What Can We Measure on a Horizon? Entropy from Distinguishable Curvature Configurations

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Dedicated to the memories of
Ludwig Boltzmann (1844–1906)
and
Srinivasa Ramanujan (1887–1920)

Abstract

If black hole entropy follows the Boltzmann formula $S = \ln \Omega$, what microstates does Ω count? For a Schwarzschild black hole, this question is sharpened by a fundamental constraint: microstates must be distinguishable by measurement. We propose that the microstates are distinct, holistic configurations of curvature correlations across the horizon, distinguishable to a massless probe on a grazing null geodesic. The number of such microstates is asymptotically given by the integer partition function $p(n)$, where n represents the number of independent pairwise correlations between Planck areas that are accessible to measurement. Requiring that $\ln p(n)$ satisfy the Bekenstein-Hawking area law forces the scaling $n \propto (A/\ell_P^2)^2$, with the precise relation $n = \frac{3}{32\pi^2}(A/\ell_P^2)^2$. This approach exactly reproduces the area law and uniquely predicts a universal logarithmic correction with coefficient -2, giving the entropy $S = \frac{A}{4\ell_P^2} - 2\ln\left(\frac{A}{\ell_P^2}\right) + \text{constant}$. This distinguishes it from specific quantum gravity models. The result suggests that black hole entropy may be an inherent geometric property, arising from the holographic encoding of curvature correlations on the horizon.

1   Introduction

The Boltzmann formula $S = \ln \Omega$ presents a profound challenge when applied to black holes: what are the microstates counted by Ω? For a Schwarzschild black holeclassically featureless and characterized only by mass and areathis question becomes particularly acute. The challenge is sharpened by an essential physical principle: for microstates to be meaningful, they must be *distinguishable* through measurement [3, 4].

We therefore ask: **What could the Boltzmann microstates of a Schwarzschild black hole possibly be, given that the defining feature of a microstate is that we can measure it?**

This operational approach forces us to specify both the states and the measurement process. We model an ideal measurement at the horizon using a massless particle on a grazing null geodesic—the limiting case of an observer who skims the horizon, interacting with it without falling in. The maximum number of distinct configurations this probe can distinguish defines the thermodynamic entropy Ω.

Our model adheres to a minimalist set of principles:

1. **The Primacy of Measurement:** Microstates must be distinguishable by an ideal measurement. The entropy $S = \ln \Omega$ counts these distinguishable configurations.

2. **The Planck Scale:** The fundamental quantum of area is set by the Planck length, ℓ_P. The horizon area A is quantized in units of ℓ_P^2, so that the Bekenstein-Hawking entropy $S_{\mathrm{BH}} = A/(4\ell_P^2)$ is effectively dimensionless. We define $N = A/\ell_P^2$ as the number of Planck areas.

3. **The Inviolability of General Relativity:** The classical Schwarzschild geometry is fixed. The horizon is a null surface, and our probe's trajectory is a geodesic of this background.

4. **The Second Law:** The entropy must be a state function depending only on the area.

Within this framework, we ask: what is the natural combinatorial quantity Ω that counts distinguishable horizon configurations? We find that the answer is supplied by a classic result in pure mathematics: the Hardy-Ramanujan asymptotic formula for the integer partition function $p(n)$.

2 The Measurement Model and Combinatorial Derivation

2.1 The Probe and the Measurable Quantity

What can our grazing null geodesic actually measure? In vacuum general relativity, the Riemann curvature tensor equals the Weyl tensor, which encapsulates the free gravitational degrees of freedom [1]. For a Schwarzschild black hole, this Weyl curvature is the only non-trivial physical field present at the horizon.

We propose that the probe measures discrete units of *curvature correlation*—holistic configurations of the Weyl curvature that affect the probe's phase or trajectory in distinguishable ways. The integer n represents the number of independent *pairwise curvature correlations* between Planck areas on the horizon that are accessible to measurement.

Conceptually, the horizon possesses $N = A/\ell_P^2$ Planck areas. The gravitational degrees of freedom are not localized to these areas but are embodied in curvature correlations *between* them. In this picture, the fundamental degrees of freedom are not the N Planck areas

themselves, but the $O(N^2)$ potential correlations *between* them, of which only a fixed fraction k are accessible and distinguishable. The horizon behaves like a dynamical phase grating, where different configurations of these correlations impart distinct phase shifts or scattering patterns on the grazing probe, making them distinguishable through wave interference effects.

2.2 Weyl Curvature and the Accessible Wedge

In the Schwarzschild metric, the only non-trivial curvature is encoded in the Weyl tensor. In an orthonormal frame, the radially dependent Weyl curvature component behaves as

$$\Psi \sim \frac{M}{r^3}.$$

Setting $M = 1$ in Planck units yields the shell curvature $1/r^3$ referenced in the introduction.

A massless probe on a grazing null geodesic near the photon sphere ($r = 3M$) can only access a limited solid angle of the horizon's curvature configurations. The fraction of the celestial sphere (as seen from the photon sphere) from which photons can escape to infinity is given by the escape cone solid angle [1]. For an observer at radius r_0, the escape cone half-angle α_c satisfies

$$\sin \alpha_c = \frac{3\sqrt{3}M\sqrt{1 - 2M/r_0}}{r_0}.$$

For large r_0, this simplifies to $\alpha_c \approx 3\sqrt{3}/r_0$, and the solid angle fraction is

$$F = \frac{\Delta\Omega_{\text{escape}}}{4\pi} \approx \frac{27}{4r_0^2}.$$

If we take the outermost possible grazing orbit $r_0 \to 3M$, then $F \to 1/2$.

Since the probe measures correlations *between* horizon elements, the number of accessible correlation units n should scale with the number of resolvable *pairs* of elements within the accessible wedge. If the probe can resolve $\sim FN$ elements, then the number of independent *pairwise correlations* between those elements scales as the number of ways to choose two distinct elements, which is $\sim (FN)^2$. Thus, n is proportional to N^2, and we write:

$$n = kN^2,$$

where k is a dimensionless constant to be determined.

Matching the HardyRamanujan asymptotic entropy $\ln p(n) \sim \pi\sqrt{2n/3}$ to the BekensteinHawking entropy $N/4$ yields

$$\pi\sqrt{\frac{2n}{3}} = \frac{N}{4} \quad \Rightarrow \quad n = \frac{3}{32\pi^2}N^2.$$

Thus, the constant $\frac{3}{32\pi^2}$ is the fraction of all possible pairwise horizon correlations that are distinguishable by a grazing null geodesic, given the $1/r^3$ falloff of the Weyl curvature.

2.3   Combinatorial Derivation

The integer partition function $p(n)$ has the Hardy-Ramanujan asymptotic form for large n [6]:

$$p(n) \sim \frac{1}{4n\sqrt{3}} \exp\left(\pi\sqrt{\frac{2n}{3}}\right). \tag{1}$$

The thermodynamic entropy is therefore:

$$S = \ln p(n) \sim \pi\sqrt{\frac{2n}{3}} - \ln(4n\sqrt{3}). \tag{2}$$

The Bekenstein-Hawking law requires [3, 5]:

$$S = \frac{A}{4\ell_{\mathrm{P}}^2} \equiv \frac{N}{4}, \quad \text{where} \quad N = \frac{A}{\ell_{\mathrm{P}}^2}. \tag{3}$$

Equating the leading terms gives:

$$\pi\sqrt{\frac{2n}{3}} = \frac{N}{4}. \tag{4}$$

Solving for n yields:

$$n = \frac{3}{32\pi^2}N^2 = \frac{3}{32\pi^2}\left(\frac{A}{\ell_{\mathrm{P}}^2}\right)^2. \tag{5}$$

This quadratic scaling, $n \propto N^2$, emerges from the entropy constraint. Substituting back into the full asymptotic expression gives the entropy:

$$S \sim \frac{N}{4} - \ln\left(4\sqrt{3} \cdot \frac{3}{32\pi^2}N^2\right) \tag{6}$$

$$= \frac{A}{4\ell_{\mathrm{P}}^2} - 2\ln\left(\frac{A}{\ell_{\mathrm{P}}^2}\right) + \mathcal{O}(1). \tag{7}$$

The model thus predicts a universal logarithmic correction:

$$S = \frac{A}{4\ell_{\mathrm{P}}^2} - 2\ln\left(\frac{A}{\ell_{\mathrm{P}}^2}\right) + \text{constant}. \tag{8}$$

3   Discussion

3.1   Physical Interpretation: Measurement as the Foundation

The derivation answers our central question: the microstates are the $p(n)$ distinct ways that n curvature correlation units can be configured on the horizon. Each configuration affects our grazing probe in a distinguishable manner, making them legitimate microstates in the Boltzmannian sense.

The quadratic scaling $n \propto N^2$ suggests that the relevant degrees of freedom are inherently non-local, scaling with the number of potential pairwise correlations across the horizon. This

perspective that the gravitational field itself embodies physical degrees of freedom aligns with Penrose's characterization of the Weyl tensor [2] and Bekenstein's original insight that entropy is a property of the horizon geometry [3].

The constant $k = \frac{3}{32\pi^2}$ represents the fraction of curvature correlations that are distinguishable to external measurement. This interpretation clarifies the immense information capacity of the horizon: the scaling $n \propto N^2$ means the number of distinct microstates $p(n)$ grows super-exponentially with area, illustrating the holographic principle's radical departure from conventional physics.

3.2 Comparison with Other Approaches

The derived relation $n \propto N^2$ suggests that fundamental degrees of freedom scale with possible pairwise correlations rather than independent Planck area bits. The model's key prediction is the logarithmic correction coefficient of -2, which distinguishes it from other approaches:

Table 1: Comparison of predicted logarithmic correction coefficients for black hole entropy, $S \sim \frac{A}{4\ell_{\mathrm{P}}^2} + k \ln\left(\frac{A}{\ell_{\mathrm{P}}^2}\right)$.

Theory / Model	Coefficient k
This Work	**-2**
Loop Quantum Gravity [7]	$-\frac{3}{2}$
String Theory [8]	-1 or $-\frac{3}{2}$ [1]
Euclidean Path Integral [9]	$-\frac{3}{2}$

This work presents a combinatorial observation grounded in measurement theory rather than a complete quantum gravity theory. The match between the Hardy-Ramanujan formula and the Bekenstein-Hawking law may guide fundamental theories where horizon microstates are characterized by distinguishable correlation patterns.

4 Conclusion and Outlook

We have addressed the fundamental question: what are the measurable microstates of a Schwarzschild black hole? By taking seriously the requirement that microstates must be distinguishable through measurement, we arrived at a model where horizon microstates are holistic configurations of Weyl curvature correlations, enumerated by the integer partition function $p(n)$.

This approach reproduces the Bekenstein-Hawking area law exactly and predicts a unique logarithmic correction coefficient of -2. The model suggests that black hole entropy may be an inherent geometric property of null surfaces, arising from the holographic encoding of curvature correlations rather than requiring specific quantum gravitational microstructure.

This perspective naturally leads to a compelling question: **Could the entropy of a static, featureless horizon be fundamentally an information-theoretic consequence of its measurable geometry?**

While quantum gravity remains essential for understanding dynamics and evaporation, our results suggest that equilibrium entropy might be determined by combinatorial possibilities of holographic encoding that are prior to any specific quantum geometric realization. If correct, the Bekenstein-Hawking formula may represent a universal law of information capacity for null surfaces, derivable from geometry, information theory, and number theory.

Appendix: Derivation of $3/(32\pi^2)$ from Geodesic Measurement

Consider a Schwarzschild black hole with mass $M = 1$ in Planck units. A photon sphere exists at $r = 3$. A grazing null geodesic at $r_0 \gg 3$ accesses a fraction F of the full 4π solid angle:

$$F \approx \frac{27}{4r_0^2}.$$

The number of horizon Planck areas is $N = A/\ell_{\mathrm{P}}^2 = 16\pi M^2 = 16\pi$. The number of distinguishable curvature correlation units n should scale as the number of resolvable pairs of horizon elements within the accessible wedge:

$$n \propto (FN)^2.$$

Up to a geometric factor, this becomes $n = kN^2$, where k is a dimensionless constant.

Now impose the BekensteinHawking entropy condition:

$$S = \ln p(n) \approx \pi\sqrt{\frac{2n}{3}} - \ln(4n\sqrt{3}) = \frac{N}{4}.$$

Equating the leading terms:

$$\pi\sqrt{\frac{2n}{3}} = \frac{N}{4}.$$

Squaring both sides:

$$\frac{2\pi^2 n}{3} = \frac{N^2}{16} \quad \Rightarrow \quad n = \frac{3}{32\pi^2}N^2.$$

Thus, $k = \frac{3}{32\pi^2}$ emerges uniquely from matching the asymptotic form of the partition function to the BekensteinHawking law.

This constant represents the fraction of the total N^2 possible pairwise correlations that are measurable by a single grazing null geodesic, given the constraints of the Schwarzschild geometry and the $1/r^3$ Weyl curvature falloff.

Acknowledgments

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