Lecture 21

Tuesday, March 12, 2024 11:42 AM

$$B = \{ x \in \{], [\}^*, \; x \text{ is "balanced"} \}$$

* x is said to be balanced if:

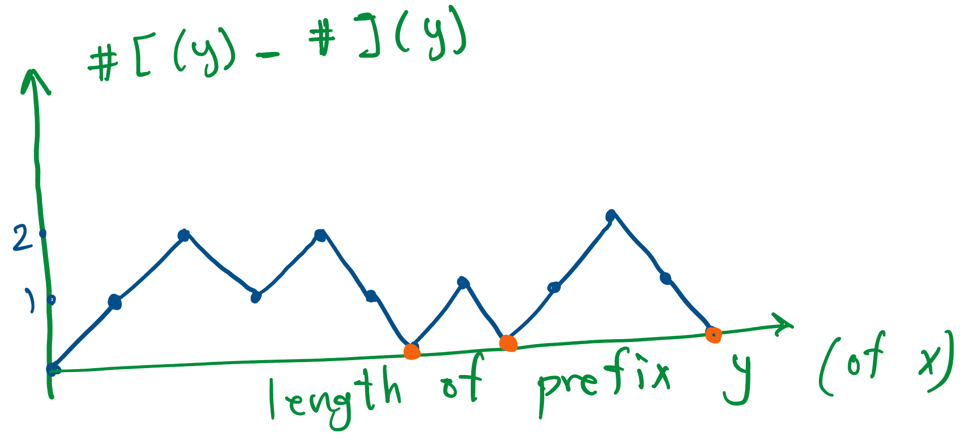
1. $\#[(x) = \#](x)$
2. $\forall$ prefix y of x, $\#[(y) \geq \#](y)$

Proposed Grammar G:

$$S \to SS \mid [S] \mid \varepsilon$$

Question: prove $L(G) = B$?!

$$x = \underbrace{[[][]]}\underbrace{[]}\underbrace{[[]]} \quad \text{3 blocks!}$$



Proof steps:

1. $L(G) \subseteq B$
2. $B \subseteq L(G)$

We can extend the defenition of "balanced strings" to $(N \cup \Sigma)^*$ as well by just ignoring nonterminals.

$[S]$, valid

$S[S]S[[S]S]$, ... valid , ...

Proof of $L(G) \subseteq B$:

We need to show that if $S \xrightarrow[G]{*} \alpha$ then α is balanced.

Base case: $S \xrightarrow[G]{0} \alpha \implies \alpha = S$ and "S" is balanced

Inductive hypothesis:

if $S \xrightarrow[G]{n} \alpha$ then α is balanced

Now show if $S \xrightarrow[G]{n+1} \beta$ then β is balanced.

we know α is balanced

$S \xrightarrow[G]{n} \alpha \xrightarrow[G]{1} \beta$, $\alpha = \beta_1 S \beta_2$

these could include S, [,], ...

$$\text{cases:} \begin{cases} \beta = \beta_1 [S] \beta_2 \ (?) \checkmark \\ \beta = \beta_1 \beta_2 \ \checkmark \\ \beta = \beta_1 S S \beta_2 \ \checkmark \end{cases}$$

Step 2: $B \subseteq L(G)$

induction on $|x|$ $(x \in B)$

base case: $|x| = 0 \Rightarrow x = \varepsilon \in L(G)$

Inductive hypothesis:

$\forall x \in B$ that $|x| \leq n$, we have $S \xrightarrow[G]{*} x$

Now show that

$\forall y \in B$ that $|y| = n+1$ we have $S \xrightarrow[G]{*} y$

* If y has only "one block"

(there is no proper prefix of y that is valid) then

$y = [x]$ where x is valid (why?)

and so $S \xrightarrow[G]{1} [S] \xrightarrow[G]{*} [x]$

so $S \xrightarrow[G]{*} y$ ✓

* If y has more than one block (there exist a proper and balanced prefix of y called z)

then $y = zw$ where

$z \in B$, $w \in B$, $z \neq \varepsilon$, $w \neq \varepsilon$

$z \in B$, $w \in B$, $z \neq \varepsilon$, $w \neq \varepsilon$

so $\quad S \xrightarrow[G]{1} SS \xrightarrow[G]{*} zS \xrightarrow[G]{*} zw$

so $\quad S \xrightarrow[G]{*} y$ ✓