Topology

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Definition 1. *let X be a not empty set and $\mathcal{T}$ a collection of its subsets. $\mathcal{T}$ is said topology on X if it has the following propeties:*

- $\varnothing, X \in \mathcal{T}$
- *the union of a whatever family of elements of $\mathcal{T}$ is in $\mathcal{T}$. In simbols:* $\forall \{A_i\}_{i\in I},\ A_i \in \mathcal{T},\ \bigcup_{i\in I} A_i \in \mathcal{T}$
- *the intersection of elements of $\mathcal{T}$ is in $\mathcal{T}$:* $\forall A_1, A_2 \in \mathcal{T},\ A_1 \cap A_2 \in \mathcal{T}$

Then $(X, \mathcal{T})$ its said to be topological space and X is called support.

Example 1. *Let $X = \mathbb{R}^2$ with the euclidean distance and $\forall c \in \mathbb{R}^2$ and $\forall r > 0$ the set $B_r(c) = \{x \in \mathbb{R}^2 : d(x, c) < r\}$ its said open spherical neighbourhood of center c and radius r. Let $\mathcal{T}$ be the totality of all the possible unions of open spherical neighbourhoods. $\mathcal{T}$ is a topology.*

Definition 2. *$(X_1, \mathcal{T}_1),\ (X_2, \mathcal{T}_2)$ topological spaces with $X_1 \cap X_2 = \varnothing$.*
let $X = X_1 \cup X_2, \mathcal{T} = \{A_1 \cup A_2 \mid A_i \in \mathcal{T}_i\}$.

Lemma 1. *$(X, \mathcal{T})$ is a topological space.*

Proof. Lets verify the axioms.

1. $\varnothing \in \mathcal{T}_1,\ \varnothing \in \mathcal{T}_2 \Rightarrow \varnothing = \varnothing \cup \varnothing$
   $X_i \in \mathcal{T}_i \Rightarrow X = X_1 \cup X_2 \in \mathcal{T}$
2. $\{A\}_{i\in I} \in \mathcal{T} \Rightarrow A_i = A_{1,i} \cup A_{2,i}$
   $\bigcup_{i\in I} A_i = \bigcup_{i\in I}(A_{1,i} \cup A_{2,i}) = \bigcup_{i\in I}(A_{1,i}) \cup \bigcup_{i\in I}(A_{2,i}) \in \mathcal{T}_1 \cup \mathcal{T}_2$
3. $A, A' \in \mathcal{T} \Rightarrow A = A_1 \cup A_2, A' = A'_1 \cup A'_2$
   $A \cap A' = (A_1 \cup A_2) \cap (A'_1 \cup A'_2) = (A_1 \cap A'_1) \cup (A_2 \cap A'_2) \in \mathcal{T}$

$\square$

Definition 3. *$\mathcal{T}$ is defined as a sum of Topologies $\mathcal{T}_1$ and $\mathcal{T}_2$.*
We notice that $\forall A_i \in \mathcal{T}_i,\ A_i \cup \varnothing \in \mathcal{T}$ so $\mathcal{T}$ contains $\mathcal{T}_1$ and $\mathcal{T}_2$.

Definition 4. *$(X, \mathcal{T})$ topological space $\mathsf{Y} \subseteq X,\ \mathsf{Y} \neq \varnothing$*
$\mathcal{T}_{/Y} = \{A \cap \mathsf{Y} \mid A \in \mathcal{T}\}$ prove that $\mathcal{T}_{/Y}$ is a topology defined as Inducted topology on Y

Definition 5. *$(X, \mathcal{T})$ topological space. $\mathcal{B} = \{B_j\}_{j \in J}$ with $B_j \in \mathcal{T}$. $\mathcal{B}$ is said basis for the topology $\mathcal{T}$ if all open sets are union of elements of $\mathcal{B}$*

Lemma 2. *Let $X \neq \varnothing$ and $\mathcal{B} \in \mathcal{P}(X)$. Let $A \subseteq X$ then the following affirmations are equivalent:*

(a) A is union of elements of $\mathcal{B}$

(b) $\forall x \in A \ \exists B \in \mathcal{B} : x \in B \subseteq A$

Proof. **(a $\Rightarrow$ b)** Let $x \in A = \bigcup_{i \in I} B_i$ with $B_i \in \mathcal{B}$. Therefore $\exists B_i : x \in B_i \subseteq A$
(b $\Rightarrow$ a) $(\subseteq)$ $A = \bigcup_{i \in I} B_i \Rightarrow \forall x \in A \exists B_x : x \in B_x \Rightarrow A \subseteq \bigcup_{i \in I} B_i$.
$(\supseteq)$ $\forall x \in A \ \exists B_i \in \mathcal{B} : x \in B_i \subseteq A \Rightarrow A \subseteq \bigcup_{i_x \in I} B_{i_x} \subseteq A \Rightarrow A = \bigcup_{x \in A} B_x$ □

Theorem 1. *Let X be a not empty set and $\mathcal{B} \in \mathcal{P}(X)$. $\mathcal{B}$ is a basis if:*

1. $X = \bigcup_{B \in \mathcal{B}} B$
2. *$\forall B_1, B_2 \in \mathcal{B}$ and $\forall x \in B_1 \cap B_2$ there exists $B_3 \in \mathcal{B} : x \in B_3 \subset B_1 \cap B_2$.*

Also, for a Basis $\mathcal{B}$ that satisfies the condition 1. and 2. there exists a topology on whitch $\mathcal{B}$ is a basis.

Proof. $(\Rightarrow)$

Let $\mathcal{B}$ a basis such that every open set A is union of elements of $\mathcal{B}$, in particular $X = \bigcup_{B \in \mathcal{B}} B$ moreover because $\mathcal{B} \subseteq \mathcal{T}$ we can say that $B_1, B_2 \in \mathcal{T}$ so is union of elements of $\mathcal{B}$. The last Lemma implies $\forall x \in B_1 \cap B_2 \ \exists B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$.

$(\Leftarrow)$

Let $\mathcal{B} \subseteq \mathcal{P}(X)$ that satisfies 1. and 2. and let $\mathcal{T}$ the totality of the unions of $\mathcal{B}$. It has to be proven that $\mathcal{T}$ is a topology on X.

i $\varnothing \in \mathcal{T}$ because $\varnothing$ is the empty union and $X \in \mathcal{T}$ for the 1.

ii $\mathcal{T}$ is closed with respect to the union by definition.

iii $A_1, A_2 \in \mathcal{T}$ one has $A_1 = \bigcup_{i \in I_1} B_i^{(1)}$ and $A_2 = \bigcup_{j \in I_2} B_i^{(2)} \Rightarrow$
$\Rightarrow A_1 \cap A_2 = \left(\bigcup_{i \in I_1} B_i^{(1)}\right) \cap \left(\bigcup_{j \in I_2} B_j^{(2)}\right) = \bigcup_{i \in I_1, j \in I_2} \left(B_i^{(1)} \cap B_j^{(2)}\right)$.

By the 2. and the last lemma $B_i^{(1)} \cap B_j^{(2)} \in \mathcal{T}$ this implies that $A_1 \cap A_2 \in \mathcal{T}$. □

Corollary 1. *Let X be a set and $\mathcal{B} \in \mathcal{P}(X)$, if $\mathcal{B}$ is an overlay of X $(X = \bigcup_{B \in \mathcal{B}} B)$ and its closed with respect to the intersection, then $\mathcal{B}$ is a topology on X and also a basis.*

Proof. The condition 1. and 2. of the last theorem are satisfied. □

Definition 6. *$(X, \mathcal{T})$ topological space, X verifies the second axiom of numerability if posseses a finite base or numerable, in that case $(X, \mathcal{T})$ is said $\mathcal{N}_2$*

Proposition 1. *Let $\mathbb{R}$ be gifted by the topology with a base of the following type:*

$$[a, b], \ a < b$$

Then $(\mathbb{R}, \mathcal{T})$ is not $\mathcal{N}_2$

Proof. Let $\mathcal{B}$ a base for $\mathcal{T}$. Let $a > 0 \in \mathbb{R}$ then $\forall x \in \mathbb{R}$, there exists $B_x \in \mathcal{B}$ with $x \in B_x \subseteq [x, x+a]$. If $y \in \mathbb{R}$ with $y > x$ then $x \notin [y, y+a]$ so $x \notin B_y$. The application $x \in \mathbb{R} \longmapsto B_x \in \mathcal{B}$ is injective so $\mathcal{B}$ has the continuum order. $\square$

Proposition 2. *Let $(X, \mathcal{T})$ be a topological space and S a subset of X.*

a) A point $x \in X$ is adherent to S if and only if $N \cap S \neq \varnothing$ for all $N \in \mathcal{N}(x)$

b) A point $x \in X$ is adherent to S if there exists a successor function $\{x_n\}$ of elements in S that converges to x. If X satisfies the second axiom of numerability then also the other implication is true.

Proof. **(a)**

Let's suppose that $x \in \overline{S}$. If $x \in S$ then the condition is satisfied because every $N \in \mathcal{N}(x)$ contains x. If $x \in D(S)$ again, the condition is satisfied because $N \setminus \{x\} \cap S \neq \varnothing$ for all $N \in \mathcal{N}(x)$. So we suppose that the condition of the statement is true, that implies that $x \notin Est(S)$ because $Est(S) \cap S = \varnothing$ and $Est(S)$, since is open, is a neighbourhood of every its point. Therefore $x \in \bar{S}$.

(b)

Let's suppose that $\{x_n\}$ is a successor function of elements of S such that $\lim_{x\to\infty} x_n = x$. By definition of limit fo all $N \in \mathcal{N}(x)$ there exist $x_n \in N$ and so the condition of the part (a) is satisfied. So $x \in \overline{S}$. Let's suppose instead that $x \in \overline{S}$ and let $\{N_n : n = 1, 2, \ldots\}$ be a fundamental system of neighbourhoods of x that satisfies the condition $N_{n+1} \subset N_n$ for all n. By the (a) for all $n \geq 1$ we can find a point $x_n \in N_n \cap S$. The successor function $\{x_n\}$ converges to x. $\square$

Zarinski's topology

Let $\mathbb{F}$ be a field with $\mathbb{F}^n = \{(x_1, \dots, x_n) : x_i \in \mathbb{F}\}$ and $\mathbb{F}[X_1, \dots, X_n]$ the ring of the polynomials in the unkowns $X_1, \dots, X_n$. Let $f(\mathbf{x}) = f(x_1, \dots, x_n) \in \mathbb{F}[\mathbf{X}]$, we define $\mathcal{V}(f) = \{\mathbf{x} \in \mathbb{F}^n : f(\mathbf{x}) = 0\}$ as an algebraic varaiety. More in general $\mathfrak{F} = \{f_i(\mathbf{x})\}_{i \in I}$ and $\mathcal{V}(\mathfrak{F}) = \{\mathbf{x} \in \mathbb{F}^n : f_i(\mathbf{x}) = 0, \forall i \in I\}$

Theorem 2. *The totality of all the algebraic varaieties on $\mathbb{F}^n$ satisfy the closed sets axioms.*

Proof. **(1)**

$\varnothing \in \mathcal{C}$ because if we take $f = 1$ then $\mathcal{V}(f) = \varnothing$. Also for $f = 0$ one has $\mathcal{V}(f) = \mathbb{F}^n$

(2)

Let $V = \mathcal{V}(\mathfrak{F})$, $W = \mathcal{W}(\mathfrak{G}) \in \mathcal{C}$ with $\mathfrak{F} = \{f_i\}_{i \in I}$ and $\mathfrak{G} = \{g_j\}_{j \in J}$. One has $V \cup W = \{\mathbf{x} \in \mathbb{F}^n : f_i(\mathbf{x}) = 0 \vee g_i(\mathbf{x}) = 0\} = \mathcal{V}(\mathfrak{K})$ with $\mathfrak{K} = \{f_i g_i\}_{i \in I, j \in J}$, infact if $\mathbf{x} \in V \cup W$ then $(f_i g_i)(\mathbf{x}) = 0$ for all i and j so $V \cup W \subseteq \mathcal{V}(\mathfrak{K})$. Let $\mathbf{x} \in \mathcal{V}(\mathfrak{K})$ and lets suppose that $\mathbf{x} \in V$ then there exists $\bar{i} \in I : f_{\bar{i}}(\mathbf{x}) \neq 0$ but $\mathbf{x} \in \mathcal{V}(\mathfrak{K})$ and that implies $f_{\bar{i}} g_j(\mathbf{x}) = 0\ \forall j \in J$ with $f_{\bar{i}}(\mathbf{x}) \neq 0 \Rightarrow g_j(\mathbf{x}) = 0\ \forall j \in J$ that means $\mathbf{x} \in W$. we can conclude that $V \cup W \supseteq \mathcal{V}(\mathfrak{K})$ such that $V \cup W = \mathcal{V}(\mathfrak{K})$

(3)

Let $\{\mathcal{V}^{(k)} = \mathcal{V}(\mathfrak{F}^{(k)})\}_{k \in K}$ a family of **a.v.** $\mathfrak{F}^{(k)} = \{f_i^{(k)}\}_{i \in I^{(k)}} \subseteq \mathbb{F}[\mathbf{X}]$ and define $V^{(k)} = \{\mathbf{x} \in \mathbb{F}^n : f_i^{(k)}(\mathbf{x}) = 0\ \forall i \in I^{(k)}\}$ and $W = \{\mathbf{x} \in \mathbb{F}^n : f_i^{(k)}(\mathbf{x}) = 0\ \forall i \in I^{(k)}\ \forall k \in K\}$, then $W = \bigcap_{k \in K} V^{(k)}$. $\square$

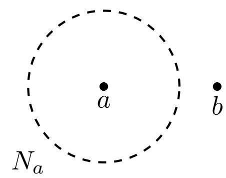
Proposition 3. *In the Zarinski's topology every point is a closed set. So if $\mathbb{F}$ is finite this topology coincides with the discrete topology on $\mathbb{F}^n$*

Proof. Let $\mathbf{a} = (a_1, \dots, a_n)$ be a point in $\mathbb{F}^n$ and consider $f_i(\mathbf{x}) = x_i - a_i,\ \forall i = 1, \dots, n$. If $\mathfrak{F} = \{f_1, \dots, f_n\}$ then $\mathcal{V}(\mathfrak{F}) = \{\mathbf{x} \in \mathbb{F}^n : f_i(\mathbf{x}) = 0,\ i = 1, \dots, n\} = \{(a_1, \dots, a_n)\}$. We can observe thtat the union of two points is again a closed set. If $\mathbb{F}$ is a finite field, also $\mathbb{F}^n$ is finite, so if we take a subset $U \in \mathbb{F}$ ($U \in \mathbb{F}^n$) $\Rightarrow U = \bigcup_{p \in U} \{p\}$ therefore one has $U = \mathbb{F} \setminus (\mathbb{F} \setminus U)$ ($U = \mathbb{F}^n \setminus (\mathbb{F}^n \setminus U)$). $\square$

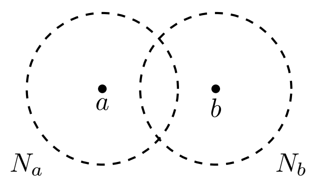
Separation axioms

Let $(X, \mathcal{T})$ be a topological space.

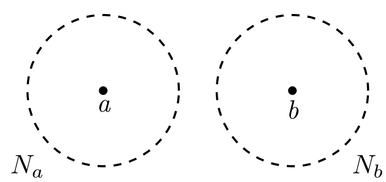
Definition 7. *$(X, \mathcal{T})$ is said to be $\mathbf{T}_0$ if $\forall a, b \in X$ with $a \neq b$ there exists a neighbourhood N_a that doesn't contain b.*



Definition 8. *$(X, \mathcal{T})$ is said to be $\mathbf{T}_1$ if $\forall a, b \in X$ with $a \neq b$ if $\exists N_a, N_b$ such that $b \notin N_a$ and $a \notin N_b$*



Definition 9. *$(X, \mathcal{T})$ is said to be $\mathbf{T}_2$ if $\exists N_a, N_b$ with $N_a \cap N_b = \varnothing$*



Observation 1. *In a topological space $\mathbf{T}_2 \Rightarrow \mathbf{T}_1 \Rightarrow \mathbf{T}_0$*

Proposition 4. *Every subspace Y of an Hausdorff ($\mathbf{T}_2$) Space X is again a an Hausdorff space*

Proof. If a and b are two distinct points of Y there exist N_a, N_b neighbourhoods of X such that $a \in N_a$ and $b \in N_b$ with $N_a \cap N_b \neq \varnothing$. Let N_a^1, N_b^1 be neighbourhoods of a and b respectively with $N_a^1 = N_a \cap Y$ and $N_b^1 = N_b \cap Y$ by construction we can say $N_a^1 \cap N_b^1 = \varnothing$. $\square$

Connected Topological Spaces

Definition 10. *Let* $(X, \mathcal{T})$ *ba a topological space.* X *is said Connected if there not exist two open sets of* X *such that their union is all the space and also the two subsets are disjointed. In symbols:* X *is connected if* $\nexists A, A_1 \in \mathcal{T} : A \cup A_1 = X$ *with* $A \cap A_1 = \varnothing$.

Proposition 5. *Let* $(X, \mathcal{T})$ *and* $a, a_1 \in X$*, if there exist a* $Y \subseteq X$ *connected such that* $a, a_1 \in Y \Rightarrow X$ *is connected.*

Proof. Lets suppose that X is disconnected by $A, A_1 \in \mathcal{T}$ and let $a \in A$ and $a_1 \in A_1$. If we take $Y \subset X$ connected such that $a, a_1 \in Y$. Then $A \cap Y \neq \varnothing$ and $A_1 \cap Y \neq \varnothing$ also $(A \cap Y) \cup (A_1 \cap Y) \subseteq A \cup A_1 \neq \varnothing$. one has $(A \cap Y) \cup (A_1 \cap Y) = (A \cup A_1) \cap Y = X \cap Y = Y$, that implies is connected, but since a, a_1 and Y are arbitrary, this is absurd. $\square$

Example 2. *Let* $\mathbb{R}$ *be gifted by the natural topology, then all the connected sets of* $\mathbb{R}$ *are the intervals:* $[a, b], (a, b], [a, b), (a, b)$.

Proposition 6. *In a topological space* $(X, \mathcal{T})$ *every subset* $Y \subseteq X$ *that is connected implies its closure is connected.*

Proof. Lets suppose that $\overline{Y}$ is disconnected and lets suppose that $\exists A_1, A_2 \in \mathcal{T}$ such that:

1. $A_1 \cap \overline{Y}, A_2 \cap \overline{Y} \neq \varnothing$
2. $\left(A_1 \cap \overline{Y}\right) \cap \left(A_2 \cap \overline{Y}\right) = \varnothing$
3. $\left(A_1 \cap \overline{Y}\right) \cup \left(A_2 \cap \overline{Y}\right) = \overline{Y}$

It can be proven that A_1 and A_2 disconnect Y, infact $\overline{Y} = Y \cup \partial Y$ [1] and if $A_1 \cap Y = \varnothing$ then $\exists a \in \partial Y$ such that the neighbourhood A_1 would have null intersection with Y, but that is impossible, so $A_1 \cap Y$. The same reasoning can be applied to $A_2 \cap Y$. We also have 2. $\Rightarrow (A_1 \cap Y) \cap (A_1 \cap Y) = \varnothing$, therefore 3. $\Rightarrow (A_1 \cap Y) \cup (A_1 \cap Y) \Rightarrow A_1 \cap \overline{Y} = (A_1 \cap Y) \cup (A_1 \cap \partial Y) \Rightarrow \left(A_1 \cap \overline{Y}\right) \cup \left(A_2 \cap \overline{Y}\right) = (A_1 \cap Y) \cup (A_2 \cap Y) \cup ((A_1 \cup A_2) \cap \partial Y)$ but that means $Y \cap \partial Y \Rightarrow Y = (A_1 \cap Y) \cup (A_2 \cap Y)$. $\square$

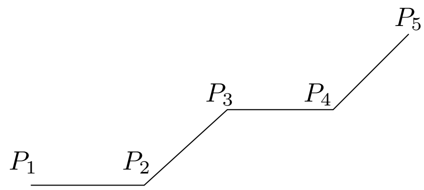
Let $(X, \mathcal{T})$ be a topological space, then we can define an equivalence relations between its points by sayng $a \sim b$ if and only if a and b are in the same connected space.

Definition 11. $\mathcal{K}(a)$ *is the equivalence class of* a*, so if* $a \in Y_a$ *such that* Y_a *is connected in symbols we have:* $\mathcal{K}(a) = \{x \in X | a \sim x\} = \{x \in X | x \in Y_a\}$

[1] $\partial Y = D(Y)$

Definition 12. *Let $C \subseteq \mathbb{R}^n$ be a set in whitch $\forall \vec{x}, \vec{y} \in \mathbb{R}^n$ the segment $\overline{XY} \subset C$. C is said to be convex.*

Definition 13. *A polygonal in $\mathbb{R}^n$ is the union of a finite successor function of segments $\{S_i\}_{i \in I}$ with $S_i = \overline{P_i P_{i+1}}$.*



Definition 14. *A set $Y \subseteq \mathbb{R}^n$ is said to be connected by polygonals if $\forall a, b \in Y$ exist a polygonal of extremes a and b contained in Y.*

Proposition 7. *Let $\mathbb{R}^n$ be gifted by the natural topology $\mathcal{T}$ and let $A \in \mathcal{T}$ then A is connected by polygonals $\Longleftrightarrow$ A is connected.*

Proof. ($\Longrightarrow$) since every component of a line (line included) in $\mathbb{R}^n$ is connected and since for two arbitrary points in the same connected subspace the space that contains it is connected, the first implication is proved.
($\Longleftarrow$) Lets suppose that A is connected but not Connected by polygonals so $\exists\ x, y \in A$ with $x \neq y$ that are not joinable by a polygonal so then $\forall x \in A$ let $E_x \subset A$ the totality of the points of A that are joinable to x by a polygonal. The relation $x \backsim y \Longleftrightarrow x$ and y are joinable by a polygonal, is an equivalence relation.
The classes of equivalence are $E_x\ \forall x$ therefore $A = E_x \cup \bigcup_{y \neq x} E_y$ would be sconnected, infact every E_x is an open set because A is open, so $\forall a \in E_x$ there exist a neighbourhood U_a with center a in whitch for all $b \in U_a$ $\overline{ab} \subset U_a$ so $b \in E_x$. $\square$

Proposition 8. *For all $a \in X$, $\mathcal{K}(a)$ is the biggest connected subset that contains a, also every connected component is a closed set in $\mathcal{T}$.*

Proof. Let $a \in X$ and $Y \subset X$ conncted and $a \in Y$, Lets prove that $Y \subseteq \mathcal{K}(a)$. Infact if $x \in \mathcal{K}(a)$ there exists a connected subset Y_x such that $x \in Y_x$ that means $Y_x \subset \mathcal{K}(a)$. One has that $\mathcal{K}(a) = \bigcup_{x \in \mathcal{K}(a)} Y_x$ with $a \in \bigcap_{x \in \mathcal{K}(a)} Y_x$ ($\neq \varnothing$) so $\mathcal{K}(a)$ is connected. Finally $\mathcal{K}(a)$ is a closed set because $\overline{\mathcal{K}(a)}$ is connected and for the first part of the theorem $\mathcal{K}(a) \subseteq \overline{\mathcal{K}(a)}$ but $\mathcal{K}(a) \supseteq \overline{\mathcal{K}(a)}$ so $\mathcal{K}(a) = \overline{\mathcal{K}(a)}$. $\square$

Continous functions and homeomorphisms

Definition 15. *Let $(X, \mathcal{T}_x)$, $(Y, \mathcal{T}_y)$ be topological spaces, $\Omega : X \to \mathsf{Y}$ is continous in $a \in X$ if $\forall I$ neighbourhood of $\Omega(a)$, $\exists K$ neighbourhood of a s.t. $\Omega(K) \subseteq I$.*

We'll say that a function is continous if it is continous in every point.

Proposition 9. *Let $\Omega : (X, \mathcal{T}_x) \to (Y, \mathcal{T}_y)$ so then the following affirmations are equivalent:*

i. *Ω is continous.*

ii. $\forall A \in \mathcal{T}_y,\ \Omega^{-1}(A) \in \mathcal{T}_x$.

iii. $\forall c \in \mathcal{C}(Y),\ \Omega^{-1}(c) \in \mathcal{C}(X)$.

iv. *The counterimages of opens under a selected base of Y are opens of X.*

v. *$\forall b = \Omega(a) \in Im\Omega = \Omega(X)$ the counterimage of every neighbourhood K' of b is a neighbourhood of a*

Proof. *i.* $\Rightarrow$ *ii.* Let $A \in \mathcal{T}_y$ we have to prove that $\forall a \in \Omega^{-1}(A)$ exists a neighbourhood of a contained in $\Omega^{-1}(A)$. For $a \in \Omega^{-1}(A)$ one has $\Omega(a) \in \Omega(\Omega^{-1}(A)) = A$. A is open and is a neighbourhood of $\Omega(a)$ so if Ω is continous there exists a neighbourhood K of a s.t. $\Omega(K) \subseteq A$ hence $K \subseteq \Omega^{-1}(A)$ and $a \in K$.
ii. $\Rightarrow$ *i.* Let $a \in X,\ I$ neighbourhood of $\Omega(a)$ let $A \in \mathcal{T}_y$ s.t. $\Omega(a) \in A \subseteq K$ for the *ii.* $\Omega^{-1}(A)$ is open and is a neighbourhood of a s.t. its image contains $\Omega(a)$ and A is contained in I. $\square$

Definition 16. *Let $(X, \mathcal{T}_x)$ and $(Y, \mathcal{T}_y)$ be topological spaces, then $\Omega : X \to Y$ is defined open function (respectively closed function) if for every open subset A of X, $\Omega(A)$ is an open subset (closed subset) of Y.*

Definition 17. *A function $\Omega : X \to Y$ is said to be an homeomorphism if it is continous and Ω, Ω^{-1} are continous. The two topological spaces are said to be homeomorphic if that application exists and we write: $X \approx Y$.*