Homework 1

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1.

a.

```
def isClockwise(poly):
    sum = 0
    for i in range(0, poly.n):
        p = (poly.x[i], poly.y[i])
        next = (i + 1) % poly.n
        q = (poly.x[next], poly.y[next])
        if cross(p, q) > 0: sum += 1
        elif cross(p, q) < 0: sum -=1
    return sum <= 0
```

The algorithm takes the cross product of every edge and checks the polarity of the result. It sums all these polarities and if it is < 0 it is clockwise.

The algorithm only iterates through all points once, and calculating cross product is done in constant time. Therefore the runtime $= O(n)$.

b.

```
def isSimple(poly):
    prev_x_prod = cross((poly.x[0], poly.y[0]), (poly.x[1], poly.y[1]))
    max_change = 1
    for i in range(1, poly.n):
        p = (poly.x[i], poly.y[i])
        next = (i + 1) % poly.n
        q = (poly.x[next], poly.y[next])
        current_cross = cross(p,q)
        if (prev_x_prod * current_cross) < 0:
            max_change -= 1
        prev_x_prod = current_cross
    return max_change >= 0
```

Similar to algorithm 1.a., this algorithm checks the polarity of cross products. Instead of keeping track of all previous polarities, it only cares about the last polarity. Since simple convex polygons contain only one polarity change when iterating through the points successively, a convex polygon with more than one polarity change is complex.

The algorithm only iterates through all points once, and calculating cross product is done in constant time. Therefore the runtime $= O(n)$.

2.

a.

A triangle or polygon of size $n = 3$ has a unique triangulation. Adding a vertex to this polygon will create a quadrilateral. This vertex can be placed anywhere inside the triangle and produce a unique triangulation, but this is not the case for polygons of size $n > 3$ with unique triangulation.

If we look outside the triangle we find areas that guarantee a quadrilateral with unique triangulation. A vertex placed inside these regions will not produce a diagonal in the new quadrilateral.

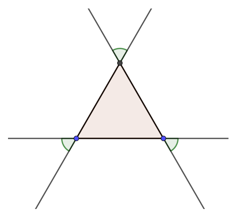
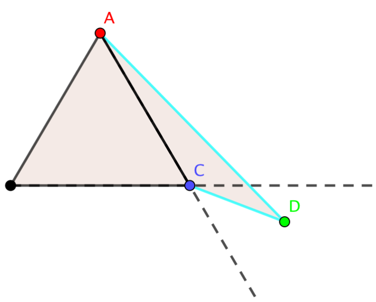
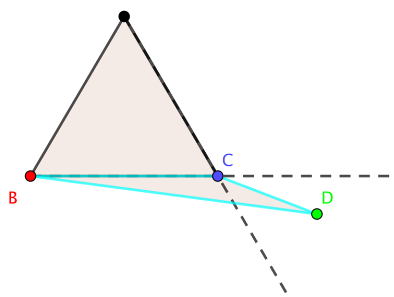


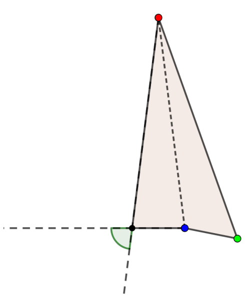
Figure 1: Regions marked with green angles are valid.

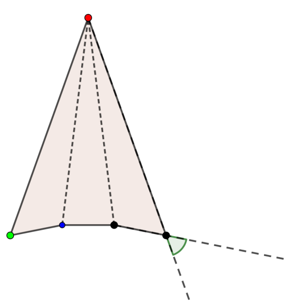


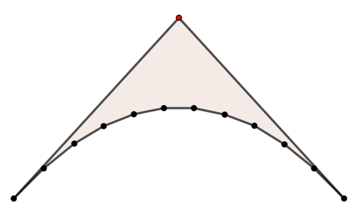


As you can see, point C prevents a diagonal with the non-neighboring vertex.

Using these regions, you can construct polygons of size $n > 3$. For example:







For a polygon of this standard shape of size n, there exists valid regions outside its two ears. Since the lower $n-1$ vertices are strictly-convex, no diagonal exists between these points. Therefore each must have only a diagonal with the top vertex (the red vertex in the upper examples), and therefore there exists only one triangulation.

More generally, the sum of interior angles is equal to $\pi(n-2)$. Reflex vertices are interior angles with an angle $> \pi$. Therefore there are at most $n-3$ reflex angles. Increasing the number of reflex vertices in a polygon will decrease the number of triangulations. If we maximize the number of reflex vertices $(n-3)$, then there should be the least number of triangulations (one unique).

3.

The generalized 2D solution would also use a hashtable that bucketed on both x and y values The bucketing can be split using r, e.g. $Bucket_x = \lfloor \frac{x}{r} \rfloor$, $Bucket_y = \lfloor \frac{y}{r} \rfloor$. We hash each point using "x, y" as the key.

In the 1D solution, only the next bucket had to be checked. In 2D is needed to check neighboring buckets, but sufficient enough to check 4: right, bottom, bottom-left, bottom-right. Previous buckets will have already checked the upper-left buckets.

So following the 1D solution, we add all points in the current bucket that is not our current point to the result. We then check 4 other buckets and test if each point is in range. If a point is in range, add it to the result.

Since each point is hashed only once, the space complexity is $O(n)$. The runtime is the same as the 1D solution because instead of checking 1 extra bucket, we are checking 4, and that is a constant increase. Therefore, $T(n) = O(n + k)$.

This solution can be adapted for L_2 by changing the range test to the eucldian distance. Every point within the current bucket will also have to be checked. This does not effect space or runtime.

4.

a.

The cross product of two vectors v_1 and v_2 gives us the area of the parallelogram that they form. Half of this area gives us the area formed between the two vectors. The polarity of the cross product is determined by the orientation of the two vectors since $v_1 \times v_2 = -(v_2 \times v_1)$.

For each pair of points p_i and p_{i-1} that share an edge, the formula finds the cross product of the vectors from the origin to those points. e.g.

$$(x_i, y_i) \times (x_{i-1}, y_{i-1}) = (x_i y_{i-1} - x_{i-1} y_i)$$

It then sums each of these cross-products and scales by 1/2. This leaves us with the area within the polygon. The orientation of the points correctly add and subtract the areas.

For a triangle (p_1, p_2, p_3), let v_1 be the vector from the origin to p_1, v_2 be the vector from the origin to p_2, and v_3 be the vector from the origin to p_3.

The area of the triangle is calculated as follows:

$$v_1 \times v_2 = x_1 y_2 - x_2 y_1$$

$$v_2 \times v_3 = x_2 y_3 - x_3 y_2$$

$$v_3 \times v_1 = x_3 y_1 - x_1 y_3$$

$$Area = \frac{1}{2} |(v_1 \times v_2) + (v_2 \times v_3) + (v_3 \times v_1)|$$

$$= \frac{1}{2} |x_1 y_2 - x_2 y_1 + x_2 y_3 - x_3 y_2 + x_3 y_1 - x_1 y_3|$$

The other way to find the area of this triangle is by calculating the cross product between two of the edges of the triangle. Let's take, for example, the edges (p_1, p_2) and (p_1, p_3). This can be represented by our vectors from the origin as $(v_1 - v_2) = (x_1 - x_2, y_1 - y_2)$ and $(v_1 - v_3) = (x_1 - x_3, y_1 - y_3)$.

The area between these two edges is calculated as follows:

$$Area = \frac{1}{2}|(v_1 - v_2) \times (v_1 - v_3)|$$

$$= \frac{1}{2}|(x_1 - x_2)(y_1 - y_3) - (x_1 - x_3)(y_1 - y_2)|$$

$$= \frac{1}{2}|x_1y_2 - x_2y_1 + x_2y_3 - x_3y_2 + x_3y_1 - x_1y_3|$$

As you can see, this is the same calculated area.

Since this finds the area of a triangle and any polygon can be triangulated, the area of a polygon is the sum of areas of the triangles. Since the given formula can be transformed into the sum of triangles, the formula correctly calculates the area of any polygon.

b.

(a.) When P is convex, two arrays can hold all points on the positive side of the chord and on the negative side of the chord. This can be found using turns: $(C_1, C_2) \times (C_1, P_i) \forall P_i \in P$.

Iterate through the points in order of P and add them to the corresponding arrays given the turn with the chord. Insert C_1 to both arrays on the first polarity change, and C_2 on the second. Then use the 4.a. formula to calculate the area of each sub-polygon in both arrays.

Since the arrays together form a single copy of P, the space complexity is $O(n)$. Since P is iterated though once the first time, and once more when finding areas of both sub-polygon, the algorithm runs in $O(n)$ time.

(b.) Starting with any point P_i, check if either point in the chord lies on the line between P_i and P_{i+1}. This is a simple check that can be done in constant time. If it fails, add P_i to the first array and continue with P_{i+1}. If it succeeds, add both P_i and the chord point to the array. Then add the chord point and P_{i+1} to the second linked list. Continue around the polygon and add to the other list. Once the second chord point is found, do the same as before.

This will correctly close the two sub-polygons in linear time. Then calculate the area of each sub-polygon. The space required is, once again, $O(n)$, and the time is linear, $O(n)$.

5.

a.

The sum of interior angles is equal to $\pi(n-2)$. Reflex vertices are interior angles with an angle $> \pi$. Therefore there are at most $n - 3$ reflex angles. In this worst case, there is an angle that can contain a diagonal with each reflex vertex, therefore there can be $n - 3$ indispensable diagonals incident on that vertex.

b. - d.

[None.]