

The Pumping Lemma  
(or how to prove a language isn't regular)

$$\{a^n b^n : n \geq 0\}, \{c d a^n b^n e : n \geq 2\}$$

$$\{x \in \{a,b\}^* : x \text{ is a palindrome}\}$$

$$A = \{a^n b^n : n \geq 0\}$$

Let's assume someone came up with

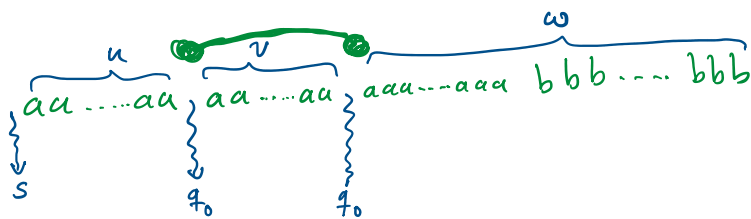
a DFA  $M$  with  $K$  states.

consider  $n > K$ :

$$a^n b^n = \underbrace{a a a \dots a a a}_n \underbrace{b b b \dots b b b}_n$$

$$s \xrightarrow{a} q_0 \xrightarrow{a} q_0 \rightarrow \dots \rightarrow q_i \xrightarrow{b} q_0 \xrightarrow{b} q_0 \rightarrow \dots \rightarrow q_0$$

by pigeonhole principle, there should  
be a repetitive state here ( $n > K$ )



$$\hat{\delta}(s, a^n b^n) \in F,$$

$$\Rightarrow \hat{\delta}(\hat{\delta}(\hat{\delta}(s, u), v), w) \in F$$

$$\hat{\delta}(q_0, v) = q_0, \quad v \neq \epsilon$$

$$\Rightarrow \hat{\delta}(s, u v^i w) \in F \quad \text{for every } i \geq 0$$

[but it shouldn't accept these for  $i \neq 1$ ]

What about other examples?

Pumping Lemma: Contrapositive form

If the following holds then set  $A$

is not regular:

For all  $K \geq 0$ ,

there exist strings  $x, y, z$ ,

such that  $xyz \in A$ ,  $|y| \geq K$ ,

and for all strings  $u, v, w$  that

satisfy  $y = uvw$ ,  $v \neq \epsilon$  there  
exists  $i \geq 0$  such that  $xuv^i wz \notin A$

$$A = \{cd a^n b^n b : n \geq 0\}$$

pick:  $x = cd$ ,  $y = a^K$ ,  $z = b^{K+1}$   
 $xyz \in A \checkmark$   $|y| \geq K \checkmark$

pick:  $i = 0$

$$xuv^0 wz \notin A \quad ?$$

$$\Leftrightarrow cd a^{\ell} a^r b^{K+1} \notin A$$

$$\Leftrightarrow cd a^{\ell+r} b^{K+1} \notin A$$

which holds since  $\ell + r < K$

$$\begin{aligned} y &= uvw \\ u &= a^{\ell}, v = a^r, \\ &w = a^m \\ r + \ell + m &= K \end{aligned}$$

## Pumping Lemma as a Game

Our goal is to prove  $A$  is not regular. The demon's goal is the opposite. The game is played as follows:

① Demon picks  $K \geq 0$

② we pick  $x, y, z$  satisfying  $\{xyz \in A$

② we pick  $x, y, z$  satisfying  $\begin{cases} xyz \in A \\ |y| \geq k \end{cases}$

③ Demon picks  $u, v, w$  that satisfy  $\begin{cases} uvw = y \\ v \neq \epsilon \end{cases}$

④ we pick  $i \in \{0, 2, 3, 4, \dots\}$

At the end if  $xuv^i w z \notin A$  then we will win.

$A$  is not regular if we have a winning strategy no matter what the demon does.

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Prove  $A = \{a^{2^n} : n \geq 0\}$  is not regular.  
 $A = \{a, aa, a^4, a^8, a^{16}, a^{32}, \dots\}$