Turning NFAs into Regular Expressions

$N = N(Q, \Sigma, \Delta, S, F)$

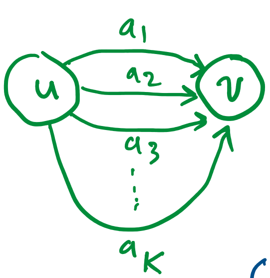
$A^R_{u,v}$: all strings that take us from state u to v by only passing through states in R on our way (even if u (v) $\notin R$, we can still start from it, but (end up in) we should never go back to it. (before)) (have visited)

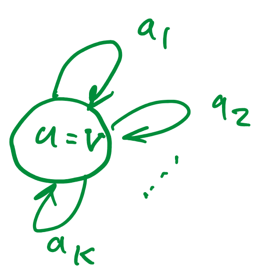
$\alpha^R_{u,v}$: a regular expression such that

$$L(\alpha^R_{u,v}) = A^R_{u,v}$$

Base case : $R = \emptyset = \{\}$

Let $a_1, a_2, \cdots, a_K$ be those symbols in Σ that $v \in \Delta(u, a_i)$.

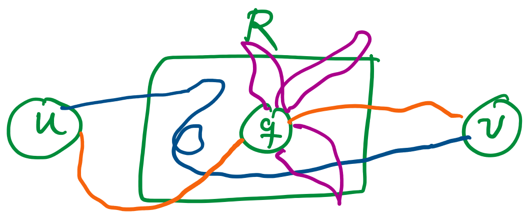




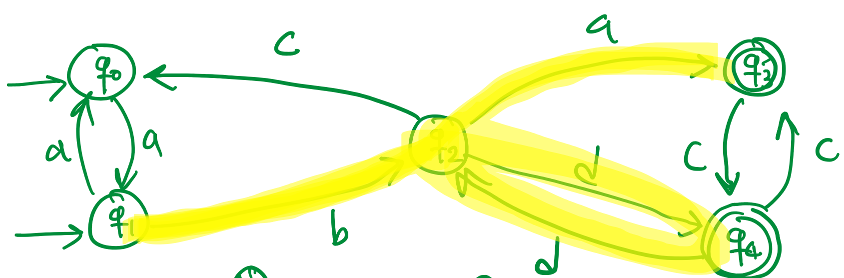
$$\alpha^{\emptyset}_{u,v} = \begin{cases} \varepsilon & u = v,\ K = 0 \\ \emptyset & u \neq v,\ K = 0 \\ \varepsilon + a_1 + \cdots + a_K & u = v,\ K > 0 \\ a_1 + \cdots + a_K & u \neq v,\ K > 0 \end{cases}$$

$$\alpha^{\emptyset}_{u,v} = \begin{cases} \varepsilon & u=v,\ k=0 \\ \emptyset & u\neq v,\ k=0 \\ \varepsilon + a_1 + \cdots + a_k & u=v,\ k>0 \\ a_1 + \cdots + a_k & u \neq v,\ k>0 \end{cases}$$

$$\alpha^{R}_{u,v} = \underline{\alpha^{R-\{q\}}_{u,v}} + \underline{\alpha^{R-\{q\}}_{u,q}} \left(\underline{\alpha^{R-\{q\}}_{q,q}} \right)^{*} \underline{\alpha^{R-\{q\}}_{q,v}}$$



How do we write a regular expression for the NFA?



$$\alpha^{Q}_{q_0,q_3} + \alpha^{Q}_{q_1,q_3} + \alpha^{Q}_{q_0,q_4} + \alpha^{Q}_{q_1,q_4}$$

In general, $S = \{s_1, \ldots, s_m\}$ and $F = \{f_1, \ldots, f_n\}$ then

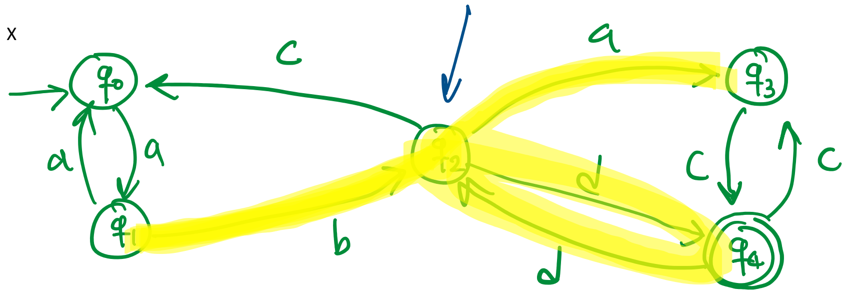
$$\alpha^{Q}_{s_1,f_1} + \alpha^{Q}_{s_1,f_2} + \cdots + \alpha^{Q}_{s_1,f_n} +$$

$\ldots s_1, f_1 \quad \ldots \quad {}_1, f_2 \quad \ldots$

$$+\alpha^{Q}_{s_2,f_1} + \cdots + \alpha^{Q}_{s_2,f_n} +$$

$$\vdots$$

$$+\alpha^{Q}_{s_m,f_1} + \cdots + \alpha^{Q}_{s_m,f_n}$$



$$\alpha^{Q}_{q_0,q_4} = \alpha^{Q-\{q_2\}}_{q_0,q_4} + \alpha^{Q-\{q_2\}}_{q_0,q_2}\left(\alpha^{Q-\{q_2\}}_{q_2,q_2}\right)^{*}\alpha^{Q-\{q_2\}}_{q_2,q_4}$$

$$= \emptyset + a(aa)^{*}b\left(\underbrace{\alpha^{Q-\{q_2\}}_{q_2,q_2}}\right)^{*}\underbrace{\alpha^{Q-\{q_2\}}_{q_2,q_4}}$$

do one more step of recursion
to make these simpler.

Simplifying Regular Expressions

(i) $\alpha + \emptyset \equiv \alpha \equiv \alpha + \alpha \equiv \alpha\varepsilon \equiv \varepsilon\alpha$

(ii) $\alpha(\beta\gamma) \equiv (\alpha\beta)\gamma \equiv \alpha\beta\gamma$

(iii) $(\alpha+\beta)\gamma \equiv \alpha\gamma + \beta\gamma$

(iv) $\alpha\emptyset \equiv \emptyset \equiv \emptyset\alpha$

(v) $\varepsilon + \alpha\alpha^* \equiv \alpha^*$

Exercise: Recall $\alpha \leq \beta \iff L(\alpha) \subseteq L(\beta)$

(i) $\beta + \alpha\gamma \leq \gamma \Longrightarrow \alpha^*\beta \leq \gamma$

(ii) $\beta + \gamma\alpha \leq \gamma \Longrightarrow \beta\alpha^* \leq \gamma$

(structural induction)

$\{a^n b^n : n \geq 0\}$ not regular