v}\underbrace{a}_{w} \longrightarrow uv^0w = aa \in A$

* what if we pick $i = 2$?

* if $u = \varepsilon$, $w = \varepsilon$, $v = y$

  then $uv^2w = v^2 = y^2 = a^{2^{K+1}} \in A$

  $y = \underbrace{aa}_{v} \longrightarrow uv^2w = aaaa \in A$

$y = \underbrace{a a \cdots}_{v} \longrightarrow u v \cdots$

* Note that for any other choice of u, v, w, we are fine.

if $uw \neq \varepsilon \xrightarrow{?} uv^2w \notin A$

Assume $v = a^{\ell}$, $\ell < 2^K$ then

$uv^2w = a^{2^K + \ell}$ but $2^K + \ell$ cannot be a power of two:

$$2^K < 2^K + \ell < 2^K + 2^K = 2^{K+1}$$

*what if we pick $i=3$?

* uv^3w

doesn't work for $w = \varepsilon$,

$u = v = a^{2^{K-1}}$

$$uv^3w = a^{2^{K-1}} a^{(2^{K-1}) \times 3}$$

$$= a^{4(2^{K-1})} = a^{2^{K+1}} \in A$$

$$\boxed{y = \underbrace{aa}_{u}\underbrace{aa}_{v} \longrightarrow uv^3w = uuaa = aaaaaaaa = a^8}$$

However, at least when $u = w = \varepsilon$ and $v = y$ then $i = 3$ works!

$$uv^3w = v^3 = a^{3 \cdot (2^K)}$$

$$2^{K+1} < 3 \cdot 2^K < 2^{K+2}$$

so $3 \cdot 2^K$ is not a power of 2

so $uv^3w \notin A$.

Our winning strategy agaist the demon:

$$\begin{cases} \text{pick } i=3 & \text{when } u=w=\varepsilon \\ \text{pick } i=2 & \text{otherwise.} \end{cases}$$