Solutions_A1

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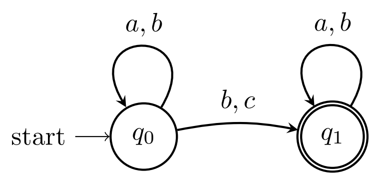
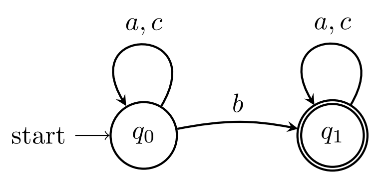
Answer 1

Part a)

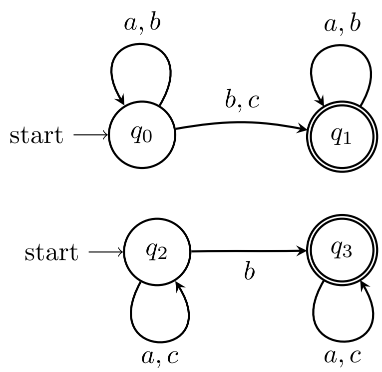


Part b)



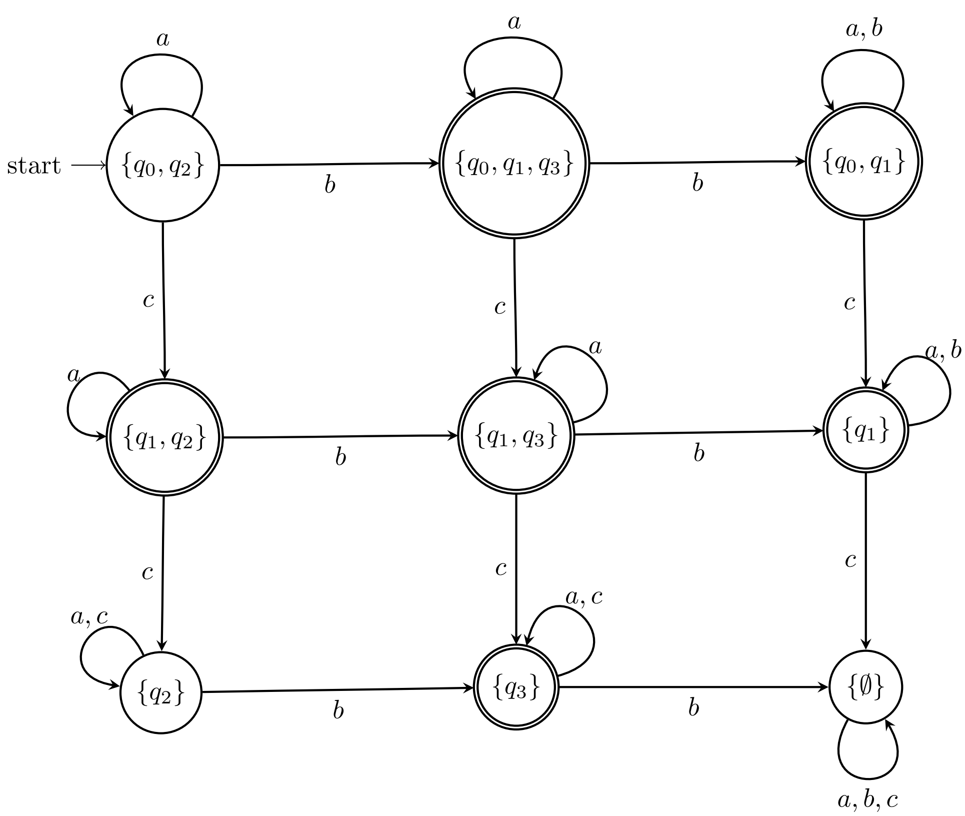
Part c)

This can have potentially two solutions:
1. In this one, you do not change the above NFA's and can just place them below with their state names changed, this will be a disjoint union. This works because a NFA can multiple start states.



Part d)

1. This is the DFA for the NFA in part 1



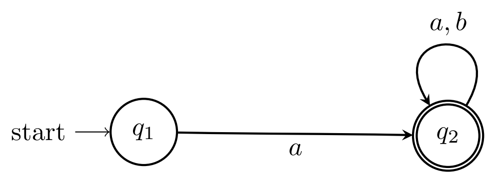
Answer2

The statement L(N_1) = $\sim L(N_2)$ is false, and this can be proven by giving a counterexample.

Let N_1 be a NFA with:

1. Q = q_1, q_2
2. $\Sigma = a, b$
3. $S = q_1$
4. $F = q_2$
5. $L(N_1)$ = { all strings starting with a } = { a,aa,aaa,aaaa,ab,aab ...... }

This is the NFA for the respective language given above, i.e this is N_1 :

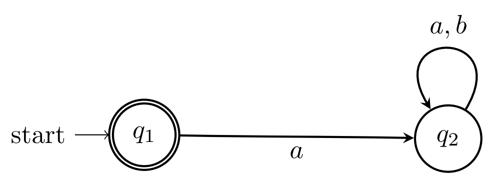


From the question given we can see, that N_2 has the final states as $Q - F$, which is basically the non-accept states of N_1. Hence, the states get flipped, which means that all the accept states of N_1 become the non-accept ones for N_2, and the non-accept ones become the accept ones.

Thus, N_2 be a NFA with:

1. Q = q_1, q_2
2. $\Sigma = a, b$
3. $S = q_1$
4. $F = q_1$
5. $L(N_2)$ = { only accepts the null string } = { ϵ }

This is the NFA, for the respective language given above, i.e for N_2:



Furthermore, we have to find the complement of the language of N_2, as is given in the question.

$\sim L(N_2) = \Sigma^* - \{\ \epsilon\} \ = \Sigma^+$

From the above statement, we have proved that $L(N_1) \neq \ \sim L(N_2)$, because $\sim L(N_2)$, is the set of all strings excluding the empty string which might contain strings like "ba,bb" and so on. However $L(N_1)$ contains all strings starting with "a", and wont contain strings like "ba,bb, bba". Thus both of them are not equal.

Answer 3

In order to prove A is regular, we can first show that A is essentially the set $\{a, b\}^*$. Then it is very easy to show that $\{a, b\}^*$ is regular.
Let A $= \{xy|x, y \in \{a, b\}^*, \#a(x) = \#b(y)\}$, and B $= \{a, b\}^*$. We can then show by induction, on the length of the strings n, that all strings of length n present in L(A), are equivalent to all strings of length n present in L(B). Thus proving that both languages are equivalent, and hence A = B.

Proof. **Base Case:** When $n = 0$, L(A) contains ϵ, which can be achieved by setting $x = \epsilon$ and $y = \epsilon$, thus holding the property $\#a(x) = \#b(y)$. This is also equivalent to $\{a, b\}^0 = \epsilon$. Hence the base case holds.
Induction Hypothesis: We assume that all the strings of length n present in L(A) are equivalent to all the strings of length n present in L(B). Which means that $\{x \in \{a, b\}^* \mid |x| = n\} = \{z \in \{xy \mid x, y \in \{a, b\}^*, \#a(x) = \#b(y)\} \mid |z| = n\}$
Inductive Step: We now need to prove that all strings of length $n + 1$ are equivalent in the languages of both the sets i.e $\{x \in \{a, b\}^* \mid |x| = n + 1\} = \{z \in \{xy \mid x, y \in \{a, b\}^*, \#a(x) = \#b(y)\} \mid |z| = n + 1\} - - - - - > 1$
From the definition of set concatenation and powers of sets as defined in the book we know that:
$\{a, b\}^{n+1} \stackrel{\text{def}}{=} \{a, b\}\{a, b\}^n$
$\{xy \mid x, y \in \{a, b\}^*, \#a(x) = \#b(y)\}^{n+1} \stackrel{\text{def}}{=} \{xy \mid x, y \in \{a, b\}^*, \#a(x) = \#b(y)\}\{xy \mid x, y \in \{a, b\}^*, \#a(x) = \#b(y)\}^n$

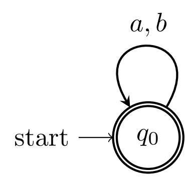
Hence Equation 1 from the above can be broken down as follows:
$\{x \in \{a, b\}\}\{x \in \{a, b\}^* \mid |x| = n\} = \{z \in \{xy \mid x, y \in \{a, b\}^*, \#a(x) = \#b(y)\}\}\{z \in \{xy \mid x, y \in \{a, b\}^*, \#a(x) = \#b(y)\} \mid |z| = n + 1\}$
In the above equation, we already know that the sets $\{x \in \{a, b\}^* \mid |x| = n\}, \{z \in \{xy \mid x, y \in \{a, b\}^*, \#a(x) = \#b(y)\} \mid |z| = n\}$ are equivalent, from our Induction Hypothesis.
Thus we only need to show that $\{x \in \{a, b\}\} = \{z \in \{xy \mid x, y \in \{a, b\}^*, \#a(x) = \#b(y)\}\}$. This is because, concatenation of the equivalent sets on LHS and RHS is equal.

To prove: $\{x \in \{a, b\}\} = \{z \in \{xy \mid x, y \in \{a, b\}^*, \#a(x) = \#b(y)\}\}$

Subproof. The above statement implies that we have to prove that all strings which are of length 1 in both the sets are equivalent.
Case 1: For our set $\{z \in \{xy \mid x, y \in \{a, b\}^*, \#a(x) = \#b(y)\}\}$, z = a, which can be generated by $x = \epsilon$ and $y = a$. Thus z = a, is equivalent to x = a from our set $\{x \in \{a, b\}\}$.
Case 2: For our set $\{z \in \{xy \mid x, y \in \{a, b\}^*, \#a(x) = \#b(y)\}\}$, z = b, which can be generated by $x = b$ and $y = \epsilon$. Thus z = b, is equivalent to x = b from our set $\{x \in \{a, b\}\}$.
Hence all strings which are of length 1 in both the sets are equivalent. Thus the sets are equivalent.

From the subproof above and our IH, we have proved that languages for both sets contains the same sets of strings.Thus the sets are equivalent. □

Thus now if we can prove that $B = \{a, b\}^*$ is regular, then we have also proved that A is regular. We can prove the following by making a DFA for the same, and proving its correctness by induction.
DFA:



To prove: $B = \{a, b\}^*$ is regular by Induction.

Proof. Firstly we assume one set as follows:
$C = \{\{a, b\}^n \mid n \geq 0\}$
We claim that $\forall x$,

$$\hat{\delta}(s, x) = \begin{cases} q_0 & x \in C \end{cases}$$

We prove it by induction on the length of x.
Base case: $|x| = 0$, $\hat{\delta}(s, \epsilon) = q_0 \rightarrow \epsilon \in C$
Inductive Step: The induction assumption holds for $|x| = n$, so we prove it $|x| = n + 1$.

$$\hat{\delta}(s, xc) = \begin{cases} \delta(q_0, c) & x \in C \end{cases}$$

Since there is only state in the DFA, thus there is only one case.
Case 1: $x \in A$

$$\delta(q_0, c) = \begin{cases} q_0 & c = a \\ q_0 & c = b \end{cases}$$

$\square$

Hence proved.