Topology

Alessio Esposito

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Definition 1. $(\mathsf{X}_1, \mathcal{T}_1),\ (\mathsf{X}_2, \mathcal{T}_2)$ *topological spaces with* $\mathsf{X}_1 \cap \mathsf{X}_2 = \emptyset$.
let $X = \mathsf{X}_1 \cup \mathsf{X}_2, \mathcal{T} = \{A_1 \cup A_2 \mid A_i \in \mathcal{T}_i\}$.

Lemma 1. $(\mathsf{X}, \mathcal{T})$ *is a topological space.*

Proof. Lets verify the axioms.

1. $\emptyset \in \mathcal{T}_1,\ \emptyset \in \mathcal{T}_2 \Rightarrow \emptyset = \emptyset \cup \emptyset$
   $\mathsf{X}_i \in \mathcal{T}_i \Rightarrow \mathsf{X} = \mathsf{X}_1 \cup \mathsf{X}_2 \in \mathcal{T}$
2. $\{A\}_{i \in I} \in \mathcal{T} \Rightarrow A_i = A_{1,i} \cup A_{2,i}$
   $\bigcup_{i \in I} A_i = \bigcup_{i \in I}(A_{1,i} \cup A_{2,i}) = \bigcup_{i \in I}(A_{1,i}) \cup \bigcup_{i \in I}(A_{2,i}) \in \mathcal{T}_1 \cup \mathcal{T}_2$
3. $A, A' \in \mathcal{T} \Rightarrow A = A_1 \cup A_2, A' = A'_1 \cup A'_2$
   $A \cap A' = (A_1 \cup A_2) \cap (A'_1 \cup A'_2) = (A_1 \cap A'_1) \cup (A_2 \cap A'_2) \in \mathcal{T}$

$\square$

Definition 2. $\mathcal{T}$ *is defined as a sum of Topologies* $\mathcal{T}_1$ *and* $\mathcal{T}_2$.
We notice that $\forall A_i \in \mathcal{T}_i,\ A_i \cup \emptyset \in \mathcal{T}$ *so* $\mathcal{T}$ *contains* $\mathcal{T}_1$ *and* $\mathcal{T}_2$.

Definition 3. $(\mathsf{X}, \mathcal{T})$ *topological space* $\mathsf{Y} \subseteq \mathsf{X},\ \mathsf{Y} \neq \emptyset$
$\mathcal{T}_{/Y} = \{A \cap \mathsf{Y} \mid A \in \mathcal{T}\}$ *prove that* $\mathcal{T}_{/Y}$ *is a topology defined as Inducted topology on* Y

Definition 4. $(\mathsf{X}, \mathcal{T})$ *topological space,* X *verifies the second axiom of numerability if posseses a finite base or numerable, in that case* $(\mathsf{X}, \mathcal{T})$ *is said* $\mathcal{N}_2$

Proposition 1. *Let* $\mathbb{R}$ *be gifted by the topology with a base of the following type:*

$$[a, b],\ a < b$$

Then $(\mathbb{R}, \mathcal{T})$ *is not* $\mathcal{N}_2$

Proof. Let $\mathcal{B}$ a base for $\mathcal{T}$. Let $a > 0 \in \mathbb{R}$ then $\forall x \in \mathbb{R}$, there exists $B_x \in \mathcal{B}$ with $x \in B_x \subseteq [x, x+a]$. If $y \in \mathbb{R}$ with $y > x$ then $x \notin [y, y+a]$ so $x \notin B_y$. The application $x \in \mathbb{R} \longmapsto B_x \in \mathcal{B}$ is injective so $\mathcal{B}$ has the continuum order. $\square$

Continous functions and homeomorphisms

Definition 5. *Let* $(\mathsf{X}, \mathcal{T}_x)$, $(\mathsf{Y}, \mathcal{T}_y)$ *be topological spaces,* $\Omega : \mathsf{X} \to \mathsf{Y}$ *is continous in* $a \in \mathsf{X}$ *if* $\forall I$ *neighbourhood of* $\Omega(a)$, $\exists K$ *neighbourhood of* a *s.t.* $\Omega(K) \subseteq I$.

We'll say that a function is continous if it is continous in every point.

Proposition 2. *Let* $\Omega : (\mathsf{X}, \mathcal{T}_x) \to (\mathsf{Y}, \mathcal{T}_y)$ *so then the following affirmations are equivalent:*

i. Ω *is continous.*

ii. $\forall A \in \mathcal{T}_y,\ \Omega^{-1}(A) \in \mathcal{T}_x$.

iii. $\forall c \in \mathcal{C}(\mathsf{Y}),\ \Omega^{-1}(c) \in \mathcal{C}(\mathsf{X})$.

iv. *The counterimages of opens under a selected base of* Y *are opens of* X.

v. $\forall b = \Omega(a) \in Im\Omega = \Omega(\mathsf{X})$ *the counterimage of every neighbourhood* K' *of* b *is a neighbourhood of* a

Proof. *i.* $\Rightarrow$ *ii.* Let $A \in \mathcal{T}_y$ we have to prove that $\forall a \in \Omega^{-1}(A)$ exists a neighbourhood of a contained in $\Omega^{-1}(A)$. For $a \in \Omega^{-1}(A)$ one has $\Omega(a) \in \Omega(\Omega^{-1}(A)) = A$. A is open and is a neighbourhood of $\Omega(a)$ so if Ω is continous there exists a neighbourhood K of a s.t. $\Omega(K) \subseteq A$ hence $K \subseteq \Omega^{-1}(A)$ and $a \in K$.
ii. $\Rightarrow$ *i.* Let $a \in \mathsf{X},\ I$ neighbourhood of $\Omega(a)$ let $A \in \mathcal{T}_y$ s.t. $\Omega(a) \in A \subseteq K$ for the *ii*. $\Omega^{-1}(A)$ is open and is a neighbourhood of a s.t. its image contains $\Omega(a)$ and A is contained in I. $\square$