Lecture 16

Thursday, February 15, 2024 11:26 AM

The Pumping Lemma

(or how to prove a language isn't regular)

$\{a^n b^n : n \geq 0\}$, $\{c d a^n b^n e : n \geq 2\}$

$\{x \in \{a, b\}^* : x \text{ is a palindrome}\}$

$A = \{a^n b^n \; n \geq 0\}$

Let's assume someone came up with a DFA M with K states.

consider $n > K$:

$$a^n b^n = \underbrace{aaa \cdots aaa}_{n}\underbrace{bbb \cdots bbb}_{n}$$

$$\underbrace{s \xrightarrow{a} \bigcirc \xrightarrow{a} \bigcirc \rightarrow \cdots \xrightarrow{a} q_1}\xrightarrow{b} \bigcirc \xrightarrow{b} \bigcirc \rightarrow \cdots \xrightarrow{b} \bigcirc$$

by pigonhole principle, there should be a repetative state here ($n > K$)

$$\underbrace{aa \cdots aa}_{u}\,\underbrace{aa \cdots aa}_{v}\,\underbrace{aaa \cdots aaa\; bbb \cdots bbb}_{w}$$

s $\quad q_0 \quad q_0$

$\hat{\delta}(s, a^n b^n) \in F$,

$\Rightarrow \hat{\delta}(\hat{\delta}(\hat{\delta}(s, u), v), w) \in F$

$\hat{\delta}(q_0, v) = q_0$, $v \neq \varepsilon$

$\Rightarrow \hat{\delta}(s, u v^i w) \in F$ for every $i \geq 0$

[but it shouldn't accept these for $i \neq 1$]

What about other examples?

Pumping Lemma: Contrapositive form

If the following holds then set A

is not regular:

For all $K \geq 0$,

there exist strings x, y, z,

such that $xyz \in A$, $|y| \geq K$,

and for all strings u, v, w that

satisfy $y = uvw$, $v \neq \varepsilon$ there

exists $i \geq 0$ such that $xuv^iwz \notin A$

$$A = \{ cd\, a^n b^n b : n \geq 0 \}$$

pick: $x = cd$, $y = a^K$, $z = b^{K+1}$

$xyz \in A$ ✓ $|y| \geq K$ ✓

pick: $i = 0$

$$x u v^0 w z \overset{?}{\notin} A$$

$$\Leftrightarrow cd\, a^l a^r b^{K+1} \notin A$$

$$\Leftrightarrow cd\, a^{l+r} b^{K+1} \notin A$$

$y = uvw$

$u = a^l$, $v = a^m$, $w = a^r$

$r + l + m = K$

which holds since $l + r < K$

Pumping Lemma as a Game

Our goal is to prove A is not reglar. The demon's goal is the opposite. The game is played as follows:

1. Demon picks $K \geq 0$
2. We pick x, y, z satisfying $\{ xyz \in A$

(2) we pick x, y, z satisfying $\begin{cases} xyz \in A \\ |y| \geq k \end{cases}$

(3) Demon picks u, v, w that satisfy $\begin{cases} uvw = y \\ v \neq \varepsilon \end{cases}$

(4) we pick $i \in \{0, 2, 3, 4, \dots\}$

At the end if $xuv^iwz \notin A$ then we will win.

A is not regular if we have a winning strategy no matter what the demon does.

Prove $A = \{ a^{2^n} : n \geq 0 \}$ is not regular.

$A = \{a, aa, a^4, a^8, a^{16}, a^{32}, \dots\}$