Calculus II

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Theorem 1. *A is closed $\iff$ every accumulation point for A is in A*

Proof. " $\Longrightarrow$ " Let $A \subseteq \mathbb{R}^n,\ A = A \cup \partial A$.
Then $\forall p \in \bar{\mathcal{D}}(A),\ C_r(p)_{\diagdown p} \cap A \neq \emptyset\ \forall C \in \mathcal{C}_p$.
if $p \notin A$ then $C_r(p)$ has elements that dont belong to $A \Rightarrow p \in \partial A$.
" $\Longleftarrow$ " Let $p \in \partial A \Rightarrow \forall C \in \mathcal{C}_p$ of center r with $r \in \mathbb{R}$ by definition we can find some $x \in C_{\diagdown p} \cap A$, so that means $p \in \bar{\mathcal{D}}(A) \Rightarrow p \in A$. □

1 Limits

Definition 1. *Let $A \subseteq \mathbb{R}^2$ and (x_0, y_0) an accumulation point for A. we define A^* as follows:*
$A^* = \{(\rho, \theta) \in [0, +\infty] \times [0, 2\pi] : (x_0 + \rho\cos(\theta), y_0 + \rho\sin(\theta)) \in A\}$.

Proposition 1. *Lets suppose that exist a circle C of center (x_0, y_0) such that $C_{\diagdown\{(x_0,y_0)\}} \subseteq A$ let r be the radius of the circle and as a consequence $(0, r] \times [0, 2\pi] \subseteq A^*$*

Proof. Let $C_{\diagdown\{(x_0,y_0)\}}$ and $\begin{cases} 0 < \rho \leqslant r \\ 0 \leqslant \theta \leqslant 2\pi \end{cases}$ if $(\rho, \theta) \in (0, r] \times [0, 2\pi]$
then $(x_0 + \rho\cos(\theta), y_0 + \rho\sin(\theta)) \in C_{\diagdown\{(x_0,y_0)\}} \subseteq A \Rightarrow (\rho, \theta) \in A^*$. □

Definition 2. *Let $\theta \in [0, 2\pi]$ and $\forall \rho \in (0, r]$ we define $\varphi_\theta(\rho) = F(\rho, \theta)$ if $\rho \in (0, r], (\rho, 0) \in A^*$ so the $\lim_{\rho\to 0} \varphi(\rho) = l \in \bar{\mathbb{R}}$.*
If that limit exists that means $\forall \theta \in [0, 2\pi]$ and $\forall \varepsilon > 0$, $\exists \delta > 0\ \forall \rho \in (0, r]$ with $\rho < \delta\ \ |\varphi_\theta - l| < \varepsilon$.
We say that $\lim_{\rho\to 0} \varphi(\rho) = l \in \bar{\mathbb{R}}$ Uniformly With Respect To (U.W.R.T) θ.

Theorem 2. *Let $f : A \subseteq \mathbb{R}^2 \to \mathbb{R}$ with (x_0, y_0) accumulation point for A.*
Follows the equivalence:
$\lim_{(x,y)\to(x_0,y_0)} f(x, y) = l \in \bar{\mathbb{R}} \iff \lim_{\rho\to 0} F(\rho, \theta) = l$ *U.W.R.T* θ.

Proof. Let $l \in \bar{\mathbb{R}}$.
" $\Rightarrow$ " $\lim_{(x,y)\to(x_0,y_0)} f(x, y) = l$ so $\forall \varepsilon > 0, \exists \delta > 0 : \forall\ (x, y) \in A$
with $\|(x, y) - (x_0, y_0)\| < \delta,\ |f(x, y) - l| < \varepsilon$.
We have to prove that $\forall \varepsilon > 0, \exists \delta > 0 : \forall \theta \in [0, 2\pi],\ \forall \rho(0, r]$
with $\rho < \delta\ \ |F(\rho, \theta) - l| < \varepsilon$. □