Recurrence Relations

Click on a question number to see how your answers were marked and, where available, full solutions.

Question Number	Score
Order 1	
Question 1	1 / 1
Question 2	1 / 1
Question 3	1 / 1
Order 2	
Question 4	1 / 1
Question 5	1 / 1
Total	**5 / 5 (100%)**

Well done. You have passed the self-assessment.

Please print the results summary by clicking on "Print this results summary" and saving to pdf. You will need to include it in your lesson review

Performance Summary

Exam Name:	Recurrence Relations
Session ID:	12458745751
Exam Start:	Sat Jan 06 2024 10:37:53
Exam Stop:	Sat Jan 06 2024 11:12:19
Time Spent:	0:34:25

Question 1

Find the closed form of the relations:

$$\begin{cases} a_0 & = -20 \\ a_n & = -6a_{n-1} - 14 \end{cases}$$

Note the index! $a_n =$

`-18*(-6)^n -2` $-18(-6)^n - 2$ ✔

Expected answer: `-18*(-6)^n - 2` $-18(-6)^n - 2$

✔ Your answer is numerically correct. You were awarded **1** mark.

You scored **1** mark for this part.

Score: 1/1 ✔

Advice

To solve Order 1 recurrences, we need to apply the following steps:

1. Express the recurrence relation for the first difference (which *will* be a geometric sequence).
2. Find the closed form for that geometric sequence.
3. Find the closed form of the original sequence using its first difference.

Finding the first difference.

We need two things here: the first element, and the recurrence formula for the first difference.

For the first element, we have $b_0 = a_1 - a_0$.

- $a_0 = -20$
- $a_1 = -6a_0 + (-14) = 106$.

So, $b_0 = 106 - -20 = 126$.

For the recurrence formula, we have

$$\begin{aligned} b_n &= a_{n+1} - a_n \\ &= -6a_n - 14 - (-6a_{n-1} - 14) \\ &= -6b_{n-1} \end{aligned}$$

Closed form of the first difference.

Now we know that the first difference is a geometric sequence, and we know its recurrence form. From Activity 2, we can find the closed form: $b_0 \times k^n$, where k is the multiplier in the recurrence. So

$$b_0 = 126(-6)^n$$

Closed form for the sequence a_n:

Finally we can use the closed form of the first difference to find a_n.

$$\begin{aligned} b_n &= a_{n+1} - a_n \\ 126(-6)^n &= -6a_n - 14 - a_n \\ &= -6a_n - 14 - a_n \\ 126(-6)^n + 14 &= -7a_n \\ -18(-6)^n - 2 &= a_n && \text{divide both sides by } -7 \end{aligned}$$

And so we find the closed form for a_n.

Question 2

Find the closed form of the relations:

$$\begin{cases} a_0 & = -31 \\ a_n & = -7a_{n-1} - 24 \end{cases}$$

Note the index! $a_n =$

`-28*(-7)^n -3` $-28(-7)^n - 3$ ✔

Expected answer: `-28*(-7)^n - 3` $-28(-7)^n - 3$

✔ Your answer is numerically correct. You were awarded **1** mark.

You scored **1** mark for this part.

Score: 1/1 ✔

Advice

To solve Order 1 recurrences, we need to apply the following steps:

1. Express the recurrence relation for the first difference (which *will* be a geometric sequence).
2. Find the closed form for that geometric sequence.
3. Find the closed form of the original sequence using its first difference.

Finding the first difference.

We need two things here: the first element, and the recurrence formula for the first difference.

For the first element, we have $b_0 = a_1 - a_0$.

- $a_0 = -31$
- $a_1 = -7a_0 + (-24) = 193.$

So, $b_0 = 193 - -31 = 224.$

For the recurrence formula, we have

$$\begin{aligned} b_n &= a_{n+1} - a_n \\ &= -7a_n - 24 - (-7a_{n-1} - 24) \\ &= -7b_{n-1} \end{aligned}$$

Closed form of the first difference.

Now we know that the first difference is a geometric sequence, and we know its recurrence form. From Activity 2, we can find the closed form: $b_0 \times k^n$, where k is the multiplier in the recurrence. So

$$b_0 = 224(-7)^n$$

Closed form for the sequence a_n:

Finally we can use the closed form of the first difference to find a_n.

$$\begin{aligned} b_n &= a_{n+1} - a_n \\ 224(-7)^n &= -7a_n - 24 - a_n \\ &= -7a_n - 24 - a_n \\ 224(-7)^n + 24 &= -8a_n \\ -28(-7)^n - 3 &= a_n \qquad \text{divide both sides by } -8 \end{aligned}$$

And so we find the closed form for a_n.

Question 3

Find the closed form of the relations:

$$\begin{cases} a_0 &= -11 \\ a_n &= -5a_{n-1} - 6 \end{cases}$$

Note the index! $a_n =$

`-10(-5)^n -1` $-10(-5)^n - 1$ ✔

Expected answer: `-10*(-5)^n - 1` $-10(-5)^n - 1$

✔ Your answer is numerically correct. You were awarded **1** mark.

You scored **1** mark for this part.

Score: 1/1 ✔

Advice

To solve Order 1 recurrences, we need to apply the following steps:

1. Express the recurrence relation for the first difference (which *will* be a geometric sequence).
2. Find the closed form for that geometric sequence.
3. Find the closed form of the original sequence using its first difference.

Finding the first difference.

We need two things here: the first element, and the recurrence formula for the first difference.

For the first element, we have $b_0 = a_1 - a_0$.

- $a_0 = -11$
- $a_1 = -5a_0 + (-6) = 49.$

So, $b_0 = 49 - -11 = 60.$

For the recurrence formula, we have

$$\begin{aligned} b_n &= a_{n+1} - a_n \\ &= -5a_n - 6 - (-5a_{n-1} - 6) \\ &= -5b_{n-1} \end{aligned}$$

Closed form of the first difference.

Now we know that the first difference is a geometric sequence, and we know its recurrence form. From Activity 2, we can find the closed form: $b_0 \times k^n$, where k is the multiplier in the recurrence. So

$$b_0 = 60(-5)^n$$

Closed form for the sequence a_n:

Finally we can use the closed form of the first difference to find a_n.

$$\begin{aligned} b_n &= a_{n+1} - a_n \\ 60(-5)^n &= -5a_n - 6 - a_n \\ &= -5a_n - 6 - a_n \\ 60(-5)^n + 6 &= -6a_n \\ -10(-5)^n - 1 &= a_n \qquad \text{divide both sides by } -6 \end{aligned}$$

And so we find the closed form for a_n.

Question 4

Solve the recurrence relation given by:

$$\begin{cases} a_0 &= 2 \\ a_1 &= -23 \\ a_n &= 4a_{n-2} - 3a_{n-1} \end{cases}$$

Note the index! $a_n =$

`5*(-4)^n - (3*(1)^n )` $5(-4)^n - 3 \times 1^n$ ✔

Expected answer: `5*(-4)^n - 3*1^n` $5(-4)^n - 3 \times 1^n$

✔ Your answer is numerically correct. You were awarded **1** mark.

You scored **1** mark for this part.

Score: 1/1 ✔

Advice

To find the closed form of a linear recurrence of order 2, we have to follow three steps:

1. Find the characteristic equation.
2. Solve the characteristic equation to obtain the general form of the sequence.
3. Find the specific parameters by solving a system of simultaneous equations.

Finding the characteristic equation.

To find the characteristic equation, we first need to rewrite ther recurrence relation to have 0 on the right hand side.

$$a_{n+2} = 4a_{n-2} - 3a_{n-1}$$
$$a_n - (4a_{n-2} - 3a_{n-1}) = 0$$

Since this relation must be true for every $n \geq 2$, it must also apply for $n = 2$:

$$a_2 + 3a_1 - 4a_0 = 0$$

Now, we substitute each a_i with x^i, to get the characteristic equation:

$$x^2 + 3x - 4 = 0$$

.

Solving the characteristic equation.

We can solve the characteristic equation using the quadratic formula:

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2}.$$

Here we have $a = 1$, $b = 3$ and $c = -4$. So the quadratic formula gives us two solutions to that equation: $x_1 = 1$ and $x_2 = -4$. **Make sure to verify your intermediate result at this stage by plugging these values back into your equation.**

The general form of the sequence is therefore:

$$a_n = s_1(-3)^n + s_2 \times 4^n.$$

Finding the parameters s_1 and s_2.

To find the parameters s_1 and s_2 we use the values of the elements a_0 and a_1.

- For $n = 0$, we know that $a_0 = 2$, and our general form gives us that this must be equal to $a_0 = s_0 * (1)^0 + s_2 * (-4)^0$.
- For $n = 1$, we know that $a_1 = -23$, and our general form gives us that this must be equal to $a_1 = s_1 * (1)^1 + s_2 * (-4)^1$.

And so we obtain the simultaneous equations:

$$\begin{cases} s_1 + s_2 & = 2 \\ s_1 - 4s_2 & = -23 \end{cases}.$$

We can solve this system of linear equations, for example by elimination. Let's eliminate s_2:

1. Multiply the top equation by -4 (the coefficient of s_2 in the bottom equation): $-4s_1 - 4s_2 = -8$.
2. Subtract the second equation from the first:

$$-5s_1 = 15.$$

3. Solve for s_1 using the equation we just obtained: $s_1 = -3$.
4. Substitute the value of s_1 into the top equation: $s_2 = 2 - s_1 = 5$.

We find that $s_1 = -3$ and $s_2 = 5$. **Make sure to verify your intermediate result at this stage by plugging these values back into the two simultaneous equations.**

Therefore the closed form is

$$a_n = 5(-4)^n - 3 \times 1^n.$$

Evaluate at least a_0, a_1 and a_2 with your closed form solution before entering it into the quiz.

Question 5

Solve the recurrence relation given by:

$$\begin{cases} a_0 & = -4 \\ a_1 & = 12 \\ a_n & = 7a_{n-1} + 30a_{n-2} \end{cases}$$

Note the index! $a_n =$

-4*(-3)^n $-4(-3)^n$ ✔

Expected answer: -4*(-3)^n $-4(-3)^n$

✔ Your answer is numerically correct. You were awarded **1** mark.

You scored **1** mark for this part.

Score: 1/1 ✔

Advice

To find the closed form of a linear recurrence of order 2, we have to follow three steps:

1. Find the characteristic equation.
2. Solve the characteristic equation to obtain the general form of the sequence.
3. Find the specific parameters by solving a system of simultaneous equations.

Finding the characteristic equation.

To find the characteristic equation, we first need to rewrite ther recurrence relation to have 0 on the right hand side.

$$a_{n+2} = 7a_{n-1} + 30a_{n-2}$$
$$a_n - (7a_{n-1} + 30a_{n-2}) = 0$$

Since this relation must be true for every $n \geq 2$, it must also apply for $n = 2$:

$$a_2 - 7a_1 - 30a_0 = 0$$

Now, we substitute each a_i with x^i, to get the characteristic equation:

$$x^2 - 7x - 30 = 0$$

.

Solving the characteristic equation.

We can solve the characteristic equation using the quadratic formula:

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2}.$$

Here we have $a = 1$, $b = -7$ and $c = -30$. So the quadratic formula gives us two solutions to that equation: $x_1 = 10$ and $x_2 = -3$. **Make sure to verify your intermediate result at this stage by plugging these values back into your equation.**

The general form of the sequence is therefore:

$$a_n = s_1 \times 7^n + s_2 \times 30^n.$$

Finding the parameters s_1 and s_2.

To find the parameters s_1 and s_2 we use the values of the elements a_0 and a_1.

- For $n = 0$, we know that $a_0 = -4$, and our general form gives us that this must be equal to $a_0 = s_0 * (10)^0 + s_2 * (-3)^0$.
- For $n = 1$, we know that $a_1 = 12$, and our general form gives us that this must be equal to $a_1 = s_1 * (10)^1 + s_2 * (-3)^1$.

And so we obtain the simultaneous equations:

$$\begin{cases} s_1 + s_2 & = -4 \\ 10s_1 - 3s_2 & = 12 \end{cases}.$$

We can solve this system of linear equations, for example by elimination. Let's eliminate s_2:

1. Multiply the top equation by -3 (the coefficient of s_2 in the bottom equation): $-3s_1 - 3s_2 = 12$.
2. Subtract the second equation from the first:

$$-13s_1 = 0.$$

3. Solve for s_1 using the equation we just obtained: $s_1 = 0$.
4. Substitute the value of s_1 into the top equation: $s_2 = -4 - s_1 = -4$.

We find that $s_1 = 0$ and $s_2 = -4$. **Make sure to verify your intermediate result at this stage by plugging these values back into the two simultaneous equations.**

Therefore the closed form is

$$a_n = -4(-3)^n.$$

Evaluate at least a_0, a_1 and a_2 with your closed form solution before entering it into the quiz.
