

# Topology

Alessio Esposito

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**Definition 1.** let  $X$  be a not empty set and  $\mathcal{T}$  a collection of its subsets.  $\mathcal{T}$  is said topology on  $X$  if it has the following properties:

- $\emptyset, X \in \mathcal{T}$
- the union of a whatever family of elements of  $\mathcal{T}$  is in  $\mathcal{T}$ . In symbols:  
 $\forall \{A_i\}_{i \in I}, A_i \in \mathcal{T}, \bigcup_{i \in I} A_i \in \mathcal{T}$
- the intersection of elements of  $\mathcal{T}$  is in  $\mathcal{T}$ :  $\forall A_1, A_2 \in \mathcal{T}, A_1 \cap A_2 \in \mathcal{T}$

Then  $(X, \mathcal{T})$  its said to be topological space and  $X$  is called support.

**Example 1.** Let  $X = \mathbb{R}^2$  with the euclidean distance and  $\forall c \in \mathbb{R}^2$  and  $\forall r > 0$  the set  $B_r(c) = \{x \in \mathbb{R}^2 : d(x, c) < r\}$  its said open spherical neighbourhood of center  $c$  and radius  $r$ . Let  $\mathcal{T}$  be the totality of all the possible unions of open spherical neighbourhoods.  $\mathcal{T}$  is a topology.

**Definition 2.**  $(X_1, \mathcal{T}_1), (X_2, \mathcal{T}_2)$  topological spaces with  $X_1 \cap X_2 = \emptyset$ . let  $X = X_1 \cup X_2, \mathcal{T} = \{A_1 \cup A_2 \mid A_i \in \mathcal{T}_i\}$ .

**Lemma 1.**  $(X, \mathcal{T})$  is a topological space.

*Proof.* Lets verify the axioms.

1.  $\emptyset \in \mathcal{T}_1, \emptyset \in \mathcal{T}_2 \Rightarrow \emptyset = \emptyset \cup \emptyset$   
 $X_i \in \mathcal{T}_i \Rightarrow X = X_1 \cup X_2 \in \mathcal{T}$
2.  $\{A_i\}_{i \in I} \in \mathcal{T} \Rightarrow A_i = A_{1,i} \cup A_{2,i}$   
 $\bigcup_{i \in I} A_i = \bigcup_{i \in I} (A_{1,i} \cup A_{2,i}) = \bigcup_{i \in I} (A_{1,i}) \cup \bigcup_{i \in I} (A_{2,i}) \in \mathcal{T}_1 \cup \mathcal{T}_2$
3.  $A, A' \in \mathcal{T} \Rightarrow A = A_1 \cup A_2, A' = A'_1 \cup A'_2$   
 $A \cap A' = (A_1 \cup A_2) \cap (A'_1 \cup A'_2) = (A_1 \cap A'_1) \cup (A_2 \cap A'_2) \in \mathcal{T}$

□

**Definition 3.**  $\mathcal{T}$  is defined as a sum of Topologies  $\mathcal{T}_1$  and  $\mathcal{T}_2$ .

We notice that  $\forall A_i \in \mathcal{T}_i, A_i \cup \emptyset \in \mathcal{T}$  so  $\mathcal{T}$  contains  $\mathcal{T}_1$  and  $\mathcal{T}_2$ .

**Definition 4.**  $(X, \mathcal{T})$  topological space  $Y \subseteq X, Y \neq \emptyset$

$\mathcal{T}_Y = \{A \cap Y \mid A \in \mathcal{T}\}$  prove that  $\mathcal{T}_Y$  is a topology defined as Inducted topology on  $Y$

**Definition 5.**  $(X, \mathcal{T})$  topological space.  $\mathcal{B} = \{B_j\}_{j \in J}$  with  $B_j \in \mathcal{T}$ .  $\mathcal{B}$  is said basis for the topology  $\mathcal{T}$  if all open sets are union of elements of  $\mathcal{B}$

**Lemma 2.** Let  $X \neq \emptyset$  and  $\mathcal{B} \in \mathcal{P}(X)$ . Let  $A \subseteq X$  then the following affirmations are equivalent:

- (a)  $A$  is union of elements of  $\mathcal{B}$
- (b)  $\forall x \in A \exists B \in \mathcal{B} : x \in B \subseteq A$

*Proof.* **(a  $\Rightarrow$  b)** Let  $x \in A = \bigcup_{i \in I} B_i$  with  $B_i \in \mathcal{B}$ . Therefore  $\exists B_i : x \in B_i \subseteq A$   
**(b  $\Rightarrow$  a)**  $(\subseteq)$   $A = \bigcup_{i \in I} B_i \Rightarrow \forall x \in A \exists B_x : x \in B_x \Rightarrow A \subseteq \bigcup_{i \in I} B_i$ .  
**( $\supseteq$ )**  $\forall x \in A \exists B_i \in \mathcal{B} : x \in B_i \subseteq A \Rightarrow A \subseteq \bigcup_{i \in I} B_{i_x} \subseteq A \Rightarrow A = \bigcup_{x \in A} B_x \quad \square$

**Theorem 1.** Let  $X$  be a not empty set and  $\mathcal{B} \in \mathcal{P}(X)$ .  $\mathcal{B}$  is a basis if:

1.  $X = \bigcup_{B \in \mathcal{B}} B$
2.  $\forall B_1, B_2 \in \mathcal{B}$  and  $\forall x \in B_1 \cap B_2$  there exists  $B_3 \in \mathcal{B} : x \in B_3 \subseteq B_1 \cap B_2$ .

Also, for a Basis  $\mathcal{B}$  that satisfies the condition 1. and 2. there exists a topology on which  $\mathcal{B}$  is a basis.

*Proof.*  $(\Rightarrow)$

Let  $\mathcal{B}$  a basis such that every open set  $A$  is union of elements of  $\mathcal{B}$ , in particular  $X = \bigcup_{B \in \mathcal{B}} B$  moreover because  $\mathcal{B} \subseteq \mathcal{T}$  we can say that  $B_1, B_2 \in \mathcal{T}$  so is union of elements of  $\mathcal{B}$ . The last Lemma implies  $\forall x \in B_1 \cap B_2 \exists B_3 \in \mathcal{B}$  such that  $x \in B_3 \subseteq B_1 \cap B_2$ .

$(\Leftarrow)$

Let  $\mathcal{B} \subseteq \mathcal{P}(X)$  that satisfies 1. and 2. and let  $\mathcal{T}$  the totality of the unions of  $\mathcal{B}$ . It has to be proven that  $\mathcal{T}$  is a topology on  $X$ .

- i  $\emptyset \in \mathcal{T}$  because  $\emptyset$  is the empty union and  $X \in \mathcal{T}$  for the 1.
- ii  $\mathcal{T}$  is closed with respect to the union by definition.
- iii  $A_1, A_2 \in \mathcal{T}$  one has  $A_1 = \bigcup_{i \in I_1} B_i^{(1)}$  and  $A_2 = \bigcup_{j \in I_2} B_j^{(2)} \Rightarrow$   
 $\Rightarrow A_1 \cap A_2 = \left( \bigcup_{i \in I_1} B_i^{(1)} \right) \cap \left( \bigcup_{j \in I_2} B_j^{(2)} \right) = \bigcup_{i \in I_1, j \in I_2} \left( B_i^{(1)} \cap B_j^{(2)} \right).$

By the 2. and the last lemma  $B_i^{(1)} \cap B_j^{(2)} \in \mathcal{T}$  this implies that  $A_1 \cap A_2 \in \mathcal{T}$ .  $\square$

**Corollary 1.** Let  $X$  be a set and  $\mathcal{B} \in \mathcal{P}(X)$ , if  $\mathcal{B}$  is an overlay of  $X$  ( $X = \bigcup_{B \in \mathcal{B}} B$ ) and its closed with respect to the intersection, then  $\mathcal{B}$  is a topology on  $X$  and also a basis.

*Proof.* The condition 1. and 2. of the last theorem are satisfied.  $\square$

**Definition 6.**  $(X, \mathcal{T})$  topological space,  $X$  verifies the second axiom of numerability if possesses a finite base or numerable, in that case  $(X, \mathcal{T})$  is said  $\mathcal{N}_2$

**Proposition 1.** Let  $\mathbb{R}$  be gifted by the topology with a base of the following type:

$$[a, b], \quad a < b$$

Then  $(\mathbb{R}, \mathcal{T})$  is not  $\mathcal{N}_2$

*Proof.* Let  $\mathcal{B}$  a base for  $\mathcal{T}$ . Let  $a > 0 \in \mathbb{R}$  then  $\forall x \in \mathbb{R}$ , there exists  $B_x \in \mathcal{B}$  with  $x \in B_x \subseteq [x, x+a]$ . If  $y \in \mathbb{R}$  with  $y > x$  then  $x \notin [y, y+a]$  so  $x \notin B_y$ . The application  $x \in \mathbb{R} \mapsto B_x \in \mathcal{B}$  is injective so  $\mathcal{B}$  has the continuum order.  $\square$

**Proposition 2.** Let  $(X, \mathcal{T})$  be a topological space and  $S$  a subset of  $X$ .

- a) A point  $x \in X$  is adherent to  $S$  if and only if  $N \cap S \neq \emptyset$  for all  $N \in \mathcal{N}(x)$
- b) A point  $x \in X$  is adherent to  $S$  if there exists a successor function  $\{x_n\}$  of elements in  $S$  that converges to  $x$ . If  $X$  satisfies the second axiom of numerability then also the other implication is true.

*Proof.* (a)

Let's suppose that  $x \in \bar{S}$ . If  $x \in S$  then the condition is satisfied because every  $N \in \mathcal{N}(x)$  contains  $x$ . If  $x \in D(S)$  again, the condition is satisfied because  $N \setminus \{x\} \cap S \neq \emptyset$  for all  $N \in \mathcal{N}(x)$ . So we suppose that the condition of the statement is true, that implies that  $x \notin \text{Est}(S)$  because  $\text{Est}(S) \cap S = \emptyset$  and  $\text{Est}(S)$ , since is open, is a neighbourhood of every its point. Therefore  $x \in \bar{S}$ .

(b)

Let's suppose that  $\{x_n\}$  is a successor function of elements of  $S$  such that  $\lim_{n \rightarrow \infty} x_n = x$ . By definition of limit for all  $N \in \mathcal{N}(x)$  there exist  $x_n \in N$  and so the condition of the part (a) is satisfied. So  $x \in \bar{S}$ . Let's suppose instead that  $x \in \bar{S}$  and let  $\{N_n : n = 1, 2, \dots\}$  be a fundamental system of neighbourhoods of  $x$  that satisfies the condition  $N_{n+1} \subset N_n$  for all  $n$ . By the (a) for all  $n \geq 1$  we can find a point  $x_n \in N_n \cap S$ . The successor function  $\{x_n\}$  converges to  $x$ .  $\square$

## Zarinski's topology

Let  $\mathbb{F}$  be a field with  $\mathbb{F}^n = \{(x_1, \dots, x_n) : x_i \in \mathbb{F}\}$  and  $\mathbb{F}[X_1, \dots, X_n]$  the ring of the polynomials in the unknowns  $X_1, \dots, X_n$ . Let  $f(\mathbf{x}) = f(x_1, \dots, x_n) \in \mathbb{F}[\mathbf{X}]$ , we define  $\mathcal{V}(f) = \{\mathbf{x} \in \mathbb{F}^n : f(\mathbf{x}) = 0\}$  as an algebraic variety. More in general  $\mathfrak{F} = \{f_i(\mathbf{x})\}_{i \in I}$  and  $\mathcal{V}(\mathfrak{F}) = \{\mathbf{x} \in \mathbb{F}^n : f_i(\mathbf{x}) = 0, \forall i \in I\}$

**Theorem 2.** *The totality of all the algebraic varieties on  $\mathbb{F}^n$  satisfy the closed sets axioms.*

*Proof.* (1)

$\emptyset \in \mathcal{C}$  because if we take  $f = 1$  then  $\mathcal{V}(f) = \emptyset$ . Also for  $f = 0$  one has  $\mathcal{V}(f) = \mathbb{F}^n$

(2)

Let  $V = \mathcal{V}(\mathfrak{F})$ ,  $W = \mathcal{V}(\mathfrak{G}) \in \mathcal{C}$  with  $\mathfrak{F} = \{f_i\}_{i \in I}$  and  $\mathfrak{G} = \{g_j\}_{j \in J}$ . One has  $V \cup W = \{\mathbf{x} \in \mathbb{F}^n : f_i(\mathbf{x}) = 0 \vee g_j(\mathbf{x}) = 0\} = \mathcal{V}(\mathfrak{K})$  with  $\mathfrak{K} = \{f_i g_j\}_{i \in I, j \in J}$ , infact if  $\mathbf{x} \in V \cup W$  then  $(f_i g_j)(\mathbf{x}) = 0$  for all  $i$  and  $j$  so  $V \cup W \subseteq \mathcal{V}(\mathfrak{K})$ . Let  $\mathbf{x} \in \mathcal{V}(\mathfrak{K})$  and lets suppose that  $\mathbf{x} \in V$  then there exists  $\bar{i} \in I : f_{\bar{i}}(\mathbf{x}) \neq 0$  but  $\mathbf{x} \in \mathcal{V}(\mathfrak{K})$  and that implies  $f_{\bar{i}} g_j(\mathbf{x}) = 0 \forall j \in J$  with  $f_{\bar{i}}(\mathbf{x}) \neq 0 \Rightarrow g_j(\mathbf{x}) = 0 \forall j \in J$  that means  $\mathbf{x} \in W$ . we can conclude that  $V \cup W \supseteq \mathcal{V}(\mathfrak{K})$  such that  $V \cup W = \mathcal{V}(\mathfrak{K})$

(3)

Let  $\{\mathcal{V}^{(k)} = \mathcal{V}(\mathfrak{F}^{(k)})\}_{k \in K}$  a family of a.v.  $\mathfrak{F}^{(k)} = \{f_i^{(k)}\}_{i \in I^{(k)}} \subseteq \mathbb{F}[\mathbf{X}]$  and define  $V^{(k)} = \{\mathbf{x} \in \mathbb{F}^n : f_i^{(k)}(\mathbf{x}) = 0 \forall i \in I^{(k)}\}$  and  $W = \{\mathbf{x} \in \mathbb{F}^n : f_i^{(k)}(\mathbf{x}) = 0 \forall i \in I^{(k)} \forall k \in K\}$ , then  $W = \bigcup_{k \in K} V^{(k)}$ .  $\square$

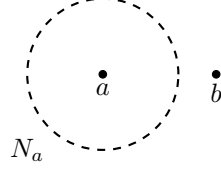
**Proposition 3.** *In the Zarinski's topology every point is a closed set. So if  $\mathbb{F}$  is finite this topology coincides with the discrete topology on  $\mathbb{F}^n$*

*Proof.* Let  $\mathbf{a} = (a_1, \dots, a_n)$  be a point in  $\mathbb{F}^n$  and consider  $f_i(\mathbf{x}) = x_i - a_i$ ,  $\forall i = 1, \dots, n$ . If  $\mathfrak{F} = \{f_1, \dots, f_n\}$  then  $\mathcal{V}(\mathfrak{F}) = \{\mathbf{x} \in \mathbb{F}^n : f_i(\mathbf{x}) = 0, i = 1, \dots, n\} = \{(a_1, \dots, a_n)\}$ . We can observe that the union of two points is again a closed set. If  $\mathbb{F}$  is a finite field, also  $\mathbb{F}^n$  is finite, so if we take a subset  $U \subseteq \mathbb{F}^n$  ( $U \subseteq \mathbb{F}^n \Rightarrow U = \bigcup_{p \in U} \{p\}$ ) therefore one has  $U = \mathbb{F}^n \setminus (\mathbb{F}^n \setminus U)$  ( $U = \mathbb{F}^n \setminus (\mathbb{F}^n \setminus U)$ ).  $\square$

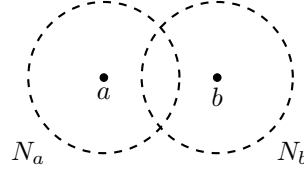
## Separation axioms

Let  $(X, \mathcal{T})$  be a topological space.

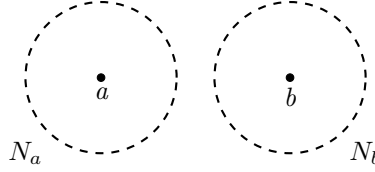
**Definition 7.**  $(X, \mathcal{T})$  is said to be  $\mathbf{T}_0$  if  $\forall a, b \in X$  with  $a \neq b$  there exists a neighbourhood  $N_a$  that doesn't contain  $b$ .



**Definition 8.**  $(X, \mathcal{T})$  is said to be  $\mathbf{T}_1$  if  $\forall a, b \in X$  with  $a \neq b$  if  $\exists N_a, N_b$  such that  $b \notin N_a$  and  $a \notin N_b$ .



**Definition 9.**  $(X, \mathcal{T})$  is said to be  $\mathbf{T}_2$  if  $\exists N_a, N_b$  with  $N_a \cap N_b = \emptyset$ .



**Observation 1.** In a topological space  $\mathbf{T}_2 \Rightarrow \mathbf{T}_1 \Rightarrow \mathbf{T}_0$

**Proposition 4.** Every subspace  $Y$  of an Hausdorff ( $\mathbf{T}_2$ ) Space  $X$  is again a Hausdorff space

*Proof.* If  $a$  and  $b$  are two distinct points of  $Y$  there exist  $N_a, N_b$  neighbourhoods of  $X$  such that  $a \in N_a$  and  $b \in N_b$  with  $N_a \cap N_b = \emptyset$ . Let  $N_a^1, N_b^1$  be neighbourhoods of  $a$  and  $b$  respectively with  $N_a^1 = N_a \cap Y$  and  $N_b^1 = N_b \cap Y$  by construction we can say  $N_a^1 \cap N_b^1 = \emptyset$ .  $\square$

## Connected Topological Spaces

**Definition 10.** Let  $(X, \mathcal{T})$  be a topological space.  $X$  is said Connected if there not exist two open sets of  $X$  such that their union is all the space and also the two subsets are disjoint. In symbols:  $X$  is connected if  $\nexists A, A_1 \in \mathcal{T} : A \cup A_1 = X$  with  $A \cap A_1 = \emptyset$ .

**Proposition 5.** Let  $(X, \mathcal{T})$  and  $a, a_1 \in X$ , if there exist a  $Y \subseteq X$  connected such that  $a, a_1 \in Y \Rightarrow X$  is connected.

*Proof.* Lets suppose that  $X$  is disconnected by  $A, A_1 \in \mathcal{T}$  and let  $a \in A$  and  $a_1 \in A_1$ . If we take  $Y \subset X$  connected such that  $a, a_1 \in Y$ . Then  $A \cap Y \neq \emptyset$  and  $A_1 \cap Y \neq \emptyset$  also  $(A \cap Y) \cup (A_1 \cap Y) \subseteq A \cup A_1 \neq \emptyset$ . one has  $(A \cap Y) \cup (A_1 \cap Y) = (A \cup A_1) \cap Y = X \cap Y = Y$ , that implies is connected, but since  $a, a_1$  and  $Y$  are arbitrary, this is absurd.  $\square$

**Example 2.** Let  $\mathbb{R}$  be gifted by the natural topology, then all the connected sets of  $\mathbb{R}$  are the intervals:  $[a, b], (a, b], [a, b), (a, b)$ .

**Proposition 6.** In a topological space  $(X, \mathcal{T})$  every subset  $Y \subseteq X$  that is connected implies its closure is connected.

*Proof.* Lets suppose that  $\overline{Y}$  is disconnected and lets suppose that  $\exists A_1, A_2 \in \mathcal{T}$  such that:

1.  $A_1 \cap \overline{Y}, A_2 \cap \overline{Y} \neq \emptyset$
2.  $(A_1 \cap \overline{Y}) \cap (A_2 \cap \overline{Y}) = \emptyset$
3.  $(A_1 \cap \overline{Y}) \cup (A_2 \cap \overline{Y}) = \overline{Y}$

It can be proven that  $A_1$  and  $A_2$  disconnect  $Y$ , infact  $\overline{Y} = Y \cup \partial Y$ <sup>1</sup> and if  $A_1 \cap Y = \emptyset$  then  $\exists a \in \partial Y$  such that the neighbourhood  $A_1$  would have null intersection with  $Y$ , but that is impossible, so  $A_1 \cap Y \neq \emptyset$ . The same reasoning can be applied to  $A_2 \cap Y$ . We also have 2.  $\Rightarrow (A_1 \cap Y) \cap (A_2 \cap Y) = \emptyset$ , therefore 3.  $\Rightarrow (A_1 \cap Y) \cup (A_2 \cap Y) \Rightarrow A_1 \cap \overline{Y} = (A_1 \cap Y) \cup (A_1 \cap \partial Y) \Rightarrow (A_1 \cap \overline{Y}) \cup (A_2 \cap \overline{Y}) = (A_1 \cap Y) \cup (A_2 \cap Y) \cup ((A_1 \cup A_2) \cap \partial Y)$  but that means  $Y \cap \partial Y \Rightarrow Y = (A_1 \cap Y) \cup (A_2 \cap Y)$ .  $\square$

Let  $(X, \mathcal{T})$  be a topological space, then we can define an equivalence relations between its points by saying  $a \sim b$  if and only if  $a$  and  $b$  are in the same connected space.

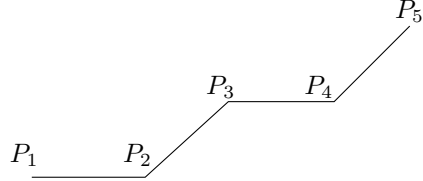
**Definition 11.**  $\mathcal{K}(a)$  is the equivalence class of  $a$ , so if  $a \in Y_a$  such that  $Y_a$  is connected in symbols we have:  $\mathcal{K}(a) = \{x \in X | a \sim x\} = \{x \in X | x \in Y_a\}$

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<sup>1</sup> $\partial Y = D(Y)$

**Definition 12.** Let  $C \subseteq \mathbb{R}^n$  be a set in which  $\forall \vec{x}, \vec{y} \in \mathbb{R}^n$  the segment  $\overline{XY} \subset C$ .  $C$  is said to be convex.

**Definition 13.** A polygonal in  $\mathbb{R}^n$  is the union of a finite successor function of segments  $\{S_i\}_{i \in I}$  with  $S_i = \overline{P_i P_{i+1}}$ .



**Definition 14.** A set  $Y \subseteq \mathbb{R}^n$  is said to be connected by polygonals if  $\forall a, b \in Y$  exist a polygonal of extremes  $a$  and  $b$  contained in  $Y$ .

**Proposition 7.** Let  $\mathbb{R}^n$  be gifted by the natural topology  $\mathcal{T}$  and let  $A \in \mathcal{T}$  then  $A$  is connected by polygonals  $\iff A$  is connected.

*Proof.* ( $\implies$ ) since every component of a line (line included) in  $\mathbb{R}^n$  is connected and since for two arbitrary points in the same connected subspace the space that contains it is connected, the first implication is proved.

( $\impliedby$ ) Lets suppose that  $A$  is connected but not Connected by polygonals so  $\exists x, y \in A$  with  $x \neq y$  that are not joinable by a polygonal so then  $\forall x \in A$  let  $E_x \subset A$  the totality of the points of  $A$  that are joinable to  $x$  by a polygonal. The relation  $x \sim y \iff x$  and  $y$  are joinable by a polygonal, is an equivalence relation.

The classes of equivalence are  $E_x \forall x$  therefore  $A = E_x \cup \bigcup_{y \neq x} E_y$  would be sconnected, infact every  $E_x$  is an open set because  $A$  is open, so  $\forall a \in E_x$  there exist a neighbourhood  $U_a$  with center  $a$  in which for all  $b \in U_a$   $\overline{ab} \subset U_a$  so  $b \in E_x$ .  $\square$

**Proposition 8.** For all  $a \in X$ ,  $\mathcal{K}(a)$  is the biggest connected subset that contains  $a$ , also every connected component is a closed set in  $\mathcal{T}$ .

*Proof.* Let  $a \in X$  and  $Y \subset X$  conncted and  $a \in Y$ , Lets prove that  $Y \subseteq \mathcal{K}(a)$ . Infact if  $x \in \mathcal{K}(a)$  there exists a connected subset  $Y_x$  such that  $x \in Y_x$  that means  $Y_x \subset \mathcal{K}(a)$ . One has that  $\mathcal{K}(a) = \bigcup_{x \in \mathcal{K}(a)} Y_x$  with  $a \in \bigcap_{x \in \mathcal{K}(a)} Y_x$  ( $\neq \emptyset$ ) so  $\mathcal{K}(a)$  is connected. Finally  $\mathcal{K}(a)$  is a closed set because  $\overline{\mathcal{K}(a)}$  is connected and for the first part of the theorem  $\mathcal{K}(a) \subseteq \overline{\mathcal{K}(a)}$  but  $\mathcal{K}(a) \supseteq \overline{\mathcal{K}(a)}$  so  $\mathcal{K}(a) = \overline{\mathcal{K}(a)}$ .  $\square$

## Continuous functions and homeomorphisms

**Definition 15.** Let  $(X, \mathcal{T}_x)$ ,  $(Y, \mathcal{T}_y)$  be topological spaces,  $\Omega : X \rightarrow Y$  is continuous in  $a \in X$  if  $\forall I$  neighbourhood of  $\Omega(a)$ ,  $\exists K$  neighbourhood of  $a$  s.t.  $\Omega(K) \subseteq I$ .

We'll say that a function is continuous if it is continuous in every point.

**Proposition 9.** Let  $\Omega : (X, \mathcal{T}_x) \rightarrow (Y, \mathcal{T}_y)$  so then the following affirmations are equivalent:

- i.  $\Omega$  is continuous.
- ii.  $\forall A \in \mathcal{T}_y, \Omega^{-1}(A) \in \mathcal{T}_x$ .
- iii.  $\forall c \in \mathcal{C}(Y), \Omega^{-1}(c) \in \mathcal{C}(X)$ .
- iv. The counterimages of opens under  $\Omega$  are opens of  $X$ .
- v.  $\forall b = \Omega(a) \in \text{Im } \Omega = \Omega(X)$  the counterimage of every neighbourhood  $K'$  of  $b$  is a neighbourhood of  $a$

*Proof.* i.  $\Rightarrow$  ii. Let  $A \in \mathcal{T}_y$  we have to prove that  $\forall a \in \Omega^{-1}(A)$  exists a neighbourhood of  $a$  contained in  $\Omega^{-1}(A)$ . For  $a \in \Omega^{-1}(A)$  one has  $\Omega(a) \in \Omega(\Omega^{-1}(A)) = A$ .  $A$  is open and is a neighbourhood of  $\Omega(a)$  so if  $\Omega$  is continuous there exists a neighbourhood  $K$  of  $a$  s.t.  $\Omega(K) \subseteq A$  hence  $K \subseteq \Omega^{-1}(A)$  and  $a \in K$ .

ii.  $\Rightarrow$  i. Let  $a \in X$ ,  $I$  neighbourhood of  $\Omega(a)$  let  $A \in \mathcal{T}_y$  s.t.  $\Omega(a) \in A \subseteq I$  for the ii.  $\Omega^{-1}(A)$  is open and is a neighbourhood of  $a$  s.t. its image contains  $\Omega(a)$  and  $A$  is contained in  $I$ .  $\square$

**Definition 16.** Let  $(X, \mathcal{T}_x)$  and  $(Y, \mathcal{T}_y)$  be topological spaces, then  $\Omega : X \rightarrow Y$  is defined open function (respectively closed function) if for every open subset  $A$  of  $X$ ,  $\Omega(A)$  is an open subset (closed subset) of  $Y$ .

**Definition 17.** A function  $\Omega : X \rightarrow Y$  is said to be a homeomorphism if it is continuous and  $\Omega, \Omega^{-1}$  are continuous. The two topological spaces are said to be homeomorphic if that application exists and we write:  $X \approx Y$ .