Calculus II

Alessio Esposito

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Theorem 1. *A is closed $\iff$ every accumulation point for A is in A*

Proof. " $\implies$ " Let $A \subseteq \mathbb{R}^n,\ A = A \cup \partial A$.
Then $\forall p \in \bar{\mathcal{D}}(A),\ C_r(p)\diagdown_p \cap A \neq \emptyset\ \forall C \in \mathcal{C}_p$.
if $p \notin A$ then $C_r(p)$ has elements that dont belong to $A \Rightarrow p \in \partial A$.
" $\impliedby$ " Let $p \in \partial A \Rightarrow \forall C \in \mathcal{C}_p$ of center r with $r \in \mathbb{R}$ by definition we can find some $x \in C\diagdown_p \cap A$, so that means $p \in \bar{\mathcal{D}}(A) \Rightarrow p \in A$. □

1 Limits

Definition 1. *Let $A \subseteq \mathbb{R}^2$ and (x_0, y_0) an accumulation point for A. we define A^* as follows:*
$A^* = \{(\rho, \theta) \in [0, +\infty] \times [0, 2\pi] : (x_0 + \rho\cos(\theta), y_0 + \rho\sin(\theta)) \in A\}$.

Proposition 1. *Lets suppose that exist a circle C of center (x_0, y_0) such that $C\diagdown_{\{(x_0,y_0)\}} \subseteq A$ let r be the radius of the circle and as a consequence $(0, r] \times [0, 2\pi] \subseteq A^*$*

Proof. Let $C\diagdown_{\{(x_0,y_0)\}}$ and $\begin{cases} 0 < \rho \leqslant r \\ 0 \leqslant \theta \leqslant 2\pi \end{cases}$ if $(\rho, \theta) \in (0, r] \times [0, 2\pi]$
then $(x_0 + \rho\cos(\theta), y_0 + \rho\sin(\theta)) \in C\diagdown_{\{(x_0,y_0)\}} \subseteq A \Rightarrow (\rho, \theta) \in A^*$. □

Definition 2. *Let $\theta \in [0, 2\pi]$ and $\forall \rho \in (0, r]$ we define $\varphi_\theta(\rho) = F(\rho, \theta)$ if $\rho \in (0, r], (\rho, \theta) \in A^*$ so the $\lim_{\rho\to 0} \varphi(\rho) = l \in \bar{\mathbb{R}}$.*
If that limit exists that means $\forall \theta \in [0, 2\pi]$ and $\forall \varepsilon > 0$, $\exists \delta > 0\ \forall \rho \in (0, r]$ with $\rho < \delta\ |\varphi_\theta - l| < \varepsilon$.
We say that $\lim_{\rho\to 0} \varphi(\rho) = l \in \bar{\mathbb{R}}$ Uniformly With Respect To (U.W.R.T.) θ.

Theorem 2. *Let $f : A \subseteq \mathbb{R}^2 \to \mathbb{R}$ with (x_0, y_0) accumulation point for A.*
Follows the equivalence:
$\lim_{(x,y)\to(x_0,y_0)} f(x, y) = l \in \bar{\mathbb{R}} \iff \lim_{\rho\to 0} F(\rho, \theta) = l$ *U.W.R.T.* θ.

Proof. Let $l \in \bar{\mathbb{R}}$.
" $\implies$ " $\lim_{(x,y)\to(x_0,y_0)} f(x, y) = l$ so $\forall \varepsilon > 0, \exists \delta > 0 : \forall\ (x, y) \in A$
with $\|(x, y) - (x_0, y_0)\| < \delta,\ |f(x, y) - l| < \varepsilon$.

We have to prove that $\forall \varepsilon > 0,\ \exists \delta > 0 : \forall \theta \in [0, 2\pi],\ \forall \rho(0, r]$
with $\rho < \delta\ |F(\rho, \theta) - l| < \varepsilon$.
Let $\varepsilon > 0,\ \theta \in [0, 2\pi],\ \rho \in (0, r]$ with $\rho < \delta$. we create the system that changes the coordinates from cartesians to polars:

$$\begin{cases} x = x_0 + \rho\cos(\theta) \\ y = y_0 + \rho\sin(\theta) \end{cases} \quad \rho = \sqrt{(x - x_0)^2 + (y - y_0)^2}$$

$\rho \in (0, r],\ \theta \in [0, 2\pi] \in (0, r] \times [0, 2\pi] \subseteq A^*,\ (\rho, \theta) \in A^* \Rightarrow (x, y) \in A$.

Now $0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} = \rho < \delta \Rightarrow |f(x, y) - l| < \varepsilon$.
$\Rightarrow |f(x_0 + \rho\cos(\theta), y_0 + \rho\sin(\theta)) - l| < \varepsilon \Rightarrow |F(\rho, \theta) - l| < \varepsilon$.

" $\impliedby$ " $\forall \varepsilon > 0, \exists \delta \leq r : \forall \theta \in [0, 2\pi]$ and $\forall \rho$ with $0 < \rho < \delta \Rightarrow$
$|F(\rho, \theta) - l| < \varepsilon$.
We have to prove that $\forall \varepsilon > 0, \exists \delta > 0, \forall (x, y) \in A$ with
$\sqrt{(x - x_0)^2 + (y - y_0)^2} = \|(x, y) - (x_0, y_0)\| < \delta \Rightarrow |f(x, y) - l| < \varepsilon$.

Let $\varepsilon > 0,\ \delta \leq r,\ (x, y) \in A,\ \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta$, we switch coordinates with ρ and θ as follows:

$$\begin{cases} x = x_0 + \rho\cos(\theta) \\ y = y_0 + \rho\sin(\theta) \end{cases} \quad \rho = \sqrt{(x - x_0)^2 + (y - y_0)^2}$$

$0 < \rho < \delta \leq r \Rightarrow \rho \in (0, r),\ \theta \in [0, 2\pi]$.
We notice that $|F(\rho,\theta) - l| < \varepsilon$, so $|f(x_0 + \rho\cos(\theta), y_0 + \rho\sin(\theta)) - l| < \varepsilon$
$\Rightarrow |f(x,y) - l| < \varepsilon$. □

Definition 3. *We say that $\theta \in [0, 2\pi]$ is admissible if $0 \in \bar{\mathcal{D}}(A_\theta)$.*

Definition 4. *Let's suppose that $\lim_{\rho\to 0} F(\rho,\theta) = l \in \mathbb{R}$ then $\forall \rho \in (0, r],\ \varphi$*

Theorem 3. $\lim_{\rho\to 0} F(\rho,\theta) = l \in \bar{\mathbb{R}}$ *U.W.R.T.* $\theta \iff \lim_{\rho\to 0} \varphi(\rho) = 0$.

Corollary 1. $\lim_{\rho\to 0} F(\rho,\theta) = l \in \bar{\mathbb{R}}$ *U.W.R.T.* $\theta \iff \exists$ *a function $\psi(\rho)$ such that $\lim_{\rho\to 0} \psi(\rho) = 0$ and $\forall\theta\ \ |F(\rho,\theta) - l| \leqslant \psi(\rho)$.*

Corollary 2. *Let's suppose that $\lim_{\rho\to 0} F(\rho,\theta) = +\infty$.*
$\forall \rho \in (0, r]$ let $h(\rho) = inf\{F(\rho,\theta) : \theta \in [0, 2\pi]\}$ so then $\lim_{\rho\to 0} F(\rho,\theta) = +\infty$ U.W.R.T. $\theta \iff \lim_{\rho\to 0} h(\rho) = +\infty$

Obs 1. $\lim_{\rho\to 0} F(\rho,\theta) = +\infty$ *U.W.R.T.* $\theta \iff \exists$ *a function $K(\rho)$ s.t.*
$\lim_{\rho\to 0} K(\rho) = +\infty$ and $F(\rho,\theta) \geq K(\rho)$

Corollary 3. *Let's suppose that $\lim_{\rho\to 0} F(\rho,\theta) = -\infty$.*
$\forall \rho \in (0, r]$ let $g(\rho) = sup\{F(\rho,\theta) : \theta \in [0, 2\pi]\}$ so then $\lim_{\rho\to 0} F(\rho,\theta) = -\infty$ U.W.R.T. $\theta \iff \lim_{\rho\to 0} g(\rho) = -\infty$

Definition 5. *Let $f : A \subseteq \mathbb{R}^2 \to \mathbb{R}$ with A open.*
let $(x_0, y_0) \in A$, $\varphi(x) = f(x, y_0)$ and $\psi = f(x_0, y)$. A is open that means that those two functions are well defined.

Differentiability

Definition 6. *Let be $f : A \subseteq \mathbb{R}^n \to \mathbb{R}$ with A Open. Let $\bar{x} \in A$ and let $i \leq n$, we denote as $\varphi_i(x_i) = f(\bar{x}_1, \bar{x}_2, ..., \bar{x}_{i-1}, \bar{x}_i, \bar{x}_{i+1}, ..., \bar{x}_n)$. Notice that $\bar{x}$ is an internal point so then it exist an interval where φ_i is well defined.*

Definition 7. *We say that f is partially derivable with respect to the variable x_i in the point $\bar{x}$ if φ_i is derivable in that point. We denote as $\frac{\partial f}{\partial x_i}$ the partial derivative with respsect to x_i in the point $\bar{x}$.*

Definition 8. *The gradient of a function in n variables is defined as follows:*

$$\nabla f : \bar{x} \in A \longmapsto \left(\frac{\partial f}{\partial x_1}, ..., \frac{\partial f}{\partial x_n}\right) \in \mathbb{R}^n$$

Let $f : A \subseteq \mathbb{R}^2 \to \mathbb{R}$ with A open, and let $(x_0, y_0) \in A$.
$(x_0, y_0, f(x_0, y_0)) \in \mathcal{G}(f)$. The equation of the plane that passes for
$(x_0, y_0, f(x_0, y_0))$ is $Z = f(x_0, y_0) + a(X - x_0) + b(Y - y_0)$ where $a, b \in \mathbb{R}$.

Definition 9. *We say that f is partially derivable with respect to x in (x_0, y_0) if φ is differentiable in x_0. in that case we φ is the partial derivative of f in the variable x and its written $\frac{\partial f}{\partial x}$*

Definition 10. *We define the gradient as $\nabla f : (x, y) \in A \longmapsto \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) \in \mathbb{R}^2$*

Definition 11. *We say that f is differentiable in the point (x_0, y_0) if exists $a, b \in \mathbb{R}$ such that:*

$$\lim_{(x,y)\to(x_0,y_0)} \frac{f(x,y) - f(x_0,y_0) - a(X - x_0) - b(Y - y_0)}{\|(x,y) - (x_0,y_0)\|} = 0\ (\triangle)$$

f is differentiable in the point (x_0, y_0) if exists a plane that passes in the point $(x_0, y_0, f(x_0, y_0))$ that approximates the graph of the function f.

Proposition 2. *If f is differentiable in the point (x_0, y_0), f is partially derivable with respsect to x and y such that $a = \frac{\partial f(x_0,y_0)}{\partial x}$ and $b = \frac{\partial f(x_0,y_0)}{\partial y}$*

Definition 12. *if f is differentiable in a point $(x, y) \in A$, the differential in the point is defined as follows:*

$$d_{(x,y)} f : (h, k) \in \mathbb{R}^2 \longmapsto \frac{\partial f(x,y)}{\partial x} h + \frac{\partial f(x,y)}{\partial y} k \in \mathbb{R}$$

Definition 13. *More in general if $f : A \subseteq \mathbb{R}^n \to \mathbb{R}$ and $\boldsymbol{x} \in A$:*

$$d^r_{\boldsymbol{x}} f : h \in \mathbb{R}^n \longmapsto \sum_{\substack{i_1,\dots,i_n \geq 0 \\ i_1+\dots+i_n = r}} \frac{r!}{i_1! \dots i_n!} \frac{\partial^r f}{\partial x^{i_1} \dots \partial x^{i_n}}(\boldsymbol{x}) h_1^{i_1}, \dots, h_n^{i_n} \in \mathbb{R}$$

Corollary 4. *f is differentiable in the point $(x_0, y_0) \iff f$ is partially derivable in the point (x_0, y_0) and the $(\triangle)$ is true.*

Theorem 4. *Let $f : A \subseteq \mathbb{R}^2 \to \mathbb{R}$ with A. If $\exists$ $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ in A and are continuos in a point (x_0, y_0), then the function is differentiable in (x_0, y_0).*

Proof. We have to prove that:

$$\lim_{(x,y)\to(x_0,y_0)} \frac{f(x,y) - f(x_0,y_0) - \frac{\partial f(x_0,y_0)}{\partial x}(x-x_0) - \frac{\partial f(x_0,y_0)}{\partial y}(y-y_0)}{\sqrt{(x-x_0)^2 + (y-y_0)^2}} = 0$$

Lets add and subtract $f(x, y_0)$, so one has:

$$f(x,y) - f(x_0,y_0) = f(x,y) - f(x,y_0) + f(x,y_0) - f(x_0,y_0)$$

We call $\varphi(t) = f(x,t)$ where $t \in I[y,y_0]$ and $I[y,y_0] = \begin{cases} [y,y_0] \ y \le y_0 \\ [y_0,y] \ y_0 \le y \end{cases}$
φ is derivable and for the Lagrange theorem $\exists y_1 \in I[y,y_0] : \varphi(y) - \varphi(y_0) = \dot{\varphi}(y_1)(y-y_0)$. So one has $f(x,y) - f(x,y_0) = \frac{\partial f(x,y_1)}{\partial y}(y-y_0)$, and we repeat the same reasoning for the other variable and one will have $f(x,y_0) - f(x_0,y_0) = \frac{\partial f(x_1,y_0)}{\partial x}(x-x_0)$
We have then:

$$\left| \frac{f(x,y) - f(x_0,y_0) - \frac{\partial f(x_0,y_0)}{\partial x}(x-x_0) - \frac{\partial f(x_0,y_0)}{\partial y}(y-y_0)}{\sqrt{(x-x_0)^2 + (y-y_0)^2}} \right| =$$

$$= \left| \frac{\frac{\partial f(x_1,y_0)}{\partial x}(x-x_0) - \frac{\partial f(x,y_1)}{\partial y}(y-y_0) - \frac{\partial f(x_0,y_0)}{\partial x}(x-x_0) - \frac{\partial f(x_0,y_0)}{\partial y}(y-y_0)}{\sqrt{(x-x_0)^2 + (y-y_0)^2}} \right| =$$

$$= \left| \frac{\left(\frac{\partial f(x_1,y_0)}{\partial x} - \frac{\partial f(x_0,y_0)}{\partial x}\right)(x-x_0) + \left(\frac{\partial f(x,y_1)}{\partial x} - \frac{\partial f(x_0,y_0)}{\partial x}\right)(y-y_0)}{\sqrt{(x-x_0)^2 + (y-y_0)^2}} \right|$$

The last member is increased by the following:

$$\left| \frac{\partial f(x_1,y_0)}{\partial x} - \frac{\partial f(x_0,y_0)}{\partial x} \right| \frac{|x-x_0|}{\|(x-x_0, y-y_0)\|} + \left| \frac{\partial f(x,y_1)}{\partial y} - \frac{\partial f(x_0,y_0)}{\partial y} \right| \frac{|y-y_0|}{\|(x-x_0, y-y_0)\|} \le$$
$$\le \left| \frac{\partial f(x_1,y_0)}{\partial x} - \frac{\partial f(x_0,y_0)}{\partial x} \right| + \left| \frac{\partial f(x,y_1)}{\partial y} - \frac{\partial f(x_0,y_0)}{\partial y} \right|$$

And since $x \to x_0 \Rightarrow x_1 \to x_0$ and $y \to y_1 \Rightarrow y_1 \to y_0$ so the second member of the inequality is equal to zero. □

Theorem 5. *Let $f : A \subseteq \mathbb{R}^n \to \mathbb{R}$ with A open. If the function is differentiable in a point $\boldsymbol{x} \in A$ then is continuos in that point.*

Proof. Since we have:

$$\lim_{x\to\mathbf{x}} \frac{f(x) - f(\mathbf{x}) - \sum_{i=1}^n \frac{\partial f(\mathbf{x})}{\partial x_i}(x_i - \mathbf{x}_i)}{\|x - \mathbf{x}\|} = 0$$

If we fix an $\varepsilon = 1$ there exists $\delta > 0 : \forall x \in A$ with $0 < \|x - \mathbf{x}\| < \delta$ one has:

$$\left| \frac{f(x) - f(\mathbf{x}) - \nabla f(\mathbf{x})(x-\mathbf{x})}{\|x-\mathbf{x}\|} \right| < \varepsilon \Longrightarrow |f(x) - f(\mathbf{x}) - \nabla f(\mathbf{x})(x-\mathbf{x})| < \|x - \mathbf{x}\|$$

□

Theorem (Schwartz). [1] *Let $f : \Omega \subseteq \mathbb{R}^2 \to \mathbb{R}$ be a function in two variables defined on a open set Ω. If f admits continous second derivatives in the point ($f \in C^2(\Omega)$) then $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$.*

Proof. Let $p = (x_0, y_0) \in \Omega$ and chose two real numbers $\varepsilon, \delta > 0$ such that $(x_0 - \varepsilon, x_0 + \delta) \times (y_0 - \delta, y_0 + \delta) \subset \Omega$. That is possible since Ω is Open. Lets also define the two functions F and G as follows:

$$F : (-\varepsilon, \varepsilon) \subset \mathbb{R} \to \mathbb{R}$$
$$G : (-\delta, \delta) \subset \mathbb{R} \to \mathbb{R}$$

In the way that:

$$F(t) = f(x_0 + t, y_0 + s) - f(x_0 + t, y_0) \quad \forall s \in (-\delta, \delta)$$
$$G(s) = f(x_0 + t, y_0 + s) - f(x_0, y_0 + s) \quad \forall t \in (-\varepsilon, \varepsilon)$$

It can be easily proved that: $F(t) - F(0) = G(s) - G(0)$ also if we apply the Lagrange theorem two times one has: $F(t) - F(0) = t\dot{F}(\xi_1)$ with $t\dot{F}(\xi_1)$ equal to: $t\left[\frac{\partial f}{\partial x}(x_0 + \xi_1, y_0 + s) - \frac{\partial f}{\partial x}(x_0 + \xi_1, y_0)\right] = ts\frac{\partial^2 f}{\partial y \partial x}(x_0 + \xi_1, y_0 + \sigma_1)$. The same reasoning can be applied to $G(s) - G(0)$ obtaining: $st\frac{\partial^2 f}{\partial x \partial y}(x_0 + \xi_2, y_0 + \sigma_2)$ with $\xi_i \in (0, t)$ and $\sigma_i \in (0, s)$ where without loss of generality we can say $t, s > 0$. Thinking about $t \to 0$ and $s \to 0 \Rightarrow \xi_i \to 0$ and $\sigma_i \to 0$ with the continuity of the two derivatives one has: $\frac{\partial^2 f}{\partial y \partial x}(x_0, y_0) = \frac{\partial^2 f}{\partial x \partial y}(x_0, y_0)$. ☐

Directional Derivatives

If we take $f : A \subseteq \mathbb{R}^2 \to \mathbb{R}$ and it's partial derivatives, we can take for example $\frac{\partial f}{\partial x}$ as the direction of the function calculated on the line $y = y_0$. So let a function be defined like the one before and let $(\lambda, \mu) \in \mathbb{R}^2$ with $\sqrt{\lambda^2 + \mu^2} = 1$. Let r the line with the following equations:

$$\begin{cases} x = x_0 + \lambda t \\ y = y_0 + \mu t \end{cases}$$

(x_0, y_0) is internal to A so there exists a rectangle R_0 of center (x_0, y_0), so every line that passes in this point encounters a segment of the rectangle.

Study of the maxima and minima

Definition 14. *If a partial derivative $\frac{\partial f}{\partial x}$ of a function $f : A \subseteq \mathbb{R}^2 \to \mathbb{R}$ is partially derivable with respect to x in a point $(x_0, y_0) \in A$ we say that f is partially derivable two times with respect to x in the point (x_0, y_0) ad it will be denoted as $\frac{\partial f}{\partial x} f_x(x_0, y_0) = \frac{\partial^2 f(x_0, y_0)}{\partial x^2} = \frac{\partial^2 f}{\partial x^2}(x_0, y_0)$.*
The same goes for the other partial derivatives: $\frac{\partial}{\partial y} f_x = \frac{\partial^2 f}{\partial x \partial y}$, $\frac{\partial}{\partial x} f_y = \frac{\partial^2 f}{\partial y \partial x}, \ldots$

Definition 15. *We define the hessian matrix as follows:*

$$\mathcal{D}^2 f = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix}$$

Definition 16. *The determinant of $\mathcal{D}^2 f$ is:*

$$\mathcal{H}(x, y) = \begin{vmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{vmatrix}$$

and is called the Hessian determinant.

Definition 17. *Let $f : A \subseteq \mathbb{R}^2 \to \mathbb{R}$, we say that $(x_0, y_0) \in A$ is maxima (minima) for f if $\forall (x, y) \in A$, $f(x, y) \leqslant f(x_0, y_0)$ ($f(x, y) \geqslant f(x_0, y_0)$).*

Theorem 6. *If f is continous and A is compact, f admits minima and maxima.*

Theorem 7. *Let $f : A \subseteq \mathbb{R}^2 \to \mathbb{R}$ and let $(x_0, y_0) \in \dot{A}$ a relative extreme and let f be partially derivable in (x_0, y_0), so then $\frac{\partial f(x_0, y_0)}{\partial x} = 0$ and $\frac{\partial f(x_0, y_0)}{\partial y} = 0$.*
The points where the partial derivatives are 0 are said "critical points" of f, $(x_0, y_0) \in \dot{A}$ is an extreme ralative $\Rightarrow$ (x_0, y_0) is a critical point for f ($\not\Leftarrow$).

Obs 2. *Let $(x_0, y_0) \in A$ and let $g(x, y) = f(x, y) - f(x_0, y_0)$, (x_0, y_0) is a relative minimum (relative maximum) for $f \iff \exists$ a circle C of center (x_0, y_0) such that $g \geq 0$ ($g \leq 0$).*

[1] Ω this time is used instead of A

Theorem 8. *Let* $f : A \subseteq \mathbb{R}^2 \to \mathbb{R}$ *and let* $(x_0, y_0) \in \dot{A}$ *a relative extreme* $\Longrightarrow \mathcal{H}(x_0, y_0) \geqslant 0$.

Theorem 9. *Let* $f : A \subseteq \mathbb{R}^2 \to \mathbb{R}$, $f \in \mathbf{C}^2$. *Let* $(x_0, y_0) \in \dot{A}$ *a critical point and lets suppose that* $\mathcal{H}(x_0, y_0) > 0 \Longrightarrow (x_0, y_0)$ *is a relative extreme and is maximum or minimum depending on* $\frac{\partial^2 f(x_0,y_0)}{\partial x^2}$ *be* < 0 *or* > 0.