Calculus II

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Theorem 1. *A is closed $\Longleftrightarrow$ every accumulation point for A is in A*

Proof. " $\Longrightarrow$ " Let $A \subseteq \mathbb{R}^n$, $A = A \cup \partial A$.
Then $\forall p \in \bar{\mathcal{D}}(A)$, $C_r(p)_{\diagdown p} \cap A \neq \emptyset \ \forall C \in \mathcal{C}_p$.
if $p \notin A$ then $C_r(p)$ has elements that dont belong to $A \Rightarrow p \in \partial A$.
" $\Longleftarrow$ " Let $p \in \partial A \Rightarrow \forall C \in \mathcal{C}_p$ of center r with $r \in \mathbb{R}$ by definition we can find some $x \in C_{\diagdown p} \cap A$, so that means $p \in \bar{\mathcal{D}}(A) \Rightarrow p \in A$. $\square$

1 Limits

Definition 1. *Let $A \subseteq \mathbb{R}^2$ and (x_0, y_0) an accumulation point for A. we define A^* as follows:*
$A^* = \{(\rho, \theta) \in [0, +\infty] \times [0, 2\pi] : (x_0 + \rho\cos(\theta), y_0 + \rho\sin(\theta)) \in A\}$.

Proposition 1. *Lets suppose that exist a circle C of center (x_0, y_0) such that $C_{\diagdown\{(x_0,y_0)\}} \subseteq A$ let r be the radius of the circle and as a consequence $(0, r] \times [0, 2\pi] \subseteq A^*$*

Proof. Let $C_{\diagdown\{(x_0,y_0)\}}$ and $\begin{cases} 0 < \rho \leqslant r \\ 0 \leqslant \theta \leqslant 2\pi \end{cases}$ if $(\rho, \theta) \in (0, r] \times [0, 2\pi]$
then $(x_0 + \rho\cos(\theta), y_0 + \rho\sin(\theta)) \in C_{\diagdown\{(x_0,y_0)\}} \subseteq A \Rightarrow (\rho, \theta) \in A^*$. $\square$

Definition 2. *Let $\theta \in [0, 2\pi]$ and $\forall \rho \in (0, r]$ we define $\varphi_\theta(\rho) = F(\rho, \theta)$ if $\rho \in (0, r], (\rho, 0) \in A^*$ so the $\lim_{\rho\to 0} \varphi(\rho) = l \in \bar{\mathbb{R}}$.*
If that limit exists that means $\forall \theta \in [0, 2\pi]$ and $\forall \varepsilon > 0$, $\exists \delta > 0$ $\forall \rho \in (0, r]$ with $\rho < \delta$ $|\varphi_\theta - l| < \varepsilon$.
We say that $\lim_{\rho\to 0} \varphi(\rho) = l \in \bar{\mathbb{R}}$ Uniformly With Respect To (U.W.R.T) θ.

Theorem 2. *Let $f : A \subseteq \mathbb{R}^2 \to \mathbb{R}$ with (x_0, y_0) accumulation point for A.*
Follows the equivalence:
$\lim_{(x,y)\to(x_0,y_0)} f(x, y) = l \in \bar{\mathbb{R}} \Longleftrightarrow \lim_{\rho\to 0} F(\rho, \theta) = l$ U.W.R.T θ.

Proof. Let $l \in \bar{\mathbb{R}}$.
" $\Longrightarrow$ " $\lim_{(x,y)\to(x_0,y_0)} f(x, y) = l$ so $\forall \varepsilon > 0, \exists \delta > 0 : \forall\ (x, y) \in A$
with $\|(x, y) - (x_0, y_0)\| < \delta$, $|f(x, y) - l| < \varepsilon$.

We have to prove that $\forall\varepsilon > 0,\ \exists\delta > 0 : \forall\theta \in [0, 2\pi],\ \forall\rho(0, r]$
with $\rho < \delta\ |F(\rho, \theta) - l| < \varepsilon$.
Let $\varepsilon > 0,\ \theta \in [0, 2\pi],\ \rho \in (0, r]$ with $\rho < \delta$. we create the system that changes the coordinates from cartesians to polars:

$$\begin{cases} x = x_0 + \rho\cos(\theta) \\ y = y_0 + \rho\sin(\theta) \end{cases} \quad \rho = \sqrt{(x - x_0)^2 + (y - y_0)^2}$$

$\rho \in (0, r],\ \theta \in [0, 2\pi] \in (0, r] \times [0, 2\pi] \subseteq A^*,\ (\rho, \theta) \in A^* \Rightarrow (x, y) \in A.$

Now $0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} = \rho < \delta \Rightarrow |f(x, y) - l| < \varepsilon.$
$\Rightarrow |f(x_0 + \rho\cos(\theta), y_0 + \rho\sin(\theta)) - l| < \varepsilon \Rightarrow |F(\rho, \theta) - l| < \varepsilon.$

" $\Longleftarrow$ " $\forall\varepsilon > 0, \exists\delta \leq r : \forall\theta \in [0, 2\pi]$ and $\forall\rho$ with $0 < \rho < \delta \Rightarrow$
$|F(\rho, \theta) - l| < \varepsilon.$
We have to prove that $\forall\varepsilon > 0, \exists\delta > 0, \forall(x, y) \in A$ with
$\sqrt{(x - x_0)^2 + (y - y_0)^2} = \|(x, y) - (x_0, y_0)\| < \delta \Rightarrow |f(x, y) - l| < \varepsilon.$

Let $\varepsilon > 0,\ \delta \leq r,\ (x, y) \in A,\ \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta$, we switch coordinates with ρ and θ as follows:

$$\begin{cases} x = x_0 + \rho\cos(\theta) \\ y = y_0 + \rho\sin(\theta) \end{cases} \quad \rho = \sqrt{(x - x_0)^2 + (y - y_0)^2}$$

$0 < \rho < \delta \leq r \Rightarrow \rho \in (0, r],\ \theta \in [0, 2\pi].$
We notice that $|F(\rho, \theta) - l| < \varepsilon$, so $|f(x_0 + \rho\cos(\theta), y_0 + \rho\sin(\theta)) - l| < \varepsilon$
$\Rightarrow |f(x, y) - l| < \varepsilon.$ ☐

Definition 3. *We say that $\theta \in [0, 2\pi]$ is admissible if $0 \in \bar{\mathcal{D}}(A_\theta)$.*

Theorem 3. $\lim_{\rho\to 0} F(\rho, \theta) = l \in \bar{\mathbb{R}}$ *U.W.R.T* $\theta \Longleftrightarrow \lim_{\rho\to 0} \varphi(\rho) = 0.$

Definition 4. *Let $f : A \subseteq \mathbb{R}^2 \to \mathbb{R}$ with A open.*
let $(x_0, y_0) \in A$, $\varphi(x) = f(x, y_0)$ and $\psi = f(x_0, y)$. A is open that means that those two functions are well defined.

Definition 5. *We say that f is differentiable with respect to x in (x_0, y_0) if φ is differentiable in x_0. in that case we φ is the partial derivative of f in the variable x and its written $\frac{\partial f}{\partial x}$*

Definition 6. *We define the gradient as $\nabla f : (x, y) \in A \mapsto (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}) \in \mathbb{R}^2$*

Obviusly we can generalize to any real vectorial space.

Definition 7. *Let be $f : A \subseteq \mathbb{R}^n \to \mathbb{R}$ with A Open. Let $\bar{x} \in A$ and let $i \leq n$, we denote as $\varphi_i(x_i) = f(\bar{x}_1, \bar{x}_2, ..., \bar{x}_{i-1}, \bar{x}_i, \bar{x}_{i+1}, ..., \bar{x}_n)$. Notice that $\bar{x}$ is an internal point so then it exist an interval where φ_i is well defined.*

Definition 8. *We say that f is partially derivable with respect to the variable x_i in the point $\bar{x}$ if φ_i is derivable in that point. We denote as $\frac{\partial f}{\partial x_i}$ the partial derivative with respsect to x_i in the point $\bar{x}$.*

Definition 9. *The gradient of a function in n variables is defined as follows:*

$$\nabla f : \bar{x} \in A \mapsto (\frac{\partial f}{\partial x_1}, ..., \frac{\partial f}{\partial x_n}) \in \mathbb{R}^n$$

Directional Derivatives

If we take $f : A \subseteq \mathbb{R}^2$ and it's partial derivatives, we can take for example $\frac{\partial f}{\partial x}$ as the direction of the function calculated on the line $y = y_0$.. So let a function be defined like the one before and let $(\lambda, \mu) \in \mathbb{R}^2$ with $\sqrt{\lambda^2 + \mu^2} = 1$. Let r the line with the following equations:

$$\begin{cases} x = x_0 + \lambda t \\ y = y_0 + \mu t \end{cases}$$

(x_0, y_0) is internal to A so there exists a rectangle R_0 of center (x_0, y_0), so every line that passes in this point encounters a segment of the rectangle.