Multiple Linear Regression: Categorical Predictors

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This material is part of the ***statsTeachR*** *project*

Multiple Linear Regression: recapping model definition

In matrix notation...

$$\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\epsilon}$$

where $E(\boldsymbol{\epsilon}) = 0$ and $Cov(\boldsymbol{\epsilon}) = \sigma^2 I$

In individual observation notation...

$$y_i = \beta_0 + \beta_1 x_{1,i} + \ldots + \beta_p x_{p,i} + \epsilon_i$$

where $\epsilon_i \overset{iid}{\sim} (0, \sigma^2)$

Categorical predictors

- Assume X is a categorical / nominal / factor variable with k levels
- With only one categorical X, we have classic one-way ANOVA design
- Can't use a single predictor with levels $1, 2, \dots, K$ – this has the wrong interpretation
- Need to create *indicator* or *dummy* variables

Indicator variables

$y_i = \beta_0 + \beta_1 \cdot x_i$

$y_i = \beta_0 + \beta_1 x_{1,i} + \beta_2 x_{2,i}$

- Let x be a categorical variable with k levels (e.g. with $k = 3$ "low", "med", "hi").
- Choose one group as the baseline (e.g. "low")
- Create $(k-1)$ binary terms to include in the model:

$$x_{1,i} = \mathbb{1}(x_i = \text{"med"})$$
$$x_{2,i} = \mathbb{1}(x_i = \text{"hi"})$$

$\mathbb{1}(x_i = \text{"hi"}) = \begin{cases} 1 & \text{if } x_i = \text{"hi"} \\ 0 & \text{o.w.} \end{cases}$

- For a model with no additional predictors, pose the model

$$y_i = \beta_0 + \beta_1 x_{1,i} + \ldots + \beta_{k-1} x_{k-1,i} + \epsilon_i$$

  and estimate parameters using least squares
- Note distinction between *predictors* and *terms*

Categorical predictor design matrix

Which of the following is a "correct" design matrix for a categorical predictor with 3 levels?

$$\mathbf{X}_1 = \begin{array}{c} \begin{matrix} x_1 & x_2 & x_3 & x_4 \end{matrix} \\ \begin{bmatrix} 1 & 1 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & 0 & 1 \end{bmatrix} \end{array} \quad \text{or} \quad \mathbf{X}_2 = \begin{bmatrix} 1 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 1 & 0 & 0 \\ 1 & 1 & 0 \\ \vdots & \vdots & \vdots \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ \vdots & \vdots & \vdots \\ 1 & 0 & 1 \end{bmatrix} \begin{array}{l} \left.\begin{matrix} \\ \text{low} \\ \\ \end{matrix}\right\} \\ \left.\begin{matrix} \\ \text{med} \\ \\ \end{matrix}\right\} \\ \left.\begin{matrix} \\ \text{hi} \\ \\ \end{matrix}\right\} \end{array} \quad \text{or} \quad \mathbf{X}_3 = \begin{bmatrix} 1 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 1 \end{bmatrix}$$

x_{1i}
x_{2i}
x_{3i}

$X_4 = 2 - X_3 - X_2 - X_1$

ANOVA model interpretation

Using the model $y_i = \beta_0 + \beta_1 x_{1,i} + \ldots + \beta_{k-1} x_{k-1,i} + \epsilon_i$, interpret

$\beta_0 = E(y \mid \text{all } x\text{'s} = 0)$

$= E(y \mid x_1 = 0, x_2 = 0)$

$=$ expected/average y for baseline group

$\beta_1 =$ difference in $E(y)$ between baseline group and group 1.

$\beta_0 + \beta_1 = E(y \mid x_1 = 1)$

Equivalent model

$x_1 \quad x_2 \quad x_3$

Define the model $y_i = \beta_1 x_{i1} + \ldots + \beta_k x_{i,k} + \epsilon_i$ where there are indicators for each possible group

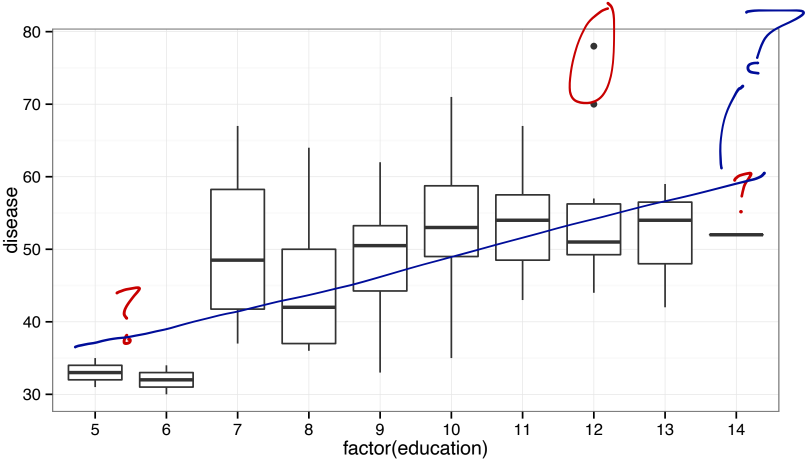
$\beta_1 = E(y | x_1 = 1)$

$=$ expected value of y in group 1

$\beta_2 = E(y | x_2 = 1)$

Categorical predictor example: lung data

```
qplot(factor(education), disease, geom="boxplot", data=dat)
```



Categorical predictor example: lung data

$$dis_i = \beta_0 + \beta_1 educ_{6,i} + \beta_2 educ_{7,i} + \cdots + \beta_{14} educ_{14,i}$$

```
mlr7 <- lm(disease ~ factor(education), data=dat)
summary(mlr7)$coef

##                      Estimate Std. Error t value  Pr(>|t|)
## (Intercept)             33.00      4.913  6.7173 1.689e-09
## factor(education)6      -1.00      7.768 -0.1287 8.979e-01
## factor(education)7      17.33      6.017  2.8808 4.969e-03
## factor(education)8      11.18      5.329  2.0975 3.879e-02
## factor(education)9      15.50      5.353  2.8953 4.765e-03
## factor(education)10     20.38      5.188  3.9289 1.683e-04
## factor(education)11     20.53      5.382  3.8155 2.505e-04
## factor(education)12     22.20      5.601  3.9633 1.489e-04
## factor(education)13     18.67      6.948  2.6868 8.609e-03
## factor(education)14     19.00      9.825  1.9338 5.632e-02
```

$\beta_0 + \beta_4$

Categorical predictor releveling

$X_1 \leftarrow$ cont   class(X_2)
$X_2 \leftarrow$ categ.
lm($y \sim X_1 + X_2$)

$$dis_i = \beta_0 + \beta_1 educ_{5,i} + \beta_2 educ_{6,i} + \beta_1 educ_{7,i} + \beta_2 educ_{9,i} + \cdots + \beta_{14} educ_{14,i}$$

```
dat$educ_new <- relevel(factor(dat$education), ref="8")
mlr8 <- lm(disease ~ educ_new, data=dat)
summary(mlr8)$coef

##             Estimate Std. Error t value  Pr(>|t|)
## (Intercept)   44.176      2.064 21.4059 7.303e-37
## educ_new5    -11.176      5.329 -2.0975 3.879e-02
## educ_new6    -12.176      6.361 -1.9143 5.880e-02
## educ_new7      6.157      4.041  1.5238 1.311e-01
## educ_new9      4.324      2.964  1.4588 1.482e-01
## educ_new10     9.208      2.654  3.4695 8.059e-04
## educ_new11     9.357      3.014  3.1042 2.559e-03
## educ_new12    11.024      3.391  3.2507 1.626e-03
## educ_new13     7.490      5.329  1.4057 1.633e-01
## educ_new14     7.824      8.756  0.8935 3.740e-01
```

8?

Categorical predictor: no baseline group

$$dis_i = \beta_1 educ_{5,i} + \beta_2 educ_{6,i} + \cdots + \beta_{14} educ_{14,i}$$

$+0$

```
mlr9 <- lm(disease ~ factor(education) - 1, data=dat)
summary(mlr9)$coef

##                      Estimate Std. Error t value  Pr(>|t|)
## factor(education)5      33.00      4.913   6.717 1.689e-09
## factor(education)6      32.00      6.017   5.318 7.716e-07
## factor(education)7      50.33      3.474  14.489 3.846e-25
## factor(education)8      44.18      2.064  21.406 7.303e-37
## factor(education)9      48.50      2.127  22.799 6.282e-39
## factor(education)10     53.38      1.669  31.991 1.359e-50
## factor(education)11     53.53      2.197  24.366 3.801e-41
## factor(education)12     55.20      2.691  20.514 1.713e-35
## factor(education)13     51.67      4.913  10.517 2.758e-17
## factor(education)14     52.00      8.509   6.111 2.561e-08
```

Creating categories using cut()

continuous

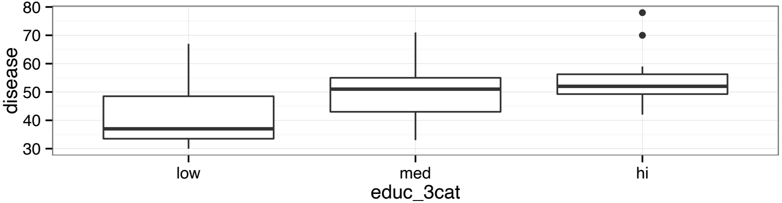
$$dis_i = \beta_1 educ_{low,i} + \beta_2 educ_{med,i} + \cdots + \beta_{\underset{3}{\cancel{14}}} educ_{hi,i}$$

?cut

```r
dat$educ_3cat <- cut(dat$education, breaks=3,
                     labels=c("low", "med", "hi"))
mlr10 <- lm(disease ~ educ_3cat - 1, data=dat)
coef(mlr10)

## educ_3catlow educ_3catmed  educ_3cathi
##        42.27        50.24        54.21

qplot(educ_3cat, disease, geom="boxplot", data=dat)
```



Today's big ideas

- Multiple linear regression: categorical variables