Expected values

Statistical Inference

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Expected values

The **expected value** or **mean** of a random variable is the center of its distribution
For discrete random variable with PMF , it is defined as follows

where the sum is taken over the possible values of
represents the center of mass of a collection of locations and weights,

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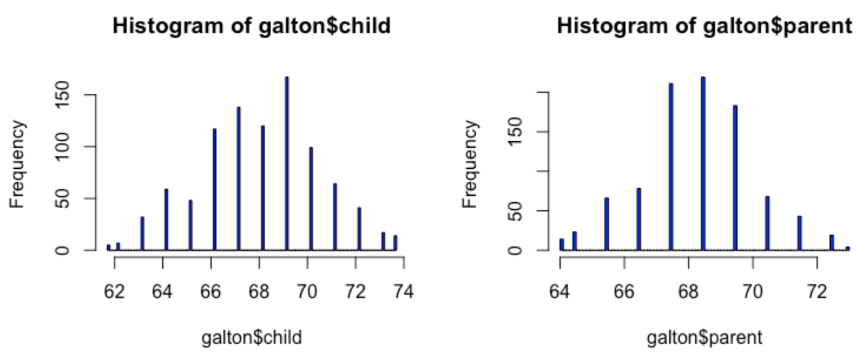
Example

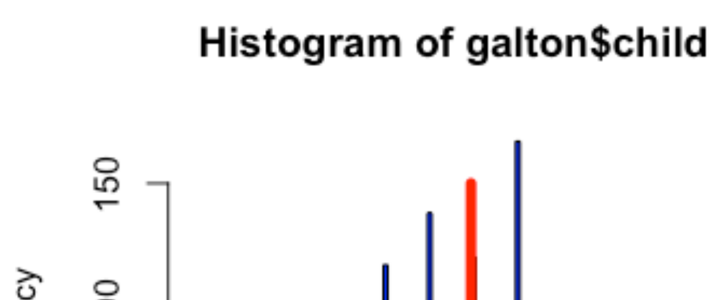
Find the center of mass of the bars

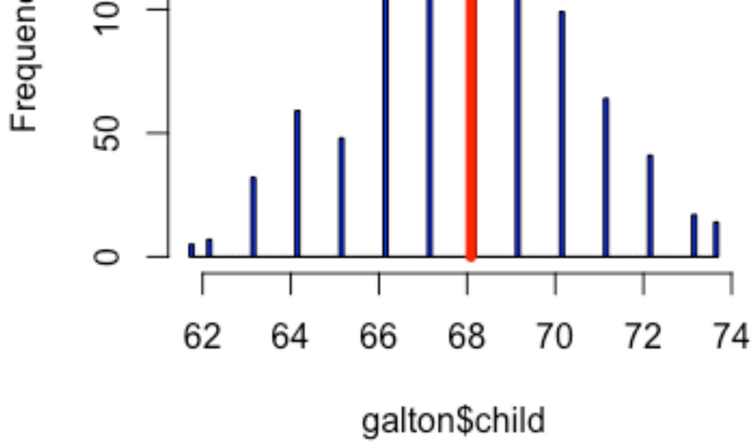
```
## Loading required package: MASS
## Loading required package: HistData
## Loading required package: Hmisc
## Loading required package: lattice
## Loading required package: survival
## Loading required package: Formula
## Loading required package: ggplot2
## 
## Attaching package: 'Hmisc'
## The following objects are masked from 'package:base':
## 
##     format.pval, round.POSIXt, trunc.POSIXt, units
## 
## Attaching package: 'UsingR'
## The following object is masked from 'package:survival':
## 
##     cancer
```



The center of mass is the empirical mean

```r
hist(galton$child,col="blue",breaks=100)
meanChild <- mean(galton$child)
lines(rep(meanChild,100),seq(0,150,length=100),col="red",lwd=5)
```





Example

Suppose that a die is rolled and is the number face up
What is the expected value of ?

Again, the geometric argument makes this answer obvious without calculation.

Continuous random variables

For a continuous random variable, , with density, , the expected value is defined as follows

This definition borrows from the definition of center of mass for a continuous body

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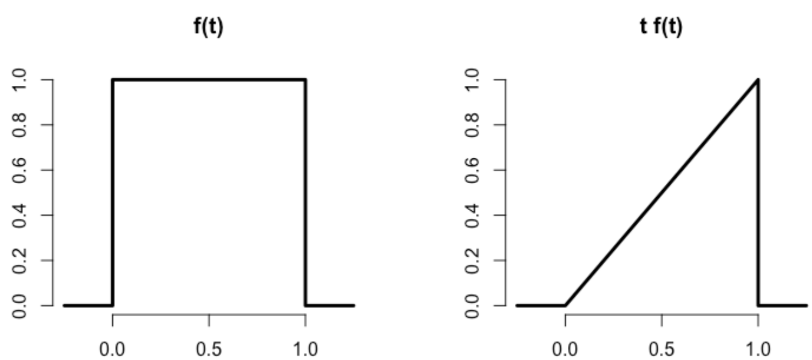
Example

Consider a density where for between zero and one
(Is this a valid density?)
Suppose that follows this density; what is its expected value?



Rules about expected values

The expected value is a linear operator
If and are not random and and are two random variables then

Example

You flip a coin, and simulate a uniform random number , what is the expected value of their sum?

Another example, you roll a die twice. What is the expected value of the average?
Let and be the results of the two rolls

Example

Let for be a collection of random variables, each from a distribution with mean
Calculate the expected value of the sample average of the

Remark

Therefore, the expected value of the **sample mean** is the population mean that it's trying to estimate

When the expected value of an estimator is what its trying to estimate, we say that the estimator is **unbiased**

The variance

The variance of a random variable is a measure of *spread*

If is a random variable with mean , the variance of is defined as

the expected (squared) distance from the mean - Densities with a higher variance are more spread out than densities with a lower variance

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Convenient computational form

If is constant then

The square root of the variance is called the **standard deviation**

The standard deviation has the same units as

Example

What's the sample variance from the result of a toss of a die?

Example

What's the sample variance from the result of the toss of a coin with probability of heads (1) of ?

Interpreting variances

Chebyshev's inequality is useful for interpreting variances

This inequality states that

For example, the probability that a random variable lies beyond standard deviations from its mean is less than

Note this is only a bound; the actual probability might be quite a bit smaller

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Example

IQs are often said to be distributed with a mean of and a sd of

What is the probability of a randomly drawn person having an IQ higher than or below ?

Thus we want to know the probability of a person being more than standard deviations from the mean

Thus Chebyshev's inequality suggests that this will be no larger than 6%

IQs distributions are often cited as being bell shaped, in which case this bound is very conservative

The probability of a random draw from a bell curve being standard deviations from the mean is on the order of (one thousandth of one percent)

Example

A former buzz phrase in industrial quality control is Motorola's "Six Sigma" whereby businesses are suggested to control extreme events or rare defective parts

Chebyshev's inequality states that the probability of a "Six Sigma" event is less than

If a bell curve is assumed, the probability of a "six sigma" event is on the order of (one ten millionth of a percent)