

Topology

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Definition 1. let X be a not empty set and \mathcal{T} a collection of its subsets. \mathcal{T} is said topology on X if it has the following properties:

- $\emptyset, X \in \mathcal{T}$
- the union of a whatever family of elements of \mathcal{T} is in \mathcal{T} . In symbols:
 $\forall \{A_i\}_{i \in I}, A_i \in \mathcal{T}, \bigcup_{i \in I} A_i \in \mathcal{T}$
- the intersection of elements of \mathcal{T} is in \mathcal{T} : $\forall A_1, A_2 \in \mathcal{T}, A_1 \cap A_2 \in \mathcal{T}$

Then (X, \mathcal{T}) is said to be topological space and X is called support.

Example 1. Let $X = \mathbb{R}^2$ with the euclidean distance and $\forall c \in \mathbb{R}^2$ and $\forall r > 0$ the set $B_r(c) = \{x \in \mathbb{R}^2 : d(x, c) < r\}$ its said open spherical neighbourhood of center c and radius r . Let \mathcal{T} be the totality of all the possible unions of open spherical neighbourhoods. \mathcal{T} is a topology.

Definition 2. $(X_1, \mathcal{T}_1), (X_2, \mathcal{T}_2)$ topological spaces with $X_1 \cap X_2 = \emptyset$. let $X = X_1 \cup X_2, \mathcal{T} = \{A_1 \cup A_2 \mid A_i \in \mathcal{T}_i\}$.

Lemma 1. (X, \mathcal{T}) is a topological space.

Proof. Lets verify the axioms.

1. $\emptyset \in \mathcal{T}_1, \emptyset \in \mathcal{T}_2 \Rightarrow \emptyset = \emptyset \cup \emptyset$
 $X_i \in \mathcal{T}_i \Rightarrow X = X_1 \cup X_2 \in \mathcal{T}$
2. $\{A_i\}_{i \in I} \in \mathcal{T} \Rightarrow A_i = A_{1,i} \cup A_{2,i}$
 $\bigcup_{i \in I} A_i = \bigcup_{i \in I} (A_{1,i} \cup A_{2,i}) = \bigcup_{i \in I} (A_{1,i}) \cup \bigcup_{i \in I} (A_{2,i}) \in \mathcal{T}_1 \cup \mathcal{T}_2$
3. $A, A' \in \mathcal{T} \Rightarrow A = A_1 \cup A_2, A' = A'_1 \cup A'_2$
 $A \cap A' = (A_1 \cup A_2) \cap (A'_1 \cup A'_2) = (A_1 \cap A'_1) \cup (A_2 \cap A'_2) \in \mathcal{T}$

□

Definition 3. \mathcal{T} is defined as a sum of Topologies \mathcal{T}_1 and \mathcal{T}_2 .

We notice that $\forall A_i \in \mathcal{T}_i, A_i \cup \emptyset \in \mathcal{T}$ so \mathcal{T} contains \mathcal{T}_1 and \mathcal{T}_2 .

Definition 4. (X, \mathcal{T}) topological space $Y \subseteq X, Y \neq \emptyset$

$\mathcal{T}_Y = \{A \cap Y \mid A \in \mathcal{T}\}$ prove that \mathcal{T}_Y is a topology defined as Inducted topology on Y

Definition 5. (X, \mathcal{T}) topological space. $\mathcal{B} = \{B_j\}_{j \in J}$ with $B_j \in \mathcal{T}$. \mathcal{B} is said basis for the topology \mathcal{T} if all open sets are union of elements of \mathcal{B}

Lemma 2. Let $X \neq \emptyset$ and $\mathcal{B} \in \mathcal{P}(X)$. Let $A \subseteq X$ then the following affirmations are equivalent:

- (a) A is union of elements of \mathcal{B}
- (b) $\forall x \in A \exists B \in \mathcal{B} : x \in B \subseteq A$

Proof. (a) \Rightarrow (b)

Let $x \in A = \bigcup_{i \in I} B_i$ with $B_i \in \mathcal{B}$. Therefore $\exists B_i : x \in B_i \subseteq A$

(b) \Rightarrow (a)

(\subseteq) $A = \bigcup_{i \in I} B_i \Rightarrow \forall x \in A \exists B_x : x \in B_x \Rightarrow A \subseteq \bigcup_{i \in I} B_i$.

(\supseteq) $\forall x \in A \exists B_i \in \mathcal{B} : x \in B_i \subseteq A \Rightarrow A \subseteq \bigcup_{i \in I} B_i \subseteq A \Rightarrow A = \bigcup_{x \in A} B_x \quad \square$

Theorem 1. Let X be a not empty set and $\mathcal{B} \in \mathcal{P}(X)$. \mathcal{B} is a basis if:

1. $X = \bigcup_{B \in \mathcal{B}} B$
2. $\forall B_1, B_2 \in \mathcal{B}$ and $\forall x \in B_1 \cap B_2$ there exists $B_3 \in \mathcal{B} : x \in B_3 \subseteq B_1 \cap B_2$.

Proof. (\Rightarrow)

Let \mathcal{B} a basis such that every open set A is union of elements of \mathcal{B} , in particular $X = \bigcup_{B \in \mathcal{B}} B$ moreover because $\mathcal{B} \subseteq \mathcal{T}$ we can say that $B_1, B_2 \in \mathcal{T}$ so is union of elements of \mathcal{B} . The last Lemma implies $\forall x \in B_1 \cap B_2 \exists B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$.

(\Leftarrow)

Let $\mathcal{B} \subseteq \mathcal{P}(X)$ that satisfies 1. and 2. and let \mathcal{T} the totality of the unions of \mathcal{B} . It has to be proven that \mathcal{T} is a topology on X .

i $\emptyset \in \mathcal{T}$ because \emptyset is the empty union and $X \in \mathcal{T}$ for the 1.

ii \mathcal{T} is closed with respect to the union by definition.

iii $A_1, A_2 \in \mathcal{T}$ one has $A_1 = \bigcup_{i \in I_1} B_i^{(1)}$ and $A_2 = \bigcup_{j \in I_2} B_j^{(2)} \Rightarrow$
 $\Rightarrow A_1 \cap A_2 = (\bigcup_{i \in I_1} B_i^{(1)}) \cap (\bigcup_{j \in I_2} B_j^{(2)}) = \bigcup_{i \in I_1, j \in I_2} (B_i^{(1)} \cap B_j^{(2)}).$

By the 2. and the last lemma $B_i^{(1)} \cap B_j^{(2)} \in \mathcal{T}$ this implies that $A_1 \cap A_2 \in \mathcal{T}$. \square

Corollary 1. Let X be a set and $\mathcal{B} \in \mathcal{P}(X)$, if \mathcal{B} is an overlay of X ($X = \bigcup_{B \in \mathcal{B}} B$) and its closed with respect to the intersection, then \mathcal{B} is a topology on X and also a basis.

Proof. The condition 1. and 2. of the last theorem are satisfied. \square

Definition 6. (X, \mathcal{T}) topological space, X verifies the second axiom of numerability if possesses a finite base or numerable, in that case (X, \mathcal{T}) is said \mathcal{N}_2

Proposition 1. Let \mathbb{R} be gifted by the topology with a base of the following type:

$$[a, b], \quad a < b$$

Then $(\mathbb{R}, \mathcal{T})$ is not \mathcal{N}_2

Proof. Let \mathcal{B} a base for \mathcal{T} . Let $a > 0 \in \mathbb{R}$ then $\forall x \in \mathbb{R}$, there exists $B_x \in \mathcal{B}$ with $x \in B_x \subseteq [x, x+a]$. If $y \in \mathbb{R}$ with $y > x$ then $x \notin [y, y+a]$ so $x \notin B_y$. The application $x \in \mathbb{R} \mapsto B_x \in \mathcal{B}$ is injective so \mathcal{B} has the continuum order. \square

Proposition 2. Let (X, \mathcal{T}) be a topological space and S a subset of X .

- a) A point $x \in X$ is adherent to S if and only if $N \cap S \neq \emptyset$ for all $N \in \mathcal{N}(x)$
- b) A point $x \in X$ is adherent to S if there exists a successor function $\{x_n\}$ of elements in S that converges to x . If X satisfies the second axiom of numerability then also the other implication is true.

Proof. (a)

Let's suppose that $x \in \bar{S}$. If $x \in S$ then the condition is satisfied because every $N \in \mathcal{N}(x)$ contains x . If $x \in D(S)$ again, the condition is satisfied because $N \setminus \{x\} \cap S \neq \emptyset$ for all $N \in \mathcal{N}(x)$. So we suppose that the condition of the statement is true, that implies that $x \notin \text{Est}(S)$ because $\text{Est}(S) \cap S = \emptyset$ and $\text{Est}(S)$, since is open, is a neighbourhood of every its point. Therefore $x \in \bar{S}$.

(b)

Let's suppose that $\{x_n\}$ is a successor function of elements of S such that $\lim_{n \rightarrow \infty} x_n = x$. By definition of limit for all $N \in \mathcal{N}(x)$ there exist $x_n \in N$ and so the condition of the part (a) is satisfied. So $x \in \bar{S}$. Let's suppose instead that $x \in \bar{S}$ and let $\{N_n : n = 1, 2, \dots\}$ be a fundamental system of neighbourhoods of x that satisfies the condition $N_{n+1} \subset N_n$ for all n . By the (a) for all $n \geq 1$ we can find a point $x_n \in N_n \cap S$. The successor function $\{x_n\}$ converges to x . \square

Zarinski's topology

Let \mathbb{F} be a field with $\mathbb{F}^n = \{(x_1, \dots, x_n) : x_i \in \mathbb{F}\}$ and $\mathbb{F}[X_1, \dots, X_n]$ the ring of the polynomials in the unknowns X_1, \dots, X_n . Let $f(\mathbf{x}) = f(x_1, \dots, x_n) \in \mathbb{F}[\mathbf{X}]$, we define $\mathcal{V}(f) = \{\mathbf{x} \in \mathbb{F}^n : f(\mathbf{x}) = 0\}$ as an algebraic variety. More in general $\mathfrak{F} = \{f_i(\mathbf{x})\}_{i \in I}$ and $\mathcal{V}(\mathfrak{F}) = \{\mathbf{x} \in \mathbb{F}^n : f_i(\mathbf{x}) = 0, \forall i \in I\}$

Theorem 2. *The totality of all the algebraic varieties on \mathbb{F}^n satisfy the closed sets axioms.*

Proof. (1)

$\emptyset \in \mathcal{C}$ because if we take $f = 1$ then $\mathcal{V}(f) = \emptyset$. Also for $f = 0$ one has $\mathcal{V}(f) = \mathbb{F}^n$

(2)

Let $V = \mathcal{V}(\mathfrak{F})$, $W = \mathcal{V}(\mathfrak{G}) \in \mathcal{C}$ with $\mathfrak{F} = \{f_i\}_{i \in I}$ and $\mathfrak{G} = \{g_j\}_{j \in J}$. One has $V \cup W = \{\mathbf{x} \in \mathbb{F}^n : f_i(\mathbf{x}) = 0 \vee g_j(\mathbf{x}) = 0\} = \mathcal{V}(\mathfrak{K})$ with $\mathfrak{K} = \{f_i g_j\}_{i \in I, j \in J}$, infact if $\mathbf{x} \in V \cup W$ then $(f_i g_j)(\mathbf{x}) = 0$ for all i and j so $V \cup W \subseteq \mathcal{V}(\mathfrak{K})$. Let $\mathbf{x} \in \mathcal{V}(\mathfrak{K})$ and lets suppose that $\mathbf{x} \in V$ then there exists $\bar{i} \in I : f_{\bar{i}}(\mathbf{x}) \neq 0$ but $\mathbf{x} \in \mathcal{V}(\mathfrak{K})$ and that implies $f_{\bar{i}} g_j(\mathbf{x}) = 0 \forall j \in J$ with $f_{\bar{i}}(\mathbf{x}) \neq 0 \Rightarrow g_j(\mathbf{x}) = 0 \forall j \in J$ that means $\mathbf{x} \in W$. we can conclude that $V \cup W \supseteq \mathcal{V}(\mathfrak{K})$ such that $V \cup W = \mathcal{V}(\mathfrak{K})$

(3)

Let $\{\mathcal{V}^{(k)} = \mathcal{V}(\mathfrak{F}^{(k)})\}_{k \in K}$ a family of a.v. $\mathfrak{F}^{(k)} = \{f_i^{(k)}\}_{i \in I^{(k)}} \subseteq \mathbb{F}[\mathbf{X}]$ and define $V^{(k)} = \{\mathbf{x} \in \mathbb{F}^n : f_i^{(k)}(\mathbf{x}) = 0 \forall i \in I^{(k)}\}$ and $W = \{\mathbf{x} \in \mathbb{F}^n : f_i^{(k)}(\mathbf{x}) = 0 \forall i \in I^{(k)} \forall k \in K\}$, then $W = \bigcup_{k \in K} V^{(k)}$. \square

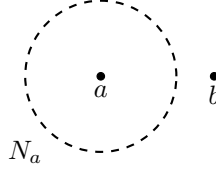
Proposition 3. *In the Zarinski's topology every point is a closed set. So if \mathbb{F} is finite this topology coincides with the discrete topology on \mathbb{F}^n*

Proof. Let $\mathbf{a} = (a_1, \dots, a_n)$ be a point in \mathbb{F}^n and consider $f_i(\mathbf{x}) = x_i - a_i$, $\forall i = 1, \dots, n$. If $\mathfrak{F} = \{f_1, \dots, f_n\}$ then $\mathcal{V}(\mathfrak{F}) = \{\mathbf{x} \in \mathbb{F}^n : f_i(\mathbf{x}) = 0, i = 1, \dots, n\} = \{(a_1, \dots, a_n)\}$. We can observe that the union of two points is again a closed set. If \mathbb{F} is a finite field, also \mathbb{F}^n is finite, so if we take a subset $U \subseteq \mathbb{F}^n$ ($U \subseteq \mathbb{F}^n \Rightarrow U = \bigcup_{p \in U} \{p\}$) therefore one has $U = \mathbb{F}^n \setminus (\mathbb{F}^n \setminus U)$ ($U = \mathbb{F}^n \setminus (\mathbb{F}^n \setminus U)$). \square

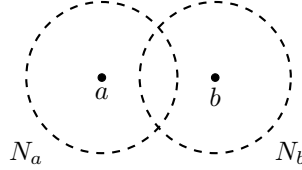
Separation axioms

Let (X, \mathcal{T}) be a topological space.

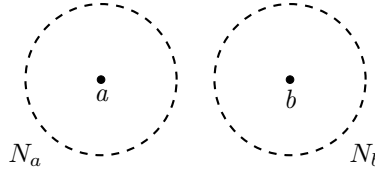
Definition 7. (X, \mathcal{T}) is said to be \mathbf{T}_0 if $\forall a, b \in X$ with $a \neq b$ there exists a neighbourhood N_a that doesn't contain b .



Definition 8. (X, \mathcal{T}) is said to be \mathbf{T}_1 if $\forall a, b \in X$ with $a \neq b$ if $\exists N_a, N_b$ such that $b \notin N_a$ and $a \notin N_b$.



Definition 9. (X, \mathcal{T}) is said to be \mathbf{T}_2 if $\exists N_a, N_b$ with $N_a \cap N_b = \emptyset$.



Observation 1. In a topological space $\mathbf{T}_2 \Rightarrow \mathbf{T}_1 \Rightarrow \mathbf{T}_0$

Proposition 4. Every subspace Y of an Hausdorff (\mathbf{T}_2) Space X is again a an Hausdorff space

Proof. If a and b are two distinct points of Y there exist N_a, N_b neighbourhoods of X such that $a \in N_a$ and $b \in N_b$ with $N_a \cap N_b = \emptyset$. Let N_a^1, N_b^1 be neighbourhoods of a and b respectively with $N_a^1 = N_a \cap Y$ and $N_b^1 = N_b \cap Y$ by construction we can say $N_a^1 \cap N_b^1 = \emptyset$. \square

Connected Topological Spaces

Definition 10. Let (X, \mathcal{T}) be a topological space. X is said *Connected* if there not exist two open sets of X such that their union is all the space and also the two subsets are disjoint. In symbols: X is connected if $\nexists A, A_1 \in \mathcal{T} : A \cup A_1 = X$ with $A \cap A_1 = \emptyset$.

Proposition 5. Let (X, \mathcal{T}) and $a, a_1 \in X$, if there exist a $Y \subseteq X$ connected such that $a, a_1 \in Y \Rightarrow X$ is connected.

Proof. Lets suppose that X is sconnected by $A, A_1 \in \mathcal{T}$ and let $a \in A$ and $a_1 \in A_1$. If we take $Y \subset X$ connected such that $a, a_1 \in Y$. Then $A \cap Y \neq \emptyset$ and $A_1 \cap Y \neq \emptyset$ also $(A \cap Y) \cup (A_1 \cap Y) \subseteq A \cup A_1 \neq \emptyset$. one has $(A \cap Y) \cup (A_1 \cap Y) = (A \cup A_1) \cap Y = X \cap Y = Y$, that implies is connected, but since a, a_1 and Y are arbitrary, this is absurd. \square

Example 2. Let \mathbb{R} be gifted by the natural topology, then all the connected sets of \mathbb{R} are the intervals: $[a, b], (a, b], [a, b), (a, b)$.

Continuous functions and homeomorphisms

Definition 11. Let (X, \mathcal{T}_x) , (Y, \mathcal{T}_y) be topological spaces, $\Omega : X \rightarrow Y$ is continuous in $a \in X$ if $\forall I$ neighbourhood of $\Omega(a)$, $\exists K$ neighbourhood of a s.t. $\Omega(K) \subseteq I$.

We'll say that a function is continuous if it is continuous in every point.

Proposition 6. Let $\Omega : (X, \mathcal{T}_x) \rightarrow (Y, \mathcal{T}_y)$ so then the following affirmations are equivalent:

- i. Ω is continuous.
- ii. $\forall A \in \mathcal{T}_y, \Omega^{-1}(A) \in \mathcal{T}_x$.
- iii. $\forall c \in \mathcal{C}(Y), \Omega^{-1}(c) \in \mathcal{C}(X)$.
- iv. The counterimages of opens under Ω are opens of X .
- v. $\forall b = \Omega(a) \in \text{Im } \Omega = \Omega(X)$ the counterimage of every neighbourhood K' of b is a neighbourhood of a

Proof. i. \Rightarrow ii. Let $A \in \mathcal{T}_y$ we have to prove that $\forall a \in \Omega^{-1}(A)$ exists a neighbourhood of a contained in $\Omega^{-1}(A)$. For $a \in \Omega^{-1}(A)$ one has $\Omega(a) \in \Omega(\Omega^{-1}(A)) = A$. A is open and is a neighbourhood of $\Omega(a)$ so if Ω is continuous there exists a neighbourhood K of a s.t. $\Omega(K) \subseteq A$ hence $K \subseteq \Omega^{-1}(A)$ and $a \in K$.

ii. \Rightarrow i. Let $a \in X$, I neighbourhood of $\Omega(a)$ let $A \in \mathcal{T}_y$ s.t. $\Omega(a) \in A \subseteq I$ for the ii. $\Omega^{-1}(A)$ is open and is a neighbourhood of a s.t. its image contains $\Omega(a)$ and A is contained in I . \square

Definition 12. Let (X, \mathcal{T}_x) and (Y, \mathcal{T}_y) be topological spaces, then $\Omega : X \rightarrow Y$ is defined open function (respectively closed function) if for every open subset A of X , $\Omega(A)$ is an open subset (closed subset) of Y .

Definition 13. A function $\Omega : X \rightarrow Y$ is said to be an homeomorphism if it is continuous and Ω, Ω^{-1} are continuous. The two topological spaces are said to be homeomorphic if that application exists and we write: $X \approx Y$.