Calculus II

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Theorem 1. *A is closed $\iff$ every accumulation point for A is in A*

Proof. " $\implies$ " Let $A \subseteq \mathbb{R}^n,\ A = A \cup \partial A$.
Then $\forall p \in \bar{\mathcal{D}}(A),\ C_r(p)_{\setminus p} \cap A \neq \emptyset\ \forall C \in \mathcal{C}_p$.
if $p \notin A$ then $C_r(p)$ has elements that dont belong to $A \Rightarrow p \in \partial A$.
" $\impliedby$ " Let $p \in \partial A \Rightarrow \forall C \in \mathcal{C}_p$ of center r with $r \in \mathbb{R}$ by definition we can find some $x \in C_{\setminus p} \cap A$, so that means $p \in \bar{\mathcal{D}}(A) \Rightarrow p \in A$. □

1 Limits

Definition 1. *Let $A \subseteq \mathbb{R}^2$ and (x_0, y_0) an accumulation point for A. we define A^* as follows:*
$A^* = \{(\rho, \theta) \in [0, +\infty] \times [0, 2\pi] : (x_0 + \rho\cos(\theta), y_0 + \rho\sin(\theta)) \in A\}$.

Proposition 1. *Lets suppose that exist a circle C of center (x_0, y_0) such that $C_{\setminus\{(x_0,y_0)\}} \subseteq A$ let r be the radius of the circle and as a consequence $(0, r] \times [0, 2\pi] \subseteq A^*$*

Proof. Let $C_{\setminus\{(x_0,y_0)\}}$ and $\begin{cases} 0 < \rho \leqslant r \\ 0 \leqslant \theta \leqslant 2\pi \end{cases}$ if $(\rho, \theta) \in (0, r] \times [0, 2\pi]$
then $(x_0 + \rho\cos(\theta), y_0 + \rho\sin(\theta)) \in C_{\setminus\{(x_0,y_0)\}} \subseteq A \Rightarrow (\rho, \theta) \in A^*$. □

Definition 2. *Let $\theta \in [0, 2\pi]$ and $\forall \rho \in (0, r]$ we define $\varphi_\theta(\rho) = F(\rho, \theta)$ if $\rho \in (0, r], (\rho, \theta) \in A^*$ so the $\lim_{\rho\to 0} \varphi(\rho) = l \in \bar{\mathbb{R}}$.*
If that limit exists that means $\forall\theta \in [0, 2\pi]$ and $\forall\varepsilon > 0$, $\exists\delta > 0\ \forall\rho \in (0, r]$ with $\rho < \delta\ |\varphi_\theta - l| < \varepsilon$.
We say that $\lim_{\rho\to 0} \varphi(\rho) = l \in \bar{\mathbb{R}}$ Uniformly With Respect To (U.W.R.T.) θ.

Theorem 2. *Let $f : A \subseteq \mathbb{R}^2 \to \mathbb{R}$ with (x_0, y_0) accumulation point for A. Follows the equivalence:*
$\lim_{(x,y)\to(x_0,y_0)} f(x, y) = l \in \bar{\mathbb{R}} \iff \lim_{\rho\to 0} F(\rho, \theta) = l$ *U.W.R.T. θ.*

Proof. Let $l \in \bar{\mathbb{R}}$.
" $\implies$ " $\lim_{(x,y)\to(x_0,y_0)} f(x, y) = l$ so $\forall\varepsilon > 0, \exists\delta > 0 : \forall\ (x, y) \in A$
with $\|(x, y) - (x_0, y_0)\| < \delta,\ |f(x, y) - l| < \varepsilon$.

We have to prove that $\forall \varepsilon > 0,\ \exists \delta > 0 : \forall \theta \in [0, 2\pi],\ \forall \rho(0, r]$
with $\rho < \delta\ |F(\rho, \theta) - l| < \varepsilon$.
Let $\varepsilon > 0,\ \theta \in [0, 2\pi],\ \rho \in (0, r]$ with $\rho < \delta$. we create the system that changes the coordinates from cartesians to polars:

$$\begin{cases} x = x_0 + \rho\cos(\theta) \\ y = y_0 + \rho\sin(\theta) \end{cases} \quad \rho = \sqrt{(x - x_0)^2 + (y - y_0)^2}$$

$\rho \in (0, r],\ \theta \in [0, 2\pi] \in (0, r] \times [0, 2\pi] \subseteq A^*,\ (\rho, \theta) \in A^* \Rightarrow (x, y) \in A$.

Now $0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} = \rho < \delta \Rightarrow |f(x, y) - l| < \varepsilon$.
$\Rightarrow |f(x_0 + \rho\cos(\theta), y_0 + \rho\sin(\theta)) - l| < \varepsilon \Rightarrow |F(\rho, \theta) - l| < \varepsilon$.

" $\Longleftarrow$ " $\forall \varepsilon > 0, \exists \delta \leq r : \forall \theta \in [0, 2\pi]$ and $\forall \rho$ with $0 < \rho < \delta \Rightarrow$
$|F(\rho, \theta) - l| < \varepsilon$.
We have to prove that $\forall \varepsilon > 0, \exists \delta > 0, \forall (x, y) \in A$ with
$\sqrt{(x - x_0)^2 + (y - y_0)^2} = \|(x, y) - (x_0, y_0)\| < \delta \Rightarrow |f(x, y) - l| < \varepsilon$.

Let $\varepsilon > 0,\ \delta \leq r,\ (x, y) \in A,\ \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta$, we switch coordinates with ρ and θ as follows:

$$\begin{cases} x = x_0 + \rho\cos(\theta) \\ y = y_0 + \rho\sin(\theta) \end{cases} \quad \rho = \sqrt{(x - x_0)^2 + (y - y_0)^2}$$

$0 < \rho < \delta \leq r \Rightarrow \rho \in (0, r],\ \theta \in [0, 2\pi]$.
We notice that $|F(\rho, \theta) - l| < \varepsilon$, so $|f(x_0 + \rho\cos(\theta), y_0 + \rho\sin(\theta)) - l| < \varepsilon$
$\Rightarrow |f(x, y) - l| < \varepsilon$. $\square$

Definition 3. *We say that* $\theta \in [0, 2\pi]$ *is admissible if* $0 \in \bar{\mathcal{D}}(A_\theta)$.

Definition 4. *Let's suppose that* $\lim_{\rho\to 0} F(\rho, \theta) = l \in \mathbb{R}$ *then* $\forall \rho \in (0, r],\ \varphi$

Theorem 3. $\lim_{\rho\to 0} F(\rho, \theta) = l \in \bar{\mathbb{R}}$ *U.W.R.T.* $\theta \Longleftrightarrow \lim_{\rho\to 0} \varphi(\rho) = 0$.

Corollary 1. $\lim_{\rho\to 0} F(\rho, \theta) = l \in \bar{\mathbb{R}}$ *U.W.R.T.* $\theta \Longleftrightarrow \exists$ *a function* $\psi(\rho)$ *such that* $\lim_{\rho\to 0} \psi(\rho) = 0$ *and* $\forall \theta\ \ |F(\rho, \theta) - l| \leqslant \psi(\rho)$.

Corollary 2. *Let's suppose that* $\lim_{\rho\to 0} F(\rho, \theta) = +\infty$.
$\forall \rho \in (0, r]$ *let* $h(\rho) = inf\{F(\rho, \theta) : \theta \in [0, 2\pi]\}$ *so then* $\lim_{\rho\to 0} F(\rho, \theta) = +\infty$ *U.W.R.T.* $\theta \Longleftrightarrow \lim_{\rho\to 0} h(\rho) = +\infty$

Obs 1. $\lim_{\rho\to 0} F(\rho, \theta) = +\infty$ *U.W.R.T.* $\theta \Longleftrightarrow \exists$ *a function* $K(\rho)$ *s.t.*
$\lim_{\rho\to 0} K(\rho) = +\infty$ *and* $F(\rho, \theta) \geq K(\rho)$

Corollary 3. *Let's suppose that* $\lim_{\rho\to 0} F(\rho, \theta) = -\infty$.
$\forall \rho \in (0, r]$ *let* $g(\rho) = sup\{F(\rho, \theta) : \theta \in [0, 2\pi]\}$ *so then* $\lim_{\rho\to 0} F(\rho, \theta) = -\infty$ *U.W.R.T.* $\theta \Longleftrightarrow \lim_{\rho\to 0} g(\rho) = -\infty$

Definition 5. *Let* $f : A \subseteq \mathbb{R}^2 \to \mathbb{R}$ *with* A *open.*
let $(x_0, y_0) \in A$, $\varphi(x) = f(x, y_0)$ *and* $\psi = f(x_0, y)$. A *is open that means that those two functions are well defined.*

Differentiability

Definition 6. *Let be $f : A \subseteq \mathbb{R}^n \to \mathbb{R}$ with A Open. Let $\bar{x} \in A$ and let $i \leq n$, we denote as $\varphi_i(x_i) = f(\bar{x}_1, \bar{x}_2, ..., \bar{x}_{i-1}, \bar{x}_i, \bar{x}_{i+1}, ..., \bar{x}_n)$. Notice that $\bar{x}$ is an internal point so then it exist an interval where φ_i is well defined.*

Definition 7. *We say that f is partially derivable with respect to the variable x_i in the point $\bar{x}$ if φ_i is derivable in that point. We denote as $\frac{\partial f}{\partial x_i}$ the partial derivative with respsect to x_i in the point $\bar{x}$.*

Definition 8. *The gradient of a function in n variables is defined as follows:*

$$\nabla f : \bar{x} \in A \longmapsto \left(\frac{\partial f}{\partial x_1}, ..., \frac{\partial f}{\partial x_n} \right) \in \mathbb{R}^n$$

The case of two variables

Let $f : A \subseteq \mathbb{R}^2 \to \mathbb{R}$ with A open, and let $(x_0, y_0) \in A$.
$(x_0, y_0, f(x_0, y_0)) \in \mathcal{G}(f)$. The equation of the plane that passes for
$(x_0, y_0, f(x_0, y_0))$ is $Z = f(x_0, y_0) + a(X - x_0) + b(Y - y_0)$ where $a, b \in \mathbb{R}$.

Definition 9. *We say that f is partially derivable with respect to x in (x_0, y_0) if φ is differentiable in x_0. in that case we φ is the partial derivative of f in the variable x and its written $\frac{\partial f}{\partial x}$*

Definition 10. *We define the gradient as $\nabla f : (x, y) \in A \longmapsto \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right) \in \mathbb{R}^2$*

Definition 11. *We say that f is differentiable in the point (x_0, y_0) if exists $a, b \in \mathbb{R}$ such that:*

$$\lim_{(x,y) \to (x_0, y_0)} \frac{f(x, y) - f(x_0, y_0) - a(X - x_0) - b(Y - y_0)}{\|(x, y) - (x_0, y_0)\|} = 0 \ (\triangle)$$

f is differentiable in the point (x_0, y_0) if exists a plane that passes in the point $(x_0, y_0, f(x_0, y_0))$ that approximates the graph of the function f.

Proposition 2. *If f is differentiable in the point (x_0, y_0), f is partially derivable with respsect to x and y such that $a = \frac{\partial f(x_0, y_0)}{\partial x}$ and $b = \frac{\partial f(x_0, y_0)}{\partial y}$*

Definition 12. *if f is differentiable in a point $(x, y) \in A$, the differential in the point is defined as follows:*

$$d_{(x,y)} f : (h, k) \in \mathbb{R}^2 \longmapsto \frac{\partial f(x, y)}{\partial x} h + \frac{\partial f(x, y)}{\partial y} k \in \mathbb{R}$$

Corollary 4. *f is differentiable in the point $(x_0, y_0) \iff f$ is partially derivable in the point (x_0, y_0) and the $(\triangle)$ is true.*

Theorem 4. *Let $f : A \subseteq \mathbb{R}^2 \to \mathbb{R}$ with A. If $\exists$ $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ in A and are continuos in a point (x_0, y_0), then the function is differentiable in (x_0, y_0).*

Theorem 5. *Let $f : A \subseteq \mathbb{R}^2 \to \mathbb{R}$ with A open. If the function is differentiable in a point $(x_0, y_0) \in A$ then is continuos in that point.*

Directional Derivatives

If we take $f : A \subseteq \mathbb{R}^2 \to \mathbb{R}$ and it's partial derivatives, we can take for example $\frac{\partial f}{\partial x}$ as the direction of the function calculated on the line $y = y_0$. So let a function be defined like the one before and let $(\lambda, \mu) \in \mathbb{R}^2$ with $\sqrt{\lambda^2 + \mu^2} = 1$. Let r the line with the following equations:

$$\begin{cases} x = x_0 + \lambda t \\ y = y_0 + \mu t \end{cases}$$

(x_0, y_0) is internal to A so there exists a rectangle R_0 of center (x_0, y_0), so every line that passes in this point encounters a segment of the rectangle.

Study of the maxima and minima

Definition 13. *If a partial derivative $\frac{\partial f}{\partial x}$ of a function $f : A \subseteq \mathbb{R}^2 \to \mathbb{R}$ is partially derivable with respect to x in a point $(x_0, y_0) \in A$ we say that f is partially derivable two times with respect to x in the point (x_0, y_0) ad it will be denoted as $\frac{\partial f}{\partial x} f_x(x_0, y_0) = \frac{\partial^2 f(x_0,y_0)}{\partial x^2} = \frac{\partial^2 f}{\partial x^2}(x_0, y_0)$.*
The same goes for the other partial derivatives: $\frac{\partial}{\partial y} f_x = \frac{\partial^2 f}{\partial x \partial y}$, $\frac{\partial}{\partial x} f_y = \frac{\partial^2 f}{\partial y \partial x}, \ldots$

Definition 14. *We define the hessian matrix as follows:*

$$\mathcal{D}^2 f = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix}$$

Definition 15. *The determinant of $\mathcal{D}^2 f$ is:*

$$\mathcal{H}(x, y) = \begin{vmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{vmatrix}$$

and is called the Hessian determinant.

Theorem 6. *Let $f : A \subseteq \mathbb{R}^2 \to \mathbb{R}$ with A open. Let's suppose that $\exists$ the partial derivatives $\frac{\partial^2 f}{\partial x \partial y}$ and $\frac{\partial^2 f}{\partial y \partial x}$ continous in a point $(x_0, y_0) \in A$. So then $\frac{\partial^2 f(x_0,y_0)}{\partial x \partial y} = \frac{\partial^2 f(x_0,y_0)}{\partial y \partial x}$.*

Definition 16. *Let $f : A \subseteq \mathbb{R}^2 \to \mathbb{R}$, we say that $(x_0, y_0) \in A$ is maxima (minima) for f if $\forall (x, y) \in A,\ f(x, y) \leqslant f(x_0, y_0) \quad (f(x, y) \geqslant f(x_0, y_0))$.*

Theorem 7. *If f is continous and A is compact, f admits minima and maxima.*

Theorem 8. *Let $f : A \subseteq \mathbb{R}^2 \to \mathbb{R}$ and let $(x_0, y_0) \in \dot{A}$ a relative extreme and let f be partially derivable in (x_0, y_0), so then $\frac{\partial f(x_0,y_0)}{\partial x} = 0$ and $\frac{\partial f(x_0,y_0)}{\partial y} = 0$.*
The points where the partial derivatives are 0 are said "critical points" of f, $(x_0, y_0) \in \dot{A}$ is an extreme ralative $\Rightarrow (x_0, y_0)$ is a critical point for f ($\not\Leftarrow$).

Obs 2. *Let* $(x_0, y_0) \in A$ *and let* $g(x, y) = f(x, y) - f(x_0, y_0)$, (x_0, y_0) *is a relative minimum (relative maximum) for* $f \iff \exists$ *a circle* C *of center* (x_0, y_0) *such that* $g \geq 0$ *(*$g \leq 0$*).*

Theorem 9. *Let* $f : A \subseteq \mathbb{R}^2 \to \mathbb{R}$ *and let* $(x_0, y_0) \in \dot{A}$ *a relative extreme* $\implies \mathcal{H}(x_0, y_0) \geqslant 0$.

Theorem 10. *Let* $f : A \subseteq \mathbb{R}^2 \to \mathbb{R}$, $f \in \mathbf{C}^2$. *Let* $(x_0, y_0) \in \dot{A}$ *a critical point and lets suppose that* $\mathcal{H}(x_0, y_0) > 0 \implies (x_0, y_0)$ *is a relative extreme and is maximum or minimum depending on* $\frac{\partial^2 f(x_0, y_0)}{\partial x^2}$ *be* < 0 *or* > 0.