

# Calculus II

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May 21, 2023

**Theorem 1.**  $A$  is closed  $\iff$  every accumulation point for  $A$  is in  $A$

*Proof.* "  $\implies$  " Let  $A \subseteq \mathbb{R}^n$ ,  $A = A \cup \partial A$ .

Then  $\forall p \in \bar{\mathcal{D}}(A)$ ,  $C_r(p) \setminus_p \cap A \neq \emptyset \ \forall C \in \mathcal{C}_p$ .

if  $p \notin A$  then  $C_r(p)$  has elements that don't belong to  $A \Rightarrow p \in \partial A$ .

"  $\Leftarrow$  " Let  $p \in \partial A \Rightarrow \forall C \in \mathcal{C}_p$  of center  $r$  with  $r \in \mathbb{R}$  by definition we can find some  $x \in C \setminus_p \cap A$ , so that means  $p \in \bar{\mathcal{D}}(A) \Rightarrow p \in A$ .  $\square$

## Limits

**Definition 1.** Let  $A \subseteq \mathbb{R}^2$  and  $(x_0, y_0)$  an accumulation point for  $A$ . we define  $A^*$  as follows:

$A^* = \{(\rho, \theta) \in [0, +\infty] \times [0, 2\pi] : (x_0 + \rho \cos(\theta), y_0 + \rho \sin(\theta)) \in A\}$ .

**Proposition 1.** Lets suppose that exist a circle  $C$  of center  $(x_0, y_0)$  such that  $C \setminus_{\{(x_0, y_0)\}} \subseteq A$  let  $r$  be the radius of the circle and as a consequence  $(0, r] \times [0, 2\pi] \subseteq A^*$

*Proof.* Let  $C \setminus_{\{(x_0, y_0)\}}$  and  $\begin{cases} 0 < \rho \leq r \\ 0 \leq \theta \leq 2\pi \end{cases}$  if  $(\rho, \theta) \in (0, r] \times [0, 2\pi]$

then  $(x_0 + \rho \cos(\theta), y_0 + \rho \sin(\theta)) \in C \setminus_{\{(x_0, y_0)\}} \subseteq A \Rightarrow (\rho, \theta) \in A^*$ .  $\square$

**Definition 2.** Let  $\theta \in [0, 2\pi]$  and  $\forall \rho \in (0, r]$  we define  $\varphi_\theta(\rho) = F(\rho, \theta)$  if  $\rho \in (0, r]$ ,  $(\rho, \theta) \in A^*$  so the  $\lim_{\rho \rightarrow 0} \varphi(\rho) = l \in \bar{\mathbb{R}}$ .

If that limit exists that means  $\forall \theta \in [0, 2\pi]$  and  $\forall \varepsilon > 0$ ,  $\exists \delta > 0 \ \forall \rho \in (0, r]$  with  $\rho < \delta \ \ |\varphi_\theta - l| < \varepsilon$ .

We say that  $\lim_{\rho \rightarrow 0} \varphi(\rho) = l \in \bar{\mathbb{R}}$  Uniformly With Respect To (U.W.R.T.)  $\theta$ .

**Theorem 2.** Let  $f : A \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}$  with  $(x_0, y_0)$  accumulation point for  $A$ .

Follows the equivalence:

$\lim_{(x,y) \rightarrow (x_0, y_0)} f(x, y) = l \in \bar{\mathbb{R}} \iff \lim_{\rho \rightarrow 0} F(\rho, \theta) = l \text{ U.W.R.T. } \theta$ .

*Proof.* Let  $l \in \bar{\mathbb{R}}$ .

"  $\implies$  "  $\lim_{(x,y) \rightarrow (x_0, y_0)} f(x, y) = l$  so  $\forall \varepsilon > 0$ ,  $\exists \delta > 0 : \forall (x, y) \in A$

with  $\|(x, y) - (x_0, y_0)\| < \delta$ ,  $|f(x, y) - l| < \varepsilon$ .

We have to prove that  $\forall \varepsilon > 0$ ,  $\exists \delta > 0 : \forall \theta \in [0, 2\pi]$ ,  $\forall \rho \in (0, r]$

with  $\rho < \delta \ \ |F(\rho, \theta) - l| < \varepsilon$ .

Let  $\varepsilon > 0$ ,  $\theta \in [0, 2\pi]$ ,  $\rho \in (0, r]$  with  $\rho < \delta$ . we create the system that changes the coordinates from cartesian to polar:

$$\begin{cases} x = x_0 + \rho \cos(\theta) \\ y = y_0 + \rho \sin(\theta) \end{cases} \quad \rho = \sqrt{(x - x_0)^2 + (y - y_0)^2}$$

$\rho \in (0, r]$ ,  $\theta \in [0, 2\pi] \Rightarrow (\rho, \theta) \in A^* \Rightarrow (x, y) \in A$ .

Now  $0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} = \rho < \delta \Rightarrow |f(x, y) - l| < \varepsilon$ .

$\Rightarrow |f(x_0 + \rho \cos(\theta), y_0 + \rho \sin(\theta)) - l| < \varepsilon \Rightarrow |F(\rho, \theta) - l| < \varepsilon$ .

"  $\Leftarrow$  "  $\forall \varepsilon > 0$ ,  $\exists \delta \leq r : \forall \theta \in [0, 2\pi]$  and  $\forall \rho$  with  $0 < \rho < \delta \Rightarrow$

$|F(\rho, \theta) - l| < \varepsilon$ .

We have to prove that  $\forall \varepsilon > 0$ ,  $\exists \delta > 0$ ,  $\forall (x, y) \in A$  with

$\sqrt{(x - x_0)^2 + (y - y_0)^2} = \|(x, y) - (x_0, y_0)\| < \delta \Rightarrow |f(x, y) - l| < \varepsilon$ .

Let  $\varepsilon > 0$ ,  $\delta \leq r$ ,  $(x, y) \in A$ ,  $\sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta$ , we switch coordinates with  $\rho$  and  $\theta$  as follows:

$$\begin{cases} x = x_0 + \rho \cos(\theta) \\ y = y_0 + \rho \sin(\theta) \end{cases} \quad \rho = \sqrt{(x - x_0)^2 + (y - y_0)^2}$$

$0 < \rho < \delta \leq r \Rightarrow \rho \in (0, r)$ ,  $\theta \in [0, 2\pi]$ .

We notice that  $|F(\rho, \theta) - l| < \varepsilon$ , so  $|f(x_0 + \rho \cos(\theta), y_0 + \rho \sin(\theta)) - l| < \varepsilon$

$\Rightarrow |f(x, y) - l| < \varepsilon$ .  $\square$

**Definition 3.** We say that  $\theta \in [0, 2\pi]$  is admissible if  $0 \in \bar{\mathcal{D}}(A_\theta)$ .

**Definition 4.** Let's suppose that  $\lim_{\rho \rightarrow 0} F(\rho, \theta) = l \in \mathbb{R}$  then  $\forall \rho \in (0, r]$ ,  $\varphi(\rho) = \sup \{|F(\rho, \theta) - l| : \theta \in [0, 2\pi]\}$

**Theorem 3.**  $\lim_{\rho \rightarrow 0} F(\rho, \theta) = l \in \bar{\mathbb{R}}$  U.W.R.T.  $\theta \iff \lim_{\rho \rightarrow 0} \varphi(\rho) = 0$ .

*Proof.*  $(\Rightarrow) \forall \varepsilon > 0 \exists \delta > 0 : \forall \theta \in [0, 2\pi]$  and  $\forall \rho \in (0, r]$  with  $\rho < \delta$   $|F(\rho, \theta) - l| < \frac{\varepsilon}{2}$  so  $|\varphi(\rho)| \leq \frac{\varepsilon}{2} < \varepsilon \Rightarrow \lim_{\rho \rightarrow 0} \varphi(\rho) = 0$ .  $(\Leftarrow) \forall \varepsilon > 0 \exists \delta > 0 : \forall \rho \in (0, r]$  with  $\rho < \delta$   $\varphi(\rho) < \varepsilon$  but  $|F(\rho, \theta) - l| \leq \varphi(\rho) \forall \theta$  so if  $\rho \in (0, r]$  and  $\rho < \delta$   $|F(\rho, \theta) - l| < \varepsilon \Rightarrow \lim_{\rho \rightarrow 0} F(\rho, \theta) = l$  U.W.R.T.  $\theta$  □

**Corollary 1.**  $\lim_{\rho \rightarrow 0} F(\rho, \theta) = l \in \bar{\mathbb{R}}$  U.W.R.T.  $\theta \iff \exists$  a function  $\psi(\rho)$  such that  $\lim_{\rho \rightarrow 0} \psi(\rho) = 0$  and  $\forall \theta$   $|F(\rho, \theta) - l| \leq \psi(\rho)$ .

**Corollary 2.** Let's suppose that  $\lim_{\rho \rightarrow 0} F(\rho, \theta) = +\infty$ .

$\forall \rho \in (0, r]$  let  $h(\rho) = \inf\{F(\rho, \theta) : \theta \in [0, 2\pi]\}$  so then  $\lim_{\rho \rightarrow 0} F(\rho, \theta) = +\infty$  U.W.R.T.  $\theta \iff \lim_{\rho \rightarrow 0} h(\rho) = +\infty$

**Obs 1.**  $\lim_{\rho \rightarrow 0} F(\rho, \theta) = +\infty$  U.W.R.T.  $\theta \iff \exists$  a function  $K(\rho)$  s.t.

$\lim_{\rho \rightarrow 0} K(\rho) = +\infty$  and  $F(\rho, \theta) \geq K(\rho)$

**Corollary 3.** Let's suppose that  $\lim_{\rho \rightarrow 0} F(\rho, \theta) = -\infty$ .

$\forall \rho \in (0, r]$  let  $g(\rho) = \sup\{F(\rho, \theta) : \theta \in [0, 2\pi]\}$  so then  $\lim_{\rho \rightarrow 0} F(\rho, \theta) = -\infty$  U.W.R.T.  $\theta \iff \lim_{\rho \rightarrow 0} g(\rho) = -\infty$

**Definition 5.** Let  $f : A \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}$  with  $A$  open.

let  $(x_0, y_0) \in A$ ,  $\varphi(x) = f(x, y_0)$  and  $\psi = f(x_0, y)$ .  $A$  is open that means that those two functions are well defined.

## Differentiability

**Definition 6.** Let be  $f : A \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$  with  $A$  Open. Let  $\bar{x} \in A$  and let  $i \leq n$ , we denote as  $\varphi_i(x_i) = f(\bar{x}_1, \bar{x}_2, \dots, \bar{x}_{i-1}, \bar{x}_i, \bar{x}_{i+1}, \dots, \bar{x}_n)$ . Notice that  $\bar{x}$  is an internal point so then it exist an interval where  $\varphi_i$  is well defined.

**Definition 7.** We say that  $f$  is partially derivable with respect to the variable  $x_i$  in the point  $\bar{x}$  if  $\varphi_i$  is derivable in that point. We denote as  $\frac{\partial f}{\partial x_i}$  the partial derivative with respect to  $x_i$  in the point  $\bar{x}$ .

**Definition 8.** The gradient of a function in  $n$  variables is defined as follows:

$$\nabla f : \bar{x} \in A \mapsto \left( \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n} \right) \in \mathbb{R}^n$$

Let  $f : A \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}$  with  $A$  open, and let  $(x_0, y_0) \in A$ .  
 $(x_0, y_0, f(x_0, y_0)) \in \mathcal{G}(f)$ . The equation of the plane that passes for  
 $(x_0, y_0, f(x_0, y_0))$  is  $Z = f(x_0, y_0) + a(X - x_0) + b(Y - y_0)$  where  $a, b \in \mathbb{R}$ .

**Definition 9.** We say that  $f$  is partially derivable with respect to  $x$  in  $(x_0, y_0)$  if  $\varphi$  is differentiable in  $x_0$ . in that case we  $\varphi$  is the partial derivative of  $f$  in the variable  $x$  and its written  $\frac{\partial f}{\partial x}$

**Definition 10.** We define the gradient as  $\nabla f : (x, y) \in A \mapsto \left( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right) \in \mathbb{R}^2$

**Definition 11.** We say that  $f$  is differentiable in the point  $(x_0, y_0)$  if exists  $a, b \in \mathbb{R}$  such that:

$$\lim_{(x, y) \rightarrow (x_0, y_0)} \frac{f(x, y) - f(x_0, y_0) - a(X - x_0) - b(Y - y_0)}{\|(x, y) - (x_0, y_0)\|} = 0 \quad (\Delta)$$

$f$  is differentiable in the point  $(x_0, y_0)$  if exists a plane that passes in the point  $(x_0, y_0, f(x_0, y_0))$  that approximates the graph of the function  $f$ .

**Proposition 2.** If  $f$  is differentiable in the point  $(x_0, y_0)$ ,  $f$  is partially derivable with respect to  $x$  and  $y$  such that  $a = \frac{\partial f(x_0, y_0)}{\partial x}$  and  $b = \frac{\partial f(x_0, y_0)}{\partial y}$

**Definition 12.** if  $f$  is differentiable in a point  $(x, y) \in A$ , the differential in the point is defined as follows:

$$d_{(x, y)} f : (h, k) \in \mathbb{R}^2 \mapsto \frac{\partial f(x, y)}{\partial x} h + \frac{\partial f(x, y)}{\partial y} k \in \mathbb{R}$$

**Definition 13.** More in general if  $f : A \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$  and  $\mathbf{x} \in A$ :

$$d_{\mathbf{x}}^r f : h \in \mathbb{R}^n \mapsto \sum_{\substack{i_1, \dots, i_n \geq 0 \\ i_1 + \dots + i_n = r}} \frac{r!}{i_1! \dots i_n!} \frac{\partial^r f}{\partial x^{i_1} \dots \partial x^{i_n}}(\mathbf{x}) h_1^{i_1}, \dots, h_n^{i_n} \in \mathbb{R}$$

**Corollary 4.**  $f$  is differentiable in the point  $(x_0, y_0) \iff f$  is partially derivable in the point  $(x_0, y_0)$  and the  $(\Delta)$  is true.

**Definition 14.** Let  $f : A \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}$  with  $A$  open and  $k > 0$  a positive integer. let  $(x_0, y_0) \in A$  and if  $f$  has differentiable derivatives of order  $k - 1$  we define the "k-grade Taylor polinomia" as follows:

$$P_k(x, y) = f(x_0, y_0) + \sum_{i=1}^k \frac{1}{i!} d_{(x_0, y_0)}^i f(x - x_0, y - y_0)$$

**Theorem 4.** Let  $f : A \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}$  with  $A$ . If  $\exists \frac{\partial f}{\partial x}$  and  $\frac{\partial f}{\partial y}$  in  $A$  and are continuos in a point  $(x_0, y_0)$ , then the function is differentiable in  $(x_0, y_0)$ .

*Proof.* We have to prove that:

$$\lim_{(x, y) \rightarrow (x_0, y_0)} \frac{f(x, y) - f(x_0, y_0) - \frac{\partial f(x_0, y_0)}{\partial x}(x - x_0) - \frac{\partial f(x_0, y_0)}{\partial y}(y - y_0)}{\sqrt{(x - x_0)^2 + (y - y_0)^2}} = 0$$

Lets add and subtract  $f(x, y_0)$ , so one has:

$$f(x, y) - f(x_0, y_0) = f(x, y) - f(x, y_0) + f(x, y_0) - f(x_0, y_0)$$

We call  $\varphi(t) = f(x, t)$  where  $t \in I[y, y_0]$  and  $I[y, y_0] = \begin{cases} [y, y_0] & y \leq y_0 \\ [y_0, y] & y_0 \leq y \end{cases}$

$\varphi$  is derivable and for the Lagrange theorem  $\exists y_1 \in I[y, y_0] : \varphi(y) - \varphi(y_0) = \dot{\varphi}(y_1)(y - y_0)$ . So one has  $f(x, y) - f(x, y_0) = \frac{\partial f(x, y_1)}{\partial y}(y - y_0)$ , and we repeat the same reasoning for the other variable and one will have  $f(x, y_0) - f(x_0, y_0) = \frac{\partial f(x_1, y_0)}{\partial x}(x - x_0)$  We have then:

$$\begin{aligned} & \left| \frac{f(x, y) - f(x_0, y_0) - \frac{\partial f(x_0, y_0)}{\partial x}(x - x_0) - \frac{\partial f(x_0, y_0)}{\partial y}(y - y_0)}{\sqrt{(x - x_0)^2 + (y - y_0)^2}} \right| = \\ &= \left| \frac{\frac{\partial f(x_1, y_0)}{\partial x}(x - x_0) - \frac{\partial f(x, y_1)}{\partial y}(y - y_0) - \frac{\partial f(x_0, y_0)}{\partial x}(x - x_0) - \frac{\partial f(x_0, y_0)}{\partial y}(y - y_0)}{\sqrt{(x - x_0)^2 + (y - y_0)^2}} \right| = \\ &= \left| \frac{\left( \frac{\partial f(x_1, y_0)}{\partial x} - \frac{\partial f(x_0, y_0)}{\partial x} \right)(x - x_0) + \left( \frac{\partial f(x, y_1)}{\partial y} - \frac{\partial f(x_0, y_0)}{\partial y} \right)(y - y_0)}{\sqrt{(x - x_0)^2 + (y - y_0)^2}} \right| \end{aligned}$$

The last member is increased by the following:

$$\begin{aligned} & \left| \frac{\partial f(x_1, y_0)}{\partial x} - \frac{\partial f(x_0, y_0)}{\partial x} \right| \frac{|x - x_0|}{\|(x - x_0, y - y_0)\|} + \left| \frac{\partial f(x, y_1)}{\partial y} - \frac{\partial f(x_0, y_0)}{\partial y} \right| \frac{|y - y_0|}{\|(x - x_0, y - y_0)\|} \leq \\ & \leq \left| \frac{\partial f(x_1, y_0)}{\partial x} - \frac{\partial f(x_0, y_0)}{\partial x} \right| + \left| \frac{\partial f(x, y_1)}{\partial y} - \frac{\partial f(x_0, y_0)}{\partial y} \right| \end{aligned}$$

And since  $x \rightarrow x_0 \Rightarrow x_1 \rightarrow x_0$  and  $y \rightarrow y_1 \Rightarrow y_1 \rightarrow y_0$  so the second member of the inequality is equal to zero. □

**Theorem 5.** Let  $f : A \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$  with  $A$  open. If the function is differentiable in a point  $\mathbf{x} \in A$  then is continuos in that point.

*Proof.* Since we have:

$$\lim_{x \rightarrow \mathbf{x}} \frac{f(x) - f(\mathbf{x}) - \sum_{i=1}^n \frac{\partial f(\mathbf{x})}{\partial x_i} (x_i - \mathbf{x}_i)}{\|x - \mathbf{x}\|} = 0$$

If we fix an  $\varepsilon = 1$  there exists  $\delta > 0 : \forall x \in A$  with  $0 < \|x - \mathbf{x}\| < \delta$  one has:

$$\left| \frac{f(x) - f(\mathbf{x}) - \nabla f(\mathbf{x})(x - \mathbf{x})}{\|x - \mathbf{x}\|} \right| < \varepsilon \implies |f(x) - f(\mathbf{x}) - \nabla f(\mathbf{x})(x - \mathbf{x})| < \|x - \mathbf{x}\|$$

So we have the the following:

$$|f(x) - f(\mathbf{x})| - |\nabla f(\mathbf{x})(x - \mathbf{x})| \leq |f(x) - f(\mathbf{x}) - \nabla f(\mathbf{x})(x - \mathbf{x})| < \|x - \mathbf{x}\| \triangle$$

The  $\triangle$  implies that  $|f(x) - f(\mathbf{x})| - |\nabla f(\mathbf{x})(x - \mathbf{x})| \leq |\nabla f(\mathbf{x})(x - \mathbf{x})| + \|x - \mathbf{x}\| \leq \|\nabla f(\mathbf{x})\| \|x - \mathbf{x}\| + \|x - \mathbf{x}\|$ . The last member of the inequality is equal to  $\|x - \mathbf{x}\| (\|\nabla f(\mathbf{x})\| + 1) \triangle$  so, if  $0 < \|x - \mathbf{x}\| < \delta$  then by calling the  $\triangle = c$  one finally has:

$$0 < |f(x) - f(\mathbf{x})| \leq c \|x - \mathbf{x}\| \rightarrow 0 \Leftarrow x \rightarrow \mathbf{x} \Rightarrow |f(x) - f(\mathbf{x})| \rightarrow 0$$

Or equivalently:  $\lim_{x \rightarrow \mathbf{x}} f(x) = f(\mathbf{x})$ . □

**Theorem (Schwartz).**<sup>1</sup> Let  $f : \Omega \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}$  be a function in two variables defined on a open set  $\Omega$ .

If  $f$  admits continuous second derivatives in the point  $(f \in C^2(\Omega))$  then  $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$ .

*Proof.* Let  $p = (x_0, y_0) \in \Omega$  and chose two real numbers  $\varepsilon, \delta > 0$  such that  $(x_0 - \varepsilon, x_0 + \delta) \times (y_0 - \delta, y_0 + \delta) \subset \Omega$ . That is possible since  $\Omega$  is Open. Lets also define the two functions  $F$  and  $G$  as follows:

$$F : (-\varepsilon, \varepsilon) \subset \mathbb{R} \rightarrow \mathbb{R}$$

$$G : (-\delta, \delta) \subset \mathbb{R} \rightarrow \mathbb{R}$$

In the way that:

$$F(t) = f(x_0 + t, y_0 + s) - f(x_0 + t, y_0) \quad \forall s \in (-\delta, \delta)$$

$$G(s) = f(x_0 + t, y_0 + s) - f(x_0, y_0 + s) \quad \forall t \in (-\varepsilon, \varepsilon)$$

It can be easily proved that:  $F(t) - F(0) = G(s) - G(0)$  also if we apply the Lagrange theorem two times one has:  $F(t) - F(0) = t\dot{F}(\xi_1)$  with  $t\dot{F}(\xi_1)$  equal to:  $t \left[ \frac{\partial f}{\partial x}(x_0 + \xi_1, y_0 + s) - \frac{\partial f}{\partial x}(x_0 + \xi_1, y_0) \right] = ts \frac{\partial^2 f}{\partial y \partial x}(x_0 + \xi_1, y_0 + \sigma_1)$ . The same reasoning can be applied to  $G(s) - G(0)$  obtaining:  $st \frac{\partial^2 f}{\partial x \partial y}(x_0 + \xi_2, y_0 + \sigma_2)$  with  $\xi_i \in (0, t)$  and  $\sigma_i \in (0, s)$  where without loss of generality we can say  $t, s > 0$ .

Thinking about  $t \rightarrow 0$  and  $s \rightarrow 0 \Rightarrow \xi_i \rightarrow 0$  and  $\sigma_i \rightarrow 0$  with the continuity of the two derivatives one has:  $\frac{\partial^2 f}{\partial y \partial x}(x_0, y_0) = \frac{\partial^2 f}{\partial x \partial y}(x_0, y_0)$ .  $\square$

## Directional Derivatives

If we take  $f : A \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}$  defined on an open set  $A$ ,  $(x_0, y_0) \in A$  and a vector of unitary norm  $\vec{v} = (v_1, v_2)$ , the Directional derivative of  $f(x_0, y_0)$  along the direction  $\vec{v}$  can be defined as the limit if it exists and its finite:

$$\frac{\partial f}{\partial \vec{v}}(x_0, y_0) = \lim_{t \rightarrow 0} \frac{f(x_0 + tv_1, y_0 + tv_2) - f(x_0, y_0)}{t}$$

## Study of the maxima and minima

**Definition 15.** If a partial derivative  $\frac{\partial f}{\partial x}$  of a function  $f : A \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}$  is partially derivable with respect to  $x$  in a point  $(x_0, y_0) \in A$  we say that  $f$  is partially derivable two times with respect to  $x$  in the point  $(x_0, y_0)$  ad it will be denoted as  $\frac{\partial f}{\partial x} f_x(x_0, y_0) = \frac{\partial^2 f(x_0, y_0)}{\partial x^2} = \frac{\partial^2 f}{\partial x^2}(x_0, y_0)$ .

The same goes for the other partial derivatives:  $\frac{\partial}{\partial y} f_x = \frac{\partial^2 f}{\partial x \partial y}$ ,  $\frac{\partial}{\partial x} f_y = \frac{\partial^2 f}{\partial y \partial x}, \dots$

**Definition 16.** We define the hessian matrix as follows:

$$\mathcal{D}^2 f = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix}$$

**Definition 17.** The determinant of  $\mathcal{D}^2 f$  is:

$$\mathcal{H}(x, y) = \begin{vmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{vmatrix}$$

and is called the Hessian determinant.

**Definition 18.** Let  $f : A \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}$ , we say that  $(x_0, y_0) \in A$  is maxima (minima) for  $f$  if  $\forall (x, y) \in A$ ,  $f(x, y) \leq f(x_0, y_0)$  ( $f(x, y) \geq f(x_0, y_0)$ ).

**Theorem 6.** If  $f$  is continuous and  $A$  is compact,  $f$  admits minima and maxima.

**Theorem 7.** Let  $f : A \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}$  and let  $(x_0, y_0) \in \dot{A}$  a relative extreme and let  $f$  be partially derivable in  $(x_0, y_0)$ , so then  $\frac{\partial f(x_0, y_0)}{\partial x} = 0$  and  $\frac{\partial f(x_0, y_0)}{\partial y} = 0$ .

The points where the partial derivatives are 0 are said "critical points" of  $f$ ,  $(x_0, y_0) \in \dot{A}$  is an extreme relative  $\Rightarrow (x_0, y_0)$  is a critical point for  $f$  ( $\Leftarrow$ ).

**Obs 2.** Let  $(x_0, y_0) \in A$  and let  $g(x, y) = f(x, y) - f(x_0, y_0)$ ,  $(x_0, y_0)$  is a relative minimum (relative maximum) for  $f \iff \exists$  a circle  $C$  of center  $(x_0, y_0)$  such that  $g \geq 0$  ( $g \leq 0$ ).

**Theorem 8.** Let  $f : A \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}$  and let  $(x_0, y_0) \in \dot{A}$  a relative extreme  $\implies \mathcal{H}(x_0, y_0) \geq 0$ .

**Theorem 9.** Let  $f : A \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}$ ,  $f \in \mathbf{C}^2$ . Let  $(x_0, y_0) \in \dot{A}$  a critical point and lets suppose that  $\mathcal{H}(x_0, y_0) > 0 \implies (x_0, y_0)$  is a relative extreme and is maximum or minimum depending on  $\frac{\partial^2 f(x_0, y_0)}{\partial x^2}$  be  $< 0$  or  $> 0$ .

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<sup>1</sup> $\Omega$  this time is used instead of  $A$

## Vectorial Functions

A vectorial function is defined as follows:  $f : A \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^m$ .  $\forall x \in A$ ,  $f(x) \in \mathbb{R}^m$  and  $f(x) = (f_1(x), \dots, f_m(x))$  with  $f_i : A \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$

**Definition 19.** Let  $f : A \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^m$  and let  $x_0$  be an accumulation point for  $A$ , we say that  $\lim_{x \rightarrow x_0} f(x) = l \in \mathbb{R}^m$  if  $\forall \varepsilon > 0 \exists \delta > 0 : \forall x \in A$  with  $0 < \|x - x_0\| < \delta \Rightarrow \|f(x) - l\| < \varepsilon$ .

**Lemma 1.** Let  $a_1, \dots, a_n \in \mathbb{R}$ , then  $\forall j \leq n \quad |a_j| \leq \sqrt{\sum_{i=1}^n a_i^2} \leq \sum_{i=1}^n |a_i|$ .

*Proof.* Let  $j \leq n$ .  $|a_j| = \sqrt{a_j^2} \leq \sqrt{\sum_{i=1}^n a_i^2}$ . We have to prove that  $\sum_{i=1}^n a_i^2 \leq (\sum_{i=1}^n |a_i|)^2$  and that's true for  $n = 2$  in fact:  $(|a_1| + |a_2|)^2 = a_1^2 + a_2^2 + 2|a_1||a_2| \geq a_1^2 + a_2^2$ . Lets suppose that's true for  $n - 1$ , that means  $\sum_{i=1}^{n-1} a_i^2 \leq (\sum_{i=1}^{n-1} |a_i|)^2$  because  $\sum_{i=1}^n a_i^2 = \sum_{i=1}^{n-1} a_i^2 + a_n^2 \leq \left(\sum_{i=1}^{n-1} |a_i|\right)^2 + a_n^2 \leq \left(\sum_{i=1}^{n-1} |a_i| + |a_n|\right)^2 = (\sum_{i=1}^n |a_i|)^2$ .  $\square$

**Theorem 10.** Let  $f : A \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}^m$  with  $f = (f_1, \dots, f_m)$ . Let  $x_0$  be an accumulation point for  $A$  and let  $l = (l_1, \dots, l_m) \in \mathbb{R}^m$ . Then  $\lim_{x \rightarrow x_0} f(x) = l \iff \forall i \leq m, \lim_{x \rightarrow x_0} f_i = l_i$ .

*Proof.* We know that if  $a_i = f_i(x) - l_i$  then  $\forall j \leq m, |f_j(x) - l_j| \leq \sqrt{\sum_{i=1}^m (f_i(x) - l_i)^2} = \|f(x) - l\| \leq \sum_{i=1}^m |f_i(x) - l_i|$ . Lets suppose that  $\lim_{x \rightarrow x_0} f(x) = l$ , then  $\|f(x) - l\| \rightarrow 0$  for  $x \rightarrow x_0$  and for the sandwich theorem  $|f_j(x) - l_j| \rightarrow 0$  for  $x \rightarrow x_0 \quad \forall j \leq m \implies \forall j \leq m \lim_{x \rightarrow x_0} f_j(x) = l_j$ . Viceversa lets suppose that  $\forall i \leq m \lim_{x \rightarrow x_0} f_i(x) = l_i \Rightarrow |f_i(x) - l_i| \rightarrow 0$  for  $x \rightarrow x_0 \quad \forall i \leq m \Rightarrow \sum_{i=1}^m |f_i(x) - l_i| \rightarrow 0$  for  $x \rightarrow x_0 \Rightarrow \|f(x) - l\| \rightarrow 0$  for  $x \rightarrow x_0 \Rightarrow \lim_{x \rightarrow x_0} f(x) = l$ .  $\square$

**Definition 20.**  $f(x)$  is continuous in a point  $x_0 \in A$  if  $\forall \varepsilon > 0, \exists \delta$  such that  $\forall x \in A, \|x - x_0\| < \delta \Rightarrow \|f(x) - f(x_0)\| < \varepsilon$ .

## Implicit functions

By defining a function  $f : A \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}$  the expression:  $f(x, y) = 0 \diamond$  means that one can consider the variable  $x$  as a parameter and  $y$  as unknown and the question is when,  $\forall x \exists! y$  such that the  $\diamond$  is true.

**Definition 21.** The equation defines implicitly  $y$  as a function of  $x$  if  $\forall x, \exists! y : f(x, y) = 0$  in that case the function  $g$  is defined as:

$$g : x \mapsto y \Rightarrow f(x, y) = 0$$

**Proposition 3.** The equation  $f(x_0, y_0) = 0$  defines implicitly  $y$  as a function of  $x$ , the set of all the zeros of  $f$  is equal to the graph of the implicit function.

*Proof.*  $(x_0, y_0)$  is a zero of  $f \iff f(x_0, y_0) = 0 \iff y_0 = g(x_0) \iff (x_0, y_0) \in Gr(g)$   $\square$

**Theorem (Implicit Functions).** Let  $f : A \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}$  lets suppose that  $f$  and  $\frac{\partial f}{\partial y}$  are continuous. Let  $(x_0, y_0) \in A$  be a zero of the function where  $\frac{\partial f}{\partial y}(x_0, y_0) \neq 0$ . **i.** Then there exists an open interval  $I$  of center  $x_0$  and an open interval  $J$  of center  $y_0 : I \times J \subseteq A$  and  $\forall x \in I, \exists! y \in J : f(x, y) = 0$ . **ii.** Also if  $g : I \rightarrow J$  is the implicit function,  $g$  is continuous and  $g(x_0) = y_0$ . **iii.** In addition if  $\exists \frac{\partial f}{\partial x}$ ,  $g$  is derivable and  $\forall x \in I, \dot{g}(x) = -\frac{\frac{\partial f}{\partial x}(x, g(x))}{\frac{\partial f}{\partial y}(x, g(x))}$ . **iv.** Furthermore  $f \in C^k$ , then  $g \in C^k$ .

*Proof.* **i.** By hypotesis  $\frac{\partial f}{\partial y} \neq 0$ . Lets suppose that  $\frac{\partial f}{\partial y}(x_0, y_0) > 0$ , so for the sign permanence theorem there exists a rectangle  $R_0 = [x_0 - \alpha, x_0 + \alpha] \times [y_0 - \beta, y_0 + \beta] \subseteq A : \forall (x, y) \in R_0, \frac{\partial f}{\partial y}(x, y) > 0$ . Let  $\varphi : y \in [y_0 - \beta, y_0 + \beta] \mapsto f(x_0, y)$ , by hypotesis  $\exists \frac{\partial f}{\partial y}$  so, by definition  $\varphi$  is derivable and  $\dot{\varphi}(y) = \frac{\partial f}{\partial y}(x_0, y) > 0 \Rightarrow \varphi$  is strictly growing.  $\varphi(y_0) = f(x_0, y_0) = 0$ ,  $\varphi(y_0 + b) > \varphi(y_0) = 0 \Rightarrow f(x_0, y_0 + b) > 0$  and  $f(x_0, y_0 - b) < 0$ . Lets define:

$$\varphi_1 : x \in [x_0 - \alpha, x_0 + \alpha] \mapsto f(x, y_0 - b)$$

$$\varphi_2 : x \in [x_0 - \alpha, x_0 + \alpha] \mapsto f(x, y_0 + b)$$

$\varphi_1$  and  $\varphi_2$  are continuous because  $f$  is continuous and  $\varphi_1(x_0) = f(x_0, y_0 - b) < 0$  and  $\varphi_2(x_0) = f(x_0, y_0 + b) > 0$  so there exists an interval  $[x_0 - \delta, x_0 + \delta] \subseteq [x_0 - \alpha, x_0 + \alpha] : \forall x \in [x_0 - \delta, x_0 + \delta], \varphi_1(x) < 0$  and  $\varphi_2(x) > 0$ . Now  $\forall x \in [x_0 - \delta, x_0 + \delta], f(x, y_0 - b) < 0$  and  $f(x, y_0 + b) > 0$  if we take an  $x \in (x_0 - \delta, x_0 + \delta)$  and define:

$$\psi : y \in [y_0 - b, y_0 + b] \mapsto f(x, y)$$

One has that  $\psi$  is derivable and  $\dot{\psi}(y) = \frac{\partial f}{\partial y}(x, y) > 0$  that implies  $\psi$  is strictly growing and also continuous.  $\psi(y_0 - b) = f(x, y_0 - b) < 0$  and  $\psi(y_0 + b) = f(x, y_0 + b) > 0$  for the zeros theorem,  $\exists y \in (y_0 - b, y_0 + b)$  where  $\psi(y) = 0 \Rightarrow f(x, y) = 0$ . Also  $y$  is unique since  $\psi$  is strictly growing and that means it can't become zero in two different points. **ii.** Let  $g : I \mapsto J$  the implicit function defined by the equation  $f(x, y) = 0$ .  $\forall x \in I, f(x, g(x)) = 0$  and  $f(x_0, y_0) \Rightarrow y_0 = g(x_0)$  we have to prove that  $g$  is continuous, so let  $\bar{x} \in I$  and the claim is  $\lim_{x \rightarrow \bar{x}} g(x) = g(\bar{x})$ .  $\forall x \in I, g(x)$  and  $g(\bar{x})$  are two distinct points of  $J \Rightarrow K[g(x), g(\bar{x})] \subseteq J$ .

$\forall x \in I$ , let  $\psi : y \in K[g(x), g(\bar{x})] \mapsto f(x, y)$  with  $\dot{\psi}(y) = \frac{\partial f}{\partial y}(x, y) > 0$ , for the lagrange theorem,  $\exists$  a point  $\xi_x \in K[g(x), g(\bar{x})]$ :

$$\psi(g(x)) - \psi(g(\bar{x})) = \dot{\psi}(\xi_x)(g(x) - g(\bar{x}))$$

That becomes:  $f(x, g(x)) - f(x, g(\bar{x})) = \frac{\partial f}{\partial y}(x, \xi_x)(g(x) - g(\bar{x})) \Rightarrow g(x) - g(\bar{x}) = -\frac{f(x, g(\bar{x}))}{\frac{\partial f}{\partial y}(x, \xi_x)} \Rightarrow |g(x) - g(\bar{x})| = \frac{|f(x, g(\bar{x}))|}{\left|\frac{\partial f}{\partial y}(x, \xi_x)\right|}$ .  $\frac{\partial f}{\partial y}$  is continuous and  $R_0$  is compact. By the Weierstrass theorem there exists  $m = \min \left\{ \left| \frac{\partial f}{\partial y}(x, y) \right| : (x, y) \in R_0 \right\}$ ,  $\frac{\partial f}{\partial y} > 0$  in  $R_0$  and that means  $m > 0$  so  $\left| \frac{\partial f}{\partial y}(x, y) \right| \geq m$  therefore one has:

$$|g(x) - g(\bar{x})| = \frac{|f(x, g(\bar{x}))|}{\frac{\partial f}{\partial y}(x, \xi_x)} \leq \frac{f(x, g(\bar{x}))}{m}$$

$x \rightarrow \bar{x} \Rightarrow (x, g(\bar{x})) \rightarrow (\bar{x}, g(\bar{x}))$ ,  $f$  is continuous  $\Rightarrow f(x, g(\bar{x})) \rightarrow f(\bar{x}, g(\bar{x})) = 0$  so  $\lim_{x \rightarrow \bar{x}} g(x) = g(\bar{x})$ . **iii.** Let  $\bar{x} \in I$  we have to prove that  $g$  is derivable in  $\bar{x}$ . If  $x \in I$

$$g(x) - g(\bar{x}) = -\frac{f(x, g(\bar{x}))}{\frac{\partial f}{\partial y}(x, \xi_x)} \Rightarrow \frac{g(x) - g(\bar{x})}{x - \bar{x}} = -\frac{1}{\frac{\partial f}{\partial y}(x, \xi_x)} \frac{f(x, g(\bar{x})) - f(\bar{x}, g(\bar{x}))}{x - \bar{x}} \blacktriangle$$

And since there exists the partial derivative of  $f$  with respect to  $x$  the last member of the  $\blacktriangle$  is the incremental ratio of the function  $f(x, g(\bar{x}))$  in the point  $f(\bar{x}, g(\bar{x}))$  and by examining  $\frac{\partial f}{\partial y}(x, \xi_x)$  with  $\xi_x \in K[g(x), g(\bar{x})]$ ,  $x \rightarrow \bar{x} \Rightarrow g(x) \rightarrow g(\bar{x}) \Rightarrow \xi_x \rightarrow g(\bar{x}) \Rightarrow (x, \xi_x) \rightarrow (\bar{x}, g(\bar{x}))$ .  $\frac{\partial f}{\partial y}$  is continuous  $\Rightarrow \lim_{x \rightarrow \bar{x}} \frac{\partial f}{\partial y}(x, \xi_x) = \frac{\partial f}{\partial y}(\bar{x}, g(\bar{x}))$  finally  $\exists \lim_{x \rightarrow \bar{x}} \frac{g(x) - g(\bar{x})}{x - \bar{x}} = -\frac{\frac{\partial f}{\partial x}(\bar{x}, g(\bar{x}))}{\frac{\partial f}{\partial y}(\bar{x}, g(\bar{x}))}$ . **iv.** is proved by induction infact, if we take  $k = 1$  if  $f \in C^1$  its partial derivatives are continuous so  $g$  is continuous, the rest can be easily verified.  $\square$

## Bound extremes

Let  $f : A \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$  with  $A$  open. If  $g : A \rightarrow \mathbb{R}^m$  then we define  $E_0 = \{x \in A : g(x) = 0\}$ .  $g = (g_1, \dots, g_m)$  so  $g(x) = 0$  is equivalent to say:

$$\begin{cases} g_1(x) = 0 \\ g_2(x) = 0 \\ \vdots \\ g_m(x) = 0 \end{cases}$$

**Definition 22.** A point  $x_0 \in E_0$  is said to be bound maximum (bound minimum) if  $\forall x \in E_0$   $f(x) \leq f(x_0)$  ( $f(x) \geq f(x_0)$ ).

**Theorem (Lagrange Multipliers).** Let  $f : A \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}$  and  $g : A \rightarrow \mathbb{R}$  with  $g \in C^1$  in an open set  $A$ .

Let  $E_0 = \{(x, y) \in A : g(x, y) = 0\}$  and  $(x_0, y_0) \in E_0$  a bound relative extreme for  $f$  and suppose that  $\nabla g(x_0, y_0) \neq (0, 0)$  then  $\exists! \lambda \in \mathbb{R} : (x_0, y_0)$  be a critical point for the function  $\mathcal{L}(x, y) = f(x, y) + \lambda g(x, y)$   $\left( \frac{\partial f}{\partial x}(x_0, y_0) + \lambda \frac{\partial g}{\partial x}(x_0, y_0) = 0 \wedge \frac{\partial f}{\partial y}(x_0, y_0) + \lambda \frac{\partial g}{\partial y}(x_0, y_0) = 0 \right)$ .

*Proof. Existence* Since  $\nabla g(x_0, y_0) \neq (0, 0)$  we can assume  $\frac{\partial g}{\partial y}(x_0, y_0) \neq 0$ .  $g(x, y) = 0$  verifies the conditions of the implicit functions theorem ( $g \in C^1$ ,  $g(x_0, y_0) = 0$ ,  $\frac{\partial g}{\partial y}(x_0, y_0) \neq 0$ ), so for that theorem  $\exists$  an open interval  $I$  of center  $x_0$  and an open interval  $J$  of center  $y_0 : I \times J \subseteq A$  and  $\forall x \in I \exists! y \in J : g(x, y) = 0$  also the implicit function  $\varphi : I \rightarrow J$  is  $C^1$ .  $\varphi(x_0) = y_0$  and  $\forall x \in I$  one has:

$$\dot{\varphi}(x) = -\frac{\frac{\partial g}{\partial x}(x, \varphi(x))}{\frac{\partial g}{\partial y}(x, \varphi(x))}$$

Lets suppose for example that  $(x_0, y_0)$  is a bound relative minimum, then there exists a rectangle  $I' \times J' \subseteq I \times J$  of center  $(x_0, y_0)$  such that  $\forall (x, y) \in (I' \times J') \cap E_0 \Rightarrow f(x, y) \leq f(x_0, y_0)$ .  $\varphi$  is continuous in  $x_0$ ,  $J'$  is an interval of center  $y_0 = \varphi(x_0)$   $\exists I'' \subseteq I'$  of center  $x_0 : \forall x \in I'' \varphi(x) \in J'$ .  $\forall x \in I'' (x, \varphi(x)) \in (I' \times J') \cap E_0$ .  $\varphi : x \mapsto y$  such that  $g(x_0, y_0) = 0$  so  $g(x, \varphi(x)) = 0 \forall x \in I$ .  $\forall x \in I'' f(x, \varphi(x)) \leq f(x_0, y_0)$ .  $\forall x \in I''$  we define  $\psi(x) = f(x, \varphi(x))$  such that  $\forall x \in I''$  we have  $(x, \varphi(x)) \in I' \times J' \subseteq I \times J \subseteq A$  and that means  $\forall x \psi(x) \leq f(x_0, y_0) = \psi(x_0) \Rightarrow x_0$  is a bound relative maximum for  $\psi$  and is internal to  $I''$ .  $\psi \in C^1$  since is a composition of  $C^1$  functions so is derivable and  $\forall x \in I''$ :

$$\dot{\psi}(x) = \frac{\partial f}{\partial x}(x, \varphi(x)) + \frac{\partial f}{\partial y}(x, \varphi(x))\dot{\varphi}(x) = \frac{\partial f}{\partial x}(x, \varphi(x)) - \frac{\partial f}{\partial y}(x, \varphi(x)) \frac{\frac{\partial g}{\partial x}(x, \varphi(x))}{\frac{\partial g}{\partial y}(x, \varphi(x))}$$

Since  $\psi$  is derivable, and  $x_0$  is a maximum, for fermat theorem one has  $\dot{\psi}(x_0) = 0$  with:

$$\dot{\psi}(x_0) = \frac{\partial f}{\partial x}(x_0, y_0) - \frac{\partial f}{\partial y}(x_0, y_0) \frac{\frac{\partial g}{\partial x}(x_0, y_0)}{\frac{\partial g}{\partial y}(x_0, y_0)} = 0 \iff \frac{\partial f}{\partial x}(x_0, y_0) \frac{\partial g}{\partial y}(x_0, y_0) - \frac{\partial f}{\partial y}(x_0, y_0) \frac{\partial g}{\partial x}(x_0, y_0) = 0$$

Equivalently we can use this expression:

$$\begin{vmatrix} \frac{\partial f}{\partial x}(x_0, y_0) & \frac{\partial f}{\partial y}(x_0, y_0) \\ \frac{\partial g}{\partial x}(x_0, y_0) & \frac{\partial g}{\partial y}(x_0, y_0) \end{vmatrix} = 0$$

And if thats is true that implies  $\exists(\lambda_1, \lambda_2) \neq (0, 0)$  such that one of the two coloums is **l.d.** <sup>2</sup>  $\implies \lambda_1 \frac{\partial f}{\partial x}(x_0, y_0) + \lambda_2 \frac{\partial g}{\partial x}(x_0, y_0) = 0$ .

If  $\lambda_1 = 0 \Rightarrow \lambda_2 \neq 0 \Rightarrow \frac{\partial g}{\partial x}(x_0, y_0) = 0$  but that's impossible since the hypotesis impose it to be different from zero, so it must be  $\lambda_1 \neq 0$  and if we chose  $\lambda = \frac{\lambda_2}{\lambda_1}$  we have  $\mathcal{L}(x, y) = f(x, y) + \lambda g(x, y) = 0$ .

**Uniqueness** Let  $\mathcal{L}(x, y) = f(x, y) + \lambda g(x, y) = 0$  and  $\mathcal{L}(x, y) = f(x, y) + \bar{\lambda} g(x, y) = 0$ . If we subtract member to member the last equations one has:

$$(\lambda - \bar{\lambda}) \frac{\partial g}{\partial y}(x_0, y_0) = 0 \Rightarrow \lambda = \bar{\lambda}$$

□

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<sup>2</sup>Linearly dependent