Automata: Short Course

COMP SCI 2SD3

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Sequences

Alphabet: an *arbitrary* (usually finite) set of elements, often denoted by the symbol Σ.

Sequence:

- an element $x = (a_1, a_2, \ldots, a_k) \in \Sigma^k$, where Σ^k is a Cartesian product of Σ's.
  For convenience we write $x = a_1 a_2 \ldots a_k$.
- a function $\phi : \{1, \ldots, k\} \to \Sigma$, such that $\phi(1) = a_1, \ldots, \phi(k) = a_k$.

♣ The two above definitions are in a sense identical since: $\underbrace{\Sigma \times \ldots \times \Sigma}_{n} \equiv \{f \mid f : \{1, \ldots, k\} \to \Sigma\}$.

- Frequently a *sequence* is considered as a primitive undefined concept that is understood and does not need any explanation.

Sequences and strings

- If the elements of Σ are *symbols*, then a *finite* sequence of symbols is often called a *string* or a *word*.

♣ In concurrency theory *sequences* are often called **traces** (for example in the textbook for this course).

- The *length* of a sequence x, denoted $|x|$, is the number of elements composing the sequence. For example $|aba| = 3$, $|aabbc| = 5$.
- The *empty sequence*, ε, is the sequence consisting of zero symbols, i.e. $|\varepsilon| = 0$.
- A *prefix* of a sequence is any number of leading symbols of that sequence, and a *suffix* is any number of trailing symbols (any number means 'zero included'). For example a sequence (word, trace) *abca* has the following prefixes: ε, *a*, *ab*, *abc*, *abca*, and the following suffixes: *abca*, *bca*, *ca*, *a*, ε.

Concatenation

- *Concatenation* (operation)
  Let $x = a_1 \dots a_k$, $y = b_1 \dots b_l$. Then
  $$x \circ y = a_1 \dots a_k b_1 \dots b_l.$$
  We usually write xy instead of $x \circ y$.
- Properties of concatenation:
  1. $x(yz) = (xy)z$
  2. $\varepsilon x = x\varepsilon = x$

  Fact. A triple $(\Sigma, \circ, \varepsilon)$ is a *monoid* (recall 2LC3).
- Power operator: $x^0 = \varepsilon$, $x^1 = x$ and $x^k = \underbrace{x \dots x}_{k}$.
- Recursive definition of power:
  $$\begin{aligned} x^0 &= \varepsilon \\ x^{k+1} &= x^k x. \end{aligned}$$

Σ^* and Formal Language

- Let Σ be a finite alphabet. Then we define Σ^* as:

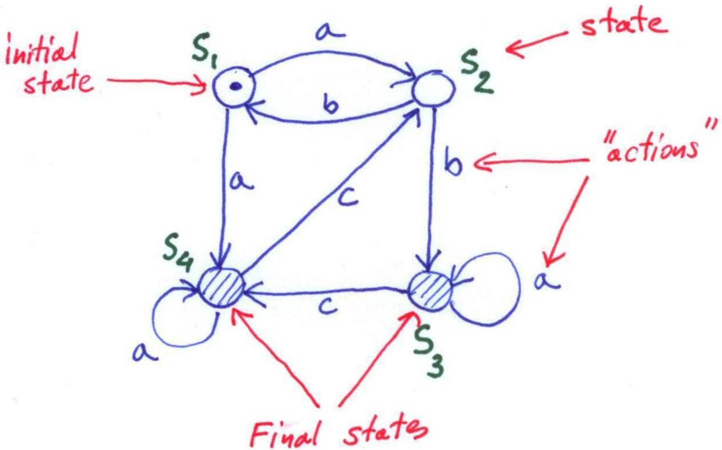
$$\Sigma^* = \{a_1 \dots a_k \mid a_i \in \Sigma \wedge k \geq 0\},$$

  i.e. the set of all sequences, including ε, built from the elements of Σ.
- For example $\{a, b\}^* = \{\varepsilon, a, b, aa, ab, ba, bb, aaa, aba, aab, ...\}$, all sequences built from a and b.
- $\emptyset^* = \{\varepsilon\}$
- If $\Sigma \neq \emptyset$ then $|\Sigma^*| = \infty$.
- A (*formal*) *language* over Σ is any subset of Σ^*, including the empty set $\emptyset$ and Σ^*.
- For example $\{ab, cba, ba, bbbb\} \subseteq \{a, b, c\}^*$ is a finite language, while $\{abc, ba, ab, abb, abbb, \dots, ab^k, \dots\} \subseteq \{a, b\}^*$ in an infinite language.

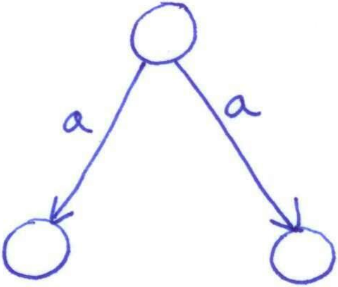
Automata or State Machines

- There is a set of **states** Q.
  Q may be finite, then we have finite state machines.
- There is a set of **actions/operations** that allow to move from one state to another state.
- There is a **transition function/relation** that allow movement from one state to another state using actions/operations.
- There is an **initial state**.
- There *might be* **final states**.
- The concept of a **current state** may easily be introduced.
- The set of actions/operations is finite.
- There is a concept of **nondeterministic choice**.

(Finite) Automata or (Finite) State Machines: an example



Automata: Non-determinism



Deterministic Automata

Definition

A **deterministic (finite) automaton** (**state machine**) is a 5-tuple:

$$M = (\Sigma, Q, \delta, s_0, F),$$

where: Σ is the **alphabet** (finite) (**input alphabet**),
Q is the **set of states** (finite),
$\delta : Q \times \Sigma \rightarrow Q$ is the **transition function**,
$s_0 \in Q$ is the **initial state**,
$F \subseteq Q$ is the set of **final states**.

Definition ($\hat{\delta}$ function, also often denoted as δ^*)

We extend the function δ to $\hat{\delta} : Q \times \Sigma^* \rightarrow Q$ as follows:

- $\forall q \in Q.\ \ \hat{\delta}(q, \varepsilon) = q$
- $\forall q \in Q. \forall x \in \Sigma^*. \forall a \in \Sigma.\ \ \hat{\delta}(q, xa) = \delta(\hat{\delta}(q, x), a).$

$\hat{\delta}$ function, also often denoted as δ^*

Definition

We extend the function δ to $\hat{\delta} : Q \times \Sigma^* \to Q$ as follows:

- $\forall q \in Q.\ \ \hat{\delta}(q, \varepsilon) = q$
- $\forall q \in Q. \forall x \in \Sigma^*. \forall a \in \Sigma.\ \ \hat{\delta}(q, xa) = \delta(\hat{\delta}(q, x), a)$.

- The above definition of $\hat{\delta}$ is *recursive* and the recursion is *on the length of* x.
- Intuitively $\delta(q, a)$ is the state that can be reached from q in one step, while $\hat{\delta}(q, x)$ is the state that can be reached from q in $|x|$ steps by using δ in each step.
- For example $\hat{\delta}(q, abcd) = \delta(\delta(\delta(\delta(q, a), b), c), d)$.
- We usually write δ instead of $\hat{\delta}$ when it does not lead to any misunderstanding.
- For example $\delta(q, abcd) = \delta(\delta(\delta(\delta(q, a), b), c), d)$.

Language Accepted/Generated by an Automaton

Definition (Language)

For every automaton M, the set

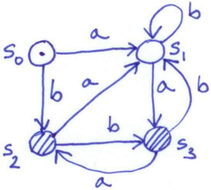
$$L(M) = \{x \mid \hat{\delta}(s_0, x) \in F\}$$

is called the language accepted/generated by M.

- The language is just a set of all sequences (words, traces) that can be derived by starting from the initial state travelling trough automaton (using δ as *next state* function) and ending in some *final state*.
- It is possible to leave a final state!
- In concurrency we often do not have final states! In such a case we assume that each state is a final state, i.e. $F = Q$!

Deterministic Automaton: an Example

- Consider the following deterministic automaton M

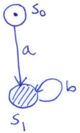


- We have: $\Sigma = \{a, b\}$, $Q = \{s_0, s_1, s_2, s_3\}$, $F = \{s_2, s_3\}$ and the below table shows the transition function δ.

δ	a	b
s_0	s_1	s_2
s_1	s_3	s_1
s_2	s_1	s_3
s_3	s_2	s_1

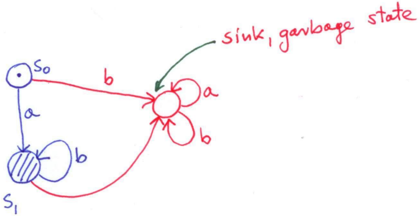
- For example $ab, bbaa \notin L(M)$, while $aaa, abbbaab \in L(M)$.

(1) We **cannot** specify:

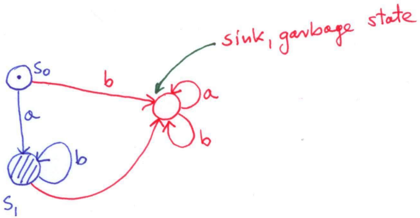


The meaning is pretty simple: "*first execute a and next execute any number of b, including none.*" However, for the first definition we have $\delta(s_0, a) = s_1$ and $\delta(s_1, b) = s_1$, but what about $\delta(s_0, b) =?$ and $\delta(s_1, a) =?$. For the second definition we have $a(s_0) = s_1$ and $b(s_1) = s_1$ but still $b(s_0) =?$ and $a(s_1) =?$.

UGLY SOLUTION



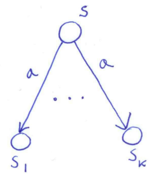
UGLY SOLUTION



- This solution is good for illustration, problematics for real systems as we have to introduce entities that may not exist in the real system!
- The standard solution involves the concept of **non-determinism**.

Non-determinism: Problem #1

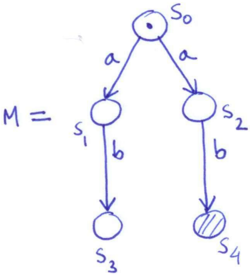
- Notation for 'power set': $2^Q = \mathcal{P}(Q) = \{X \mid X \subseteq Q\}$, and clearly $\emptyset \in 2^Q$.
- Problem #1: How to model the below situation?



- The standard solution
  $\delta(s, a) = \{s_1, \ldots, s_k\}$, which implies $\delta : Q \times \Sigma \to 2^Q$,

Non-determinism: Problem #2

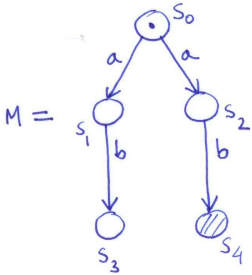
- Consider the following automaton:



- Which is true? $ab \in L(M)$ or $ab \notin L(M)$?

Non-determinism: Problem #2

- Consider the following automaton:

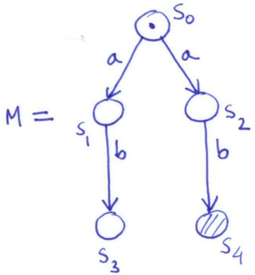


- Which is true? $ab \in L(M)$ or $ab \notin L(M)$?

  Usually it is assumed that $ab \in L(M)$. It is called **angelic semantics**.

Non-determinism: Problem #2

- Consider the following automaton:



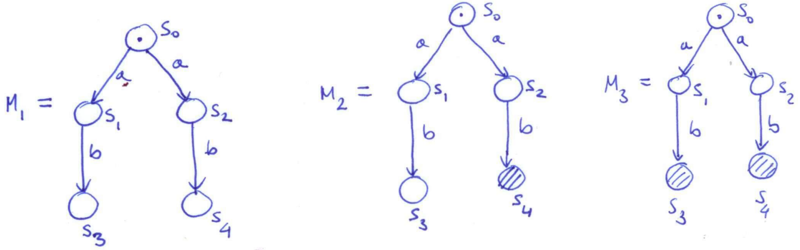
- Which is true? $ab \in L(M)$ or $ab \notin L(M)$?

  Usually it is assumed that $ab \in L(M)$. It is called **angelic semantics**.

Angelic vs Demonic Semantics

- **Angelic**: At each state an *angel* will tell you where to go, so if there is a good choice you will make it. The only bad case is when all choices are bad.
- **Demonic**: At each state a *demon* will tell you where to go, so if there is a bad choice you will make it. The only good case is when all choices are good.
- *Demonic semantics* is much less popular. It is relatively new and was motivated by *fault tolerant systems*. In this class we will use only *angelic* semantics. I have mentioned demonic, to show that non-determinism is more complex than the one presented in most textbooks.

- Consider the three automata below:



- Let $L_A(M_1)$, $i = 1, 2, 3$ denote a language defined by M_i under *angelic* semantics, and let $L_D(M_1)$, $i = 1, 2, 3$ denote a language defined by M_i under *demonic* semantics. Note that $L_A(M_1) = L_D(M_1) = \emptyset$, $L_A(M_2) = \{ab\}$, $L_D(M_2) = \emptyset$ and $L_A(M_3) = L_D(M_3) = \{ab\}$.

Definition (Non-deterministic Automaton)

A **non-deterministic (finite) automaton** (**state machine**) is a 5-tuple:

$$M = (\Sigma, Q, \delta, s_0, F),$$

where: Σ is the **alphabet** (finite) (**input alphabet**),
Q is the **set of states** (finite),
$\delta : Q \times \Sigma \to 2^Q$ is the **transition function**,
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Definition (non-deterministic $\hat{\delta}$ function)

We extend the function δ to $\hat{\delta} : Q \times \Sigma^* \to 2^Q$ as follows:

- $\forall q \in Q.\ \ \hat{\delta}(q, \varepsilon) = \{q\}$
- $\forall q \in Q. \forall x \in \Sigma^*. \forall a \in \Sigma.\ \ \hat{\delta}(q, xa) = \bigcup_{s \in \hat{\delta}(q,x)} \delta(s, a).$

Sometimes, by a small abuse of notation, we write

$$\hat{\delta}(q, xa) = \delta(\hat{\delta}(q, x), a).$$

Language Defined by a Non-deterministic Automaton

Definition (Angelic semantics)

For every automaton M, the set

$$L(M) = \{x \mid \hat{\delta}(s_0, x) \cap F \neq \emptyset\}$$

is called the language accepted/generated by M.

Definition (Demonic semantics)

For every automaton M, the set

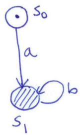
$$L(M) = \{x \mid \hat{\delta}(s_0, x) \subseteq F\}$$

is called the language accepted/generated by M.

We will not consider demonic semantics in this course.

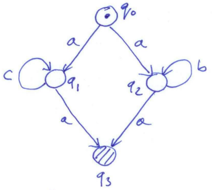
Non-determinism: Example 1

- Consider the following example:



- Standard model: $\Sigma = \{a, b\}$, $Q = \{s_0, s\}$, $F = \{s_1\}$.
  $\delta(s_0, a) = \{s_1\}$, $\delta(s_0, b) = \emptyset$,
  $\delta(s_1, a) = \emptyset$, $\delta(s_1, b) = \{s_1\}$.
- In this case we do not have 'splits' like the one from page 15, all outcomes of the function δ are either singletons or empty set, so intuitively this is rather a deterministic system.
- However formally the automaton is non-deterministic.

- Consider the following example:



- Classical model: $\Sigma = \{a, b, c\}$, $Q = \{q_0, q_1, q_2, q_3\}$, $F = \{q_3\}$.
  $\delta(q_0, a) = \{q_1, q_2\}$, $\delta(q_0, b) = \delta(q_0, c) = \emptyset$,
  $\delta(q_1, a) = \{q_3\}$, $\delta(q_1, b) = \emptyset$, $\delta(q_1, c) = \{q_1\}$,
  $\delta(q_2, a) = \{q_3\}$, $\delta(q_2, b) = \{q_2\}$, $\delta(q_2, c) = \emptyset$,
  $\delta(q_3, a) = \delta(q_3, b) = \delta(q_3, c) = \emptyset$
- 'Local' model: $\Sigma = \{a, b, c\}$, $Q = \{q_0, q_1, q_2, q_3\}$, $F = \{q_3\}$.
  $a = \{(q_0, q_1), (q_0, q_2), (q_1, q_3), (q_2, q_3)\}$, $b = \{(q_2, q_2)\}$,
  $c = \{(q_1, q_1)\}$.

Another Approach to Non-determinism

Definition

An **automaton** (**state machine**) is a 5-tuple:

$$M = (\Sigma, Q, \delta, s_0, F),$$

where: Σ is the **alphabet** (finite),
Q is the **set of states** (finite),
$\delta : Q \times \Sigma \rightarrow 2^Q$ is the **transition function**,
$s_0 \in Q$ is the **initial state**,
$F \subseteq Q$ is the set of **final states**.

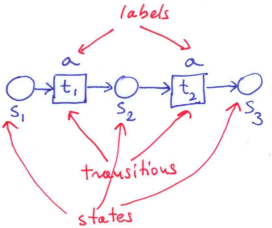
Definition

- M is **deterministic** iff $\forall q \in Q.\forall a \in \Sigma.\ |\delta(q, a)| \leq 1$
- M is **strictly deterministic** iff $\forall q \in Q.\forall a \in \Sigma.\ |\delta(q, a)| = 1$

These definitions are often used in the papers that deal with applications rather than theory.

Labelled Transition Systems

- **Transition** (from Collins Dictionary):
  "*a passing or change from one place, state, condition, etc., to another.*"
- Consider the case: $s_1 \xrightarrow{a} s_2 \xrightarrow{a} s_3$
  Can "*a*" be called a transition?
- Transitions, state and labels:



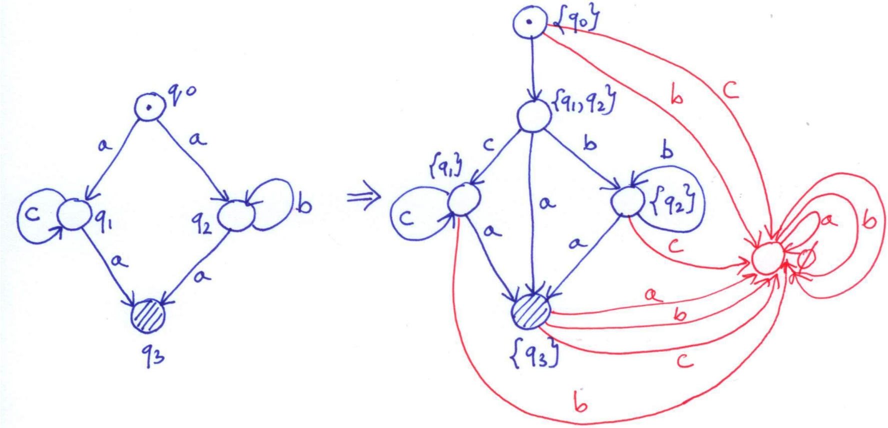
- Transitions are **unique**: $\boxed{t_1} \iff s_1 \xrightarrow{a} s_2$
- Transitions (labelled) are usually needed and used for modelling concurrency.

Non-determinism vs Determinism: Example

- Does non-determinism increases the descriptive power?

Non-determinism vs Determinism: Example

- Does non-determinism increases the descriptive power?
  NO. See an example below and Theorem on next page.

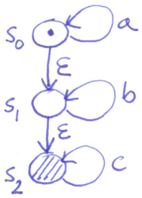


Non-determinism vs Determinism

Theorem

Let L be a language accepted by a **non-deterministic finite automaton** *M, i.e. $L = L(M)$. There exists a* **deterministic finite automaton** *M' such that $L(M') = L$.*

Invisible (silent) Actions or ε-moves



- In many cases we need to model **invisible actions**!

Definition

A **non-deterministic automaton with ε-moves** is:

$$M = (Q, \Sigma, \delta, q_0, F),$$

where: Q, Σ, q_0, F are as usual, and:

$$\delta : Q \times (\Sigma \cup \{\varepsilon\}) \rightarrow 2^Q.$$

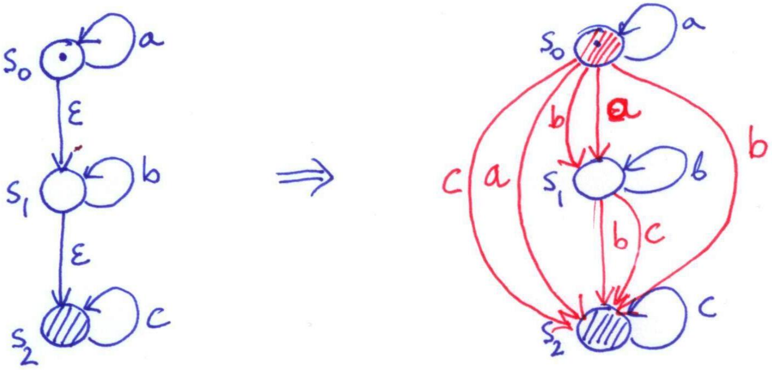
- **Problem**: ε has **two** interpretations:
  1. Do nothing,
  2. Go to another state by executing invisible (silent) action.

Removal of ε-moves

Theorem

If L is accepted by a nondeterministic automaton **with** *ε-moves, then L is also accepted by a nondeterministic automaton* **without** *ε-moves.*

Removal of ε-moves: An example



- We may then transform the non-deterministic automaton from the right hand side into appropriate deterministic one.

Definition

Two automata M_1 and M_2 are **equivalent** if and only if $L(M_1) = L(M_2)$.

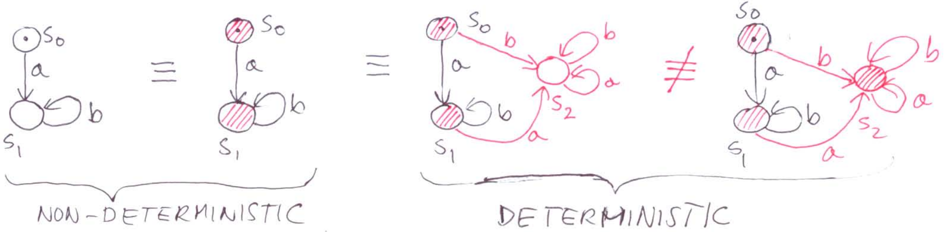
Conclusion

Non-deterministic automata, nondeterministic automata with ε-moves, and deterministic automata are all equivalent.

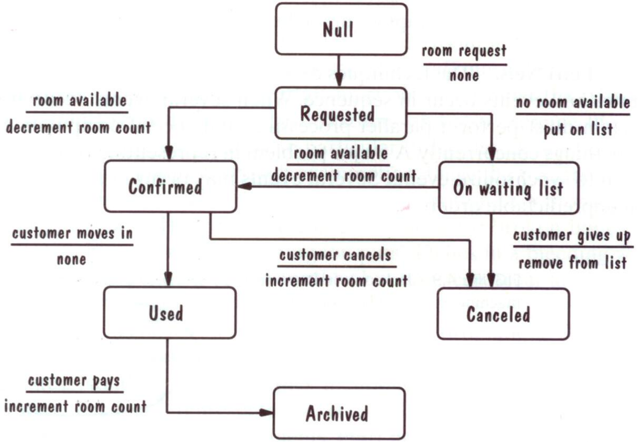
- *Equivalence* means only the same language, other properties may differ.
- For example the concept of 'demonic semantics' does not make much sense for deterministic automata as we do not have non-deterministic choices.
- For any deterministic automaton M, if all states are finite then then $L(M) = \Sigma^*$, so this concept also has very little sense.

No Final States

- No final states is equivalent to all states are final, i.e. $F = \emptyset \equiv F = Q$.
- But this makes sense only for nondeterministic automata.

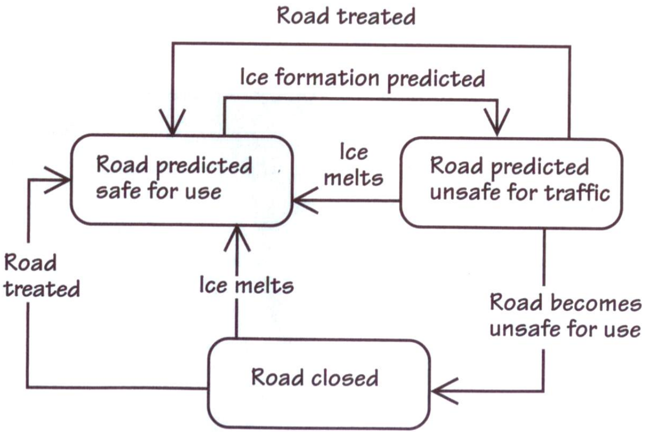


Modelling Dynamic Systems With Automata: Hotel Reservation



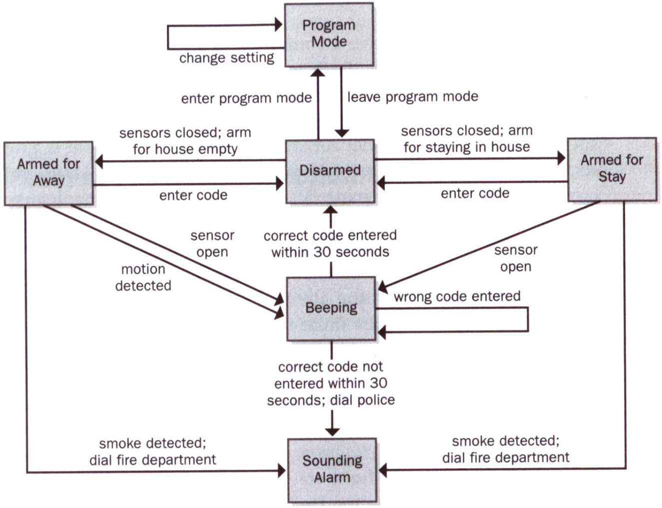
- No arrow to initial state and no arrow from final state.

Road Deicing



- Neither initial nor final state specified.

Simplified Home Security



- Neither initial nor final state specified.

- **FOR THIS KIND OF APPLICATIONS AUTOMATA ARE USUALLY NONDETERMINISTIC**, or **deterministic in the sense of the definition from page 23**.

Regular Expressions: Intuition

$(0 \cup 1)0^* \quad \rightarrow \quad \{0, 00, 000, \dots, 1, 10, 100, 1000, \dots\}$

'zero or one followed by any number of zeros including none'

$ab^* \quad \rightarrow \quad \{a, ab, abb, abbb, \dots\}$

$(a \cup b)^* \quad \rightarrow \quad \{a, b\}^*$

'all strings (including ε) that can be built from a and b'

$(a \cup \varepsilon)(b \cup \varepsilon) =$

$ab \cup \varepsilon b \cup a\varepsilon \cup \varepsilon\varepsilon =$

$ab \cup b \cup a \cup \varepsilon \quad \rightarrow \quad \{\varepsilon, a, b, ab\}$

Definition (Formal Definition of Regular Expressions)

Let Σ be an alphabet. A string R built from the elements of $\Sigma \cup \{\varepsilon, \emptyset, (,), \cup, ^*\}$ is a **regular expression**, if it is defined by the following rules:

1. $\emptyset, \varepsilon$ and each $a \in \Sigma$ are *regular expressions*.
2. $(R_1 \cup R_2)$ is a *regular expression* if R_1 and R_2 are regular expressions.
3. $(R_1 R_2)$ is a *regular expression* if R_1 and R_2 are regular expressions.
4. $(R)^*$ is a *regular expression* if R is a regular expression.
5. There are no other regular expressions.

The set of all regular expressions over the alphabet Σ will be denoted by $Rex(\Sigma)$.

- We usually skip some parenthesis.
- Rules: * first, followed by concatenation, and finally $\cup$, unless parentheses say differently.

Definition (Interpretation)

Let $L : Rex(\Sigma) \to 2^{\Sigma^*}$ be the following function called **interpretation**:

1. $L(\emptyset) = \emptyset,\ L(\varepsilon) = \varepsilon,\ L(a) = \{a\}$
2. $L((R_1 \cup R_2)) = L(R_1) \cup L(R_2)$
3. $L((R_1 R_2)) = L(R_1)L(R_2)$
4. $L((R)^*) = L(R)^*$

Definition (Language)

For every regular expression R, $L(R)$ is a **language defined by** R.

A class of all languages defined by regular expressions will be denoted by $\mathcal{L}_{REX}$.

Languages Defined by Regular Expressions: Examples

- $L((0 \cup 1)0^*) = \{0, 00, 000, \ldots, 1, 10, 100, 1000, \ldots\}$
- $L(ab^*) = \{a, ab, abb, abbb, abbbb, \ldots\}$
- $L((a \cup \varepsilon)(b \cup \varepsilon)) = \{\varepsilon, a, b, ab\}$
- We customarily often identify a regular expression R with $L(R)$ but technically R **is not** $L(R)$.

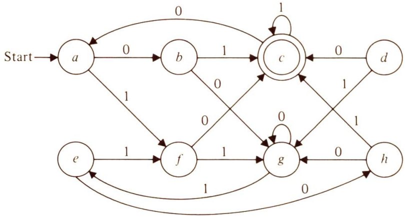
♣ **A question:** What is the relationship between $\mathcal{L}_R$ and $\mathcal{L}_{REX}$?

Theorem

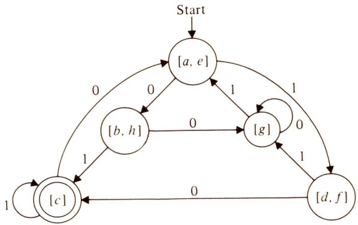
$$\mathcal{L}_R = \mathcal{L}_{REX}$$

Minimization of Deterministic Finite Automata

- Consider the following two deterministic automata:



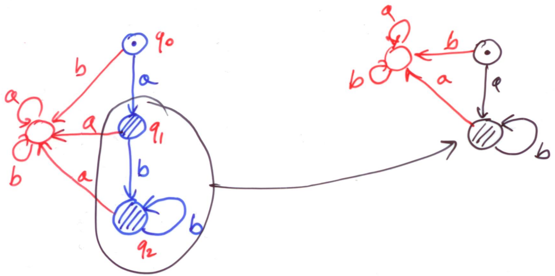
M_1



M_2

- It can be proved that $L(M_1) = L(M_2)$, and clearly M_2 has less states.
- It can be proved that M_2 is the minimum state automaton that is equivalent to M_1, i.e. $L(M_1) = L(M_2)$.

Consider the following two automata, both generating the language ab^*:



The states q_1 and q_2 of the left automaton and *equivalent*, so they, and appropriate arrows, can be glued together.

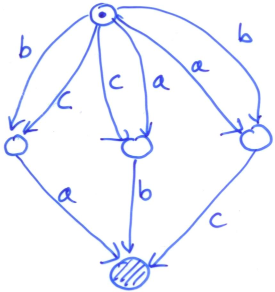
Minimum State Deterministic Finite Automata

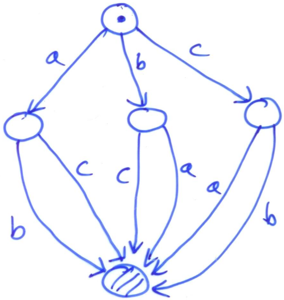
Theorem

For every deterministic automaton M_1 there is the minimum state deterministic automaton M_2 such that $L(M_1) = L(M_2)$.
The automaton M_2 is unique up to the names of states.

- The word 'deterministic' is important and cannot be omitted!
- Consider the following two non-deterministic automata, both generating the language

$$L = \{ab, ac, bc, ba, ca, cb\}$$

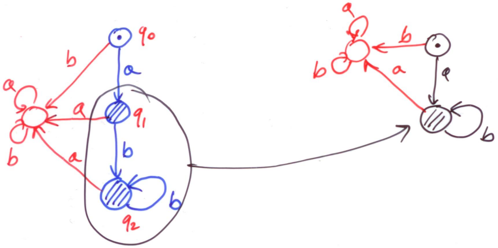




- Both automata above are minimum state and there is no way to make them identical!

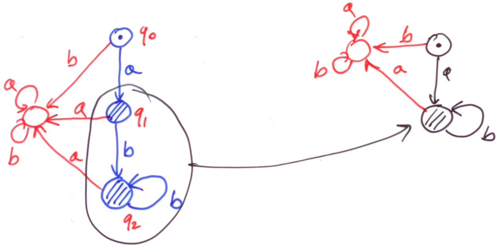
Non-deterministic Automata and Minimization Problem

- When talking about Minimum State Non-deterministic automaton (usually when we discuss some application), we usually mean the case as below (no non-deterministic splits):



- If we forgot about red part, both automata are non-deterministic and the black automaton on the right can be interpreted as the minimum state automaton equivalent to the blue automaton on the left.
- However, to derive formally the back automaton on the right from the blue on left we need to add the red parts.

Non-deterministic Automata and Minimization Problem



- Labelled Transition Systems are almost always non-deterministic and the statements 'minimal', 'minimization', etc., in the textbook, refer to the meaning from the previous slide.