Applied Bayesian Homework 2

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Problem 1

Rules of Craps

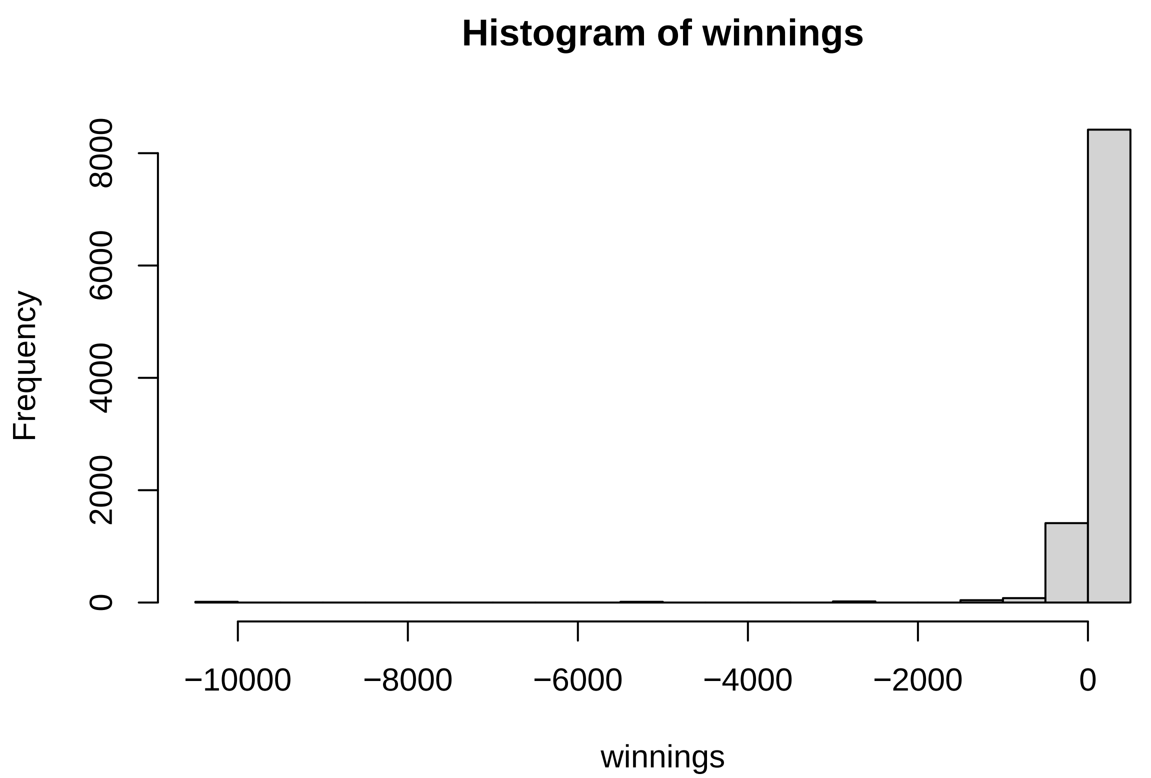
- Two fair dice are rolled. If the sum is 7 or 11, the player wins immediately; if the sum is 2, 3, or 12, the player loses immediately. Otherwise the sum becomes the *point*.
- The two dice continue to be rolled until either a sum of 7 is rolled (in which case the player loses) or a sum equal to the *point* is rolled (in which case the player wins).

Simulation

```r
# Helper function that simulates the craps game
# Returns 1 if players wins and 0 if he loses.
craps <- function(){
    first.roll = sample(1:6, 1) + sample(1:6,1)
    if(first.roll %in% c(2,3,12)){
        return(0)
    }
    else if(first.roll %in% c(7,11)){
        return(1)
    }
    else{
        point = first.roll
        next.roll = sample(1:6, 1) + sample(1:6,1)
        while(next.roll != point && next.roll != 7){
            next.roll = sample(1:6, 1) + sample(1:6,1)
        }
        if(next.roll == point){
            return(1)
        }
        return(0)
  }
}
```

```r
sims = 10000    # Number of times simulated
n = 10      # Number of games to play
i.bet = 10  # Initial bet of 10 dollars
winnings = rep(0, sims)
```

```
for(t in 1:sims){
    bet = i.bet
    wins_sim = 0
    for(i in 1:n){
        result = craps()
        if(result == 0){
            wins_sim = wins_sim - bet
            bet = 2*bet
        }
        if(result == 1){
            wins_sim = wins_sim + bet
            bet = i.bet
        }
    }
    winnings[t] = wins_sim
}
hist(winnings)
```



From the histogram we can see that most of the time we are right around 0 with our winnings, with an average winnings of -6.45 . But there is a potential of having very large losses. This shows that this betting strategy is not ideal.

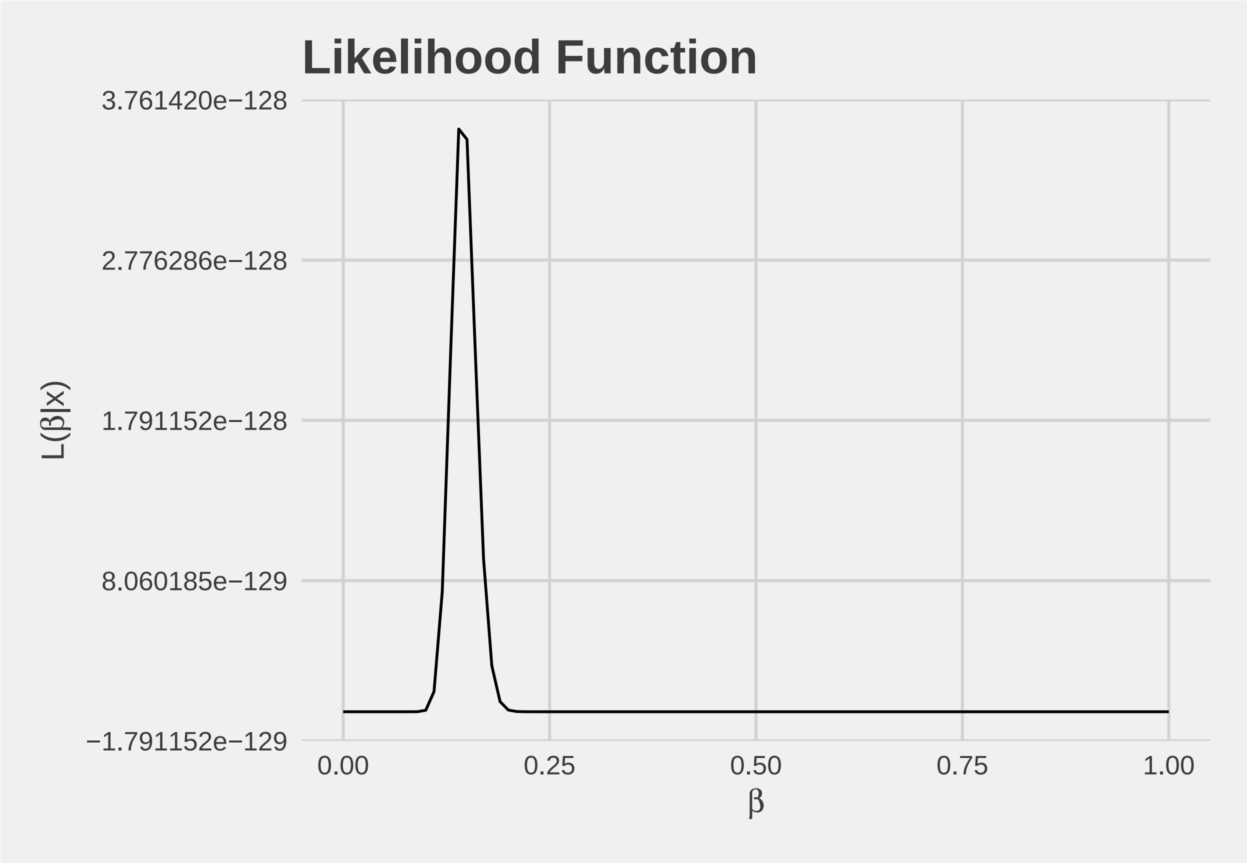
Problem 2

A manufacturer of microwave ovens is trying to determine the length of warranty period it should attach to its magnetron tube, the most critical component in the oven. Preliminary testing has shown that the **length of life (in years), x**, of a magnetron tube has an **exponential distribution probability distribution with a rate parameter** β The manufacturer has collected *100 observations* about the length of life of magnetron tubes (data set is available on blackboard).

a. Write down the likelihood function of rate parameter β and draw the likelihood function of β based on the given data set.

$$L(\beta|x) = \prod_{i=1}^{100} \beta e^{-\beta x_i} = \beta^{100} e^{-\beta \sum_{i=1}^{100} x_i} L(\beta|x) = \beta^{100} e^{-691.74\beta}$$

```
a1 = length(oven$x)
b1 = sum(oven$x)
Lx <- function(x){
  return ((x^a1)*exp(-x*b1))
}
ggplot(data.frame(x = c(0,1) ), aes(x = x )) +
  stat_function(fun = Lx) +
  labs(title = 'Likelihood Function',
       x = expression(beta),
       y = bquote(paste('L(',beta,'|x)'))) +
  theme_fivethirtyeight() +
  theme(axis.title = element_text())
```



b. Based on part a) and then find the likelihood estimate (MLE) of β.

b) From part a our likihood function is $L(\beta|\underline{x}) = \beta^{100} e^{-\beta(\ \ )}$

$$\ell(\beta) = \ln(\beta^n e^{-\beta \Sigma x_i})$$

$$= n \ln(\beta) - \beta \Sigma x_i$$

$$\frac{\partial \ell}{\partial \beta} = \frac{n}{\beta} - \Sigma x_i \overset{set}{=} 0$$

$$\frac{n}{\beta} = \Sigma x_i \Rightarrow \hat{\beta}^{MLE} = \frac{n}{\Sigma x_i}$$

confirming by taking second derivative

$$\frac{\partial^2 \ell}{\partial \beta^2} = \frac{-n}{\beta^2} < 0 \quad \text{confiming}$$

$$\hat{\beta}^{MLE} = \frac{n}{\Sigma x_i} = \frac{1}{\bar{x}} = 0.144563$$

c. Based on the uniform prior: $P(\beta) = 1$, find the posterior distribution of β and then find the posterior mean of β. Draw the posterior density function of β using R-software.

c) given a prior of $\beta \sim$ Uniform $\rightarrow$ $p(\beta) = 1$

$$p(\beta|x) \propto p(x|\beta)\,p(\beta)$$

$$\propto \beta^{n} e^{-\beta \sum x_i} (1)$$

$$\propto \beta^{(n+1)-1} e^{-\beta \sum x_i}$$

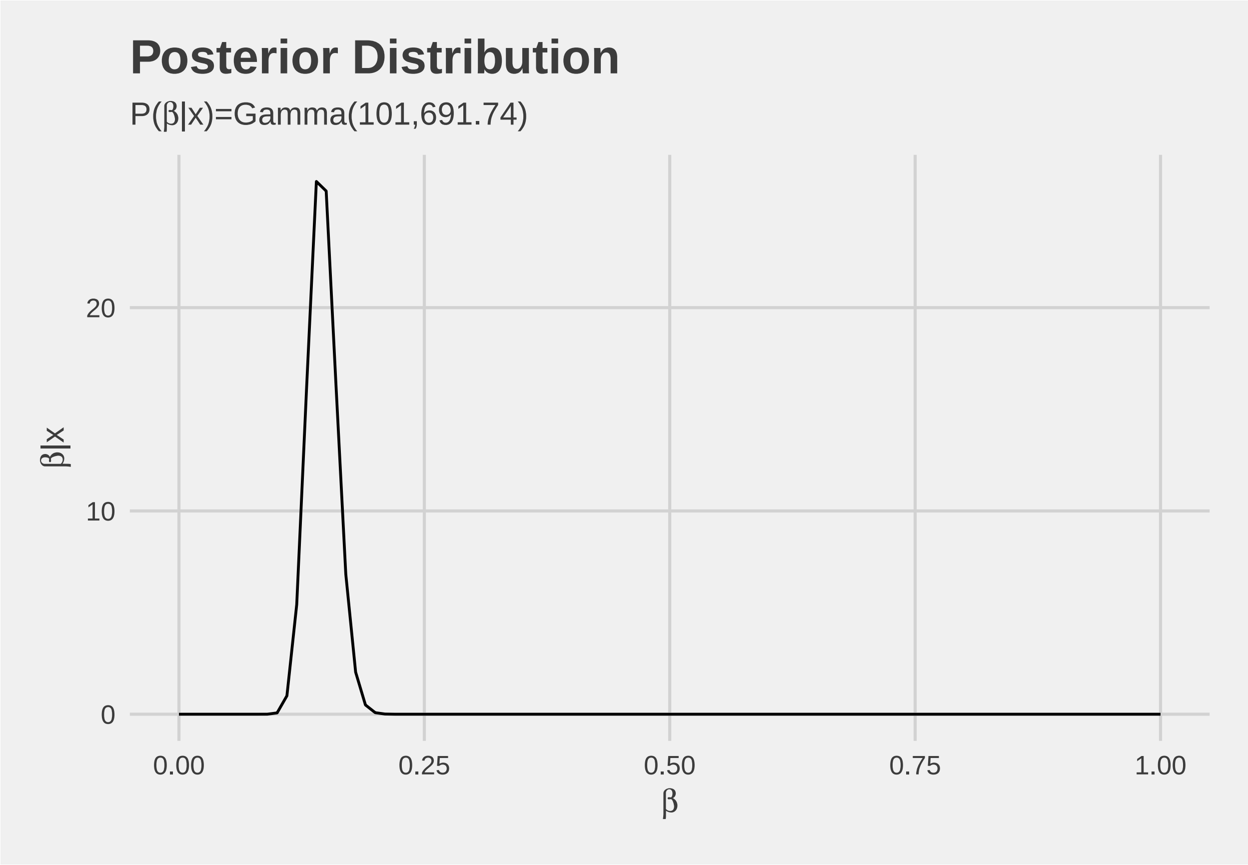
$$p(\beta|x) \propto \text{gamma}(\underbrace{n+1}_{\text{shape}}, \underbrace{\textstyle\sum x_i}_{\text{rate}})$$

plugging in our values:

$$p(\beta|x) \propto \text{gamma}(101, 691.74)$$

$$E(\beta|x) = \frac{a}{b} = \frac{101}{691.74} = 0.1460086$$

```r
a = length(oven$x) + 1
b = sum(oven$x)
fx <- function(x){
  return ((b^a/gamma(a))*x^(a-1)*exp(-b*x))
}
ggplot(data.frame(x = c(0,1) ), aes(x = x )) +
  stat_function(fun = fx) +
  labs(title = 'Posterior Distribution',
       subtitle = bquote(paste('P(',beta,'|x)=Gamma(101,691.74)')),
       x = expression(beta),
       y = bquote(paste(beta,'|',x))) +
  theme_fivethirtyeight() +
  theme(axis.title = element_text())
```



d. Based on the gamma distribution $p(\beta) \propto \beta^{a-1} exp(-\beta b)$, find the posterior distribution of β and then find the posterior mean of β.

d) given $\beta \sim gamma(a, b)$

then

$$p(\beta|x) \propto p(y|\beta)\, p(\beta)$$

$$\propto \beta^{n} e^{-\beta \Sigma x} \beta^{a-1} e^{-\beta b}$$

$$\propto \beta^{n+a-1} e^{-\beta(b+\Sigma x)}$$

$$p(\beta|x) \propto gamma(n+a, b+\Sigma x)$$

$$p(\beta|x) \propto gamma(101+a, b+691.74)$$

$$E(\beta|x) = \frac{101+a}{b+691.74}$$

e. What is a 95% Bayesian credible interval for the average length of life of a magnetron tube under the ***uniform*** prior on part c)?

```
M = 10000
beta = rgamma(M, 101, 691.74)
xast = rexp(M, beta)

quantile(xast, c(0.025,0.975))
```

```
##       2.5%      97.5%
##  0.1699791 25.4095318
```

- Our 95% Bayesian credible interval is given by (0.172, 25.73). This means that over a large amount of samples, 95% of the time the average length of life for the magnetron tube will reside in this interval.

f. What is the posterior probability that the average length of life of a magnetron tube is less than 10 years under the ***uniform*** prior on part c)?

  - Using the set up from part e we simply take the mean(xast $<$ 10) and get the probability of the magnetron tube life being less than 10 years as 0.7673.

```
mean(xast < 10)
```

```
## [1] 0.7646
```

g. Under the uniform prior, find Bayesian credible interval of β and Bayesian credible interval $1/\beta$? Is Bayesian credible interval invariant?

  - Bayesian credible for β is (0.119, 0.176)

```
qgamma(c(0.025,0.975), 101, 691.74)
```

```
## [1] 0.1189264 0.1758273
```

  - Bayesian credible for $1/\beta$ is (5.252, 7.67) $\neq$ the interval of β thus the bayesian credible interval is not invariant.

```
M = 10000
beta <- rgamma( M, shape = 110, rate = 691.74)
t <- 1 / beta

quantile(t, c(0.025,0.975))
```

```
##     2.5%    97.5%
## 5.255206 7.632088
```