UNIVERSITY OF GHANA

COLLEGE OF BASICE AND APPLIED SCIENCES

DEPARTMENT OF MATHEMATICS

BOOK OF EXERCISES - CALCULUS ONE (MATH122)

Exercises on Limits, Continuity and Intermediate Value Theorem

1. Let $f(x) = \frac{2^x - 1}{x}$, $x \neq 0$. Using numerical calculations, find the limit of $f(x)$ as $x \to 0$.

2. Let f be a real-valued function and $a \in \mathbb{R}$. Explain the meaning of the following;

   i. $\lim_{x \to a^+} f(x) = L$

   ii. $\lim_{x \to a^-} f(x) = L$

   iii. $\lim_{x \to a} f(x) = L$

3. For the given function $f(x)$ below, determine whether the limit of $f(x)$ exist as

   i. x approaches 5

   ii. $x \to -5$

$$f(x) = \begin{cases} 0 & , x \leq -5 \\ \sqrt{25 - x^2} & , -5 < x < 5 \\ 3x & , x \geq 5 \end{cases}$$

4. Find the real constants ϕ and ξ in the polynomial $p(x) = x^2 + \phi x + \xi$ such that

$$\lim_{x \to 2} \frac{p(x)}{x - 2} = 6.$$

   Are these constants unique?

5. For the function $f(x)$ defined below, find the value of k so that $\lim_{x\to 2} f(x) = \alpha$ where α is a finite real number. Hence find the value of α.

$$f(x) = \begin{cases} 3x + k & , x \leq 2 \\ x - 2 & , 2 < x \end{cases}$$

6. By using an appropriate theorem, evaluate the limit of $f(x)$ as x goes to 0.

   i. $\sqrt{5-2x^2} \leq f(x) \leq \sqrt{5-x^2}$ for $-1 \leq x \leq 1$.

   ii. $2 - x^2 \leq f(x) \leq 2\cos(x)$ for $x \in \mathbb{R}$.

7. For the function $f(x)$ defined below, find the constant a if $\lim_{x\to 1} f(x)$ exists.

$$f(x) = \begin{cases} x^2 - 2 & , x < 1 \\ ax - 4 & , x \geq 1 \end{cases}$$

8. Evaluate the following limits:

   a.

   i. $\displaystyle\lim_{x\to 1} \frac{x-1}{\sqrt{x}-1}$ ii. $\displaystyle\lim_{x\to 1} \frac{x-1}{\sqrt{4x+5}-3}$ iii. $\displaystyle\lim_{x\to 4} \frac{3(x-4)\sqrt{x+5}}{3-\sqrt{x+5}}$

   b.

   i. $\displaystyle\lim_{x\to 0} \frac{\sqrt{x+5}-\sqrt{5}}{x}$ ii. $\displaystyle\lim_{x\to -2} \frac{x+2}{x^2+5x+6}$ iii. $\displaystyle\lim_{x\to 1} \frac{x^6-1}{x^3-1}$

9. Evaluate

   a.

   i. $\displaystyle\lim_{x\to 0} \frac{\tan(x)}{x}$ ii. $\displaystyle\lim_{x\to 0} \frac{1-\cos(2x)}{x}$ iii. $\displaystyle\lim_{x\to 0} \frac{\sin(\sin(x))}{\sin(x)}$

   b.

   i. $\displaystyle\lim_{x\to \pi/4} \frac{\sin(x)-\cos(x)}{\cos(2x)}$ ii. $\displaystyle\lim_{x\to 0} \frac{\sec(x)}{\tan(x)}$ iii. $\displaystyle\lim_{x\to \pi/2} \frac{1-\sin(x)}{\cos(x)}$

10. Evaluate

    i. $\displaystyle\lim_{x\to 3} \frac{|x-3|}{3-x}$ ii. $\displaystyle\lim_{x\to 1} \frac{\sqrt[3]{x}-1}{\sqrt{x}-1}$ Hint: Let $t = x^{\frac{1}{3}}$

11. Evaluate

a.

i. $\lim\limits_{x\to\infty} \dfrac{x^2+1}{(x+2)(3x-4)}$ ii. $\lim\limits_{x\to-\infty} \dfrac{\sqrt{x^2+4x}}{4x+1}$ iii. $\lim\limits_{x\to\infty} (\sqrt{x+1}-\sqrt{x})$

b.

i. $\lim\limits_{x\to\infty} \dfrac{x^4-x^2+1}{x^5+x^3-x^2}$ ii. $\lim\limits_{x\to\infty} x\sin\left(\dfrac{1}{x}\right)$ iii. $\lim\limits_{x\to-\infty} \dfrac{x^4+x}{6x^3-x^2}$

12. Evaluate

a.

i. $\lim\limits_{x\to 0} \dfrac{\sin^3(x)}{x(1-\cos(x))}$ ii. $\lim\limits_{x\to 0} \dfrac{1-\cos(x)}{x^2}$ iii. $\lim\limits_{x\to 0} \dfrac{\sin(x)\cos(x)-\tan(x)}{x^2\sin(x)}$

b.

i. $\lim\limits_{x\to 0} \dfrac{\sin(3x)}{\tan(2x)}$ ii. $\lim\limits_{x\to 0} x\sin\left(\dfrac{1}{x}\right)$

13. For what values of x are the following functions continuous;

i. $f(x) = x^2 - 3x - 2$

ii. $f(x) = \frac{1}{x-1}$

iii. $f(x) = \frac{x}{x^3-8}$

iv. $f(x) = \frac{|x+1|}{x+1}$

14. Determine whether or not the given function is continuous on the prescribed interval.

i. $f(x) = \dfrac{1}{x}$ on $[1,2]$ ii. $f(x) = \begin{cases} 2x & , 0 < x \leq 3 \\ 15 - x^2 & , -3 \leq x \leq 0 \end{cases}$

15. Explain why the function $f(x)$ defined by

$$f(x) = \begin{cases} \frac{x^2-2x-8}{x-4} & , x \neq 4 \\ 3 & , x = 4 \end{cases}$$

is discontinuous at $x = 4$. Define a new function $f(x)$ so that f is continuous everywhere.

16. Find the value of the constant β which makes the function $f(x)$ continuous at $x = 1$, where

$$f(x) = \begin{cases} \frac{x^4-1}{x-1} & , x \neq 1 \\ \beta & , x = 1 \end{cases}$$

17. Find the value of the constant k such that $f(x)$ is continuous everywhere if

$$f(x) = \begin{cases} kx+1 & ,x \leq 3 \\ kx^2-1 & ,x > 3 \end{cases}$$

18. i. If $x^2 = \sqrt{x+1}$, show that there is a root of the equation in the interval $[1, 2]$.

    ii. Use the Intermediate Value Theorem to prove that there is a positive number c such that $c^2 = 2$.

    iii. Prove that the equation

$$x^5 - x^2 + 2x + 3 = 0$$

    has at least one real root.

Exercises on Differentiation and Applications of Differentiation

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1. Find from the first principles the derivatives of the following functions

    i. $f(x) = \frac{1}{(x-1)^2}$

    ii. $f(x) = \sqrt{x+1}$

    iii. $f(x) = \frac{4-3x}{2+x}$

    iv. $f(x) = \sin(3x)$

2. Differentiate the following functions with respect to x.

    i. $f(x) = \frac{x^2+4x+3}{\sqrt{x}}$

    ii. $f(x) = \frac{\sqrt{x}-1}{\sqrt{x}+1}$

    iii. $f(x) = (6x^2+5)^3(x^3-7)^4$

    iv. $f(x) = \sqrt{x^2+1}$

    v. $f(x) = (5-2x^2)^{-1}$

3. Find the derivative of each of the following functions:

    i. $f(x) = \tan(3x) - 4\sec(5x)$

    ii. $f(x) = \frac{\sin^2(x)}{\cos(x)}$

    iii. $f(x) = \sqrt{1+2\tan(x)}$

iv. $f(x) = \sin^3(x) + \cos^3(x)$

v. $f(x) = \sec^2(2x) - \tan^2(2x)$

vi. $f(x) = \frac{\sin(x)+\cos(x)}{\sin(x)-\cos(x)}$

4. If $F(x) = f(g(x))$ and $g(3) = 6,\ g'(3) = 4,\ f'(3) = 2$ and $f'(6) = 7$, find $F'(3)$.

5. i. If $y = \sec(x)\tan(x)$, show that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 3\sec^2(x) - \sec(x)$$

ii. If $y = x\sin(x)$, show that

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 2\cos(x) - y$$

6. i. For what values of A and B does $y = A\cos(x) + B\sin(x)$ satisfy the equation

$$y'' + 2y' + 3y = 2\sin(x)$$

ii. For what values of A and B does $y = Ax\cos(x) + Bx\sin(x)$ satisfy the equation

$$y'' + y = -3\cos(x)$$

7. If the equation $2x^2 + 6xy + 4y^2 = 3$ defines y as a twice-differentiable function of x, express $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of x and y and show that

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{3}{(3x+4y)^3}$$

8. If the equation $y^4 = 4xy + 3$ defines y as a twice-differentiable function of x, express $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of x and y and show that

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{3(1-y^4)}{2(y^3-x)^3}$$

9. Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of t when

$$x(t) = \frac{1+t}{1-2t}, \quad y(t) = \frac{1+2t}{1-t}.$$

Prove that $\frac{\mathrm{d}y}{\mathrm{d}x} = 1$ when $t = 0$ and find a second value of t for which $\frac{\mathrm{d}y}{\mathrm{d}x} = 1$. Prove that

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = -\frac{2}{3}\left(\frac{1-2t}{1-t}\right)^3$$

10. For the curve $x = t^2 + t$, $y = 3 - t^3$, show that

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = -\frac{6t(t+1)}{(2t+1)^3}, \quad t \neq -1/2.$$

11. Differentiate the following with respect to x.

    i. $f(x) = \sin^{-1}(x/3)$

    ii. $f(x) = \cos^{-1}(x)$

    iii. $f(x) = \tan^{-1}(\cot(x))$

    iv. $f(x) = (1 + x^2)\tan^{-1}(x)$

    v. $f(x) = x\cos^{-1}(x)$

    vi. $f(x) = \sqrt{1 - x^2}\sin^{-1}(x)$

12. i. If $y = (\sin^{-1}(x))^2$, prove that

$$(1 - x^2)\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - x\frac{\mathrm{d}y}{\mathrm{d}x} = 2$$

    ii. Given that $y = \sin(m\sin^{-1}(x))$, prove that

$$(1 - x^2)\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - x\frac{\mathrm{d}y}{\mathrm{d}x} + m^2 y = 0$$

13. Differentiate the following functions with respect to x.

    i. $f(x) = \mathrm{e}^{2x}$

    ii. $f(x) = \mathrm{e}^{2x^3}$

    iii. $f(x) = \mathrm{e}^{x\cos(x)}$

    iv. $f(x) = \mathrm{e}^{2x}\cos(3x)$

    v. $f(x) = x^2\mathrm{e}^{3x-2}$

    vi. $f(x) = \mathrm{e}^{x^2}\sec(x)$.

14. Differentiate the following functions with respect to x.

    i. $f(x) = \ln(x^3 - 3x)$

ii. $f(x) = \ln(\sec(x) + \tan(x))$

iii. $f(x) = \ln(\sqrt{4x+5})$

iv. $f(x) = \ln\left(\frac{\cos(x)}{\sqrt{1-x^2}}\right)$

v. $f(x) = \ln(3x^2 \mathrm{e}^{-x})$

15. If $x = 1 + \ln(t)$ and $y = \ln(t+1)$, find $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of t.

16. Differentiate the following functions with respect to x.

    i. $f(x) = \log_{10} x$

    ii. $f(x) = 5^{2x}$

    iii. $f(x) = x^{\ln(x)}$

    iv. $f(x) = \sqrt[3]{\frac{x^2-2}{x^2+4}}$

17. The radius of a circular blot is increasing at the rate of 0.15cm per second. At what rate is its area increasing at the moment when the radius is 5cm?

18. The area of a circle is increasing at the rate of 3cm^2 per second. Find

    i. the rate at which the radius is increasing at the moment when the radius is 6cm.

    ii. the rate at which the circumference is increasing when the radius is 6cm.

19. i. Show that the rate of change of the area of a circle with respect to the radius equals in magnitude the circumference. Find also the magnitude of the rate of change of the area with respect to the circumference.

    ii. Show that rate of change of the volume of sphere with respect to the radius equals in magnitude the surface area. Find also the magnitude of radius of change of the volume with respect to the surface area.

20. Assuming that a raindrop is a perfect sphere and that through condensation it accumulates moisture at a rate equal to twice its surface area, show that its radius increases at a constant rate.

21. If $(x+y)^3 - 5x + y = 1$ defines y as a twice-differentiable function of x, find $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of x and y. Find the equation of the tangent to the given curve at the point at which it meets the line $x + y = 1$.

22. If $x = \frac{1}{1+t^2}$ and $y = \frac{t}{1+t^2}$, show that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{t^2-1}{2t}.$$

Find the equation of the tangent at the point where $t = 2$.

23. Obtain the gradient at the point $P(\theta)$, to the curve whose parametric equations are given by

$$x = a\cos(\theta), \quad y = b\sin(\theta), \quad 0 \leq \theta \leq 2\pi.$$

The point A is given $\theta = \pi/4$. Find

   i. the equation of the tangent to the curve at A,

   ii. the equation of the normal to the curve at A.

24. Find the critical points of the curve

$$f(x) = x^3 + 5x^2 + 3x - 10$$

and distinguish between them. Sketch the graph of f.

25. Find the relative maximum or minimum value of the function f given by

$$f(x) = 4x^3 - 3x + 3$$

   i. by using the first derivative test,

   ii. by using the second derivative test.

26. Find the extremum points of the function

$$f(x) = 2x^3 + 3x^2 - 12x - 5$$

and identify intervals where the function is

   i. increasing

   ii. decreasing

   iii. concave upwards and

   iv. concave downward.

Sketch the graph of f.

27. For what values of x is the function f defined by

$$f(x) = x^4 - 4x^3 + 4x^2 + 1$$

an increasing function of x?. Find the local maximum and local minimum of f and sketch the graph of f.

28. Find the points of inflexion of the curve $y = a\sin^2(x) + b\cos^2(x)$.

29. Find the intervals in which the function

$$f(x) = \frac{x}{x^2 - 1}$$

is increasing or decreasing. Find also any asymptotes of the graph of f and sketch it.

30. Sketch the graph of the function

$$y = \frac{x^2 - 4x + 9}{x^2 + 4x + 9}$$

31. A wire of length $4a$ is bent to enclose a sector of a circle of radius r. Find this radius if the area of the sector is maximum.

32. A figure consists of a semi-circle with a rectangle constructed on its diameter. Given that the perimeter of the figure is 20m, find its dimensions in order that its area may be maximum.

33. i. Use calculus to calculate $f(x) = 3x^2 - 7x + 8$ approximately when $x = 2.015$.

    ii. Find the derivative of
    $$f(x) = \frac{7x - 4}{10 - 5x}$$
    and deduce the approximate numerical value of $f(x)$ when $x = 8.96$

34. Find the equations of the tangents to the curves

    i. $y = \cos^{-1}(x + 1)$ at the point $x = -3/2$

    ii. $y = \tan^{-1}(1/x)$ at the point $x = 1$.

35. i. Find the point of inflexion of the curve
    $$y = (x + 1)\tan^{-1}(x).$$

    ii. Show that $y = \tan^{-1}(x) + \cot^{-1}(x)$ is a constant by examining its derivative. What is the value of this constant?

    iii. Find in terms of x and y, the derivative of y with respect to x if
    $$x\sin^{-1} y + y\tan^{-1} x = x$$

36. A particle moves in a straight line so that at time t seconds its distance from a fixed

point 0 is s meters where $s = t^2 e^{2-t}$. Find the the distance of the particle from 0 when it first comes to rest and its acceleration at that point.

37. i. Find the values of x for which the function $(x^2 - 2x - 1)e^{2x}$ has maximum or minimum values, distinguishing between them.

    ii. Find the values of x between 0 and 2π for which the function $e^x \cos(x)$ has maximum or minimum values, distinguishing between them.

38. Find the critical point and the point of inflexion on the curve

$$y = \frac{1}{x}\ln(x), \quad x > 0$$

and sketch the curve.

Exercises on Integration and Applications of Integration

1. Integrate the following with respect to x

   i. $y = 5x^4$

   ii. $y = \sqrt{x}$

   iii. $y = x^{-\frac{3}{4}}$

   iv. $y = (x+1)^2$

   v. $y = \frac{1}{x^4} - \frac{1}{x^4} + x^2 - x^4$

   vi. $y = \frac{x+1}{x^3}$

   vii. $y = x(x^3+1)^2$

   viii. $y = x^2(1-x)^2$

2. If $\frac{dy}{dx} = 8x - \frac{4}{x^2}$ and $y = 14$ when $x = 2$, find y in terms of x.

3. If $f'(x) = 4x^3 - \frac{1}{x^3}$ and if $f(-2) = \frac{1}{4}$, find $f(x)$.

4. Given $\frac{dy}{dx} = x^{\frac{1}{2}} - x^{-\frac{1}{2}}$ and that $y = \frac{4}{3}$ when $x = 4$, find y in terms of x.

5. Integrate the following with respect to x

   i. $y = (x-1)^4$

   ii. $y = (2x-1)^5$

   iii. $y = \left(x - \frac{1}{x}\right)^4$

iv. $y = (x+1)^2$

v. $y = \frac{1}{\sqrt{4x+1}}$

6. Integrate the following with respect to x

   i. $\sin(4x)$

   ii. $\sec^2(3x)$

   iii. $\cos(x/2)$

   iv. $\cos^2(x)$

   v. $\tan^2(x)$

7. By using suitable substitution, integrate the following with respect to x.

   i. $x\sqrt{1-x^2}$

   ii. $x(1+x^2)^{\frac{3}{2}}$

   iii. $\sec^2(x)\tan(x)$

   iv. $\sin^5(x)$

   v. $\frac{2x}{\sqrt{x^2+1}}$

8. Integrate the following with respect to x

   i. e^{3x}

   ii. e^{x+3}

   iii. e^{1-x}

   iv. $(\mathrm{e}^{x} - \mathrm{e}^{-x})^2$

   v. $\frac{x^2}{x^3-1}$

   vi. $\frac{4x-5}{2x^2-5x+3}$

9. Evaluate

   i. $\int_6^8 \frac{1}{x-4}\mathrm{d}x$

   ii. $\int_0^1 5\frac{\mathrm{e}^x}{\mathrm{e}^x+1}\mathrm{d}x$

   iii. $\int_0^{\frac{\pi}{2}} \frac{\cos(x)}{4+\sin(x)}\mathrm{d}x$

   iv. $\int_0^{\frac{\pi}{2}} \frac{\cos(x)-\sin(x)}{\sin(x)+\cos(x)}\mathrm{d}x$

10. Evaluate

   i. $\int_1^2 (x^2-1)\mathrm{d}x$

ii. $\int_2^3 \frac{1}{(x-1)^3}\mathrm{d}x$

iii. $\int_{-1} \left(x^2 - \frac{1}{x^2}\right)^2 \mathrm{d}x$

iv. $\int_0^{\frac{\pi}{2}} \sin^2(x)\mathrm{d}x$

v. $\int_0^{\frac{\pi}{2}} (\cos(x) + \sin(x))^2\mathrm{d}x$

vi. $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos(3x)\mathrm{d}x$

11. Evaluate

i. $\int_0^1 (2x^2+1)(2x^3+3x+4)^{1/2}\mathrm{d}x$

ii. $\int_0^{\frac{\pi}{2}} \frac{\cos(x)}{(4+\sin(x))^2}\mathrm{d}x$

iii. $\int_0^1 x\sqrt{4-3x^2}\mathrm{d}x$

iv. $\int_0^{\frac{\pi}{2}} \sin^3(x)\mathrm{d}x$

v. $\int_{-1}^2 \sqrt{4-x^2}\mathrm{d}x$

vi. $\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{4\cos(x)}{3+\cos^2(x)}\mathrm{d}x$

vii. $\int_1^4 \frac{1}{x+\sqrt{x}}\mathrm{d}x$

viii. $\int_0^1 x^3(1-x^2)^{1/2}\mathrm{d}x$

12. The curve $y = (1+x)(3-x)$ intersects the positive side of the x-axis at A. The tangent to the curve at O, the origin, intersects the curve again at P. Calculate

i. the finite area bounded by the curve and OA,

ii. the finite area bounded by the curve and OP.

13. The straight line $y = 3x - 3$ intersects the parabola $y^2 = 12x$ at the points P and Q. Show that P and Q lie on opposite side of the x-axis and calculate the finite area bounded by the chord and the arc PQ of the parabola.

14. The curve $y = 1 - \frac{1}{4}x^2$ intersects the positive side of the x-axis at A and the y-axis at B. O is the origin. Calculate the volume generated when the finite area bounded by BO, OA, and the arc AB is rotated through four right angles

i. about the x-axis

ii. about the y-axis.

Give each answer as a multiple of π.

15. P(2, 3) lies on the curve $y = \frac{1}{2}x^2 + 1$. O is the origin, M is the foot of the perpendicular to the x-axis and the curve intersects the y-axis at A. The area bounded by AO, OM, MP and the curve PA of the curve is rotated through four right angles

about the y-axis. Find the volume of the solid formed, giving your answer as a multiple of π.

16. The curves $y = 4 - x^2$ and $y = x^2 + 2$ intersect at P and Q. The finite area bounded by the chord PQ and the curve $y = 4 - x^2$ is rotated through four right angles about the x-axis. Find the volume of the solid formed.

17. Sketch the curve $y^2 = (x-1)(x^2-1)$. If the curve is rotated about the axis of x through an angle 2π, show that the volume enclosed by the surface swept out by the loop of the curve is $\frac{4}{3}\pi$.