

# Topology

Alessio Esposito

April 5, 2023

**Definition 1.** let  $X$  be a not empty set and  $\mathcal{T}$  a collection of its subsets.  $\mathcal{T}$  is said topology on  $X$  if it has the following properties:

- $\emptyset, X \in \mathcal{T}$
- the union of a whatever family of elements of  $\mathcal{T}$  is in  $\mathcal{T}$ . In symbols:  
 $\forall \{A_i\}_{i \in I}, A_i \in \mathcal{T}, \bigcup_{i \in I} A_i \in \mathcal{T}$
- the interection of elements of  $\mathcal{T}$  is in  $\mathcal{T}$ :  $\forall A_1, A_2 \in \mathcal{T}, A_1 \cap A_2 \in \mathcal{T}$

Then  $(X, \mathcal{T})$  its said to be topological space and  $X$  is called support.

**Example 1.** Let  $X = \mathbb{R}^2$  with the euclidean distance and  $\forall c \in \mathbb{R}^2$  and  $\forall r > 0$  the set  $B_r(c) = \{x \in \mathbb{R}^2 : d(x, c) < r\}$  its said open spherical neighbourhood of center  $c$  and radius  $r$ . Let  $\mathcal{T}$  be the totality of all the possible unions of open spherical neighbourhoods.  $\mathcal{T}$  is a topology.

**Definition 2.**  $(X_1, \mathcal{T}_1), (X_2, \mathcal{T}_2)$  topological spaces with  $X_1 \cap X_2 = \emptyset$ . let  $X = X_1 \cup X_2, \mathcal{T} = \{A_1 \cup A_2 \mid A_i \in \mathcal{T}_i\}$ .

**Lemma 1.**  $(X, \mathcal{T})$  is a topological space.

*Proof.* Lets verify the axioms.

1.  $\emptyset \in \mathcal{T}_1, \emptyset \in \mathcal{T}_2 \Rightarrow \emptyset = \emptyset \cup \emptyset$   
 $X_i \in \mathcal{T}_i \Rightarrow X = X_1 \cup X_2 \in \mathcal{T}$
2.  $\{A_i\}_{i \in I} \in \mathcal{T} \Rightarrow A_i = A_{1,i} \cup A_{2,i}$   
 $\bigcup_{i \in I} A_i = \bigcup_{i \in I} (A_{1,i} \cup A_{2,i}) = \bigcup_{i \in I} A_{1,i} \cup \bigcup_{i \in I} A_{2,i} \in \mathcal{T}_1 \cup \mathcal{T}_2$
3.  $A, A' \in \mathcal{T} \Rightarrow A = A_1 \cup A_2, A' = A'_1 \cup A'_2$   
 $A \cap A' = (A_1 \cup A_2) \cap (A'_1 \cup A'_2) = (A_1 \cap A'_1) \cup (A_2 \cap A'_2) \in \mathcal{T}$

□

**Definition 3.**  $\mathcal{T}$  is defined as a sum of Topologies  $\mathcal{T}_1$  and  $\mathcal{T}_2$ .

We notice that  $\forall A_i \in \mathcal{T}_i, A_i \cup \emptyset \in \mathcal{T}$  so  $\mathcal{T}$  contains  $\mathcal{T}_1$  and  $\mathcal{T}_2$ .

**Definition 4.**  $(X, \mathcal{T})$  topological space  $Y \subseteq X, Y \neq \emptyset$

$\mathcal{T}_Y = \{A \cap Y \mid A \in \mathcal{T}\}$  prove that  $\mathcal{T}_Y$  is a topology defined as Inducted topology on  $Y$

**Definition 5.**  $(X, \mathcal{T})$  topological space,  $X$  verifies the second axiom of numerability if possesses a finite base or numerable, in that case  $(X, \mathcal{T})$  is said  $\mathcal{N}_2$

**Proposition 1.** Let  $\mathbb{R}$  be gifted by the topology with a base of the following type:

$$[a, b], \quad a < b$$

Then  $(\mathbb{R}, \mathcal{T})$  is not  $\mathcal{N}_2$

*Proof.* Let  $\mathcal{B}$  a base for  $\mathcal{T}$ . Let  $a > 0 \in \mathbb{R}$  then  $\forall x \in \mathbb{R}$ , there exists  $B_x \in \mathcal{B}$  with  $x \in B_x \subseteq [x, x+a]$ . If  $y \in \mathbb{R}$  with  $y > x$  then  $x \notin [y, y+a]$  so  $x \notin B_y$ . The application  $x \in \mathbb{R} \mapsto B_x \in \mathcal{B}$  is injective so  $\mathcal{B}$  has the continuum order.  $\square$

## Continous functions and homeomorphisms

**Definition 6.** Let  $(X, \mathcal{T}_x)$ ,  $(Y, \mathcal{T}_y)$  be topological spaces,  $\Omega : X \rightarrow Y$  is continous in  $a \in X$  if  $\forall I$  neighbourhood of  $\Omega(a)$ ,  $\exists K$  neighbourhood of  $a$  s.t.  $\Omega(K) \subseteq I$ .

We'll say that a function is continous if it is continous in every point.

**Proposition 2.** Let  $\Omega : (X, \mathcal{T}_x) \rightarrow (Y, \mathcal{T}_y)$  so then the following affirmations are equivalent:

- i.  $\Omega$  is continous.
- ii.  $\forall A \in \mathcal{T}_y, \quad \Omega^{-1}(A) \in \mathcal{T}_x$ .
- iii.  $\forall c \in \mathcal{C}(Y), \quad \Omega^{-1}(c) \in \mathcal{C}(X)$ .
- iv. The counterimages of opens under a selected base of  $Y$  are opens of  $X$ .
- v.  $\forall b = \Omega(a) \in \text{Im}\Omega = \Omega(X)$  the counterimage of every neighbourhood  $K'$  of  $b$  is a neighbourhood of  $a$

*Proof.* i.  $\Rightarrow$  ii. Let  $A \in \mathcal{T}_y$  we have to prove that  $\forall a \in \Omega^{-1}(A)$  exists a neighbourhood of  $a$  contained in  $\Omega^{-1}(A)$ . For  $a \in \Omega^{-1}(A)$  one has  $\Omega(a) \in \Omega(\Omega^{-1}(A)) = A$ .  $A$  is open and is a neighbourhood of  $\Omega(a)$  so if  $\Omega$  is continous there exists a neighbourhood  $K$  of  $a$  s.t.  $\Omega(K) \subseteq A$  hence  $K \subseteq \Omega^{-1}(A)$  and  $a \in K$ .

ii.  $\Rightarrow$  i. Let  $a \in X$ ,  $I$  neighbourhood of  $\Omega(a)$  let  $A \in \mathcal{T}_y$  s.t.  $\Omega(a) \in A \subseteq I$  for the ii.  $\Omega^{-1}(A)$  is open and is a neighbourhood of  $a$  s.t. its image contains  $\Omega(a)$  and  $A$  is contained in  $I$ .  $\square$