Calculus II

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Theorem 1. *A is closed $\iff$ every accumulation point for A is in A*

Proof. " $\implies$ " Let $A \subseteq \mathbb{R}^n,\ A = A \cup \partial A$.
Then $\forall p \in \bar{\mathcal{D}}(A),\ C_r(p)_{\setminus p} \cap A \neq \emptyset\ \forall C \in \mathcal{C}_p$.
if $p \notin A$ then $C_r(p)$ has elements that dont belong to $A \Rightarrow p \in \partial A$.
" $\impliedby$ " Let $p \in \partial A \Rightarrow \forall C \in \mathcal{C}_p$ of center r with $r \in \mathbb{R}$ by definition we can find some $x \in C_{\setminus p} \cap A$, so that means $p \in \bar{\mathcal{D}}(A) \Rightarrow p \in A$. □

1 Limits

Definition 1. *Let $A \subseteq \mathbb{R}^2$ and (x_0, y_0) an accumulation point for A. we define A^* as follows:*
$A^* = \{(\rho, \theta) \in [0, +\infty] \times [0, 2\pi] : (x_0 + \rho\cos(\theta), y_0 + \rho\sin(\theta)) \in A\}$.

Proposition 1. *Lets suppose that exist a circle C of center (x_0, y_0) such that $C_{\setminus \{(x_0,y_0)\}} \subseteq A$ let r be the radius of the circle and as a consequence $(0, r] \times [0, 2\pi] \subseteq A^*$*

Proof. Let $C_{\setminus \{(x_0,y_0)\}}$ and $\begin{cases} 0 < \rho \leqslant r \\ 0 \leqslant \theta \leqslant 2\pi \end{cases}$ if $(\rho, \theta) \in (0, r] \times [0, 2\pi]$
then $(x_0 + \rho\cos(\theta), y_0 + \rho\sin(\theta)) \in C_{\setminus \{(x_0,y_0)\}} \subseteq A \Rightarrow (\rho, \theta) \in A^*$. □

Definition 2. *Let $\theta \in [0, 2\pi]$ and $\forall \rho \in (0, r]$ we define $\varphi_\theta(\rho) = F(\rho, \theta)$ if $\rho \in (0, r], (\rho, \theta) \in A^*$ so the $\lim_{\rho \to 0} \varphi(\rho) = l \in \bar{\mathbb{R}}$.*
If that limit exists that means $\forall \theta \in [0, 2\pi]$ and $\forall \varepsilon > 0,\ \exists \delta > 0\ \forall \rho \in (0, r]$ with $\rho < \delta\ \ |\varphi_\theta - l| < \varepsilon$.
We say that $\lim_{\rho \to 0} \varphi(\rho) = l \in \bar{\mathbb{R}}$ Uniformly With Respect To (U.W.R.T.) θ.

Theorem 2. *Let $f : A \subseteq \mathbb{R}^2 \to \mathbb{R}$ with (x_0, y_0) accumulation point for A.*
Follows the equivalence:
$\lim_{(x,y) \to (x_0,y_0)} f(x, y) = l \in \bar{\mathbb{R}} \iff \lim_{\rho \to 0} F(\rho, \theta) = l$ *U.W.R.T.* θ.

Proof. Let $l \in \bar{\mathbb{R}}$.
" $\implies$ " $\lim_{(x,y) \to (x_0,y_0)} f(x, y) = l$ so $\forall \varepsilon > 0, \exists \delta > 0 : \forall\ (x, y) \in A$
with $\|(x, y) - (x_0, y_0)\| < \delta,\ |f(x, y) - l| < \varepsilon$.

We have to prove that $\forall \varepsilon > 0,\ \exists \delta > 0 : \forall \theta \in [0, 2\pi],\ \forall \rho(0, r]$
with $\rho < \delta\ |F(\rho, \theta) - l| < \varepsilon$.
Let $\varepsilon > 0,\ \theta \in [0, 2\pi],\ \rho \in (0, r]$ with $\rho < \delta$. we create the system that changes the coordinates from cartesians to polars:

$$\begin{cases} x = x_0 + \rho\cos(\theta) \\ y = y_0 + \rho\sin(\theta) \end{cases} \quad \rho = \sqrt{(x - x_0)^2 + (y - y_0)^2}$$

$\rho \in (0, r],\ \theta \in [0, 2\pi] \in (0, r] \times [0, 2\pi] \subseteq A^*,\ (\rho, \theta) \in A^* \Rightarrow (x, y) \in A$.

Now $0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} = \rho < \delta \Rightarrow |f(x, y) - l| < \varepsilon$.
$\Rightarrow |f(x_0 + \rho\cos(\theta), y_0 + \rho\sin(\theta)) - l| < \varepsilon \Rightarrow |F(\rho, \theta) - l| < \varepsilon$.

" $\impliedby$ " $\forall \varepsilon > 0, \exists \delta \leq r : \forall \theta \in [0, 2\pi]$ and $\forall \rho$ with $0 < \rho < \delta \Rightarrow$
$|F(\rho, \theta) - l| < \varepsilon$.
We have to prove that $\forall \varepsilon > 0, \exists \delta > 0, \forall (x, y) \in A$ with
$\sqrt{(x - x_0)^2 + (y - y_0)^2} = \|(x, y) - (x_0, y_0)\| < \delta \Rightarrow |f(x, y) - l| < \varepsilon$.

Let $\varepsilon > 0,\ \delta \leq r,\ (x, y) \in A,\ \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta$, we switch coordinates with ρ and θ as follows:

$$\begin{cases} x = x_0 + \rho\cos(\theta) \\ y = y_0 + \rho\sin(\theta) \end{cases} \quad \rho = \sqrt{(x - x_0)^2 + (y - y_0)^2}$$

$0 < \rho < \delta \leq r \Rightarrow \rho \in (0, r),\ \theta \in [0, 2\pi]$.
We notice that $|F(\rho, \theta) - l| < \varepsilon$, so $|f(x_0 + \rho\cos(\theta), y_0 + \rho\sin(\theta)) - l| < \varepsilon$
$\Rightarrow |f(x, y) - l| < \varepsilon$. □

Definition 3. *We say that* $\theta \in [0, 2\pi]$ *is admissible if* $0 \in \bar{\mathcal{D}}(A_\theta)$.

Definition 4. *Let's suppose that* $\lim_{\rho\to 0} F(\rho,\theta) = l \in \mathbb{R}$ *then* $\forall \rho \in (0, r],\ \varphi(\rho) = \sup\{|F(\rho,\theta) - l| : \theta \in [0, 2\pi]\}$

Theorem 3. $\lim_{\rho\to 0} F(\rho,\theta) = l \in \bar{\mathbb{R}}$ *U.W.R.T.* $\theta \iff \lim_{\rho\to 0} \varphi(\rho) = 0$.

Proof. ($\Rightarrow$) $\forall \varepsilon > 0\ \exists \delta > 0 : \forall \theta \in [0, 2\pi]$ and $\forall \rho \in (0, r]$ with $\rho < \delta\ |F(\rho,\theta) - l| < \frac{\varepsilon}{2}$ so $|\varphi(\rho)| \leq \frac{\varepsilon}{2} < \varepsilon \Rightarrow \lim_{\rho\to 0} \varphi(\rho) = 0$. ($\Leftarrow$) $\forall \varepsilon > 0\ \exists \delta > 0 : \forall \rho \in (0, r)$ with $\rho < \delta\ \varphi(\rho) < \varepsilon$ but $|F(\rho,\theta) - l| \leq \varphi(\rho) \forall \theta$ so if $\rho \in (0, r]$ and $\rho < \delta\ |F(\rho,\theta) - l| < \varepsilon \Rightarrow$
$\Rightarrow \lim_{\rho\to 0} F(\rho,\theta) = l$ *U.W.R.T.* θ □

Corollary 1. $\lim_{\rho\to 0} F(\rho,\theta) = l \in \bar{\mathbb{R}}$ *U.W.R.T.* $\theta \iff \exists$ *a function* $\psi(\rho)$ *such that* $\lim_{\rho\to 0} \psi(\rho) = 0$ *and* $\forall \theta\ |F(\rho,\theta) - l| \leqslant \psi(\rho)$.

Corollary 2. *Let's suppose that* $\lim_{\rho\to 0} F(\rho,\theta) = +\infty$.
$\forall \rho \in (0, r]$ *let* $h(\rho) = inf\{F(\rho,\theta) : \theta \in [0, 2\pi]\}$ *so then* $\lim_{\rho\to 0} F(\rho,\theta) = +\infty$ *U.W.R.T.* $\theta \iff \lim_{\rho\to 0} h(\rho) = +\infty$

Obs 1. $\lim_{\rho\to 0} F(\rho,\theta) = +\infty$ *U.W.R.T.* $\theta \iff \exists$ *a function* $K(\rho)$ *s.t.*
$\lim_{\rho\to 0} K(\rho) = +\infty$ *and* $F(\rho,\theta) \geq K(\rho)$

Corollary 3. *Let's suppose that* $\lim_{\rho\to 0} F(\rho,\theta) = -\infty$.
$\forall \rho \in (0, r]$ *let* $g(\rho) = sup\{F(\rho,\theta) : \theta \in [0, 2\pi]\}$ *so then* $\lim_{\rho\to 0} F(\rho,\theta) = -\infty$ *U.W.R.T.* $\theta \iff \lim_{\rho\to 0} g(\rho) = -\infty$

Definition 5. *Let* $f : A \subseteq \mathbb{R}^2 \to \mathbb{R}$ *with* A *open.*
let $(x_0, y_0) \in A$, $\varphi(x) = f(x, y_0)$ *and* $\psi = f(x_0, y)$. A *is open that means that those two functions are well defined.*

Differentiability

Definition 6. *Let be* $f : A \subseteq \mathbb{R}^n \to \mathbb{R}$ *with* A *Open. Let* $\bar{x} \in A$ *and let* $i \leq n$, *we denote as* $\varphi_i(x_i) = f(\bar{x}_1, \bar{x}_2, ..., \bar{x}_{i-1}, \bar{x}_i, \bar{x}_{i+1}, ..., \bar{x}_n)$. *Notice that* $\bar{x}$ *is an internal point so then it exist an interval where* φ_i *is well defined.*

Definition 7. *We say that* f *is partially derivable with respect to the variable* x_i *in the point* $\bar{x}$ *if* φ_i *is derivable in that point. We denote as* $\frac{\partial f}{\partial x_i}$ *the partial derivative with respsect to* x_i *in the point* $\bar{x}$.

Definition 8. *The gradient of a function in* n *variables is defined as follows:*

$$\nabla f : \bar{x} \in A \longmapsto \left(\frac{\partial f}{\partial x_1}, ..., \frac{\partial f}{\partial x_n}\right) \in \mathbb{R}^n$$

Let $f : A \subseteq \mathbb{R}^2 \to \mathbb{R}$ with A open, and let $(x_0, y_0) \in A$.
$(x_0, y_0, f(x_0, y_0)) \in \mathcal{G}(f)$. The equation of the plane that passes for $(x_0, y_0, f(x_0, y_0))$ is $Z = f(x_0, y_0) + a(X - x_0) + b(Y - y_0)$ where $a, b \in \mathbb{R}$.

Definition 9. *We say that* f *is partially derivable with respect to* x *in* (x_0, y_0) *if* φ *is differentiable in* x_0. *in that case we* φ *is the partial derivative of* f *in the variable* x *and its written* $\frac{\partial f}{\partial x}$

Definition 10. *We define the gradient as* $\nabla f : (x, y) \in A \longmapsto \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) \in \mathbb{R}^2$

Definition 11. *We say that* f *is differentiable in the point* (x_0, y_0) *if exists* $a, b \in \mathbb{R}$ *such that:*

$$\lim_{(x,y)\to(x_0,y_0)} \frac{f(x, y) - f(x_0, y_0) - a(X - x_0) - b(Y - y_0)}{\|(x, y) - (x_0, y_0)\|} = 0\ (\triangle)$$

f is differentiable in the point (x_0, y_0) *if exists a plane that passes in the point* $(x_0, y_0, f(x_0, y_0))$ *that approximates the graph of the function* f.

Proposition 2. *If* f *is differentiable in the point* (x_0, y_0), f *is partially derivable with respsect to* x *and* y *such that* $a = \frac{\partial f(x_0,y_0)}{\partial x}$ *and* $b = \frac{\partial f(x_0,y_0)}{\partial y}$

Definition 12. *if* f *is differentiable in a point* $(x, y) \in A$, *the differential in the point is defined as follows:*

$$d_{(x,y)} f : (h, k) \in \mathbb{R}^2 \longmapsto \frac{\partial f(x, y)}{\partial x} h + \frac{\partial f(x, y)}{\partial y} k \in \mathbb{R}$$

Definition 13. *More in general if* $f : A \subseteq \mathbb{R}^n \to \mathbb{R}$ *and* $\boldsymbol{x} \in A$:

$$d^r_{\boldsymbol{x}} f : h \in \mathbb{R}^n \longmapsto \sum_{\substack{i_1,\ldots,i_n \geqslant 0 \\ i_1+\cdots+i_n = r}} \frac{r!}{i_1! \ldots i_n!} \frac{\partial^r f}{\partial x^{i_1} \ldots \partial x^{i_n}}(\boldsymbol{x}) h_1^{i_1}, \ldots, h_n^{i_n} \in \mathbb{R}$$

Corollary 4. f *is differentiable in the point* $(x_0, y_0) \iff f$ *is partially derivable in the point* (x_0, y_0) *and the* $(\triangle)$ *is true.*

Definition 14. *Let $f : A \subseteq \mathbb{R}^2 \to \mathbb{R}$ with A open and $k > 0$ a positive integer. let $(x_0, y_0) \in A$ and if f has differentiable derivatives of order $k-1$ we define the "k-grade Taylor polinomia" as follows:*

$$P_k(x,y) = f(x_0,y_0) + \sum_{i=1}^{k} \frac{1}{i!} d^i_{(x_0,y_0)} f(x - x_0, y - y_0)$$

Theorem 4. *Let $f : A \subseteq \mathbb{R}^2 \to \mathbb{R}$ with A. If $\exists$ $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ in A and are continuos in a point (x_0, y_0), then the function is differentiable in (x_0, y_0).*

Proof. We have to prove that:

$$\lim_{(x,y)\to(x_0,y_0)} \frac{f(x,y) - f(x_0,y_0) - \frac{\partial f(x_0,y_0)}{\partial x}(x-x_0) - \frac{\partial f(x_0,y_0)}{\partial y}(y-y_0)}{\sqrt{(x-x_0)^2 + (y-y_0)^2}} = 0$$

Lets add and subtract $f(x, y_0)$, so one has:

$$f(x,y) - f(x_0,y_0) = f(x,y) - f(x,y_0) + f(x,y_0) - f(x_0,y_0)$$

We call $\varphi(t) = f(x,t)$ where $t \in I[y, y_0]$ and $I[y,y_0] = \begin{cases} [y, y_0] & y \le y_0 \\ [y_0, y] & y_0 \le y \end{cases}$

φ is derivable and for the Lagrange theorem $\exists y_1 \in I[y, y_0] : \varphi(y) - \varphi(y_0) = \dot{\varphi}(y_1)(y - y_0)$. So one has $f(x,y) - f(x,y_0) = \frac{\partial f(x,y_1)}{\partial y}(y - y_0)$, and we repeat the same reasoning for the other variable and one will have $f(x,y_0) - f(x_0,y_0) = \frac{\partial f(x_1,y_0)}{\partial x}(x - x_0)$ We have then:

$$\left| \frac{f(x,y) - f(x_0,y_0) - \frac{\partial f(x_0,y_0)}{\partial x}(x-x_0) - \frac{\partial f(x_0,y_0)}{\partial y}(y-y_0)}{\sqrt{(x-x_0)^2 + (y-y_0)^2}} \right| =$$

$$= \left| \frac{\frac{\partial f(x_1,y_0)}{\partial x}(x-x_0) - \frac{\partial f(x,y_1)}{\partial y}(y-y_0) - \frac{\partial f(x_0,y_0)}{\partial x}(x-x_0) - \frac{\partial f(x_0,y_0)}{\partial y}(y-y_0)}{\sqrt{(x-x_0)^2 + (y-y_0)^2}} \right| =$$

$$= \left| \frac{\left(\frac{\partial f(x_1,y_0)}{\partial x} - \frac{\partial f(x_0,y_0)}{\partial x} \right)(x-x_0) + \left(\frac{\partial f(x,y_1)}{\partial x} - \frac{\partial f(x_0,y_0)}{\partial x} \right)(y-y_0)}{\sqrt{(x-x_0)^2 + (y-y_0)^2}} \right|$$

The last member is increased by the following:

$$\left| \frac{\partial f(x_1,y_0)}{\partial x} - \frac{\partial f(x_0,y_0)}{\partial x} \right| \frac{|x-x_0|}{\|(x-x_0, y-y_0)\|} + \left| \frac{\partial f(x,y_1)}{\partial y} - \frac{\partial f(x_0,y_0)}{\partial y} \right| \frac{|y-y_0|}{\|(x-x_0, y-y_0)\|} \le$$
$$\le \left| \frac{\partial f(x_1,y_0)}{\partial x} - \frac{\partial f(x_0,y_0)}{\partial x} \right| + \left| \frac{\partial f(x,y_1)}{\partial y} - \frac{\partial f(x_0,y_0)}{\partial y} \right|$$

And since $x \to x_0 \Rightarrow x_1 \to x_0$ and $y \to y_1 \Rightarrow y_1 \to y_0$ so the second member of the inequality is equal to zero. ☐

Theorem 5. *Let $f : A \subseteq \mathbb{R}^n \to \mathbb{R}$ with A open. If the function is differentiable in a point $\mathbf{x} \in A$ then is continuos in that point.*

Proof. Since we have:

$$\lim_{x\to\mathbf{x}} \frac{f(x) - f(\mathbf{x}) - \sum_{i=1}^{n} \frac{\partial f(\mathbf{x})}{\partial x_i}(x_i - \mathbf{x}_i)}{\|x - \mathbf{x}\|} = 0$$

If we fix an $\varepsilon = 1$ there exists $\delta > 0 : \forall x \in A$ with $0 < \|x - \mathbf{x}\| < \delta$ one has:

$$\left| \frac{f(x) - f(\mathbf{x}) - \nabla f(\mathbf{x})(x - \mathbf{x})}{\|x - \mathbf{x}\|} \right| < \varepsilon \Longrightarrow |f(x) - f(\mathbf{x}) - \nabla f(\mathbf{x})(x - \mathbf{x})| < \|x - \mathbf{x}\|$$

So we have the the following:

$$|f(x) - f(\mathbf{x})| - |\nabla f(\mathbf{x})(x - \mathbf{x})| \le |f(x) - f(\mathbf{x}) - \nabla f(\mathbf{x})(x - \mathbf{x})| < \|x - \mathbf{x}\| \ \triangle$$

The $\triangle$ implies that $|f(x) - f(\mathbf{x})| - |\nabla f(\mathbf{x})(x - \mathbf{x})| \le |\nabla f(\mathbf{x})(x - \mathbf{x})| + \|x - \mathbf{x}\| \le \|\nabla f(\mathbf{x})\| \|x - \mathbf{x}\| + \|x - \mathbf{x}\|$. The last member of the inequality is equal to $\|x - \mathbf{x}\| (\nabla f(\mathbf{x}) + 1) \vartriangle$ so, if $0 < \|x - \mathbf{x}\| < \delta$ then by calling the $\vartriangle = c$ one finally has:

$$0 < |f(x) - f(\mathbf{x})| \le c\,\|x - \mathbf{x}\| \to 0 \Leftarrow x \to \mathbf{x} \Rightarrow |f(x) - f(\mathbf{x})| \to 0$$

Or equivalently: $\lim_{x\to\mathbf{x}} f(x) = f(\mathbf{x})$. ☐

Theorem (Schwartz). [1] *Let $f : \Omega \subseteq \mathbb{R}^2 \to \mathbb{R}$ be a function in two variables defined on a open set Ω. If f admits continuous second derivatives in the point ($f \in C^2(\Omega)$) then $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$.*

Proof. Let $p = (x_0, y_0) \in \Omega$ and chose two real numbers $\varepsilon, \delta > 0$ such that $(x_0 - \varepsilon, x_0 + \delta) \times (y_0 - \delta, y_0 + \delta) \subset \Omega$. That is possible since Ω is Open. Lets also define the two functions F and G as follows:

$$F : (-\varepsilon, \varepsilon) \subset \mathbb{R} \to \mathbb{R}$$
$$G : (-\delta, \delta) \subset \mathbb{R} \to \mathbb{R}$$

In the way that:

$$F(t) = f(x_0 + t, y_0 + s) - f(x_0 + t, y_0) \quad \forall s \in (-\delta, \delta)$$
$$G(s) = f(x_0 + t, y_0 + s) - f(x_0, y_0 + s) \quad \forall t \in (-\varepsilon, \varepsilon)$$

It can be easily proved that: $F(t) - F(0) = G(s) - G(0)$ also if we apply the Lagrange theorem two times one has: $F(t) - F(0) = t\dot{F}(\xi_1)$ with $t\dot{F}(\xi_1)$ equal to: $t\left[\frac{\partial f}{\partial x}(x_0 + \xi_1, y_0 + s) - \frac{\partial f}{\partial x}(x_0 + \xi_1, y_0)\right] = ts\frac{\partial^2 f}{\partial y \partial x}(x_0 + \xi_1, y_0 + \sigma_1)$. The same reasoning can be applied to $G(s) - G(0)$ obtaining: $st\frac{\partial^2 f}{\partial x \partial y}(x_0 + \xi_2, y_0 + \sigma_2)$ with $\xi_i \in (0, t)$ and $\sigma_i \in (0, s)$ where without loss of generality we can say $t, s > 0$. Thinking about $t \to 0$ and $s \to 0 \Rightarrow \xi_i \to 0$ and $\sigma_i \to 0$ with the continuity of the two derivatives one has: $\frac{\partial^2 f}{\partial y \partial x}(x_0, y_0) = \frac{\partial^2 f}{\partial x \partial y}(x_0, y_0)$. $\square$

Directional Derivatives

If we take $f : A \subseteq \mathbb{R}^2 \to \mathbb{R}$ defined on an open set A, $(x_0, y_0) \in A$ and a vector of unitary norm $\vec{v} = (v_1, v_2)$, the Directional derivative of $f(x_0, y_0)$ along the direction $\bar{v}$ can be defined as the limit if it exists and its finite:

$$\frac{\partial f}{\partial \bar{v}}(x_0, y_0) = \lim_{t \to 0} \frac{f(x_0 + tv_1, y_0 + tv_2) - f(x_0, y_0)}{t}$$

Study of the maxima and minima

Definition 15. *If a partial derivative $\frac{\partial f}{\partial x}$ of a function $f : A \subseteq \mathbb{R}^2 \to \mathbb{R}$ is partially derivable with respect to x in a point $(x_0, y_0) \in A$ we say that f is partially derivable two times with respect to x in the point (x_0, y_0) ad it will be denoted as $\frac{\partial f}{\partial x} f_x(x_0, y_0) = \frac{\partial^2 f(x_0, y_0)}{\partial x^2} = \frac{\partial^2 f}{\partial x^2}(x_0, y_0)$.*
The same goes for the other partial derivatives: $\frac{\partial}{\partial y} f_x = \frac{\partial^2 f}{\partial x \partial y}$, $\frac{\partial}{\partial x} f_y = \frac{\partial^2 f}{\partial y \partial x}, \ldots$

Definition 16. *We define the hessian matrix as follows:*

$$\mathcal{D}^2 f = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix}$$

Definition 17. *The determinant of $\mathcal{D}^2 f$ is:*

$$\mathcal{H}(x, y) = \begin{vmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{vmatrix}$$

and is called the Hessian determinant.

Definition 18. *Let $f : A \subseteq \mathbb{R}^2 \to \mathbb{R}$, we say that $(x_0, y_0) \in A$ is maxima (minima) for f if $\forall (x, y) \in A$, $f(x, y) \leqslant f(x_0, y_0)$ ($f(x, y) \geqslant f(x_0, y_0)$).*

Theorem 6. *If f is continuous and A is compact, f admits minima and maxima.*

Theorem 7. *Let $f : A \subseteq \mathbb{R}^2 \to \mathbb{R}$ and let $(x_0, y_0) \in \dot{A}$ a relative extreme and let f be partially derivable in (x_0, y_0), so then $\frac{\partial f(x_0, y_0)}{\partial x} = 0$ and $\frac{\partial f(x_0, y_0)}{\partial y} = 0$.*
The points where the partial derivatives are 0 are said "critical points" of f, $(x_0, y_0) \in \dot{A}$ is an extreme ralative $\Rightarrow (x_0, y_0)$ is a critical point for f ($\not\Leftarrow$).

Obs 2. *Let $(x_0, y_0) \in A$ and let $g(x, y) = f(x, y) - f(x_0, y_0)$, (x_0, y_0) is a relative minimum (relative maximum) for $f \iff \exists$ a circle C of center (x_0, y_0) such that $g \geq 0$ ($g \leq 0$).*

Theorem 8. *Let $f : A \subseteq \mathbb{R}^2 \to \mathbb{R}$ and let $(x_0, y_0) \in \dot{A}$ a relative extreme $\Longrightarrow \mathcal{H}(x_0, y_0) \geqslant 0$.*

Theorem 9. *Let $f : A \subseteq \mathbb{R}^2 \to \mathbb{R}$, $f \in \mathbf{C}^2$. Let $(x_0, y_0) \in \dot{A}$ a critical point and lets suppose that $\mathcal{H}(x_0, y_0) > 0 \Longrightarrow (x_0, y_0)$ is a relative extreme and is maximum or minimum depending on $\frac{\partial^2 f(x_0, y_0)}{\partial x^2}$ be < 0 or > 0.*

[1] Ω this time is used instead of A

Implicit functions

By defining a function $f : A \subseteq \mathbb{R}^2 \to \mathbb{R}$ the expression: $f(x, y) = 0$ $\lozenge$ means that one can consider the variable x as a parameter and y as unknown and the question is when, $\forall x\ \exists! y$ such that the $\lozenge$ is true.

Definition 19. *The equation defines implicitly y as a function of x if $\forall x, \exists! y : f(x, y) = 0$ in that case the function g is defined as:*

$$g : x \longmapsto y \Rightarrow f(x, y) = 0$$

Proposition 3. *The equation $f(x_0, y_0) = 0$ defines implicitly y as a function of x, the set of all the zeros of f is equal to the graph of the implicit function.*

Proof. (x_0, y_0) is a zero of $f \iff f(x_0, y_0) = 0 \iff y_0 = g(x_0) \iff (x_0, y_0) \in Gr(g)$ $\square$

Theorem (Implicit Functions). *Let $f : A \subseteq \mathbb{R}^2 \to \mathbb{R}$ lets suppose that f and $\frac{\partial f}{\partial y}$ are continuous. Let $(x_0, y_0) \in A$ be a zero of of the function where $\frac{\partial f}{\partial y}(x_0, y_0) \neq 0$.* ***i.*** *Then there exists an open interval I of center x_0 and an open interval J of center $y_0 : I \times J \subseteq A$ and $\forall x \in I,\ \exists! y \in J : f(x, y) = 0$.* ***ii.*** *Also if $g : I \to J$ is the implicit function, g is continuous and $g(x_0) = y_0$.* ***iii.*** *In addition if $\exists \frac{\partial f}{\partial x}$, g is derivable and $\forall x \in I,\ \dot{g}(x) = -\frac{\frac{\partial f}{\partial x}(x, g(x))}{\frac{\partial f}{\partial y}(x, g(x))}$.* ***iv.*** *Furthermore $f \in C^k$, then $g \in C^k$.*

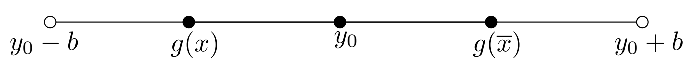
Proof. **i.** By hypotesis $\frac{\partial f}{\partial y} \neq 0$. Lets suppose that $\frac{\partial f}{\partial y}(x_0, y_0) > 0$, so for the sign permanence theorem there exists a rectangle $R_0 = [x_0 - \alpha, x_0 + \alpha] \times [y_0 - \beta, y_0 + \beta] \subseteq A : \forall (x, y) \in R_0,\ \frac{\partial f}{\partial y}(x, y) > 0$. Let $\varphi : y \in [y_0 - \beta, y_0 + \beta] \longmapsto f(x_0, y)$, by hypotesis $\exists \frac{\partial f}{\partial y}$ so, by definition φ is derivable and $\dot{\varphi}(y) = \frac{\partial f}{\partial y}(x_0, y) > 0 \Rightarrow \varphi$ is strictly growing. $\varphi(y_0) = f(x_0, y_0) = 0,\ \varphi(y_0 + b) > \varphi(y_0) = 0 \Rightarrow f(x_0, y_0 - b) < 0$ and $f(x_0, y_0 + b) > 0$. Lets define:

$$\varphi_1 : x \in [x_0 - \alpha, x_0 + \alpha] \longmapsto f(x, y_0 - b)$$
$$\varphi_2 : x \in [x_0 - \alpha, x_0 + \alpha] \longmapsto f(x, y_0 + b)$$

φ_1 and φ_2 are continuous because f is continuous and $\varphi_1(x_0) = f(x_0, y_0 - b) < 0$ and $\varphi_2(x_0) = f(x_0, y_0 + b) > 0$ so there exists an interval $[x_0 - \delta, x_0 + \delta] \subseteq [x_0 - \alpha, x_0 + \alpha] : \forall x \in [x_0 - \delta, x_0 + \delta],\ \varphi_1(x) < 0$ and $\varphi_2(x) > 0$. Now $\forall x \in [x_0 - \delta, x_0 + \delta], f(x, y_0 - b) < 0$ and $f(x, y_0 + b) > 0$ if we take an $x \in (x_0 - \delta, x_0 + \delta)$ and define:

$$\psi : y \in [y_0 - b, y_0 + b] \longmapsto f(x, y)$$

One has that ψ is derivable and $\dot{\psi}(y) = \frac{\partial f}{\partial y}(x, y) > 0$ that implies ψ is strictly growing and also continuous. $\psi(y_0 - b) = f(x, y_0 - b) < 0$ and $\psi(y_0 - b) = f(x, y_0 + b) > 0$ for the zeros theorem, $\exists y \in (y_0 - b, y_0 + b)$ where $\psi(y) = 0 \Rightarrow f(x, y) = 0$. Also y is unique since ψ is strictly growing and that means it can't become zero in two diffrent points. **ii.** Let $g : I \longmapsto J$ the implicit function defined by the equation $f(x, y) = 0$. $\forall x \in I,\ f(x, g(x)) = 0$ and $f(x_0, y_0) \Rightarrow y_0 = g(x_0)$ we have to demostrate that g is continuous, so let $\overline{x} \in I$ and the claim is $\lim_{x \to \overline{x}} g(x) = g(\overline{x})$. $\forall x \in I,\ g(x)$ and $g(\overline{x})$ are two distinct points of $J \Rightarrow K[g(x), g(\overline{x})] \subseteq J$.



$\forall x \in I$, let $\psi : y \in K[g(x), g(\overline{x})] \longmapsto f(x, y)$ $\square$

Bound extremes

Let $f : A \subseteq \mathbb{R}^n \to \mathbb{R}$ with A open. If $g : A \to \mathbb{R}^m$ then we define $E_0 = \{x \in A : g(x) = 0\}$. $g = (g_1, \ldots, g_m)$ so $g(x) = 0$ is equivalent to say:

$$\begin{cases} g_1(x) = 0 \\ g_2(x) = 0 \\ \vdots \\ g_m(x) = 0 \end{cases}$$

Definition 20. *A point $x_0 \in E_0$ is said to be bound maximum (bound minimum) if $\forall x \in E_0$ $f(x) \leq f(x_0)$ $(f(x) \geq f(x_0))$.*

Theorem (Lagrange Multipliers). *Let $f : A \subseteq \mathbb{R}^2 \to \mathbb{R}$ and $g : A \to \mathbb{R}$ with $g \in C^1$ in an open set A.*
Let $E_0 = \{(x, y) \in A : g(x, y) = 0\}$ and $(x_0, y_0) \in E_0$ a bound relative extreme for f and suppose that $\nabla g(x_0, y_0) \neq (0, 0)$ then $\exists! \lambda \in \mathbb{R}$: (x_0, y_0) be a critical point for the function $\mathcal{L}(x, y) = f(x, y) + \lambda g(x, y)$ $\left(\frac{\partial f}{\partial x}(x_0, y_0) + \lambda \frac{\partial g}{\partial x}(x_0, y_0) = 0 \wedge \frac{\partial f}{\partial y}(x_0, y_0) + \lambda \frac{\partial g}{\partial y}(x_0, y_0) = 0\right)$.

Proof. **Existence** Since $\nabla g(x_0, y_0) \neq (0, 0)$ we can assume $\frac{\partial g}{\partial y}(x_0, y_0) \neq 0$. $g(x, y) = 0$ verifies the conditions of the implicit functions theorem ($g \in C^1,\ g(x_0, y_0) = 0,\ \frac{\partial g}{\partial y}(x_0, y_0) \neq 0$), so for that theorem $\exists$ an open interval I of center x_0 and an open interval J of center $y_0 : I \times J \subseteq A$ and $\forall x \in I\ \exists!\ y \in J : g(x, y) = 0$ also the implicit function $\varphi : I \to J$ is C^1. $\varphi(x_0) = y_0$ and $\forall x \in I$ one has:

$$\dot{\varphi}(x) = -\frac{\frac{\partial g}{\partial x}(x, \varphi(x))}{\frac{\partial g}{\partial y}(x, \varphi(x))}$$

Lets suppose for example that (x_0, y_0) is a bound relative minimum, then there exists a rectangle $I' \times J' \subseteq I \times J$ of center (x_0, y_0) such that $\forall (x, y) \in (I' \times J') \cap E_0 \Rightarrow f(x, y) \leq f(x_0, y_0)$. φ is continuous in x_0, J' is an interval of center $y_0 = \varphi(x_0)$ $\exists I'' \subseteq I'$ of center $x_0 : \forall x \in I''$ $\varphi(x) \in J'$. $\forall x \in I''$ $(x, \varphi(x)) \in (I' \times J') \cap E_0$. $\varphi : x \longmapsto$ y such that $g(x_0, y_0) = 0$ so $g(x, \varphi(x)) = 0$ $\forall x \in I$. $\forall x \in I''$ $f(x, \varphi(x)) \leq f(x_0, y_0)$. $\forall x \in I''$ we define $\psi(x) = f(x, \varphi(x))$ such that $\forall x \in I''$ we have $(x, \varphi(x)) \in I' \times J' \subseteq I \times J \subseteq A$ and that means $\forall x$ $\psi(x) \leq f(x_0, y_0) = \psi(x_0) \Rightarrow x_0$ is a bound relative maximum for ψ and is internal to I''. $\psi \in C^1$ since is a composition of C^1 functions so is derivable and $\forall x \in I''$:

$$\dot{\psi}(x) = \frac{\partial f}{\partial x}(x, \varphi(x)) + \frac{\partial f}{\partial y}(x, \varphi(x))\dot{\varphi}(x) = \frac{\partial f}{\partial x}(x, \varphi(x)) - \frac{\partial f}{\partial y}(x, \varphi(x))\frac{\frac{\partial g}{\partial x}(x, \varphi(x))}{\frac{\partial g}{\partial y}(x, \varphi(x))}$$

Since ψ is derivable, and x_0 is a maximum, for fermat theorem one has $\dot{\psi}(x_0) = 0$ with:

$$\dot{\psi}(x_0) = \frac{\partial f}{\partial x}(x_0, y_0) - \frac{\partial f}{\partial y}(x_0, y_0)\frac{\frac{\partial g}{\partial x}(x_0, y_0)}{\frac{\partial g}{\partial y}(x_0, y_0)} = 0 \Longleftrightarrow \frac{\partial f}{\partial x}(x_0, y_0)\frac{\partial g}{\partial y}(x_0, y_0) - \frac{\partial f}{\partial y}(x_0, y_0)\frac{\partial g}{\partial x}(x_0, y_0) = 0$$

Equivalently we can use this expression:

$$\begin{vmatrix} \frac{\partial f}{\partial x}(x_0, y_0) & \frac{\partial f}{\partial y}(x_0, y_0) \\ \frac{\partial g}{\partial x}(x_0, y_0) & \frac{\partial g}{\partial y}(x_0, y_0) \end{vmatrix} = 0$$

And if thats is true that implies $\exists (\lambda_1, \lambda_2) \neq (0, 0)$ such that one of the two coloums is **l.d.** [2] $\Longrightarrow \lambda_1 \frac{\partial f}{\partial x}(x_0, y_0) + \lambda_2 \frac{\partial g}{\partial x}(x_0, y_0) = 0$.
If $\lambda_1 = 0 \Rightarrow \lambda_2 \neq 0 \Rightarrow \frac{\partial g}{\partial x}(x_0, y_0) = 0$ but thats impossible since the hypotesis impose it to be different from zero, so it must be $\lambda_1 \neq 0$ and if we chose $\lambda = \frac{\lambda_2}{\lambda_1}$ we have $\mathcal{L}(x, y) = f(x, y) + \lambda g(x, y) = 0$.
Uniqueness Let $\mathcal{L}(x, y) = f(x, y) + \lambda g(x, y) = 0$ and $\mathcal{L}(x, y) = f(x, y) + \overline{\lambda} g(x, y) = 0$. If we subtract member to member the last equations one has:

$$\left(\lambda - \overline{\lambda}\right) \frac{\partial g}{\partial y}(x_0, y_0) = 0 \Rightarrow \lambda = \overline{\lambda}$$

$\square$

[2] Linearly dependent