

Calculus II

Alessio Esposito

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Theorem 1. *A is closed \iff every accumulation point for A is in A*

Proof. " \implies " Let $A \subseteq \mathbb{R}^n$, $A = A \cup \partial A$.

Then $\forall p \in \bar{D}(A)$, $C_r(p) \setminus p \cap A \neq \emptyset \ \forall C \in \mathcal{C}_p$.

if $p \notin A$ then $C_r(p)$ has elements that don't belong to $A \Rightarrow p \in \partial A$.

" \impliedby " Let $p \in \partial A \Rightarrow \forall C \in \mathcal{C}_p$ of center r with $r \in \mathbb{R}$ by definition we can find some $x \in C \setminus p \cap A$, so that means $p \in \bar{D}(A) \Rightarrow p \in A$. \square

1 Limits

Definition 1. Let $A \subseteq \mathbb{R}^2$ and (x_0, y_0) an accumulation point for A. we define A^* as follows:

$$A^* = \{(\rho, \theta) \in [0, +\infty) \times [0, 2\pi] : (x_0 + \rho \cos(\theta), y_0 + \rho \sin(\theta)) \in A\}.$$

Proposition 1. Let's suppose that exist a circle C of center (x_0, y_0) such that $C \setminus \{(x_0, y_0)\} \subseteq A$ let r be the radius of the circle and as a consequence $(0, r] \times [0, 2\pi] \subseteq A^*$

Proof. Let $C \setminus \{(x_0, y_0)\}$ and $\begin{cases} 0 < \rho \leq r \\ 0 \leq \theta \leq 2\pi \end{cases}$ if $(\rho, \theta) \in (0, r] \times [0, 2\pi]$

then $(x_0 + \rho \cos(\theta), y_0 + \rho \sin(\theta)) \in C \setminus \{(x_0, y_0)\} \subseteq A \Rightarrow (\rho, \theta) \in A^*$. \square

Definition 2. Let $\theta \in [0, 2\pi]$ and $\forall \rho \in (0, r]$ we define $\varphi_\theta(\rho) = F(\rho, \theta)$ if $\rho \in (0, r]$, $(\rho, \theta) \in A^*$ so the $\lim_{\rho \rightarrow 0} \varphi(\rho) = l \in \bar{\mathbb{R}}$.

If that limit exists that means $\forall \theta \in [0, 2\pi]$ and $\forall \varepsilon > 0$, $\exists \delta > 0 \ \forall \rho \in (0, r]$ with $\rho < \delta \implies |\varphi_\theta - l| < \varepsilon$.

We say that $\lim_{\rho \rightarrow 0} \varphi(\rho) = l \in \bar{\mathbb{R}}$ Uniformly With Respect To (U.W.R.T.) θ .

Theorem 2. Let $f : A \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}$ with (x_0, y_0) accumulation point for A.

Follows the equivalence:

$$\lim_{(x,y) \rightarrow (x_0,y_0)} f(x,y) = l \in \bar{\mathbb{R}} \iff \lim_{\rho \rightarrow 0} F(\rho, \theta) = l \text{ U.W.R.T. } \theta.$$

Proof. Let $l \in \bar{\mathbb{R}}$.

" \implies " $\lim_{(x,y) \rightarrow (x_0,y_0)} f(x,y) = l$ so $\forall \varepsilon > 0, \exists \delta > 0 : \forall (x,y) \in A$ with $\|(x,y) - (x_0,y_0)\| < \delta, |f(x,y) - l| < \varepsilon$.

We have to prove that $\forall \varepsilon > 0, \exists \delta > 0 : \forall \theta \in [0, 2\pi], \forall \rho \in (0, r]$

with $\rho < \delta \implies |F(\rho, \theta) - l| < \varepsilon$.

Let $\varepsilon > 0, \theta \in [0, 2\pi], \rho \in (0, r]$ with $\rho < \delta$. we create the system that changes the coordinates from cartesian to polars:

$$\begin{cases} x = x_0 + \rho \cos(\theta) \\ y = y_0 + \rho \sin(\theta) \end{cases} \quad \rho = \sqrt{(x - x_0)^2 + (y - y_0)^2}$$

$\rho \in (0, r], \theta \in [0, 2\pi] \implies (\rho, \theta) \in A^*, (\rho, \theta) \in A^* \implies (x, y) \in A$.

Now $0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} = \rho < \delta \implies |f(x, y) - l| < \varepsilon$.
 $\implies |f(x_0 + \rho \cos(\theta), y_0 + \rho \sin(\theta)) - l| < \varepsilon \implies |F(\rho, \theta) - l| < \varepsilon$.

" \Leftarrow " $\forall \varepsilon > 0, \exists \delta \leq r : \forall \theta \in [0, 2\pi]$ and $\forall \rho$ with $0 < \rho < \delta \implies |F(\rho, \theta) - l| < \varepsilon$.

We have to prove that $\forall \varepsilon > 0, \exists \delta > 0, \forall (x, y) \in A$ with

$\sqrt{(x - x_0)^2 + (y - y_0)^2} = \|(x, y) - (x_0, y_0)\| < \delta \implies |f(x, y) - l| < \varepsilon$.

Let $\varepsilon > 0, \delta \leq r, (x, y) \in A, \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta$, we switch coordinates with ρ and θ as follows:

$$\begin{cases} x = x_0 + \rho \cos(\theta) \\ y = y_0 + \rho \sin(\theta) \end{cases} \quad \rho = \sqrt{(x - x_0)^2 + (y - y_0)^2}$$

$0 < \rho < \delta \leq r \implies \rho \in (0, r), \theta \in [0, 2\pi]$.

We notice that $|F(\rho, \theta) - l| < \varepsilon$, so $|f(x_0 + \rho \cos(\theta), y_0 + \rho \sin(\theta)) - l| < \varepsilon$
 $\implies |f(x, y) - l| < \varepsilon$. \square

Definition 3. We say that $\theta \in [0, 2\pi]$ is admissible if $0 \in \bar{D}(A_\theta)$.

Definition 4. Let's suppose that $\lim_{\rho \rightarrow 0} F(\rho, \theta) = l \in \mathbb{R}$ then $\forall \rho \in (0, r], \varphi$

Theorem 3. $\lim_{\rho \rightarrow 0} F(\rho, \theta) = l \in \bar{\mathbb{R}}$ U.W.R.T. $\theta \iff \lim_{\rho \rightarrow 0} \varphi(\rho) = 0$.

Corollary 1. $\lim_{\rho \rightarrow 0} F(\rho, \theta) = l \in \bar{\mathbb{R}}$ U.W.R.T. $\theta \iff \exists$ a function $\psi(\rho)$ such that $\lim_{\rho \rightarrow 0} \psi(\rho) = 0$ and $\forall \theta \implies |F(\rho, \theta) - l| \leq \psi(\rho)$.

Corollary 2. Let's suppose that $\lim_{\rho \rightarrow 0} F(\rho, \theta) = +\infty$.

$\forall \rho \in (0, r]$ let $h(\rho) = \inf\{F(\rho, \theta) : \theta \in [0, 2\pi]\}$ so then $\lim_{\rho \rightarrow 0} F(\rho, \theta) = +\infty$ U.W.R.T. $\theta \iff \lim_{\rho \rightarrow 0} h(\rho) = +\infty$

Obs 1. $\lim_{\rho \rightarrow 0} F(\rho, \theta) = +\infty$ U.W.R.T. $\theta \iff \exists$ a function $K(\rho)$ s.t. $\lim_{\rho \rightarrow 0} K(\rho) = +\infty$ and $F(\rho, \theta) \geq K(\rho)$

Corollary 3. Let's suppose that $\lim_{\rho \rightarrow 0} F(\rho, \theta) = -\infty$.

$\forall \rho \in (0, r]$ let $g(\rho) = \sup\{F(\rho, \theta) : \theta \in [0, 2\pi]\}$ so then $\lim_{\rho \rightarrow 0} F(\rho, \theta) = -\infty$ U.W.R.T. $\theta \iff \lim_{\rho \rightarrow 0} g(\rho) = -\infty$

Definition 5. Let $f : A \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}$ with A open.

let $(x_0, y_0) \in A, \varphi(x) = f(x, y_0)$ and $\psi = f(x_0, y)$. A is open that means that those two functions are well defined.

Differentiability

Definition 6. Let be $f : A \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$ with A Open. Let $\bar{x} \in A$ and let $i \leq n$, we denote as $\varphi_i(x_i) = f(\bar{x}_1, \bar{x}_2, \dots, \bar{x}_{i-1}, \bar{x}_i, \bar{x}_{i+1}, \dots, \bar{x}_n)$. Notice that \bar{x} is an internal point so then it exist an interval where φ_i is well defined.

Definition 7. We say that f is partially derivable with respect to the variable x_i in the point \bar{x} if φ_i is derivable in that point. We denote as $\frac{\partial f}{\partial x_i}$ the partial derivative with respect to x_i in the point \bar{x} .

Definition 8. The gradient of a function in n variables is defined as follows:

$$\nabla f : \bar{x} \in A \mapsto \left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n} \right) \in \mathbb{R}^n$$

The case of two variables

Let $f : A \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}$ with A open, and let $(x_0, y_0) \in A$.
 $(x_0, y_0, f(x_0, y_0)) \in \mathcal{G}(f)$. The equation of the plane that passes for
 $(x_0, y_0, f(x_0, y_0))$ is $Z = f(x_0, y_0) + a(X - x_0) + b(Y - y_0)$ where $a, b \in \mathbb{R}$.

Definition 9. We say that f is partially derivable with respect to x in (x_0, y_0) if φ is differentiable in x_0 . in that case we φ is the partial derivative of f in the variable x and its written $\frac{\partial f}{\partial x}$

Definition 10. We define the gradient as $\nabla f : (x, y) \in A \mapsto \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right) \in \mathbb{R}^2$

Definition 11. We say that f is differentiable in the point (x_0, y_0) if exists $a, b \in \mathbb{R}$ such that:

$$\lim_{(x,y) \rightarrow (x_0,y_0)} \frac{f(x,y) - f(x_0,y_0) - a(X - x_0) - b(Y - y_0)}{\|(x,y) - (x_0,y_0)\|} = 0 \quad (\Delta)$$

f is differentiable in the point (x_0, y_0) if exists a plane that passes in the point $(x_0, y_0, f(x_0, y_0))$ that approximates the graph of the function f .

Proposition 2. If f is differentiable in the point (x_0, y_0) , f is partially derivable with respect to x and y such that $a = \frac{\partial f(x_0, y_0)}{\partial x}$ and $b = \frac{\partial f(x_0, y_0)}{\partial y}$

Definition 12. if f is differentiable in a point $(x, y) \in A$, the differential in the point is defined as follows:

$$d_{(x,y)}f : (h, k) \in \mathbb{R}^2 \mapsto \frac{\partial f(x,y)}{\partial x}h + \frac{\partial f(x,y)}{\partial y}k \in \mathbb{R}$$

Corollary 4. f is differentiable in the point $(x_0, y_0) \iff f$ is partially derivable in the point (x_0, y_0) and the (Δ) is true.

Theorem 4. Let $f : A \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}$ with A . If $\exists \frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ in A and are continuous in a point (x_0, y_0) , then the function is differentiable in (x_0, y_0) .

Proof. We have to prove that:

$$\lim_{(x,y) \rightarrow (x_0,y_0)} \frac{f(x,y) - f(x_0,y_0) - \frac{\partial f(x_0,y_0)}{\partial x}(x-x_0) - \frac{\partial f(x_0,y_0)}{\partial y}(y-y_0)}{\sqrt{(x-x_0)^2 + (y-y_0)^2}} = 0$$

Lets add and subtract $f(x, y_0)$, so one has:

$$f(x,y) - f(x_0,y_0) = f(x,y) - f(x,y_0) + f(x,y_0) - f(x_0,y_0)$$

We call $\varphi(t) = f(x, t)$ where $t \in I[y, y_0]$ and $I[y, y_0] = \begin{cases} [y, y_0] & y \leq y_0 \\ [y_0, y] & y_0 \leq y \end{cases}$

φ is derivable and for the Lagrange theorem $\exists y_1 \in I[y, y_0] : \varphi(y) - \varphi(y_0) = \dot{\varphi}(y_1)(y - y_0)$. So one has $f(x, y) - f(x, y_0) = \frac{\partial f(x, y_1)}{\partial y}(y - y_0)$, and we repeat the same reasoning for the other variable and one will have $f(x, y_0) - f(x_0, y_0) = \frac{\partial f(x_1, y_0)}{\partial x}(x - x_0)$ We have then:

$$\begin{aligned} & \left| \frac{f(x,y) - f(x_0,y_0) - \frac{\partial f(x_0,y_0)}{\partial x}(x-x_0) - \frac{\partial f(x_0,y_0)}{\partial y}(y-y_0)}{\sqrt{(x-x_0)^2 + (y-y_0)^2}} \right| = \\ &= \left| \frac{\frac{\partial f(x_1,y_0)}{\partial x}(x-x_0) - \frac{\partial f(x,y_1)}{\partial y}(y-y_0) - \frac{\partial f(x_0,y_0)}{\partial x}(x-x_0) - \frac{\partial f(x_0,y_0)}{\partial y}(y-y_0)}{\sqrt{(x-x_0)^2 + (y-y_0)^2}} \right| = \\ &= \left| \frac{\left(\frac{\partial f(x_1,y_0)}{\partial x} - \frac{\partial f(x_0,y_0)}{\partial x} \right)(x-x_0) + \left(\frac{\partial f(x,y_1)}{\partial y} - \frac{\partial f(x_0,y_0)}{\partial y} \right)(y-y_0)}{\sqrt{(x-x_0)^2 + (y-y_0)^2}} \right| \end{aligned}$$

The last member is increased by the following:

$$\begin{aligned} & \left| \frac{\partial f(x_1,y_0)}{\partial x} - \frac{\partial f(x_0,y_0)}{\partial x} \right| \frac{|x-x_0|}{\|(x-x_0, y-y_0)\|} + \left| \frac{\partial f(x,y_1)}{\partial y} - \frac{\partial f(x_0,y_0)}{\partial y} \right| \frac{|y-y_0|}{\|(x-x_0, y-y_0)\|} \leq \\ & \left| \frac{\partial f(x_1,y_0)}{\partial x} - \frac{\partial f(x_0,y_0)}{\partial x} \right| + \left| \frac{\partial f(x,y_1)}{\partial y} - \frac{\partial f(x_0,y_0)}{\partial y} \right| \end{aligned}$$

And since $x \rightarrow x_0 \Rightarrow x_1 \rightarrow x_0$ and $y \rightarrow y_1 \Rightarrow y_1 \rightarrow y_0$ so the second member of the inequality is equal to zero. \square

Theorem 5. Let $f : A \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}$ with A open. If the function is differentiable in a point $(x_0, y_0) \in A$ then is continuous in that point.

Directional Derivatives

If we take $f : A \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}$ and it's partial derivatives, we can take for example $\frac{\partial f}{\partial x}$ as the direction of the function calculated on the line $y = y_0$. So let a

function be defined like the one before and let $(\lambda, \mu) \in \mathbb{R}^2$ with $\sqrt{\lambda^2 + \mu^2} = 1$. Let r the line with the following equations:

$$\begin{cases} x = x_0 + \lambda t \\ y = y_0 + \mu t \end{cases}$$

(x_0, y_0) is internal to A so there exists a rectangle R_0 of center (x_0, y_0) , so every line that passes in this point encounters a segment of the rectangle.

Study of the maxima and minima

Definition 13. If a partial derivative $\frac{\partial f}{\partial x}$ of a function $f : A \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}$ is partially derivable with respect to x in a point $(x_0, y_0) \in A$ we say that f is partially derivable two times with respect to x in the point (x_0, y_0) and it will be denoted as $\frac{\partial f}{\partial x} f_x(x_0, y_0) = \frac{\partial^2 f(x_0, y_0)}{\partial x^2} = \frac{\partial^2 f}{\partial x^2}(x_0, y_0)$.

The same goes for the other partial derivatives: $\frac{\partial}{\partial y} f_x = \frac{\partial^2 f}{\partial x \partial y}$, $\frac{\partial}{\partial x} f_y = \frac{\partial^2 f}{\partial y \partial x}, \dots$

Definition 14. We define the hessian matrix as follows:

$$\mathcal{D}^2 f = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix}$$

Definition 15. The determinant of $\mathcal{D}^2 f$ is:

$$\mathcal{H}(x, y) = \begin{vmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{vmatrix}$$

and is called the Hessian determinant.

Theorem 6. Let $f : A \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}$ with A open. Let's suppose that \exists the partial derivatives $\frac{\partial^2 f}{\partial x \partial y}$ and $\frac{\partial^2 f}{\partial y \partial x}$ continuous in a point $(x_0, y_0) \in A$. So then $\frac{\partial^2 f(x_0, y_0)}{\partial x \partial y} = \frac{\partial^2 f(x_0, y_0)}{\partial y \partial x}$.

Definition 16. Let $f : A \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}$, we say that $(x_0, y_0) \in A$ is maxima (minima) for f if $\forall (x, y) \in A$, $f(x, y) \leq f(x_0, y_0)$ ($f(x, y) \geq f(x_0, y_0)$).

Theorem 7. If f is continuous and A is compact, f admits minima and maxima.

Theorem 8. Let $f : A \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}$ and let $(x_0, y_0) \in \dot{A}$ a relative extreme and let f be partially derivable in (x_0, y_0) , so then $\frac{\partial f(x_0, y_0)}{\partial x} = 0$ and $\frac{\partial f(x_0, y_0)}{\partial y} = 0$. The points where the partial derivatives are 0 are said "critical points" of f , $(x_0, y_0) \in \dot{A}$ is an extreme relative $\Rightarrow (x_0, y_0)$ is a critical point for f (\Leftarrow).

Obs 2. Let $(x_0, y_0) \in A$ and let $g(x, y) = f(x, y) - f(x_0, y_0)$, (x_0, y_0) is a relative minimum (relative maximum) for $f \iff \exists$ a circle C of center (x_0, y_0) such that $g \geq 0$ ($g \leq 0$).

Theorem 9. *Let $f : A \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}$ and let $(x_0, y_0) \in A$ a relative extreme $\implies \mathcal{H}(x_0, y_0) \geq 0$.*

Theorem 10. *Let $f : A \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}$, $f \in \mathbf{C}^2$. Let $(x_0, y_0) \in A$ a critical point and lets suppose that $\mathcal{H}(x_0, y_0) > 0 \implies (x_0, y_0)$ is a relative extreme and is maximum or minimum depending on $\frac{\partial^2 f(x_0, y_0)}{\partial x^2}$ be < 0 or > 0 .*