COMPSCI 2AC3 Assignment 2 Solution

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Question 1

Let $\Sigma = \{0, 1\}$. The reverse of a string x denoted by $\mathrm{rev}(x)$ and is defined by the following recursive rule:

- $\mathrm{rev}(\epsilon) = \epsilon$
- $\forall x \in \Sigma^*, \forall a \in \Sigma, \mathrm{rev}(xa) = a.\mathrm{rev}(x)$

For any set $A \subseteq \Sigma^*$, define $\mathrm{rev}(A)$ to be:

$$\mathrm{rev}(A) = \{\mathrm{rev}(x) : x \in A\}$$

Can we *always* (i.e., for any $A \subseteq \Sigma^*$) say that if A is regular then so is $\mathrm{rev}(A)$? Prove your answer.

Solution:
Yes, if A is regular, then $\mathrm{rev}(A)$ must also regular. We prove this by showing that for any DFA M that accepts A, we can create a new NFA N that accepts $\mathrm{rev}(A)$.

Proof. If A is regular, then there exists a DFA $M = (Q, \Sigma, \delta, s, F)$ that accepts A. Then we construct an NFA $N = (Q', \Sigma, \delta', s', F')$ that accepts $\mathrm{rev}(A)$ as follows:

(a) Every final state of M becomes a start state for N. We do this by creating a new state that is the only start state for N and having epsilon transitions to all states in F. Formally:

$$\begin{aligned} s' &= t \\ \delta'(s', \epsilon) &= f \qquad\qquad \forall f \in F \end{aligned}$$

(b) Make the start state of M the sole final state of N. That is:

$$F' = s$$

(c) Reverse the direction of all transitions of M for N. We define this as follows:

$$\delta'(k,i) = \{Q_j \in Q : \delta(Q_j,i) = k\} \qquad i \in \Sigma \text{ and } k \in Q \tag{1}$$

Now we must prove that this NFA N accepts only $\text{rev}(A)$. We do this by showing that x is accepted by M iff $\text{rev}(x)$ is accepted by N. Formally,

$$\forall x \in A, \forall q \in Q, \forall R \subseteq Q, \hat{\delta}(q,x) \in R \equiv q \in \hat{\delta}'(R, \text{rev}(x)) \tag{2}$$

Let $n = |x|$. Then we will prove (2) by induction on n.

Base Case. $|x| = 0$
Since we know $x = \epsilon$, we have:

$$\begin{aligned}
\hat{\delta}(q,\epsilon) \in R &\equiv q \in R && \text{by (3.1), Kozen's book pg.16} \\
&\equiv q \in \hat{\delta}'(q,\epsilon) && \text{by (1)} \\
&\equiv q \in \hat{\delta}'(q,\text{rev}(\epsilon)) && \text{Base case of rev}(x)
\end{aligned}$$

Thus the base case holds.

Inductive Step. We assume as our inductive hypothesis that (2) holds for $|x| = n$. Then we must show that it also holds for $|xi| = n+1$, where $i \in \Sigma$. We have:

$$\begin{aligned}
\hat{\delta}(q,xi) \in R &\equiv \delta(\hat{\delta}(q,x),i) \in R && \text{by (3.2), Kozen's book pg.16} \\
&\equiv \exists t \in R, \delta(\hat{\delta}(q,x),i) = t && \text{name the state t} \\
&\equiv \exists t \in R, \hat{\delta}(q,x) \in \delta'(t,i) && \text{by (1)} \\
&\equiv \hat{\delta}(q,x) \in \bigcup_{t \in R} \delta'(t,i) && \text{by definition of } \cup \\
&\equiv \hat{\delta}(q,x) \in \delta'(R,i) && \text{by (6.2), Kozen's book pg.33} \\
&\equiv q \in \hat{\delta}'(\hat{\delta}'(R,i),\text{rev}(x)) && \text{by inductive hypothesis} \\
&\equiv q \in \hat{\delta}'(R,i.\text{rev}(x)) && \text{by Lemma 6.1, Kozen's book pg.34} \\
&\equiv q \in \hat{\delta}'(R,\text{rev}(xi)) && \text{by inductive step of rev}(x)
\end{aligned}$$

Hence the inductive step holds and we have shown that N accepts only $\text{rev}(A)$. Therefore we have shown that $\text{rev}(A)$ is regular for any regular set A. $\square$

Question 2

For each of the following subsets of $\{0,1\}^*$, tell whether the set is regular or not. Prove your answer.

2a)

$\{x \mid x \in \{0,1\}^*, \#1(x) - \#0(x) < 5\}$

Solution:
Let $A = \{x \mid x \in \{0,1\}^*, \#1(x) - \#0(x) < 5\}$. A is not regular and we will show this using pumping lemma.

Proof. For any $k \geq 0$, consider $xyz = 1^{4+k}0^k$ where $x = 1^4$, $y = 1^k$ (so $|y| \geq k$), and $z = 0^k$.
Since $\#1(x) - \#0(x) = (4+k) - k = 4 < 5$, we know that $xyz \in A$.
Observe that y consists only of 1's, so u, v, w must also contain only 1's.

For any $v = 1^j$, where $0 < j \leq k$ (since v cannot be the empty string), we have $u = 1^l$ and $w = 1^{k-l-j}$, where $0 \leq l \leq k-j$.
By choosing $i = 2$, we have

$$\begin{aligned} xuv^iwz &= xuv^2wz \\ &= xuvvwz \\ &= 1^4 1^l 1^j 1^j 1^{k-l-j} 0^k \\ &= 1^{4+k+j} 0^k. \end{aligned}$$

Then $\#1(x) - \#0(x) = (4+k+j) - k = 4 + j$.
However, $j > 0$, so it follows that $j \geq 1$ and so $4 + j \geq 4 + 1 = 5$. This means that $\#1(x) - \#0(x) \geq 5$.
Thus, $xuv^2wz \notin A$. Therefore, by pumping lemma, A is not regular. $\square$

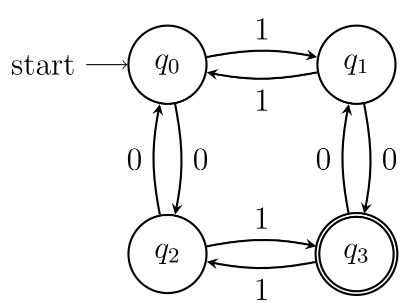
2b)

$\{x \mid x \in \{0,1\}^*, \#1(x) \cdot \#0(x) \text{ is an odd number}\}$

Solution:
Let $B = \{x \mid x \in \{0,1\}^*, \#1(x) \cdot \#0(x) \text{ is an odd number}\}$. The set B is regular and we will create a DFA that accepts B.

We first make the obversation that the product of two integers is odd if and only if both of those integers are also odd.
This means that a string x is in B iff $\#0(x)$ is odd and $\#1(x)$ is odd.

The DFA that accepts this language is



Proof. To show that the DFA accepts only the language $L(B)$, we consider all possible strings and show that a string is only accepted if $\#0(x)$ is odd and $\#1(x)$ is odd.
Let $P(n)$ be the statement that, for all strings w with $|w| = n$, the following statements hold:

- $P_0(|w|) \equiv \forall w \cdot (\hat{\delta}(q_0, w) = q_0 \Rightarrow \#0(w)$ is even and $\#1(w)$ is even)
- $P_1(|w|) \equiv \forall w \cdot (\hat{\delta}(q_0, w) = q_1 \Rightarrow \#0(w)$ is even and $\#1(w)$ is odd)
- $P_2(|w|) \equiv \forall w \cdot (\hat{\delta}(q_0, w) = q_2 \Rightarrow \#0(w)$ is odd and $\#1(w)$ is even)
- $P_3(|w|) \equiv \forall w \cdot (\hat{\delta}(q_0, w) = q_3 \Rightarrow \#0(w)$ is odd and $\#1(w)$ is odd)

We note that if $P(n)$ is true, then a string will only end in the accept state if it matches the condition given for B.

We claim then that if $P(|w|)$ is true for all strings $w \in \{0,1\}^*$, then the given DFA matches the language $L(B)$. We prove $P(|w|)$ by induction over the length of the string $|w|$.

Base Case. $|w| = 0$
For $w = \epsilon$, $\hat{\delta}(q_0, w) = q_0$. Since ϵ contains zero 0's and zero 1's, it has an even number of 1's and 0's. Thus $P_0(0)$ holds. Note that the antecedents for $P_1(0), P_2(0), P_3(0)$ are false and so the implications are true.

Inductive Step. Assume $P(n)$ holds, show that $P(n+1)$ also holds
For $|w| = n+1$, we know $w = xi$ where $x \in \{0,1\}^*$ such that $|x| = n$ and $i \in \{0,1\}$.
Then for $P(n+1)$, we get

$$\begin{aligned}\hat{\delta}(q_0, w) &= \hat{\delta}(q_0, xi) \\ &= \delta(\hat{\delta}(q_0, x), i) \\ &= q_0.\end{aligned}$$

This means that for $\delta(\hat{\delta}(q_0, x), i) = q_0$ to be true, $\hat{\delta}(q_0, x)$ must follow one of two cases:

1. $i = 0$ and $\hat{\delta}(q_0, x) = q_2$, or
2. $i = 1$ and $\hat{\delta}(q_0, x) = q_1$.

For the first case, by the inductive hypothesis, x contains an odd number of 0's, and an even number of 1's. So xi contains an even number 0's and an even number of 1's.
Similarly for the second case, by the inductive hypothesis, x contains an even number of 0's and an odd number of 1's. Then xi contains an even number of 0's and an even number of 1's.
Thus, $P_0(n+1)$ holds.

We can construct a similar argument for each of $P_1(n+1)$, $P_2(n+1)$, and $P_3(n+1)$. They are not shown here as it is trivial to explicitly write out the proofs for these cases.

Since we have proven that $P(|w|)$ is true for all strings w, by our claim, we have shown that the DFA matches the language $L(B)$. As we have constructed a DFA for B, we have shown that it is regular. □

2c)

$\{xx \mid x \in \{0,1\}^*\}$

Solution: Let $C = \{xx \mid x \in \{0,1\}^*\}$. C is not regular and we will show this using pumping lemma.

Proof. For any $k \geq 0$, consider $xyz = 0^k 1^k 0^k 1^k$ where $x = 0^k$, $y = 1^k$ (so $|y| \geq k$), and $z = 0^k 1^k$.
It is clear that any string of this form is in the set C as it is the string $0^k 1^k$ twice.
Observe that y consists only of 1's, so u, v, w must also contain only 1's.

Then, for any $v = 1^j$ for $0 < j \leq k$, we have $u = 1^l$ and $w = 1^{k-l-j}$ for $0 \leq l \leq k - j$.
We choose $i = 0$. Then we have

$$\begin{aligned} xuv^i wz &= xuv^0 wz \\ &= xuwz \\ &= 0^k 1^l 1^{k-l-j} 0^k 1^k \\ &= 0^{k-j} 1^k 0^k 1^k. \end{aligned}$$

In the first part of the string, we have $0^{k-j} 1^k$, while in the second part, we have $0^k 1^k$. Since $j > 0$, it follows that $k - j \leq k - 1 \neq k$.
Thus, $xuv^0 wz \notin C$. Therefore, by pumping lemma, C is not regular. $\square$