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Gal Gross
Eckhard Meinrenken

Manifolds, Vector Fields, and Differential Forms

An Introduction to Differential Geometry

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Preface

This book is intended as a modern introduction to Differential Geometry, at a level accessible to advanced undergraduate students or master's students. Earlier versions of this text have been used as lecture notes for an undergraduate course in Differential Geometry at the University of Toronto since 2015, taught by the second author for the first three years.

As the title of this book indicates, we take 'Differential Geometry' to mean the theory of *manifolds*. Over the past few decades, manifolds have become increasingly important in many branches of mathematics and physics. There is an enormous amount of literature on the subject, and many outstanding textbooks. However, most of these references are pitched at a graduate or postgraduate level and are not suited for a more basic course. It is this gap that this book aims to address.

Accordingly, the book will rely on a minimal set of prerequisites. The required background material is typically covered in the first two or three years of university: a solid grounding in linear algebra and multivariable calculus, and ideally a course on ordinary differential equations. We will not require knowledge of abstract algebra or point set topology, but rather develop some of the necessary notions *on the fly*, and only in the generality needed here.

A few words about the philosophy of this book. We introduce manifolds 'intrinsically' in terms of coordinate patches, glued by transition functions. From this perspective, the theory of manifolds appears as a natural continuation of multivariable calculus; the role of point set topology is kept to a minimum.

We believe that it is important to develop *intuition* for the concepts introduced, to get a feel for the subject. To a large extent, this means *visualization*, but this does not always involve drawing pictures of curves and surfaces in two or three dimensions. For example, it is not difficult to get an understanding of the projective plane intrinsically, and also to 'visualize' the projective plane, but it is relatively hard to depict the projective plane as a surface (with self-intersections) in 3-space. For surfaces such as the Klein bottle or the 2-torus, such depictions are easier, but even in those cases they are not always helpful, and can even be a little misleading.

As another example, a surface in 3-space is considered non-orientable if it ‘has only one side,’ but it is not immediately clear that this property is intrinsic to the surface, or how to generalize to two-dimensional surfaces in higher dimensional spaces.

While we are trying to develop a ‘hands-on’ approach to manifolds, we believe that a certain level of *abstraction* cannot and should not be avoided. By analogy, when students are exposed to general vector spaces in Linear Algebra, the concept may seem rather abstract at first. But it usually does not take long to absorb these ideas, and gain familiarity with them even if one cannot always draw pictures. Likewise, not all of differential geometry is accounted for by drawing pictures, and what may seem abstract initially will seem perfectly natural with some practice and experience.

Another principle that we aim to follow is to provide good *motivation* for all concepts, rather than just impose them. Some subtleties or technical points have emerged through the long development of the theory, but they exist for reasons, and we feel it is important to expose those reasons.

Finally, we consider it important to offer some *practice* with the theory. Sprinkled throughout the text are questions, indicated by a feather symbol, that are meant to engage the student, and encourage active learning. Often, these are ‘review questions’ or ‘quick questions’ which an instructor might pose to students to stimulate class participation. Sometimes they amount to routine calculations, which are more instructive if the student attempts to do them on their own. In other cases, the student is encouraged to try out a new idea or concept before moving on. Answers to these questions are provided at the end of the book.

Each chapter concludes with ‘problems,’ which are designed as homework assignments. These include simpler problems whose goal is to reinforce the material, but also a large number of rather challenging problems. Most of these problems have been tried out on students at the University of Toronto, and have been revised to make them as clear and interesting as possible. We are grateful to Boris Khesin for letting us include some of his homework problems, and we suggest his wonderful article [12] with Serge Tabachnikov, titled ‘Fun problems in geometry and beyond,’ for further readings along these lines.

Prerequisites

Students should be comfortable with fundamental concepts from multivariable calculus: open and closed subsets, directional derivative, Jacobian matrix, the inverse and implicit function theorems, the Riemann integral and the change-of-variables formula, the multivariable Taylor theorem, and (ideally) the Heine-Borel theorem. Familiarity with the classical “vector calculus” operations of div, curl, grad, and the theorems of Gauss, Green, and Stokes would be useful as motivation for Chapter 7 on differential forms. Some familiarity with differential forms as a calculational tool in $\mathbb{R}^n$ is helpful but is not necessary; we review the formal symbolic manipulation at the beginning of that chapter, and then develop the theory step by step. The discussion of vector fields uses the basic results on ordinary differential equations, and in particular, the theorem on the existence and uniqueness of solutions for systems of first-order equations. Prior knowledge of point set topology is helpful, but not required.

Suggestions for Instructors

This book contains significantly more material than can possibly be covered in a one-semester course, and the instructor will have to be selective. One possible approach is to treat the theory of vector fields in some detail, but omit the material on differential forms. On the other hand, if it is desired to cover differential forms as well, then it will be necessary to take shortcuts with some of the earlier material.

In both approaches, the course would include most of the material until Chapter 5 on tangent spaces, but not all of this material is essential. We have marked with an asterisk the sections which may be omitted on first reading. For instance, the discussion of linkages, complex projective spaces and complex Grassmannians, the material on connected sums and quotients of manifolds, and the Hopf fibration. These starred sections are useful as further source of examples and exercises and can help deepen students' understanding of the material or present some relevant constructions and generalizations.

In addition, one can postpone the sections on compactness and orientability (Sections 2.5–2.6), since these are not strictly needed until the chapter on integration (Chapter 8). In the discussion of maps of maximal rank (Chapter 4), it may suffice to prove the basic facts on submersions, but keep the discussion of immersions to a minimum, since the arguments are very similar.

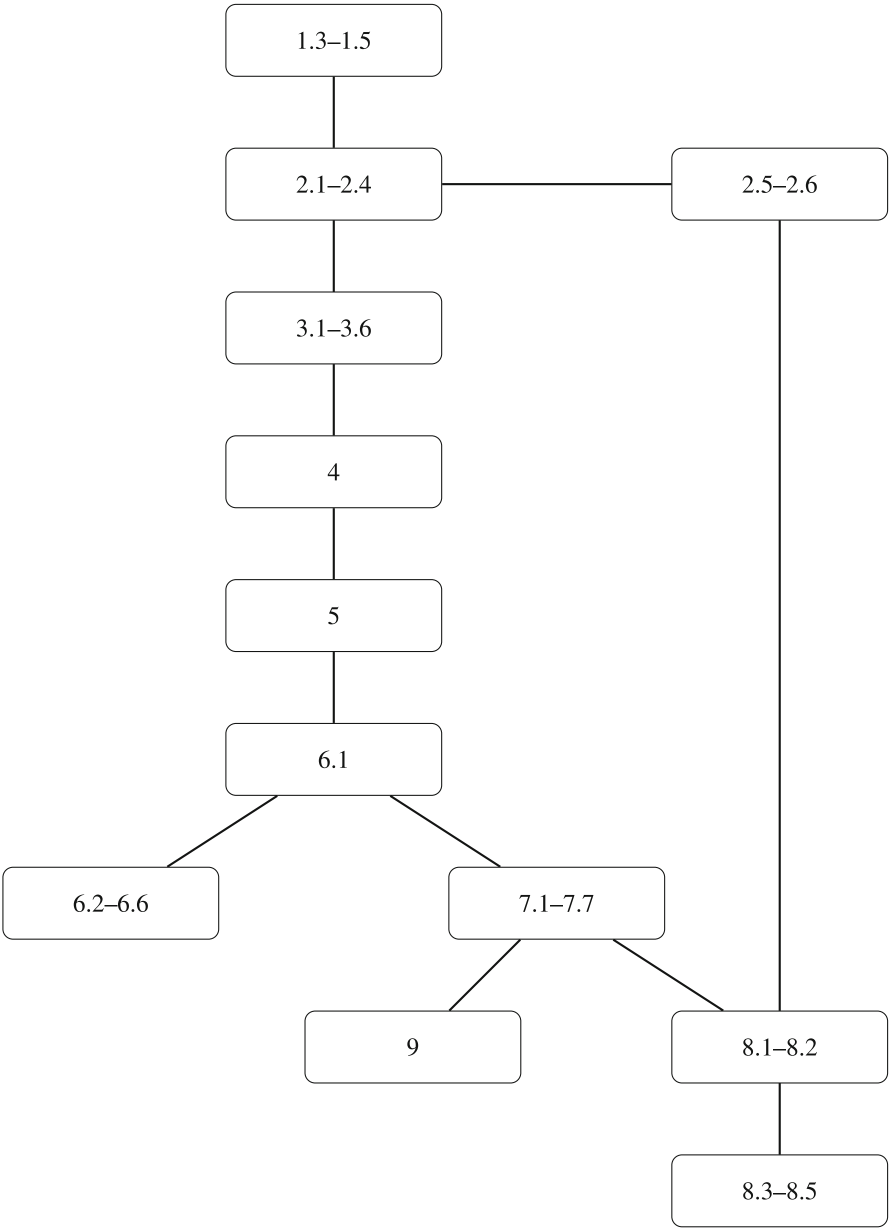
If the aim is to include differential forms and their applications, one will have to keep the discussion of vector fields rather brief—perhaps omitting the concept of related vector fields, the geometric interpretation of Lie brackets, and Frobenius' theorem. The theory of integration will require 'partitions of unity,' but one may want to skip the existence proof (which is presented in Appendix C); in contrast, 'bump functions' are used in several proofs throughout the text. Once Stokes' theorem has been reached, Chapter 8 describes many applications—an instructor will have to pick and choose.

It is also possible to leave out the material on differential forms entirely, and instead concentrate on a more detailed treatment of vector fields. In this approach, one might spend more time on flows of vector fields, the geometric interpretation of Lie brackets, and Frobenius' theorem. As an application, one could include a discussion of Poincaré's theorem in Section 8.5.3; this would require a brief discussion of winding numbers. (Winding numbers are covered in Section 8.3.2 in the context of differential forms, but it is straightforward to give a direct treatment.) Alternatively, or in addition, one might explain the construction of tangent and cotangent bundles, and finish with a discussion of general vector bundles.

The diagram below shows the logical dependency between the chapters. Any one of the terminal nodes is a fitting capstone for a course, illustrating the three approaches discussed above.

Acknowledgments

As already indicated, earlier versions of this book have been used as a textbook at the University of Toronto for several years. We thank the students participating in these courses for numerous helpful comments and excellent questions, improving the readability of this text. We are grateful to Ed Bierstone, Marco Gualtieri, Daniel Hudson, Robert Jerrard, and Boris Khesin at the University of Toronto, as well as Chenchang Zhu at Universität Göttingen and Maria Amelia Salazar at Universidade Federal Fluminense, for pointing out errors and for many suggestions. We thank Catherine Cheung for creating the illustrations, as well as Paul Nylander for permission to use some of his artwork. We are grateful to Ann Kostant for help and encouragement at the early stages of this project.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Contents

1 Introduction

1.1 A Very Short History

In the words of S.S. Chern, "*the fundamental objects of study in differential geometry are manifolds.*" [5, Page 332]. Roughly, an n-dimensional manifold is a mathematical object that "locally" looks like $\mathbb{R}^n$. The theory of manifolds has a long and complicated history. For centuries, manifolds have been studied extrinsically, as subsets of Euclidean spaces, given, for example, as level sets of equations. In this context, it is not always easy to separate the properties of a manifold from the choice of an embedding; a famous discovery in this context is Carl Friedrich Gauss' *Theorema Egregium* from 1828, proving that the Gauss curvature of embedded surfaces depends only on the choice of a metric on the surface itself . The term "manifold" goes back to the 1851 thesis [17] of Bernhard Riemann, *"Grundlagen für eine allgemeine Theorie der Functionen einer veränderlichen complexen Grösse"* ("foundations of a general theory of functions of a complex variable") and his 1854 *Habilitation* address [18] *"Über die Hypothesen, welche der Geometrie zugrunde liegen"* ("on the assumptions underlying geometry").

However, in neither reference did Riemann attempt to give a precise definition of the concept. This was done subsequently through the work of many authors, including

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Riemann himself. See, e.g., [19] for the long list of names involved. Henri Poincaré, in his 1895 work *analysis situs* [10], introduced the idea of a *manifold atlas*.

A rigorous axiomatic definition of manifolds was given by Oswald Veblen and J. H. C. Whitehead [22] only in 1931.

We will see below that the concept of a manifold is really not all that complicated; and in hindsight it may come as a surprise that it took so long to evolve. Initially, the concept may have been regarded as simply a change of perspective—describing manifolds intrinsically from the outset, rather than extrinsically, as regular level sets of functions on Euclidean space.

Developments in physics played a major role in supporting this new perspective. In Albert Einstein's theory of General Relativity from 1916, space-time is regarded as a 4-dimensional manifold with no distinguished coordinates (not even a distinguished separation into space and time directions); a local observer may want to introduce local *xyzt* coordinates to perform measurements, but all physically meaningful quantities must admit formulations that are "manifestly coordinate-independent." At the same time, it would seem unnatural to try to embed the 4-dimensional curved space-time continuum into some higher-dimensional flat space in the absence of any physical significance for the additional dimensions. For the various vector-valued functions appearing in the theory, such as electromagnetic fields, one is led to ask about their "natural" formulation consistent with their transformation properties under local coordinate changes. The theory of differential forms, introduced in its modern form by Elie Cartan in 1899, and the associated coordinate-free notions of differentiation and integration become inevitable at this stage. Many years later, *gauge theory* once again emphasized coordinate-free formulations and provided physics-based motivations for more elaborate constructions such as fiber bundles and connections.

Since the late 1940s and early 1950s, differential geometry and the theory of manifolds have become part of the basic education of any mathematician or theoretical physicist, with applications in other areas of science such as engineering and economics. There are many sub-branches, such as complex geometry, Riemannian geometry, and symplectic geometry, which further subdivide into sub-sub-branches. It continues to thrive as an active area of research, with exciting new results and deep open questions.

1.2 The Concept of Manifolds: Informal Discussion

To repeat, an n-dimensional manifold is something that "locally" looks like $\mathbb{R}^n$. The prototype of a manifold is the surface of planet Earth:

It is (roughly) a 2-dimensional sphere, but we use local charts to depict it as subsets of 2-dimensional Euclidean space. Note that such a chart will always give a somewhat distorted picture of the planet; the distances on the sphere are never quite correct, and either the areas or the angles (or both) are wrong. For example, in the standard maps of the world, Greenland always appears much bigger than it really is. (Do you know how its area compares to that of India?)

To describe the entire planet, one uses an atlas with a collection of such charts, such that every point on the planet is depicted in at least one such chart.

This idea will be used to give an "intrinsic" definition of a manifold, as essentially a collection of charts glued together in a consistent way. One then proceeds to develop analysis on such manifolds, for example, a theory of integration and differentiation, by working in charts. The task is then to understand the change of coordinates as one leaves the domain of one chart and enters the domain of another.

1.3 Manifolds in Euclidean Space

In multivariable calculus, you may have encountered manifolds as solution sets of equations. For example, the solution set $S \subseteq \mathbb{R}^3$ of an equation of the form $f(x,y,z) = a$ defines a smooth surface in $\mathbb{R}^3$, provided the gradient of f is non-vanishing at all points of S. We call such a value of f a *regular value*, and hence $S = f^{-1}(a)$ a *regular level set*.* Similarly, the joint solution set $C \subseteq \mathbb{R}^3$ of two equations

$$f(x,y,z) = a, \quad g(x,y,z) = b$$

defines a smooth curve in $\mathbb{R}^3$, provided (a,b) is a regular value† of (f,g) in the sense that the gradients of f and g are linearly independent at all points of C.

A familiar example of a manifold is the 2-dimensional sphere S^2, conveniently described as a level surface inside $\mathbb{R}^3$:

$$S^2 = \{(x,y,z) \in \mathbb{R}^3 \,|\, x^2 + y^2 + z^2 = 1\}.$$

There are many ways of introducing local coordinates on the 2-sphere: For example, one can use spherical polar coordinates, cylindrical coordinates, stereographic projections, or orthogonal projections onto the coordinate planes. We will discuss some of these coordinates below. More generally,‡ one has the n-dimensional sphere S^n inside $\mathbb{R}^{n+1}$,

$$S^n = \{(x^0, \ldots, x^n) \in \mathbb{R}^{n+1} \,|\, (x^0)^2 + \cdots + (x^n)^2 = 1\}.$$

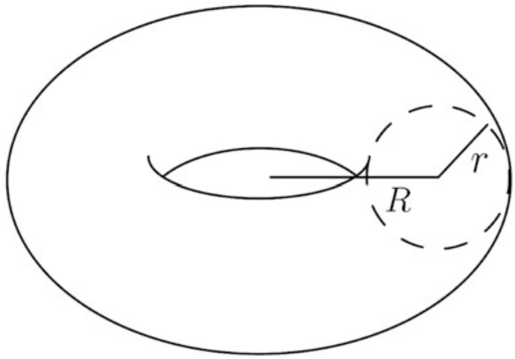
The 0-sphere S^0 consists of two points, the 1-sphere S^1 is the *unit circle*.

* Let us also take this opportunity to remind the reader of certain ambiguities of notation. Given a function $f : X \to Y$ and any subset $B \subseteq Y$, we have the preimage defined by $f^{-1}(B) = \{x \in X \,|\, f(x) \in B\}$. It is common to write $f^{-1}(a)$ for $f^{-1}(\{a\})$, so that the former is a subset of the domain of f. If it happens that f is bijective, one also has the inverse function $f^{-1} : Y \to X$ defined by $f^{-1}(y)$ is the unique $x \in X$ with $f(x) = y$ (that is, if f is given by the rule $x \mapsto y$, then f^{-1} is given by the rule $y \mapsto x$), so that the former is an element of the domain of f. One has to rely on context to distinguish between the usage of f^{-1} as the preimage and as the inverse function.

† Here $(\cdot,\cdot)$ denotes an ordered pair. Context will dictate where $(\cdot,\cdot)$ should be interpreted as an ordered pair, or as an open interval.

‡ Following common practice, we adopt the superscript notation for indices, so that a point in say $\mathbb{R}^4$ is written as $x = (x^1, x^2, x^3, x^4)$.

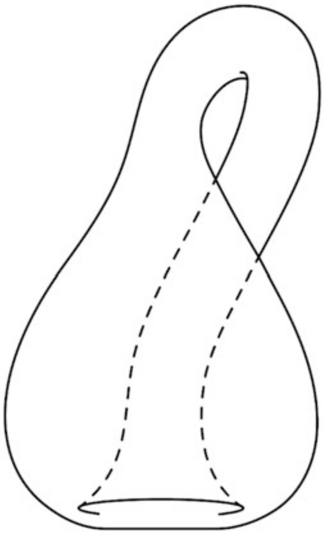
Another example is the *2-torus*, T^2. It is often depicted as a surface of revolution: Given real numbers r, R with $0 < r < R$, take a circle of radius r in the xz-plane, with center at $(R, 0)$, and rotate about the z-axis.



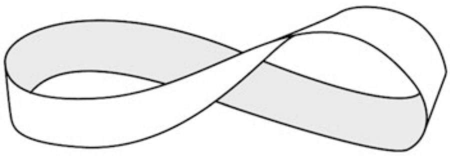
The resulting surface is given by an equation,

$$T^2 = \{(x,y,z) | \left(\sqrt{x^2+y^2} - R\right)^2 + z^2 = r^2\}. \tag{1.1}$$

Not all surfaces can be realized as "embedded" in $\mathbb{R}^3$; for non-orientable surfaces one needs to allow for self-intersections. This type of realization is referred to as an *immersion*: We do not allow edges or corners, but we do allow that different parts of the surface pass through each other. An example is the *Klein bottle*.



The Klein bottle is an example of a *non-orientable surface*: It has only one side. A simpler example of a non-orientable surface is the *open Möbius strip*.

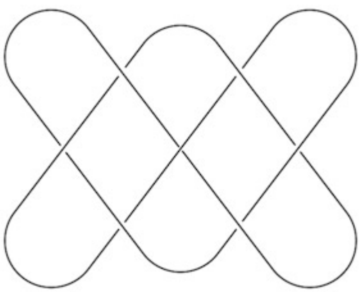


(Here, *open* means that we are excluding the boundary. Note that only at interior points the Möbius strip looks like $\mathbb{R}^2$, while at boundary points it looks like a half space $\{(x,y) \in \mathbb{R}^2 | \, x \geq 0\}$). In fact, one way of seeing that the Klein bottle is non-orientable is to show that it contains a Möbius strip (see Problem 2). A surface given as a regular level set $f^{-1}(0)$ of a function f on $\mathbb{R}^3$ is necessarily orientable: For any such surface one has one side where f is positive, and another side where f is negative.

1.4 Intrinsic Descriptions of Manifolds

In this book, we will mostly avoid concrete embeddings of manifolds into any $\mathbb{R}^N$. Here, the term "embedding" is used in an intuitive sense, for example, as the realization as the level set of some equations. (Later we will give a precise definition.) There are a number of reasons for why we prefer developing an "intrinsic" theory of manifolds.

(a) Embeddings of simple manifolds in Euclidean space can look quite complicated. The following one-dimensional manifold



is intrinsically, "as a manifold," just a closed curve, that is, a circle. The problem of distinguishing embeddings of a circle into $\mathbb{R}^3$ is one of the goals of *knot theory*, a deep and difficult area of mathematics.

(b) Such complications disappear if one goes to higher dimensions. For example, the above knot (and indeed any knot in $\mathbb{R}^3$) can be disentangled inside $\mathbb{R}^4$ (with $\mathbb{R}^3$ viewed as a subspace). Thus, in $\mathbb{R}^4$ they become *unknots*.

(c) The intrinsic description is sometimes much simpler to deal with than the extrinsic one. For instance, Equation (1.1) describing the torus $T^2 \subseteq \mathbb{R}^3$ is not especially simple or beautiful. But once we introduce the following parametrization of the torus

$$x = (R + r\cos\varphi)\cos\theta, \quad y = (R + r\cos\varphi)\sin\theta, \quad z = r\sin\varphi,$$

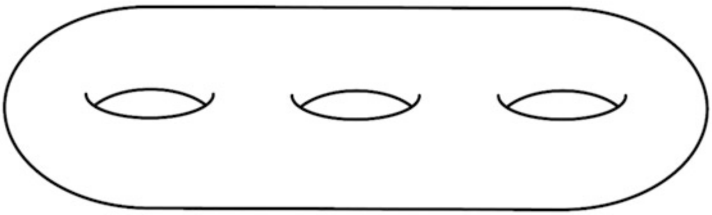
where θ, φ are determined up to multiples of 2π, we recognize that T^2 is simply a product:

$$T^2 = S^1 \times S^1. \tag{1.2}$$

That is, T^2 consists of ordered pairs of points on the circle, with the two factors corresponding to θ and φ. In contrast to (1.1), there is no distinction between a "small" (circle) (of radius r) and a "large" circle (of radius R). The new description suggests an embedding of T^2 into $\mathbb{R}^4$ which is "nicer" than the embedding into $\mathbb{R}^3$. But then again, why not just work with the description (1.2), and avoid embeddings altogether!

(d) When dealing with additional structures on manifolds, there is special interest in structures that are intrinsic to the manifold. For instance, we remarked that surfaces are either orientable or non-orientable, and we tentatively described the non-orientable surfaces in Euclidean $\mathbb{R}^3$ as having only "one side"—but this is not a very satisfactory characterization, and one would prefer a definition that does not rely on a choice of embedding.

(e) Often, there is no natural choice of an embedding of a given manifold inside $\mathbb{R}^N$, at least not in terms of concrete equations. For instance, while the triple torus



is easily pictured in 3-space $\mathbb{R}^3$, it is hard to describe it concretely as the level set of an equation.

(f) While many examples of manifolds arise naturally as level sets of equations in some Euclidean space, there are also many examples for which the initial construction is different. For example, the set M whose elements are all affine lines in $\mathbb{R}^2$ (that is, straight lines that need not go through the origin) is naturally a 2-dimensional manifold. But some thought is required to realize it as a surface in $\mathbb{R}^3$. The next section deals with other such examples.

1.5 Soccer Balls and Linkages

Mechanical systems typically have certain degrees of freedom, and hence may take on various *configurations*. The set of all possible configurations of such a system is called its *configurations space* and is often (but not always) described by a manifold.

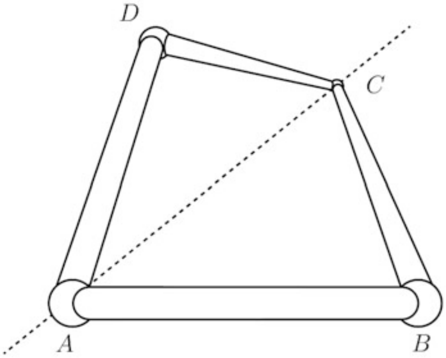
As a simple example, consider the possible configurations of a soccer ball, positioned over the some fixed point of the lawn (the penalty mark, say).

From any fixed position of the ball, any other configuration is obtained by a rotation. It takes three parameters to describe a rotation, with two parameters specifying the axis of rotation and a third parameter specifying the angle of rotation. Hence we expect the configuration space of the soccer ball to be a 3-dimensional manifold, and this turns out to be true. Note that once an initial configuration is chosen, the configuration space of the soccer ball is *identified* with the group of rotations.

As a more elaborate example, consider a *spatial linkage* given by a collection of $N \geq 3$ rods, of prescribed lengths $l_1, \ldots, l_N > 0$, joined at their end points in such a way that they close up to a loop. (This is only possible if the length of the longest rod is less than or equal to the sum of the lengths of the remaining rods. We will assume that this is the case.) The rods may move freely around the joints. We shall consider two linkage configurations to be the same if they are obtained from each other by Euclidean motions (i.e., translations and rotations of the entire linkage). Denote the configuration space by

$$M(l_1, \ldots, l_N).$$

If $N = 3$, the linkage is a triangle, and there are no possibilities of changing the linkage: The configuration space $M(l_1, l_2, l_3)$ (if non-empty) is just a point. The following picture shows a typical linkage for $N = 4$. Note that this linkage has two degrees of freedom (other than rotations and translations), given by the "bending" of the linkage along the dotted line through A, C, and a similar bending transformation along the straight line through B, D.



Hence, we expect that the configuration space $M(l_1, l_2, l_3, l_4)$ of a linkage with $N = 4$ rods, if non-empty, should typically be a 2-dimensional manifold (a surface).

To get an estimate for the number of degrees of freedom (i.e., the dimension of the configuration spaces, assuming the latter is a manifold) for general $N \geq 3$, note that the configuration of an N-linkage is realized by an ordered collection $\mathbf{u}_1, \ldots, \mathbf{u}_N$ of vectors of lengths[§] $||\mathbf{u}_1|| = l_1, \ldots,\ ||\mathbf{u}_N|| = l_N$, with the condition that the vectors add to zero:

$$\mathbf{u}_1 + \cdots + \mathbf{u}_N = \mathbf{0}. \tag{1.3}$$

Two such collections $\mathbf{u}_1, \ldots, \mathbf{u}_N$ and $\mathbf{u}'_1, \ldots, \mathbf{u}'_N$ describe the same linkage configuration if they are related by a rotation. Let us now count the number of independent parameters. Each vector $\mathbf{u}_i$ is described by two parameters (its position on a sphere of radius l_i), giving $2N$ parameters. But these are not independent, due to the condition (1.3); the three components of this equation reduce the number of independent parameters by 3. Furthermore, using rotations we may arrange that $\mathbf{u}_1$ points in the x-direction, reducing the number of parameters by another 2, and using a subsequent rotation about the x-axis, we may arrange that $\mathbf{u}_2$ lies in the xy-plane, further reducing the number of parameters by 1. Hence we expect that configurations are described by $2N - 3 - 2 - 1 = 2N - 6$ parameters, consistent with our observations for $N = 3$ and $N = 4$. Thus, letting M be the space of all configurations, and assuming this is a manifold, we expect its dimension to be

$$\dim M(l_1, \ldots, l_N) = 2N - 6. \tag{1.4}$$

1 (answer on page 267). For any straight line through non-adjacent vertices of a linkage, one can define a "bending transformation" similar to what we had for $N = 4$. How many straight lines with this property are there for $N = 5$? Does this match the expected dimension of $M(l_1, \ldots, l_5)$?

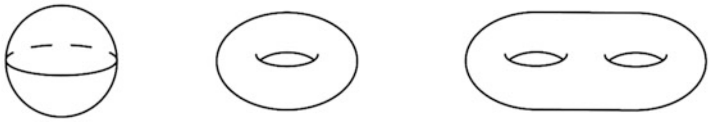
Of course, our discussion oversimplifies matters—for example, if $l_N = l_1 + \cdots + l_{N-1}$, there is only one configuration, hence our rough count is wrong in this case.

§ Here we are using $||\cdot||$ for the usual Euclidean norm.

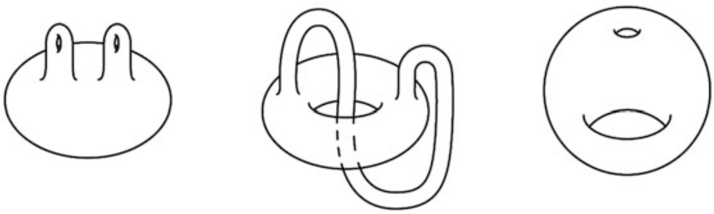
More generally, whenever it is possible to make all rods "parallel," which happens whenever there are sign choices such that $\pm l_1 \pm l_2 \pm \cdots \pm l_N = 0$, the space M will have singularities or be a manifold of a lower dimension. But for *typical* rod lengths this cannot happen, and it turns out that the configuration space $M(l_1, \ldots, l_N)$ (if non-empty) is indeed a manifold of dimension $2N - 6$. These manifolds have been much-studied, using techniques from symplectic geometry and algebraic geometry.

1.6 Surfaces

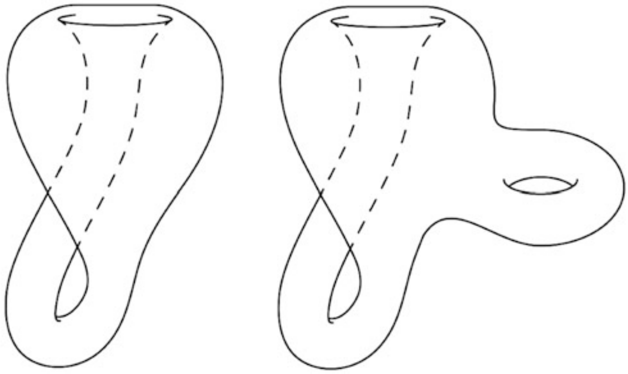
Let us briefly give a very informal discussion of *surfaces*. A surface is the same thing as a 2-dimensional manifold. We have already encountered some examples: The sphere, torus, double torus, triple torus, and so on.



All of these are "orientable" surfaces, which essentially means that they have two sides which you might paint in two different colors. It turns out that these are *all* the orientable surfaces, if we consider the surfaces "intrinsically" and only consider surfaces that are *compact* in the loose sense that they do not "go off to infinity" and do not have a boundary (thus excluding a cylinder, for example). For instance, each of the following drawings depicts a double torus.

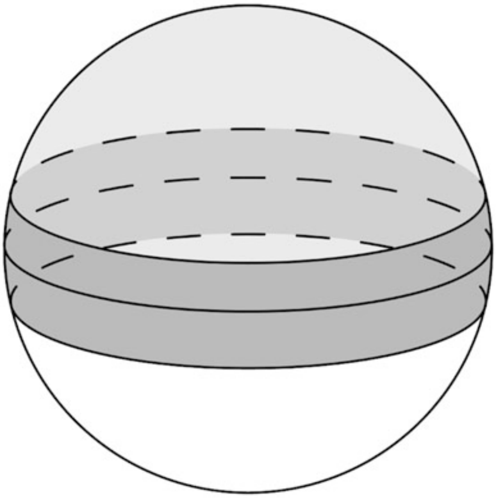


We also have one example of a non-orientable surface: The Klein bottle. More examples are obtained by attaching handles (just like we can think of the torus, double torus and so on as a sphere with handles attached).



Are these *all* the non-orientable surfaces? In fact, the answer is *no*. We have missed what is in some sense the simplest non-orientable surface. Ironically, it is the surface that is hardest to visualize in 3-space. This surface is called the *projective plane* or *projective space* and is denoted $\mathbb{R}P^2$. One can define $\mathbb{R}P^2$ as the set of all lines through the origin (i.e., 1-dimensional linear subspaces) in $\mathbb{R}^3$. It should be clear that this is a 2-dimensional manifold, since it takes 2 parameters to specify such a line. We can label such lines by their points of intersection with S^2, hence we can also think of $\mathbb{R}P^2$ as the set of antipodal (i.e., opposite) points on S^2. In other words, it is obtained from S^2 by identifying antipodal points. To get a better idea of how $\mathbb{R}P^2$ looks like, let us subdivide the sphere S^2 into two parts:

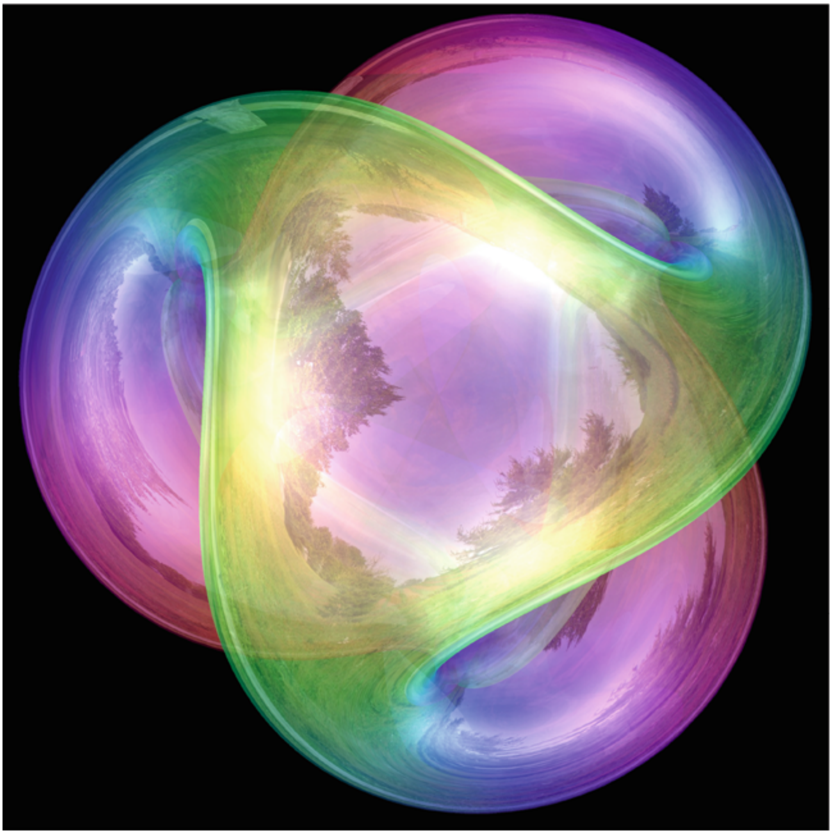
(i) Points having distance $\leq \varepsilon$ from the equator;
(ii) Points having distance $\geq \varepsilon$ from the equator.



If we perform the antipodal identification for (i), we obtain a Möbius strip. If we perform antipodal identification for (ii), we obtain a 2-dimensional disk (think of it as the points of (ii) lying in the upper hemisphere). Hence, $\mathbb{R}\mathrm{P}^2$ can also be regarded as gluing the boundary of a Möbius strip to the boundary of a disk:

Now, the question arises: *Is it possible to realize* $\mathbb{R}\mathrm{P}^2$ *smoothly as a surface (possibly with self-intersections) inside* $\mathbb{R}^3$?

Thus, we are aiming to visualize the projective plane similarly to the Klein bottle. It turns out that this is a highly non-trivial task. Simple attempts of joining the boundary circle of the Möbius strip with the boundary of the disk will always create sharp edges or corners—try it! Around 1900, David Hilbert posed this problem to his student Werner Boy, who discovered that the answer is: *yes*. Below is an artistic rendition of *Boy's surface*.



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The picture alone provides only limited insight—in particular, it does not give a clear illustration of the self-intersections of the surface. There are some nice online videos explaining the *construction* of the surface, giving a better understanding. Still, one is left with the impression that Boy's surface is pretty complicated. By contrast, if one is only interested in $\mathbb{R}\mathrm{P}^2$ itself, rather than its presentation as a surface in $\mathbb{R}^3$, it is much simpler to work with the definition (as a sphere with antipodal identification). That is, $\mathbb{R}\mathrm{P}^2$ is much easier to understand intrinsically.

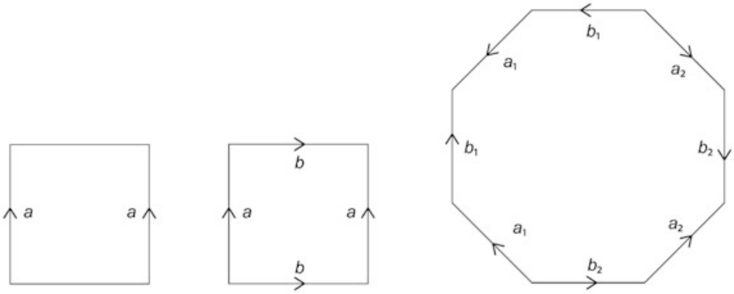
2 (answer on page 267). What surface results from "puncturing" the projective plane (i.e., removing a single point)?

Going back to the classification of surfaces, we have the following.

Fact 1.1. *All compact, connected surfaces are obtained from either the 2-sphere S^2, the Klein bottle, or the projective plane $\mathbb{R}\mathrm{P}^2$, by attaching handles.*

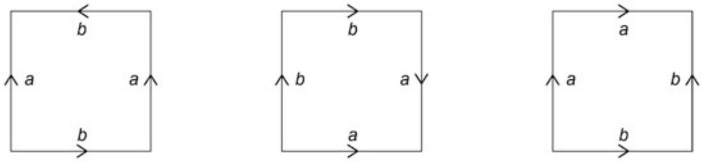
We will not give a formal proof of this fact in this book. The notion of "compactness" will be discussed in Section 2.5; it roughly means that we disallow surfaces that are open (such as an open Möbius strip) or unbounded (such as a surface with infinitely many handles).

A convenient way of representing surfaces is with a so-called gluing diagram. In the diagrams below, boundaries are identified so that the arrows (and labels) match. These diagrams represent, from left to right, a cylinder, a 2-torus, and genus 2 surface (double torus).



In applications, working with the gluing diagrams is often preferable to working with the visualizations as surfaces in $\mathbb{R}^3$, even for a 2-torus. (See the discussion in 1.4(c).)

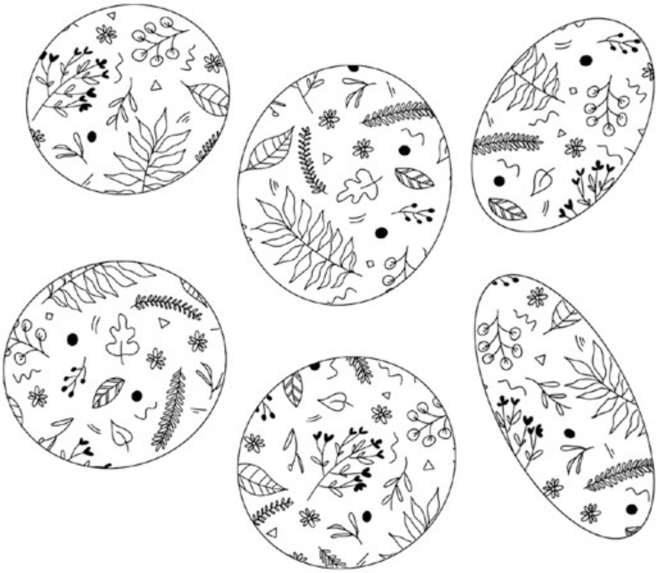
3 (answer on page 267). What surfaces are obtained from the following gluing diagrams?



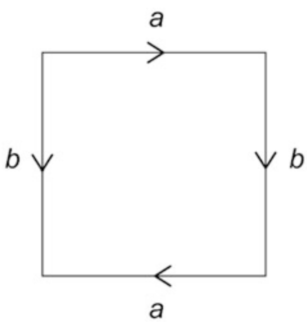
1.7 Problems

In the following problems, you are only asked to give *informal* explanations—after all, we did not yet give any formal definitions.

1. What surface is described by the following atlas?



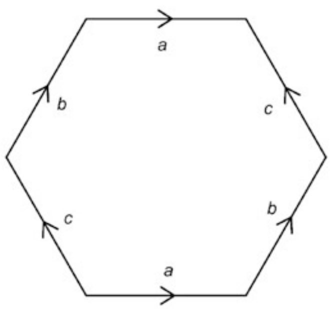
2. Regard the Klein bottle as obtained from the following gluing diagram.



Using such gluing diagrams (or perhaps similar pictures), explain how to cut the Klein bottle along a circle, in such a way that the resulting "surface with boundary" is:
(a) A disjoint union of two Möbius strips.
(b) A cylinder (with two boundary circles).
(c) A single Möbius strip.

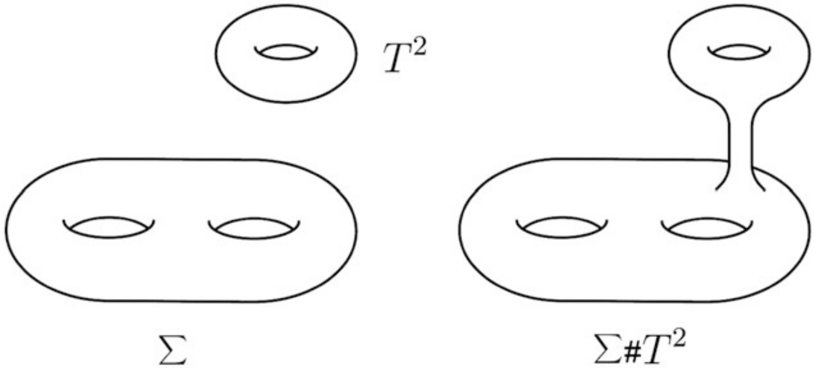
3. Explain the following fact: *Cutting a Möbius strip along its central circle gives a cylinder.* (Probably the cleanest argument uses gluing diagrams.)

4. What surface is obtained by identifying opposite sides of a regular hexagon? (You may find it helpful to consider tilings of the plane by hexagons.)



5. One way of constructing surfaces in $\mathbb{R}^3$ is to start with a reasonably nice bounded subset $\Gamma \subseteq \mathbb{R}^3$, thicken it to a 3-dimensional body B (e.g., take all points of distance $\leq \varepsilon$ from Γ, for some small $\varepsilon > 0$), and let Σ be the boundary of B. (It may help to think of Play-Doh attached to some contraption of wires.) For instance, if Γ is a point, then the resulting Σ is a 2-sphere, if Γ is a circle, then Σ is a 2-torus. What surface Σ is obtained by taking Γ to be
(a) A figure eight curve in the xy-plane?
(b) For $d \in \mathbb{N}$, the union of d distinct meridians, connecting the north and south poles $(0,0,1)$, $(0,0,-1)$ of the unit 2-sphere in $\mathbb{R}^3$? (A meridian is the intersection of the sphere with a half-plane containing the poles.)
(c) The union of the standard unit circles in the xy-plane and yz-plane?
(d) The union of the edges of a tetrahedron?

6. Explain how to realize the double torus (genus two surface) as a surface in 3-dimensional space $\mathbb{R}^3$, in such a way that it is symmetric under reflection across the xy-plane, as well as rotation by 120 degrees about the z-axis. (Hint: One strategy could be to think of possibilities for the part where $z \geq 0$. Another approach is to use the method from Problem 5.)

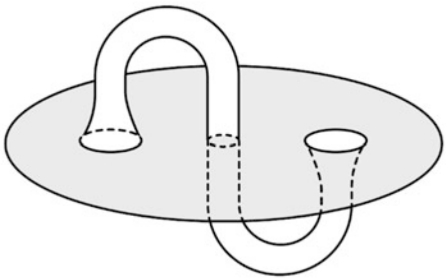


7. Given two connected surfaces Σ_1, Σ_2, one can construct a new connected surface, denoted

$$\Sigma_1 \# \Sigma_2,$$

by taking the "connected sum". First, remove two small disks from Σ_1, Σ_2, thereby creating surfaces with boundary. Then glue-in a cylinder connecting the two boundary circles, without creating edges. The resulting surface is $\Sigma_1 \# \Sigma_2$. For example, the connected sum $\Sigma \# T^2$ is Σ with a handle attached (see figure above).

   (a) Explain how a connected sum with a Klein bottle is equivalent to attaching a handle, but with one end of the handle attached "from the other side" (see figure below).



   (b) Let Σ be a connected *non-orientable* surface. Explain why the connected sum of Σ with a 2-torus is the same as the connected sum of Σ with a Klein bottle.
   (c) What is the connected sum $\mathbb{R}P^2 \# \mathbb{R}P^2$? (Note that it must be in the list of 2-dimensional surfaces from Fact 1.1.)
   (d) Show that every connected non-orientable compact surface is obtained by taking connected sums of projective planes, $\mathbb{R}P^2 \# \mathbb{R}P^2 \# \cdots \# \mathbb{R}P^2$.

8. We saw that the real projective plane $\mathbb{R}\mathrm{P}^2$ can be obtained from the 2-sphere S^2, realized as the level set $x^2+y^2+z^2=1$, by *antipodal identification*, identifying a point $p=(x,y,z)$ with the antipodal point $-p=(-x,-y,-z)$.
   What surface is obtained by antipodal identification of a 2-torus, embedded as the solution set of
$$(\sqrt{x^2+y^2}-R)^2+z^2=r^2$$
   for $0<r<R$? Explain your answer in words and/or pictures, without giving a detailed mathematical proof.

9. Show that the configuration space $M(3,3,3,1)$ of the linkage with lengths $3,3,3,1$ (see Section 1.5) is a 2-sphere. (Argue using pictures!)

2

Manifolds

One of the goals of this book is to develop the theory of manifolds in intrinsic terms, although we may occasionally use immersions or embeddings into Euclidean space in order to illustrate concepts. In physics terminology, we will formulate the theory of manifolds in terms that are "manifestly coordinate-free."

2.1 Atlases and Charts

As we mentioned above, the basic feature of manifolds is the existence of "local coordinates." The transition from one set of coordinates to another should be *smooth*. We recall the following notions from multivariable calculus.

Definition 2.1. *Let $U \subseteq \mathbb{R}^m$ and $V \subseteq \mathbb{R}^n$ be open subsets. A map $F: U \to V$ is called* smooth *if it is infinitely differentiable. The set of smooth functions from U to V is denoted $C^\infty(U,V)$. The map F is called a* diffeomorphism *from U to V if it is invertible and the inverse map $F^{-1}: V \to U$ is again smooth.*

Example 2.2. The exponential map $\exp: \mathbb{R} \to \mathbb{R}$, $x \mapsto \exp(x) = e^x$ is smooth. It may be regarded as a map onto $\mathbb{R}_{>0} = \{y | y > 0\}$, and as such, it is a diffeomorphism

$$\exp: \mathbb{R} \to \mathbb{R}_{>0}$$

with inverse $\exp^{-1} = \log$ (the natural logarithm). Similarly, the function $x \mapsto \tan(x)$ is a diffeomorphism from the open interval $(-\frac{\pi}{2}, \frac{\pi}{2})$ onto $\mathbb{R}$, with inverse the function arctan.

Definition 2.3. *For a smooth map $F \in C^\infty(U,V)$ between open subsets $U \subseteq \mathbb{R}^m$ and $V \subseteq \mathbb{R}^n$, and any $\mathbf{x} \in U$, one defines the* Jacobian matrix *$DF(\mathbf{x})$ to be the $(n \times m)$-matrix of partial derivatives*

G. Gross, E. Meinrenken, *Manifolds, Vector Fields, and Differential Forms*,
Springer Undergraduate Mathematics Series,
https://doi.org/10.1007/978-3-031-25409-3_2

$$(DF(\mathbf{x}))^i_j = \frac{\partial F^i}{\partial x^j}.$$

If $n = m$, it is a square matrix, and its determinant is called the Jacobian determinant *of F at* $\mathbf{x}$.

The *inverse function theorem* (recalled for convenience as Theorem 4.19) states that F is a diffeomorphism if and only if (i) F is invertible, and (ii) for all $\mathbf{x} \in U$, the Jacobian matrix $DF(\mathbf{x})$ is invertible. (That is, one does not actually have to check smoothness of the inverse map!)

The following definition formalizes the concept of introducing local coordinates.

Definition 2.4 (Charts). *Let M be a set.*

(a) An m-dimensional (coordinate) chart *(U, φ) on M is a subset $U \subseteq M$ together with a map $\varphi : U \to \mathbb{R}^m$, such that $\varphi(U) \subseteq \mathbb{R}^m$ is open and φ is a bijection from U to $\varphi(U)$. The set U is the* chart domain*, and φ is the* coordinate map.

(b) Two charts (U, φ) and (V, ψ) are called compatible *if the subsets $\varphi(U \cap V)$ and $\psi(U \cap V)$ are open, and the* transition map

$$\psi \circ \varphi^{-1} : \varphi(U \cap V) \to \psi(U \cap V)$$

is a diffeomorphism.

As a special case, charts with $U \cap V = \emptyset$ are always compatible.

4 (answer on page 268). The bijection requirement on φ plays an important role; hence, this may be a good opportunity to think through some set theory. (Also see Appendix A.) Prove the following (from now on, we shall use the properties below without further comment):
Let X, Y be sets, $f : X \to Y$ a map, and suppose $A, B \subseteq X$, and $C, D \subseteq Y$.
(a) Show that

$$\begin{aligned} f(A \cup B) &= f(A) \cup f(B), && \\ f(A \cap B) &= f(A) \cap f(B) && \text{if } f \text{ injective,} \\ f(A \setminus B) &= f(A) \setminus f(B) && \text{if } f \text{ injective,} \\ f(A^c) &= f(A)^c && \text{if } f \text{ bijective.} \end{aligned}$$

Here the superscript c denotes the complement. By giving counterexamples, show that the second and third equalities may fail if f is not injective and that the last equality may fail if f is only injective or only surjective.

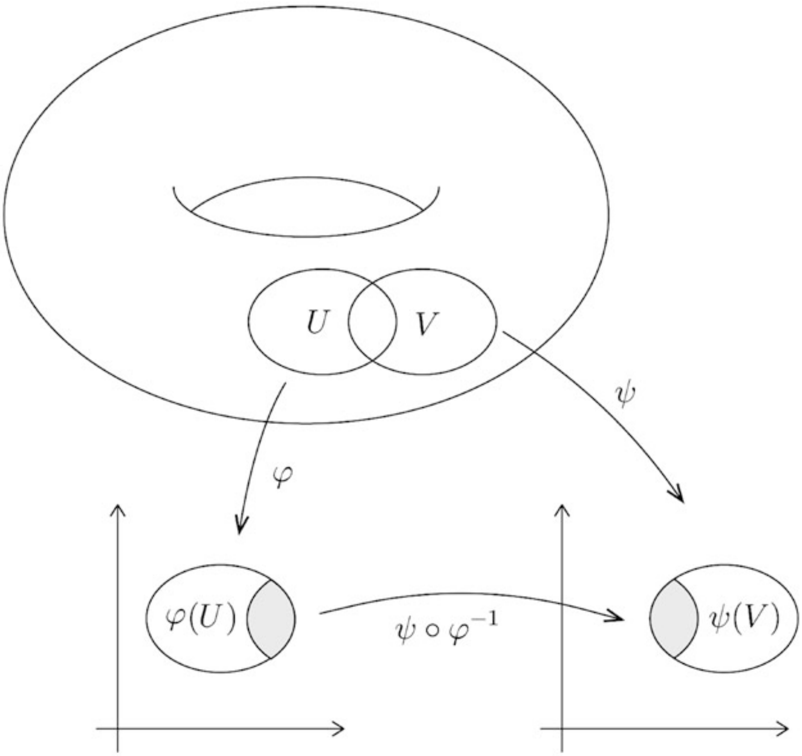
(b) Let us denote by $f^{-1}(C) = \{x : f(x) \in C\}$ the preimage of C. Show that:

$$f^{-1}(C \cup D) = f^{-1}(C) \cup f^{-1}(D),$$
$$f^{-1}(C \cap D) = f^{-1}(C) \cap f^{-1}(D),$$
$$f^{-1}(C^c) = \left(f^{-1}(C)\right)^c,$$
$$f^{-1}(C \setminus D) = f^{-1}(C) \setminus f^{-1}(D).$$

5 (answer on page 269). Is compatibility of charts an equivalence relation? (See Appendix A for a reminder on equivalence relations.)

Let (U, φ) be a coordinate chart. Given a point $p \in U$, and writing $\varphi(p) = (u^1, \ldots, u^m)$, we say that the u^i are the *coordinates of* p in the given chart. (Note the convention of indexing by superscripts; be careful not to confuse indices with powers.) Letting p vary, these become real-valued functions $p \mapsto u^i(p)$; they are simply the component functions of φ.

Transition maps $\psi \circ \varphi^{-1}$ are also called *change of coordinates*. Below is a picture of a "coordinate change".



Definition 2.5 (Atlas). *Let M be a set. An m-dimensional* atlas *on M is a collection of coordinate charts $\mathscr{A} = \{(U_\alpha, \varphi_\alpha)\}$ such that:*

(a) The U_α covers all of M, i.e., $\bigcup_\alpha U_\alpha = M$.
(b) For all indices α, β, the charts $(U_\alpha, \varphi_\alpha)$ and (U_β, φ_β) are compatible.

In this definition, $\alpha, \beta, \ldots$ are indices used to distinguish the different charts; the indexing set may be finite or infinite, perhaps even uncountable.

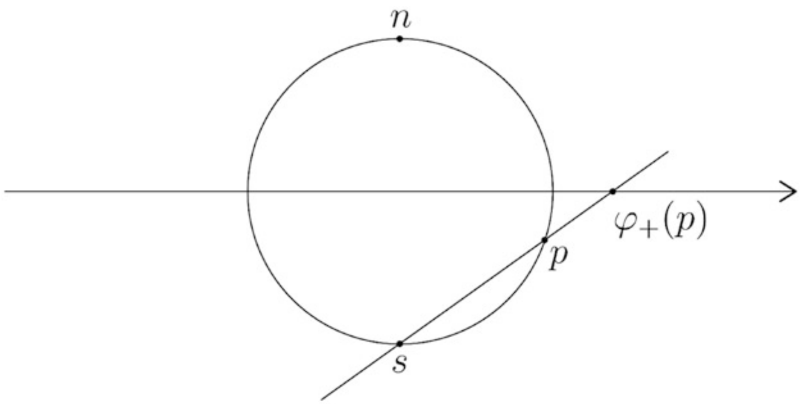
Example 2.6 (An Atlas on the 2-Sphere). Let $S^2 \subseteq \mathbb{R}^3$ be the unit sphere, consisting of all $(x,y,z) \in \mathbb{R}^3$ satisfying the equation $x^2 + y^2 + z^2 = 1$. We shall define an atlas with two charts (U_+, φ_+) and (U_-, φ_-). Let $n = (0,0,1)$ be the north pole, let $s = (0,0,-1)$ be the south pole, and put

$$U_+ = S^2 \setminus \{s\}, \quad U_- = S^2 \setminus \{n\}.$$

Regard $\mathbb{R}^2$ as the coordinate subspace of $\mathbb{R}^3$ on which $z = 0$. Let

$$\varphi_+ : U_+ \to \mathbb{R}^2, \quad p \mapsto \varphi_+(p)$$

be *stereographic projection from the south pole*. That is, $\varphi_+(p)$ is the unique point of intersection of $\mathbb{R}^2$ with the affine line passing through p and s.



Similarly,

$$\varphi_- : U_- \to \mathbb{R}^2, \quad p \mapsto \varphi_-(p)$$

is *stereographic projection from the north pole*, where $\varphi_-(p)$ is the unique point of intersection of $\mathbb{R}^2$ with the affine line passing through p and n. A calculation gives the explicit formulas, for $(x,y,z) \in S^2 \subseteq \mathbb{R}^3$ in U_+, respectively U_-:

$$\varphi_+(x,y,z) = \left(\frac{x}{1+z}, \frac{y}{1+z}\right), \quad \varphi_-(x,y,z) = \left(\frac{x}{1-z}, \frac{y}{1-z}\right). \tag{2.1}$$

6 (answer on page 269). Verify (2.1).

Both $\varphi_\pm : U_\pm \to \mathbb{R}^2$ are bijections onto $\mathbb{R}^2$. Indeed, given $(u,v) \in \mathbb{R}^2$, we may solve the equation $(u,v) = \varphi_\pm(x,y,z)$, using the conditions that $x^2+y^2+z^2 = 1$ and $z \pm 1 \neq 0$. The calculation gives

$$\varphi_\pm^{-1}(u,v) = \left(\frac{2u}{1+(u^2+v^2)}, \frac{2v}{1+(u^2+v^2)}, \pm\frac{1-(u^2+v^2)}{1+(u^2+v^2)}\right). \tag{2.2}$$

7 (answer on page 269). Verify (2.2).

Note that $\varphi_+(U_+ \cap U_-) = \mathbb{R}^2 \backslash \{(0,0)\}$. The transition map on the overlap of the two charts is

$$(\varphi_- \circ \varphi_+^{-1})(u,v) = \left(\frac{u}{u^2+v^2}, \frac{v}{u^2+v^2}\right),$$

which is smooth on $\mathbb{R}^2 \backslash \{(0,0)\}$ as required.

Here is another simple, but less familiar example where one has an atlas with two charts.

Example 2.7 (Affine Lines in $\mathbb{R}^2$). By an *affine line* in a vector space E, we mean a subset $\ell \subseteq E$ that is obtained by adding a fixed vector $\mathbf{v}_0$ to all elements of a 1-dimensional subspace. In plain terms, an affine line is simply a straight line that does not necessarily pass through the origin. (We reserve the term *line*, without prefix, for 1-dimensional subspaces, that is, for a straight line that *does* pass through the origin.) Let

$$M = \{\ell \mid \ell \text{ is an affine line in } \mathbb{R}^2\}.$$

Let $U \subseteq M$ be the subset of lines that are not vertical, and $V \subseteq M$ the lines that are not horizontal. Any $\ell \in U$ is given by an equation of the form

$$y = mx + b,$$

where m is the slope and b is the y intercept. The map $\varphi : U \to \mathbb{R}^2$ taking ℓ to (m,b) is a bijection. On the other hand, lines in V are given by equations of the form

$$x = ny + c,$$

and we also have the map $\psi : V \to \mathbb{R}^2$ taking such ℓ to (n,c). The intersection $U \cap V$ are lines ℓ that are neither vertical nor horizontal. Hence, $\varphi(U \cap V)$ is the set of all (m,b) such that $m \neq 0$, and similarly, $\psi(U \cap V)$ is the set of all (n,c) such that $n \neq 0$.

8 (answer on page 270). Compute the transition maps $\psi \circ \varphi^{-1}$, $\varphi \circ \psi^{-1}$ and show they are smooth. Conclude that (U,φ) and (V,ψ) define a 2-dimensional atlas on M.

It turns out that M is a 2-dimensional manifold—a surface. Of course, we should be able to identify this mysterious surface:

9 (answer on page 270). What is this surface?

We return to our objective of giving a general definition of the concept of manifolds. As a first approximation, we may take an m-dimensional manifold to be a set with an m-dimensional atlas. This is almost the right definition, but we will make a few adjustments. A first criticism is that we may not want any *particular* atlas as part of the definition. For example, the 2-sphere with the atlas given by stereographic projections onto the xy-plane, and the 2-sphere with the atlas given by stereographic projections onto the yz-plane, should be one and the same manifold: S^2. To resolve this problem, we will use the following notion.

Definition 2.8. *Suppose $\mathscr{A} = \{(U_\alpha, \varphi_\alpha)\}$ is an m-dimensional atlas on M, and let (U, φ) be another chart. Then (U, φ) is said to be* compatible *with $\mathscr{A}$ if it is compatible with all charts $(U_\alpha, \varphi_\alpha)$ of $\mathscr{A}$.*

Example 2.9. On the 2-sphere S^2, we have constructed the atlas

$$\mathscr{A} = \{(U_+, \varphi_+), (U_-, \varphi_-)\}$$

given by stereographic projection. Consider the chart (V, ψ), with domain V the set of all $(x, y, z) \in S^2$ such that $y < 0$, and $\psi(x, y, z) = (x, z)$. To check that it is compatible with (U_+, φ_+), note that $U_+ \cap V = V$, and

$$\varphi_+(U_+ \cap V) = \{(u, v) |\, v < 0\}, \quad \psi(U_+ \cap V) = \{(x, z) |\, x^2 + z^2 < 1\}.$$

10 (answer on page 270). Find explicit formulas for $\psi \circ \varphi_+^{-1}$ and $\varphi_+ \circ \psi^{-1}$. Conclude that (V, ψ) is compatible with (U_+, φ_+).

Note that (U, φ) is compatible with the atlas $\mathscr{A} = \{(U_\alpha, \varphi_\alpha)\}$ if and only if the union $\mathscr{A} \cup \{(U, \varphi)\}$ is again an atlas on M. This suggests defining a bigger atlas, by using *all* charts that are compatible with the given atlas. In order for this to work, we need the new charts to be compatible not only with the charts of $\mathscr{A}$, but also with each other. This is not entirely obvious, since compatibility of charts is not an equivalence relation (see 5).

Lemma 2.10. *Let $\mathscr{A} = \{(U_\alpha, \varphi_\alpha)\}$ be a given atlas on the set M. If two charts (U, φ), (V, ψ) are compatible with $\mathscr{A}$, then they are also compatible with each other.*

Proof. For every chart $(U_\alpha, \varphi_\alpha)$, the sets $\varphi_\alpha(U \cap U_\alpha)$ and $\varphi_\alpha(V \cap U_\alpha)$ are open; hence, their intersection is open. This intersection is (see 11 below)

$$\varphi_\alpha(U \cap U_\alpha) \cap \varphi_\alpha(V \cap U_\alpha) = \varphi_\alpha(U \cap V \cap U_\alpha). \tag{2.3}$$

Since $\varphi \circ \varphi_\alpha^{-1} : \ \varphi_\alpha(U \cap U_\alpha) \to \varphi(U \cap U_\alpha)$ is a diffeomorphism, it follows that

$$\varphi(U \cap V \cap U_\alpha) = (\varphi \circ \varphi_\alpha^{-1})\big(\varphi_\alpha(U \cap V \cap U_\alpha)\big)$$

is open. Taking the union over all α, we see that

$$\varphi(U \cap V) = \bigcup_\alpha \varphi(U \cap V \cap U_\alpha)$$

is open. A similar argument applies to $\psi(U \cap V)$.
The transition map $\psi \circ \varphi^{-1} : \ \varphi(U \cap V) \to \psi(U \cap V)$ is smooth since for all α, its restriction to $\varphi(U \cap V \cap U_\alpha)$ is a composition of two smooth maps $\varphi_\alpha \circ \varphi^{-1} : \ \varphi(U \cap V \cap U_\alpha) \longrightarrow \varphi_\alpha(U \cap V \cap U_\alpha)$ and $\psi \circ \varphi_\alpha^{-1} : \ \varphi_\alpha(U \cap V \cap U_\alpha) \longrightarrow \psi(U \cap V \cap U_\alpha)$. Likewise, the composition $\varphi \circ \psi^{-1} : \ \psi(U \cap V) \to \varphi(U \cap V)$ is smooth. $\square$

11 (answer on page 271). Explain why (2.3) is true.

12 (answer on page 271). Suppose (U, φ) is a chart, with image $\widetilde{U} = \varphi(U) \subseteq \mathbb{R}^m$. Let $V \subseteq U$ be a subset such that $\widetilde{V} = \varphi(V) \subseteq \widetilde{U}$ is open, and let $\psi = \varphi|_V$ be the restriction of φ. Prove that (V, ψ) is again a chart and is compatible with (U, φ). Furthermore, if (U, φ) is a chart from an atlas $\mathscr{A}$, then (V, ψ) is compatible with that atlas.

Theorem 2.11. *Given an atlas $\mathscr{A} = \{(U_\alpha, \varphi_\alpha)\}$ on M, let $\widetilde{\mathscr{A}}$ be the collection of all charts (U, φ) that are compatible with $\mathscr{A}$. Then $\widetilde{\mathscr{A}}$ is itself an atlas on M, containing $\mathscr{A}$. In fact, $\widetilde{\mathscr{A}}$ is the largest atlas containing $\mathscr{A}$.*

Proof. Note first that $\widetilde{\mathscr{A}}$ contains $\mathscr{A}$, since the set of charts compatible with $\mathscr{A}$ contains the charts from the atlas $\mathscr{A}$ itself. In particular, the charts in $\widetilde{\mathscr{A}}$ cover M. By the lemma above, any two charts in $\widetilde{\mathscr{A}}$ are compatible. Hence, $\widetilde{\mathscr{A}}$ is an atlas. If (U, φ) is a chart compatible with all charts in $\widetilde{\mathscr{A}}$, then in particular it is compatible with all charts in $\mathscr{A}$; hence, $(U, \varphi) \in \widetilde{\mathscr{A}}$ by the definition of $\widetilde{\mathscr{A}}$. This shows that $\widetilde{\mathscr{A}}$ cannot be extended to a larger atlas. $\square$

Definition 2.12. *An atlas $\mathscr{A}$ is called* maximal *if it is not properly contained in any larger atlas. Given an arbitrary atlas $\mathscr{A}$, one calls $\widetilde{\mathscr{A}}$ (as in Theorem 2.11) the* maximal atlas determined by $\mathscr{A}$.

Remark 2.13. Although we will not need it, let us briefly discuss the notion of equivalence of atlases. (For background on equivalence relations, see Appendix A.) Two atlases $\mathscr{A} = \{(U_\alpha, \varphi_\alpha)\}$ and $\mathscr{A}' = \{(U'_\alpha, \varphi'_\alpha)\}$ are called *equivalent* if every chart of $\mathscr{A}$ is compatible with every chart in $\mathscr{A}'$. For example, the atlas on the 2-sphere given by the two stereographic projections to the *xy*-plane is equivalent to the atlas $\mathscr{A}'$ given by the two stereographic projections to the *yz*-plane. Using Lemma 2.10, one sees that equivalence of atlases is indeed an equivalence relation. (In fact, two atlases are equivalent if and only if their union is an atlas.) Furthermore, two atlases are equivalent if and only if they are contained in the same maximal atlas. That is, *any maximal atlas determines an equivalence class of atlases and vice versa.*

2.2 Definition of Manifold

As our next approximation toward the right definition, we can take an m-dimensional manifold to be a set M together with an m-dimensional *maximal* atlas. This is already quite close to what we want, but for technical reasons we would like to impose two further conditions.

First of all, we insist that M can be covered by *countably many* coordinate charts. In most of our examples, M is in fact covered by finitely many coordinate charts. The countability condition is used for various arguments involving a proof by induction. (Specifically, it is needed for the construction of *partitions of unity*, discussed in Appendix C.4.)

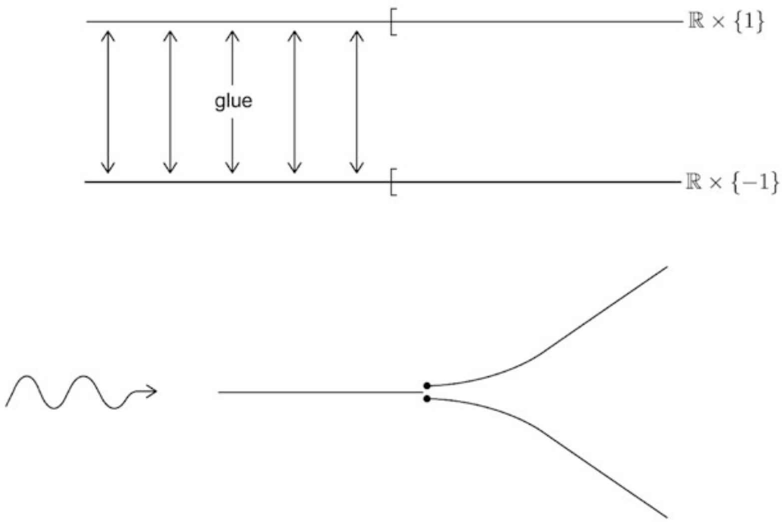
Example 2.14. A simple non-example that is not countable: Let $M = \mathbb{R}$, with $\mathscr{A} = \{(U_\alpha, \varphi_\alpha)\}$ the 0-dimensional atlas, where each U_α consists of a single point, and $\varphi_\alpha : U_\alpha \to \{0\}$ is the unique map to $\mathbb{R}^0 = \{0\}$. Compatibility of charts is obvious. But M cannot be covered by countably many of these charts. Thus, we will not consider $\mathbb{R}$ to be a zero-dimensional manifold. Just as one would expect, it will turn out to be a one-dimensional manifold (and only one-dimensional—see Problem 9 at the end of Chapter 3).

Second, we would like to avoid the following type of example.

Example 2.15. Let X be a disjoint union of two copies of the real line $\mathbb{R}$. We denote the two copies by $\mathbb{R}\times\{1\}$ and $\mathbb{R}\times\{-1\}$, just so that we can tell them apart. Consider the equivalence relation on X generated by

$$(x,1) \sim (x',-1) \;\Leftrightarrow x' = x < 0,$$

and let $M = X/\sim$ be the set of equivalence classes. That is, we “glue” the two real lines along their negative real axes (taking care that no glue gets on the origins of the axes). Below is an (not very successful) attempt to sketch the resulting space.



As a set, M is a disjoint union of $\mathbb{R}_{<0}$ with two copies of $\mathbb{R}_{\geq 0}$. Let $\pi : X \to M$ be the quotient map, and let

$$U = \pi(\mathbb{R} \times \{1\}), \quad V = \pi(\mathbb{R} \times \{-1\})$$

denote the images of the two real lines. The projection map $X \to \mathbb{R}$, $(x, \pm 1) \mapsto x$ is constant on equivalence classes; hence, it descends to a map $f : M \to \mathbb{R}$; let $\varphi : U \to \mathbb{R}$ be the restriction of f to U and $\psi : V \to \mathbb{R}$ the restriction of f to V. Then

$$\varphi(U) = \psi(V) = \mathbb{R}, \quad \varphi(U \cap V) = \psi(U \cap V) = \mathbb{R}_{<0},$$

and the transition map is the identity map. Hence, $\mathscr{A} = \{(U, \varphi), (V, \psi)\}$ is an atlas for M. A strange feature of M with this atlas is that although the points

$$p = \varphi^{-1}(\{0\}) \in U, \quad q = \psi^{-1}(\{0\}) \in V,$$

are distinct ($p \neq q$), they are "arbitrarily close": For any $\varepsilon, \delta > 0$, the preimages $\varphi^{-1}(-\varepsilon, \varepsilon) \subseteq U$ and $\psi^{-1}(-\delta, \delta) \subseteq V$ have non-empty intersection. There is no really satisfactory way of drawing M (our picture above is inadequate).

Since the behavior exhibited in Example 2.15 is inconsistent with the idea of a manifold that "locally looks like $\mathbb{R}^n$" (where, e.g., every converging sequence has a unique limit), we shall insist that for any two distinct points $p, q \in M$, there are always disjoint coordinate charts separating the two points. This is called the *Hausdorff condition*, after *Felix Hausdorff* (1868–1942).

Definition 2.16. *An m-dimensional manifold is a set M, together with a maximal atlas* $\mathscr{A} = \{(U_\alpha, \varphi_\alpha)\}$ *with the following properties:*

(a) **(Countability condition)** *M is covered by countably many coordinate charts in $\mathscr{A}$. That is, there are indices $\alpha_1, \alpha_2, \ldots$ (not necessarily distinct) with*

$$M = \bigcup_{i=1}^{\infty} U_{\alpha_i}.$$

(b) **(Hausdorff condition)** *For any two distinct points $p, q \in M$, there are coordinate charts $(U_\alpha, \varphi_\alpha)$ and (U_β, φ_β) in $\mathscr{A}$ such that $p \in U_\alpha$, $q \in U_\beta$, and*

$$U_\alpha \cap U_\beta = \emptyset.$$

The charts $(U, \varphi) \in \mathscr{A}$ are called (coordinate) charts on the manifold M.

Before giving examples, let us note the following useful fact concerning the Hausdorff condition.

Lemma 2.17. *Let M be a set with a maximal atlas $\mathscr{A} = \{(U_\alpha, \varphi_\alpha)\}$, and suppose $p, q \in M$ are distinct points contained in a single coordinate chart $(U, \varphi) \in \mathscr{A}$. Then we can find indices α, β such that $p \in U_\alpha$, $q \in U_\beta$, with $U_\alpha \cap U_\beta = \emptyset$.*

Proof. Let (U, φ) be as in the lemma, and $\widetilde{U} = \varphi(U) \subseteq \mathbb{R}^m$. Since

$$\widetilde{p} = \varphi(p), \quad \widetilde{q} = \varphi(q)$$

are distinct points in $\widetilde{U}$, we can choose disjoint open subsets $\widetilde{U}_\alpha$, $\widetilde{U}_\beta \subseteq \widetilde{U}$ containing $\widetilde{p} = \varphi(p)$ and $\widetilde{q} = \varphi(q)$, respectively. Let U_α, $U_\beta \subseteq U$ be their preimages, and take $\varphi_\alpha = \varphi|_{U_\alpha}$, $\varphi_\beta = \varphi|_{U_\beta}$. Then $(U_\alpha, \varphi_\alpha)$ and (U_β, φ_β) are charts in $\mathscr{A}$ (see ✐ 12), with disjoint chart domains, and by construction, we have that $p \in U_\alpha$ and $q \in U_\beta$. □

Example 2.18. Consider the 2-sphere S^2 with the atlas given by the two coordinate charts (U_+, φ_+) and (U_-, φ_-). This atlas extends uniquely to a maximal atlas. The countability condition is satisfied, since S^2 is already covered by two charts. The Hausdorff condition is satisfied as well: Given distinct points $p, q \in S^2$, if both are contained in U_+ or both in U_-, we can apply Lemma 2.17. The only remaining case is if one point is the north pole and the other the south pole. But here we can construct U_α, U_β by replacing U_+ and U_- with the open upper hemisphere and open lower hemisphere, respectively. Alternatively, we can use the chart given by stereographic projection to the xz-plane, noting that this is also in the maximal atlas (see ✐ 10).

Remark 2.19. As we explained above, the Hausdorff condition rules out some strange examples that do not quite fit our idea of a space that is locally like $\mathbb{R}^n$. Nevertheless, the so-called *non-Hausdorff manifolds* (with non-Hausdorff more properly called *not necessarily Hausdorff*) do arise in some important applications. Much of the theory can be developed without the Hausdorff property, but there are some complications. For instance, initial value problems for vector fields need not have unique solutions for non-Hausdorff manifolds. Let us also note that while the classification of 1-dimensional manifolds is very easy, there is no nice classification of 1-dimensional non-Hausdorff manifolds.

Remark 2.20 (Charts Taking Values in "Abstract" Vector Spaces). In the definition of an m-dimensional manifold M, rather than letting the charts $(U_\alpha, \varphi_\alpha)$ take values in $\mathbb{R}^m$, we could just as well let them take values in m-dimensional real vector spaces E_α:

$$\varphi_\alpha : U_\alpha \to E_\alpha.$$

Transition functions are defined as before, except they now take an open subset of E_β to an open subset of E_α. A choice of basis identifies $E_\alpha = \mathbb{R}^m$ and takes us back to the original definition.

As far as the definition of manifolds is concerned, nothing has been gained by adding this level of abstraction. However, it often happens that the E_α's are given to us "naturally." For example, if M is a surface inside $\mathbb{R}^3$, one would typically use xy-coordinates, or xz-coordinates, or yz-coordinates on appropriate chart domains. It can then be useful to regard the xy-plane, xz-plane, and yz-plane as the target spaces of the coordinate maps, and for notational reasons, it may be convenient not to associate them with a single $\mathbb{R}^2$.

2.3 Examples of Manifolds

We will now discuss some basic examples of manifolds. In each case, the manifold structure is given by a finite atlas; hence, the countability property is immediate. We will not spend too much time on verifying the Hausdorff property; while it may be done "by hand," we will later have better ways of doing this.

We begin the list of examples with the observation that any open subset U of $\mathbb{R}^n$ is a manifold, with atlas determined by the chart (U, id_U).

2.3.1 Spheres

The construction of an atlas for the 2-sphere S^2, by stereographic projection, also works for the n-sphere

$$S^n = \{(x^0, \ldots, x^n) \mid (x^0)^2 + \cdots + (x^n)^2 = 1\}.$$

Let $U_\pm$ be the subsets obtained by removing $(\mp 1, 0, \ldots, 0)$. Stereographic projection from these two points defines bijections $\varphi_\pm : U_\pm \to \mathbb{R}^n$, and by calculations similar to those for the 2-sphere, we see that

$$\varphi_\pm(x^0, x^1, \ldots, x^n) = \frac{1}{1 \pm x^0}(x^1, \ldots, x^n) \tag{2.4}$$

with inverse (writing $\mathbf{u} = (u^1, \ldots, u^n)$)

$$\varphi_\pm^{-1}(\mathbf{u}) = \frac{1}{1 + ||\mathbf{u}||^2}\big(\pm(1 - ||\mathbf{u}||^2), 2u^1, \ldots, 2u^n\big). \tag{2.5}$$

For the transition function, one finds

$$(\varphi_- \circ \varphi_+^{-1})(\mathbf{u}) = \frac{\mathbf{u}}{||\mathbf{u}||^2}. \tag{2.6}$$

We leave it as an exercise to check the details.

An equivalent atlas, with $2n + 2$ charts, is given by the subsets $U_0^+, \ldots, U_n^+, U_0^-, \ldots, U_n^-$, where

$$U_j^+ = \{\mathbf{x} \in S^n |\, x^j > 0\}, \quad U_j^- = \{\mathbf{x} \in S^n |\, x^j < 0\}$$

for $j = 0, \ldots, n$, with $\varphi_j^\pm : U_j^\pm \to \mathbb{R}^n$ the projection to the j-th coordinate plane (in other words, omitting the j-th component x^j):

$$\varphi_j^\pm(x^0, \ldots, x^n) = (x^0, \ldots, x^{j-1}, x^{j+1}, \ldots, x^n).$$

2.3.2 Real Projective Spaces

The n-dimensional projective space, denoted $\mathbb{R}\mathrm{P}^n$, is the set of all lines $\ell \subseteq \mathbb{R}^{n+1}$, where *line* is taken to mean "1-dimensional subspace." It may also be regarded as a quotient space (see Appendix A)

$$\mathbb{R}\mathrm{P}^n = (\mathbb{R}^{n+1} \backslash \{\mathbf{0}\})/\sim$$

for the equivalence relation

$$\mathbf{x} \sim \mathbf{x}' \Leftrightarrow \exists \lambda \in \mathbb{R} \backslash \{\mathbf{0}\} : \mathbf{x}' = \lambda \mathbf{x}.$$

Indeed, any non-zero vector $\mathbf{x} \in \mathbb{R}^{n+1} \backslash \{\mathbf{0}\}$ determines a line, while two vectors $\mathbf{x}, \mathbf{x}'$ determine the same line if and only if they agree up to a non-zero scalar multiple. The equivalence class of $\mathbf{x} = (x^0, \ldots, x^n)$ under this relation is commonly denoted

$$[\mathbf{x}] = (x^0 : \ldots : x^n).$$

The $(\cdot : \ldots : \cdot)$ are called *homogeneous coordinates*.

13 (answer on page 272). Show that the following are equivalent characterizations of $\mathbb{R}\mathrm{P}^n$ (in the sense that there are "natural" set-theoretic bijections):

(a) The sphere S^n with antipodal identification.

(b) The closed ball $B^n := \{\mathbf{x} \in \mathbb{R}^n \mid ||\mathbf{x}||^2 \leq 1\}$, with antipodal identification on its boundary sphere S^{n-1}. (That is, $\mathbf{x} \sim -\mathbf{x}$ for $\mathbf{x} \in S^{n-1}$.)

Specializing to $n = 1$, define a bijection $\mathbb{R}\mathrm{P}^1 \cong S^1$.

The projective space $\mathbb{R}\mathrm{P}^n$ has a *standard atlas*

$$\mathscr{A} = \{(U_0, \varphi_0), \ldots, (U_n, \varphi_n)\}$$

defined as follows. For $j = 0, \ldots, n$, let

$$U_j = \{(x^0 : \ldots : x^n) \in \mathbb{R}\mathrm{P}^n \mid x^j \neq 0\}$$

be the set for which the j-th coordinate is non-zero, and put

$$\varphi_j : \ U_j \to \mathbb{R}^n, \ \ (x^0 : \ldots : x^n) \mapsto \big(\frac{x^0}{x^j}, \ldots, \frac{x^{j-1}}{x^j}, \frac{x^{j+1}}{x^j}, \ldots, \frac{x^n}{x^j}\big).$$

This is well-defined, since the quotients do not change when all x^i are multiplied by a fixed scalar. Put differently, given an element $[\mathbf{x}] \in \mathbb{R}\mathrm{P}^n$ for which the j-th component x^j is non-zero, we first rescale the representative $\mathbf{x}$ to make the j-th component 1 and then use the remaining components as our coordinates. As an example (with $n = 2$),

$$\varphi_1(7:3:2) = \varphi_1\big(\frac{7}{3} : 1 : \frac{2}{3}\big) = \big(\frac{7}{3}, \frac{2}{3}\big).$$

From this description, it is immediate that φ_j is a bijection from U_j onto $\mathbb{R}^n$, with inverse map

$$\varphi_j^{-1}(u^1, \ldots, u^n) = (u^1 : \ldots : u^j : 1 : u^{j+1} : \ldots : u^n).$$

Geometrically, viewing $\mathbb{R}\mathrm{P}^n$ as the set of lines in $\mathbb{R}^{n+1}$, the subset $U_j \subseteq \mathbb{R}\mathrm{P}^n$ consists of those lines ℓ that intersect the affine hyperplane

$$H_j = \{\mathbf{x} \in \mathbb{R}^{n+1} \mid x^j = 1\},$$

and the map φ_j takes such a line ℓ to its unique point of intersection $\ell \cap H_j$, followed by the identification $H_j \cong \mathbb{R}^n$ (dropping the coordinate $x^j = 1$).

Let us verify that $\mathscr{A}$ is indeed an atlas. Clearly, the domains U_j cover $\mathbb{R}\mathrm{P}^n$, since any element $[\mathbf{x}] \in \mathbb{R}\mathrm{P}^n$ has at least one of its components non-zero. For $i \neq j$, the intersection $U_i \cap U_j$ consists of elements $\mathbf{x}$ with the property that both components x^i, x^j are non-zero.

14 (answer on page 272). Compute the transition maps $\varphi_i \circ \varphi_j^{-1}$, and verify that they are smooth. (You will need to distinguish between the cases $i < j$ and $i > j$.)

To complete the proof that this atlas (or the unique maximal atlas containing it) defines a manifold structure, it remains to check the Hausdorff property. This can be done with the help of Lemma 2.17, but we postpone the proof since we will soon have a simple argument in terms of smooth functions (see Proposition 3.6 below).

In summary, the real projective space $\mathbb{R}\mathrm{P}^n$ is a manifold of dimension n. For $n = 1$, it is called the *(real) projective line*, for $n = 2$ the *(real) projective plane*.

Remark 2.21. Geometrically, U_i consists of all lines in $\mathbb{R}^{n+1}$ meeting the affine hyperplane H_i; hence, its complement consists of all lines that are parallel to H_i, i.e., the lines in the coordinate subspace defined by $x^i = 0$. The set of such lines is $\mathbb{R}\mathrm{P}^{n-1}$. In other words, the complement of U_i in $\mathbb{R}\mathrm{P}^n$ is identified with $\mathbb{R}\mathrm{P}^{n-1}$. Thus, as sets, $\mathbb{R}\mathrm{P}^n$ is a disjoint union

$$\mathbb{R}\mathrm{P}^n = \mathbb{R}^n \sqcup \mathbb{R}\mathrm{P}^{n-1},$$

where $\mathbb{R}^n$ is identified (by the coordinate map φ_i) with the open subset U_n, and $\mathbb{R}\mathrm{P}^{n-1}$ with its complement. Inductively, we obtain a decomposition

$$\mathbb{R}\mathrm{P}^n = \mathbb{R}^n \sqcup \mathbb{R}^{n-1} \sqcup \cdots \sqcup \mathbb{R} \sqcup \mathbb{R}^0, \tag{2.7}$$

where $\mathbb{R}^0 = \{0\}$. At this stage, it is simply a decomposition into subsets; later it will be recognized as a decomposition into submanifolds (see Example 4.8).

15 (answer on page 272). Find an identification of the space of rotations in $\mathbb{R}^3$ with the 3-dimensional projective space $\mathbb{R}\mathrm{P}^3$. (Suggestion: Associate a rotation to every $\mathbf{x} \in \mathbb{R}^3$ with $||\mathbf{x}|| \le \pi$.)

2.3.3 Complex Projective Spaces*

In a similar fashion, one can define a *complex projective space* $\mathbb{C}\mathrm{P}^n$ as the set of complex 1-dimensional subspaces of $\mathbb{C}^{n+1}$. We have

$$\mathbb{C}\mathrm{P}^n = (\mathbb{C}^{n+1} \backslash \{\mathbf{0}\}) / \sim,$$

where the equivalence relation is given by the condition that $\mathbf{z} \sim \mathbf{z}'$ if and only if there exists a complex λ with $\mathbf{z}' = \lambda \mathbf{z}$. (Note that the scalar λ is then unique and is non-zero.) Identify $\mathbb{C}$ with $\mathbb{R}^2$; thus $\mathbb{C}^{n+1}$ with $\mathbb{R}^{2n+2}$. Letting $S^{2n+1} \subseteq \mathbb{C}^{n+1} = \mathbb{R}^{2n+2}$ be the "unit sphere" consisting of complex vectors of length $||\mathbf{z}|| = 1$, we have

$$\mathbb{C}\mathrm{P}^n = S^{2n+1} / \sim,$$

where $\mathbf{z}' \sim \mathbf{z}$ if and only if there exists a complex number λ with $\mathbf{z}' = \lambda \mathbf{z}$. (Note that the scalar λ is then unique and has absolute value 1.) One defines charts (U_j, φ_j) similarly to those for the real projective space:

$$U_j = \left\{ (z^0 : \ldots : z^n) \,|\, z^j \neq 0 \right\}, \quad \varphi_j : \ U_j \to \mathbb{C}^n = \mathbb{R}^{2n},$$

$$\varphi_j(z^0 : \ldots : z^n) = \left(\frac{z^0}{z^j}, \ldots, \frac{z^{j-1}}{z^j}, \frac{z^{j+1}}{z^j}, \ldots, \frac{z^n}{z^j}\right).$$

The transition maps between charts are given by similar formulas as for $\mathbb{R}\mathrm{P}^n$ (just replace $\mathbf{x}$ with $\mathbf{z}$); they are smooth maps between open subsets of $\mathbb{C}^n = \mathbb{R}^{2n}$. Thus $\mathbb{C}\mathrm{P}^n$ is a smooth manifold of dimension $2n$. As with $\mathbb{R}\mathrm{P}^n$ (see Equation (2.7)), there is a decomposition

$$\mathbb{C}\mathrm{P}^n = \mathbb{C}^n \sqcup \mathbb{C}^{n-1} \sqcup \cdots \sqcup \mathbb{C} \sqcup \mathbb{C}^0.$$

We will show later (Section 3.6.2) that $\mathbb{C}\mathrm{P}^1 \cong S^2$ as manifolds; for larger n, we obtain genuinely "new" manifolds.

Remark 2.22. (For those who know a little bit of complex analysis.) We took m-dimensional manifolds to be modeled on open subsets of $\mathbb{R}^m$, with smooth transition maps. In a similar way, one can define *complex manifolds M* of complex dimension m to be modeled on open subsets of $\mathbb{C}^m$, with transition maps that are infinitely differentiable in the complex sense, i.e., *holomorphic*. In more detail, a complex manifold of dimension m may be defined as a real manifold M of dimension $2m$, with an atlas $\mathscr{A}$, such that all coordinate charts $(U_\alpha, \varphi_\alpha)$ take values in $\mathbb{C}^m \cong \mathbb{R}^{2m}$, and all transition maps $\varphi_\alpha \circ \varphi_\beta^{-1} : \ \varphi_\beta(U_\alpha \cap U_\beta) \to \varphi_\alpha(U_\alpha \cap U_\beta)$ are holomorphic.

The complex projective space $\mathbb{C}\mathrm{P}^n$ is an important example of a complex manifold, of complex dimension n. For $n = 1$, it is called the *complex projective line*, for $n = 2$ the *complex projective plane*.

2.3.4 Real Grassmannians*

The set $\mathrm{Gr}(k,n)$ of all k-dimensional subspaces of $\mathbb{R}^n$ is called the *Grassmannian of k-planes in $\mathbb{R}^n$*. (Named after *Hermann Grassmann* (1809–1877).) As a special case, $\mathrm{Gr}(1,n) = \mathbb{R}\mathrm{P}^{n-1}$.

We will show that the Grassmannian is a manifold of dimension

$$\dim(\mathrm{Gr}(k,n)) = k(n-k).$$

An atlas for $\mathrm{Gr}(k,n)$ may be constructed as follows. The idea is to present linear subspaces $E \subseteq \mathbb{R}^n$ of dimension k as graphs of linear maps from $\mathbb{R}^k$ to $\mathbb{R}^{n-k}$. Here $\mathbb{R}^k$ is viewed as the coordinate subspace corresponding to a choice of k components from $\mathbf{x} = (x^1, \ldots, x^n) \in \mathbb{R}^n$, and $\mathbb{R}^{n-k}$ the coordinate subspace for the remaining coordinates.

We introduce some notation to help us make this idea precise. For any subset $I \subseteq \{1, \ldots, n\}$ of the set of indices, let

$$I' = \{1, \ldots, n\} \backslash I$$

be its complement. Let $\mathbb{R}^I \subseteq \mathbb{R}^n$ be the coordinate subspace

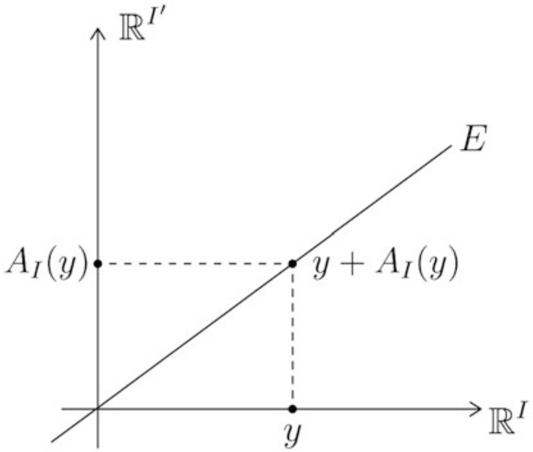
$$\mathbb{R}^I = \{\mathbf{x} \in \mathbb{R}^n |\, x^i = 0 \text{ for all } i \in I'\}.$$

If I has cardinality[*] $|I| = k$, then $\mathbb{R}^I \in \mathrm{Gr}(k,n)$. Note that $\mathbb{R}^{I'} = (\mathbb{R}^I)^\perp$. Let

$$U_I = \{E \in \mathrm{Gr}(k,n) |\ E \cap \mathbb{R}^{I'} = \{\mathbf{0}\}\},$$

the set of k-dimensional subspaces that are transverse to $\mathbb{R}^{I'}$. Each $E \in U_I$ is described as the graph of a unique linear map $A_I : \mathbb{R}^I \to \mathbb{R}^{I'}$, that is,

$$E = \{\mathbf{y} + A_I(\mathbf{y}) | \mathbf{y} \in \mathbb{R}^I\}.$$



16 (answer on page 273). Verify the claim that every $E \in U_I$ determines a unique linear map $A_I : \mathbb{R}^I \to \mathbb{R}^{I'}$ such that $E = \{\mathbf{y} + A_I(\mathbf{y}) | \mathbf{y} \in \mathbb{R}^I\}$.

Thus, we have a bijection

$$\varphi_I : U_I \to \mathrm{Hom}(\mathbb{R}^I, \mathbb{R}^{I'}),\ E \mapsto \varphi_I(E) = A_I,$$

where $\mathrm{Hom}(V,W)$ denotes the space of linear maps from a vector space V to a vector space W (also commonly denoted $\mathscr{L}(V,W)$). Note that we have a linear isomorphism $\mathrm{Hom}(\mathbb{R}^I, \mathbb{R}^{I'}) \cong \mathbb{R}^{k(n-k)}$, because the bases of $\mathbb{R}^I$ and $\mathbb{R}^{I'}$ identify the space of linear maps with $((n-k) \times k)$-matrices, which in turn is just $\mathbb{R}^{k(n-k)}$ by listing the matrix entries. On the other hand, as explained in Remark 2.20, it is not necessary to make this identification, and indeed, it is better to work with the vector space $\mathrm{Hom}(\mathbb{R}^I, \mathbb{R}^{I'})$ as the chart codomain. In terms of A_I, the subspace $E \in U_I$ is the range of the injective linear map

$$\begin{pmatrix} 1 \\ A_I \end{pmatrix} :\ \mathbb{R}^I \to \mathbb{R}^I \oplus \mathbb{R}^{I'} \cong \mathbb{R}^n, \tag{2.8}$$

[*] It is common to use $|\cdot|$ for the cardinality ("size") of a set. Context will distinguish it from absolute value or complex modulus.

where we write elements of $\mathbb{R}^n$ as column vectors. The situation may be described by the commutative diagram below.

$$\begin{tikzcd}
\mathbb{R}^I \oplus \mathbb{R}^{I'} \arrow[r, "\cong"] & \mathbb{R}^n \\
\mathbb{R}^I \arrow[u, hook, "\begin{pmatrix} 1 \\ A_I \end{pmatrix}"] \arrow[r, "\cong"] & E \arrow[u, hook]
\end{tikzcd} \tag{2.9}$$

(The inverse of the bottom map is the map $E \to \mathbb{R}^I$ given by projection on $\mathbb{R}^I$.)

To check that the charts are compatible, suppose $E \in U_I \cap U_J$, and let A_I and A_J be the linear maps describing E in the two charts. We have to show that the map

$$\varphi_J \circ \varphi_I^{-1}: \ \mathrm{Hom}(\mathbb{R}^I, \mathbb{R}^{I'}) \to \mathrm{Hom}(\mathbb{R}^J, \mathbb{R}^{J'}), \ \ A_I = \varphi_I(E) \mapsto A_J = \varphi_J(E)$$

is smooth. To find the expression for A_J in terms of A_I, combine (2.9) with a similar diagram for J.

$$\begin{tikzcd}
\mathbb{R}^I \oplus \mathbb{R}^{I'} \arrow[r, "\cong"] & \mathbb{R}^n & \mathbb{R}^J \oplus \mathbb{R}^{J'} \arrow[l, "\cong"'] \\
\mathbb{R}^I \arrow[u, hook, "\begin{pmatrix} 1 \\ A_I \end{pmatrix}"] \arrow[r, "\cong"] & E \arrow[u, hook] & \mathbb{R}^J \arrow[l, "\cong"'] \arrow[u, hook, "\begin{pmatrix} 1 \\ A_J \end{pmatrix}"']
\end{tikzcd} \tag{2.10}$$

The composition of the first upper horizontal map with the inverse of the second upper horizontal map gives an isomorphism $\mathbb{R}^I \oplus \mathbb{R}^{I'} \to \mathbb{R}^J \oplus \mathbb{R}^{J'}$, written in block form as

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}: \ \mathbb{R}^I \oplus \mathbb{R}^{I'} \to \mathbb{R}^J \oplus \mathbb{R}^{J'}.$$

For example, the lower left block "c" is the inclusion $\mathbb{R}^I \to \mathbb{R}^n$ as the corresponding coordinate subspace, followed by projection to the coordinate subspace $\mathbb{R}^{J'}$. Similarly, the composition of the first lower horizontal map with the inverse of the second lower horizontal map gives an isomorphism

$$S: \ \mathbb{R}^I \to \mathbb{R}^J.$$

Leaving out the middle map from (2.10), we obtain the commutative diagram below.

$$\begin{tikzcd}
\mathbb{R}^I \oplus \mathbb{R}^{I'} \arrow[r, "\begin{pmatrix} a & b \\ c & d \end{pmatrix}"] & \mathbb{R}^J \oplus \mathbb{R}^{J'} \\
\mathbb{R}^I \arrow[u, hook, "\begin{pmatrix} 1 \\ A_I \end{pmatrix}"] \arrow[r, "S"'] & \mathbb{R}^J \arrow[u, hook, "\begin{pmatrix} 1 \\ A_J \end{pmatrix}"']
\end{tikzcd} \tag{2.11}$$

That is,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 \\ A_I \end{pmatrix} = \begin{pmatrix} 1 \\ A_J \end{pmatrix} S$$

or equivalently

$$\begin{pmatrix} a + bA_I \\ c + dA_I \end{pmatrix} = \begin{pmatrix} S \\ A_J S \end{pmatrix}.$$

Using the first row of this equation to eliminate the second row, we obtain the desired formula for the transition function $\varphi_J \circ \varphi_I^{-1}$, expressing A_J in terms of A_I:

$$A_J = (c + dA_I)\,(a + bA_I)^{-1}.$$

The dependence of the right-hand side on the matrix entries of A_I is smooth, by Cramer's formula for the inverse matrix.

It follows that the collection of all $\varphi_I : U_I \to \mathrm{Hom}(\mathbb{R}^I, \mathbb{R}^{I'}) \cong \mathbb{R}^{k(n-k)}$ defines on $\mathrm{Gr}(k,n)$ the structure of a manifold of dimension $k(n-k)$. The number of charts of this atlas equals the number of subsets $I \subseteq \{1,\dots,n\}$ of cardinality k, that is, $\binom{n}{k}$.

The Hausdorff property may be checked in a similar fashion to $\mathbb{R}\mathrm{P}^n$. Here is a sketch of an alternative argument (later, we will have much simpler criteria for the Hausdorff property, avoiding these types of ad hoc arguments). Given distinct $E_1, E_2 \in \mathrm{Gr}(k,n)$, choose a subspace $F \in \mathrm{Gr}(k,n)$ such that $F^\perp$ has zero intersection with both E_1, E_2. (Such a subspace always exists.) One can then define a chart (U, φ), where U is the set of subspaces E transverse to $F^\perp$, and φ realizes any such map as the graph of a linear map $F \to F^\perp$. Thus $\varphi :\ U \to \mathrm{Hom}(F, F^\perp)$. As above, we can check that this is compatible with all the charts (U_I, φ_I). Since both E_1, E_2 are in this chart U, we are done by Lemma 2.17.

17 (answer on page 273). Prove the parenthetical remark above: Given $E_1, E_2 \in \mathrm{Gr}(k,n)$, there exists $F \in \mathrm{Gr}(k,n)$ such that $F^\perp \cap E_1 = \{\mathbf{0}\}$ and $F^\perp \cap E_2 = \{\mathbf{0}\}$.

Remark 2.23. As already mentioned, $\mathrm{Gr}(1,n) = \mathbb{R}\mathrm{P}^{n-1}$. One can check that our system of charts in this case is the standard atlas for $\mathbb{R}\mathrm{P}^{n-1}$.

18 (answer on page 273). This is a preparation for the following remark. Recall that a linear map $\Pi :\ \mathbb{R}^n \to \mathbb{R}^n$ is an orthogonal projection onto some subspace $E \subseteq \mathbb{R}^n$ if $\Pi(\mathbf{x}) = \mathbf{x}$ for $\mathbf{x} \in E$ and $\Pi(\mathbf{x}) = \mathbf{0}$ for $\mathbf{x} \in E^\perp$. Show that a square matrix $P \in \mathrm{Mat}_{\mathbb{R}}(n)$ is the matrix of an orthogonal projection if and only if it has the properties

$$P^\top = P,\ PP = P,$$

where the superscript $\top$ indicates "transpose." What is the matrix of the orthogonal projection onto $E^\perp$?

For any k-dimensional subspace $E \subseteq \mathbb{R}^n$, let $P_E \in \mathrm{Mat}_{\mathbb{R}}(n)$ be the matrix of the linear map given by orthogonal projection onto E. By 18,

$$P_E^\top = P_E, \;\; P_E P_E = P_E;$$

conversely, any square matrix P with the properties $P^\top = P$, $PP = P$ with $\mathrm{rank}(P) = k$ corresponds to a k-dimensional subspace $E = \{P\mathbf{x} | \, \mathbf{x} \in \mathbb{R}^n\} \subseteq \mathbb{R}^n$. This identifies the Grassmannian $\mathrm{Gr}(k,n)$ with the set of orthogonal projections of rank k. In summary, we have an inclusion

$$\mathrm{Gr}(k,n) \hookrightarrow \mathrm{Mat}_{\mathbb{R}}(n) \cong \mathbb{R}^{n^2}, \quad E \mapsto P_E. \tag{2.12}$$

Note that this inclusion takes values in the subspace $\mathrm{Sym}_{\mathbb{R}}(n) \cong \mathbb{R}^{n(n+1)/2}$ of *symmetric* $(n \times n)$-matrices.

19 (answer on page 274). Describe a natural bijection $\mathrm{Gr}(k,n) \cong \mathrm{Gr}(n-k,n)$, both in terms of subspaces and in terms of orthogonal projections.

20 (answer on page 274). Given a real n-dimensional vector space V, let $\mathrm{Gr}(k,V)$ be the set of k-dimensional subspaces of V. It is identified with $\mathrm{Gr}(k,n)$ once a basis of V is chosen; in particular, it is a manifold of dimension $k(n-k)$. Letting V^* be the dual space, describe a bijection $\mathrm{Gr}(k,V) \cong \mathrm{Gr}(n-k,V^*)$, which is "natural" in the sense that it does not depend on additional choices (such as a choice of basis).

Remark 2.24. Similarly to $\mathbb{R}\mathrm{P}^2 = S^2/\sim$, the quotient modulo antipodal identification, one can also consider

$$M = (S^2 \times S^2)/\sim$$

the quotient space by the equivalence relation

$$(\mathbf{x}, \mathbf{x}') \sim (-\mathbf{x}, -\mathbf{x}').$$

It turns out (see, e.g., [5]) that this manifold M is the same as $\mathrm{Gr}(2,4)$, in the sense that there is a bijection of sets identifying the atlases.

2.3.5 Complex Grassmannians*

Similarly to the case of projective spaces, one can also consider the *complex Grassmannian* $\mathrm{Gr}_{\mathbb{C}}(k,n)$ of complex k-dimensional subspaces of $\mathbb{C}^n$. It is a manifold of dimension $2k(n-k)$, which can also be regarded as a complex manifold of complex dimension $k(n-k)$.

2.4 Open Subsets

Let M be a set equipped with an m-dimensional maximal atlas $\mathscr{A} = \{(U_\alpha, \varphi_\alpha)\}$.

Definition 2.25. *A subset $U \subseteq M$ is* open *if and only if for all charts $(U_\alpha, \varphi_\alpha) \in \mathscr{A}$ the set $\varphi_\alpha(U \cap U_\alpha)$ is open.*

To check that a subset U is open, it is not actually necessary to verify this condition for all charts. As the following proposition shows, it is enough to check for any collection of charts whose union contains U. In particular, we may take $\mathscr{A}$ in Definition 2.25 to be any atlas, not necessarily a maximal atlas.

Proposition 2.26. *Given $U \subseteq M$, let $\mathscr{B} \subseteq \mathscr{A}$ be any collection of charts whose union contains U. Then U is open if and only if for all charts (U_β, φ_β) from $\mathscr{B}$, the sets $\varphi_\beta(U \cap U_\beta)$ are open.*

Proof. The "only if" part is obvious; let us prove the "if" part. In what follows, we reserve the index β to indicate charts (U_β, φ_β) from $\mathscr{B}$. Suppose $\varphi_\beta(U \cap U_\beta)$ is open for all such β. Let $(U_\alpha, \varphi_\alpha)$ be a given chart in the maximal atlas $\mathscr{A}$. We have that

$$\begin{aligned}
\varphi_\alpha(U \cap U_\alpha) &= \bigcup_\beta \varphi_\alpha(U \cap U_\alpha \cap U_\beta) \\
&= \bigcup_\beta (\varphi_\alpha \circ \varphi_\beta^{-1})\big(\varphi_\beta(U \cap U_\alpha \cap U_\beta)\big) \\
&= \bigcup_\beta (\varphi_\alpha \circ \varphi_\beta^{-1})\big(\varphi_\beta(U_\alpha \cap U_\beta) \cap \varphi_\beta(U \cap U_\beta)\big).
\end{aligned}$$

Since $\mathscr{B} \subseteq \mathscr{A}$, all $\varphi_\beta(U_\alpha \cap U_\beta)$ are open. Hence, the intersection with $\varphi_\beta(U \cap U_\beta)$ is open, and so is the preimage under the diffeomorphism $\varphi_\alpha \circ \varphi_\beta^{-1}$. Finally, we use that a union of open sets of $\mathbb{R}^m$ is again open. □

If $\mathscr{A}$ is an atlas on M, and $U \subseteq M$ is open, then U inherits an atlas by restriction:

$$\mathscr{A}_U = \{(U \cap U_\alpha, \varphi_\alpha|_{U \cap U_\alpha})\}.$$

21 (answer on page 274). Verify that if $\mathscr{A}$ is a maximal atlas, then so is $\mathscr{A}_U$, and if this maximal atlas $\mathscr{A}$ satisfies the countability and Hausdorff properties, then so does $\mathscr{A}_U$.

This then proves:

Proposition 2.27 (Open Subsets Are Manifolds). *An open subset of a manifold is again a manifold.*

The collection of open sets of M with respect to an atlas has properties similar to those for $\mathbb{R}^m$.

Proposition 2.28. *Let M be a set with an m-dimensional maximal atlas. The collection of all open subsets of M has the following properties:*

- *$\emptyset, M$ are open.*
- *The intersection $U \cap U'$ of any two open sets U, U' is again open.*
- *The union $\bigcup_{i \in I} U_i$ of a collection U_i, $i \in I$ of open sets is again open. Here I is any index set (not necessarily countable).*

Proof. All of these properties follow from similar properties of open subsets in $\mathbb{R}^m$. For instance, if U, U' are open, then

$$\varphi_\alpha((U \cap U') \cap U_\alpha) = \varphi_\alpha(U \cap U_\alpha) \cap \varphi_\alpha(U' \cap U_\alpha)$$

is an intersection of open subsets of $\mathbb{R}^m$; hence, it is open, and therefore, $U \cap U'$ is open. $\square$

These properties mean, by definition, that the collection of open subsets of M defines a *topology* on M. This allows us to adopt various notions from topology:

(a) If U is an open subset and $p \in U$, then U is called an *open neighborhood of p*. More generally, if $A \subseteq U$ is a subset contained in M, then U is called an *open neighborhood of A*.
(b) A subset $A \subseteq M$ is called *closed* if its complement $M \backslash A$ is open.
(c) M is called *disconnected* if it can be written as the disjoint union $M = U \sqcup V$ of two open subsets $U, V \subseteq M$ (with $U \cap V = \emptyset$). M is called *connected* if it is not disconnected; equivalently, if the only subsets $A \subseteq M$ that are both closed and open are $A = \emptyset$ and $A = M$.

The Hausdorff condition in the definition of manifolds can now be restated as the condition that *any two distinct points p, q in M have disjoint open neighborhoods*. (It is not necessary to take them to be domains of coordinate charts.)

It is immediate from the definition that domains of coordinate charts are open. Indeed, this gives an alternative way of defining the open sets.

22 (answer on page 274). Let M be a set with a maximal atlas. Show that a subset $U \subseteq M$ is open if and only if it is either empty or is a union $U = \bigcup_{i \in I} U_i$ where the U_i are domains of coordinate charts.

23 (answer on page 274). Let M be a set with an m-dimensional maximal atlas $\mathscr{A}$, and let (U, φ) be a chart in $\mathscr{A}$. Let $V \subseteq \mathbb{R}^m$ be open. Prove that $\varphi^{-1}(V)$ is open.

24 (answer on page 275). Let M be a set with an m-dimensional maximal atlas $\mathscr{A}$, and $A \subseteq M$ a subset. Show that A is *not* open if and only if there exists $a \in A$ such that every open neighborhood of a contains points of $M \setminus A$.

Regarding the notion of connectedness, we have:

25 (answer on page 275). Let M be a set with an m-dimensional maximal atlas. A function $f: M \to \mathbb{R}$ is called *locally constant* if every point $p \in M$ has an open neighborhood over which f is constant. Show that M is connected if and only if every locally constant function is in fact constant.

2.5 Compactness

A subset $K \subseteq \mathbb{R}^m$ is *compact* if it is closed as well as bounded. This property has important consequences—for instance, every continuous function on a compact set attains its maximum and minimum. How should we define compactness for subsets of manifolds? Our first approach (Definition 2.29) does not require any background in general topology and instead works with the well-known properties of closed and bounded subsets of $\mathbb{R}^m$. Students with knowledge of topology may prefer the definition in terms of open covers (Definition 2.33). Proposition 2.36 shows that the two approaches are equivalent.

To begin, note that for compact subsets of $\mathbb{R}^m$ we have the following results:

26 (answer on page 275).

(a) Let $K \subseteq \mathbb{R}^m$ be compact and $C \subseteq R^m$ closed. Show that $K \cap C$ is compact.

(b) Let $K_1, \ldots, K_n \subseteq \mathbb{R}^m$ be compact. Show that $K_1 \cup \cdots \cup K_n$ is compact.

Let M be a manifold. We would like to define compactness "in charts": If $K \subseteq M$ is contained in the domain of a coordinate chart, then this subset should be compact if its image under the coordinate map is compact. Moreover, finite unions of compact subsets should again be compact. This leads us to the following definition.

Definition 2.29 (Compact Subsets—First Definition). *Let M be a manifold. A subset $K \subseteq M$ is* compact *if it can be written as a finite union*

$$K = K_1 \cup \cdots \cup K_n,$$

where each $K_i \subseteq M$ is contained in the domain of some coordinate chart (U_i, φ_i), and each $\varphi_i(K_i)$ is a closed and bounded subset of $\mathbb{R}^m$.

In particular, the manifold M itself is called *compact* if $K = M$ has this property.

Example 2.30. We show that the n-sphere $M = S^n$ is compact. The closed upper hemisphere $\{\mathbf{x} \in S^n | \; x^0 \geq 0\}$ is compact because it is contained in the coordinate chart (U_+, φ_+) for stereographic projection, and its image under φ_+ is the closed and bounded subset $\{\mathbf{u} \in \mathbb{R}^n | \; ||\mathbf{u}|| \leq 1\}$. Likewise, the closed lower hemisphere is compact, and hence, S^n itself (as the union of the upper and lower hemispheres) is compact.

Example 2.31. We show that the real projective space $\mathbb{RP}^n$ is compact. Recall that the standard atlas for $\mathbb{RP}^n$ is $\{(U_i, \varphi_i) | \; i = 0, \ldots, n\}$, where U_i is the set of points $(x^0 : \ldots : x^n)$ with $x^i \neq 0$, and

$$\varphi_i(x^0 : \ldots : x^n) = \left(\frac{x^0}{x^i}, \ldots, \frac{x^{i-1}}{x^i}, \frac{x^{i+1}}{x^i}, \ldots, \frac{x^n}{x^i} \right).$$

This has range $\varphi_i(U_i) = \mathbb{R}^n$. For given $R > 0$, let

$$K_i = \varphi_i^{-1}(\overline{B_R(\mathbf{0})}),$$

the preimage of the closed ball of radius R. We claim that for sufficiently large R, each $p \in \mathbb{RP}^n$ is contained in one of the K_i; hence, $\mathbb{RP}^n = K_0 \cup \cdots \cup K_n$. To see this, write $p = (x^0 : \ldots : x^n)$, and let i be the index for which $|x^i|$ is maximal. Then $|x^j| \leq |x^i|$ for all j; hence,

$$||\varphi_i(p)||^2 = \sum_{j \neq i} \frac{|x^j|^2}{|x^i|^2} \leq \sum_{j \neq i} 1 = n.$$

Hence, $R = \sqrt{n}$ has the desired property. We conclude that $\mathbb{RP}^n$ is compact.

In a similar way, one can prove the compactness of $\mathbb{CP}^n$, $\mathrm{Gr}(k,n)$, and $\mathrm{Gr}_{\mathbb{C}}(k,n)$. Soon we will have a simpler way of verifying compactness for these manifolds, by showing that they are closed and bounded subsets of $\mathbb{R}^N$ for a suitable N.

The basic properties of compact subsets of $\mathbb{R}^m$ generalize to compact subsets $K \subseteq M$ of manifolds:

(a) If $C \subseteq M$ is closed, then $K \cap C$ is again compact.
(b) The union of finitely many compact subsets $K_1, \ldots, K_n \subseteq M$ is compact.

27 (answer on page 275). Prove Properties (a) and (b) above from the corresponding properties of compact subsets of $\mathbb{R}^m$.

Remark 2.32. For manifolds M, we may define a subset $A \subseteq M$ to be *bounded* if it is contained in some compact subset K of M. It is then true that closed, bounded subsets A of manifolds are compact. (But we cannot use this as a definition of compactness, since we used compactness to define boundedness.)

Let us now turn to the topologist's definition of compact subsets, in terms of open covers. While seemingly more complex, it turns out to be quite convenient for many applications.

Definition 2.33 (Compact Subsets—Second Definition). *A subset $K \subseteq M$ is called* compact *if it has the* Heine–Borel property*: For every collection $\{U_\alpha\}$ of open subsets of M whose union contains K, there are finitely many indices $\alpha_1, \ldots, \alpha_n$ such that*

$$K \subseteq U_{\alpha_1} \cup \cdots \cup U_{\alpha_n}.$$

In short, $K \subseteq M$ is compact if and only if every open cover of K admits a finite subcover. The *Heine–Borel Theorem* from multivariable calculus (see, e.g., Munkres [15], Chapter 1§4) states that a subset $A \subseteq \mathbb{R}^m$ has the Heine–Borel property if and only if it is closed and bounded.

28 (answer on page 276). Show that if $K_1, \ldots, K_n \subseteq M$ are compact in the sense of Definition 2.33, then so is their union $K = K_1 \cup \cdots \cup K_n$.

Proposition 2.34. *Let M be a manifold.*

(a) If $K \subseteq M$ is compact, then K is closed.
(b) If $K \subseteq M$ is compact, and $C \subseteq M$ is closed, then $K \cap C$ is compact. In particular, the intersection of compact sets is compact.

(Here compactness is understood in the sense of Definition 2.33.)

Proof. For Property (a), suppose K is compact but not closed. By 24, there exists $y \in M \setminus K$ such that every open neighborhood of y contains some point of K. Using the Hausdorff property, for each $x \in K$, there exist disjoint open neighborhoods of x and y. To emphasize the dependence on x, we denote these neighborhoods by U_x and V_x, i.e.,

$$x \in U_x, \quad y \in V_x, \quad U_x \cap V_x = \emptyset.$$

The collection of subsets $\{U_x \,|\, x \in K\}$ is an open cover of K and by compactness has a finite subcover:

$$K \subseteq U_{x_1} \cup \cdots \cup U_{x_n}.$$

Then $V = V_{x_1} \cap \cdots \cap V_{x_n}$ is an open neighborhood of y with no points of K, a contradiction. Property (b) is left as 29. $\square$

29 (answer on page 276). Prove Property (b) of Proposition 2.34.

Remark 2.35. The definition of compactness in terms of the Heine–Borel property (Definition 2.33) applies to arbitrary topological spaces and in particular to non-Hausdorff manifolds. However, in this more general setting, compact subsets need not be closed. See Problem 8 at the end of the chapter.

Proposition 2.36. *Definitions 2.29 and 2.33 of compact subsets of manifolds M are equivalent.*

Proof. For the purpose of this proof, let us call subsets satisfying Definition 2.29 *c-compact* (chart compact) and subsets satisfying Definition 2.29 *t-compact* (topologically compact). As a first step toward the proof, we observe:

$(*)$ If (U, φ) is a coordinate chart, then $K \subseteq U$ is t-compact if and only if $\varphi(K)$ is closed and bounded.

This follows from the Heine–Borel theorem for $\mathbb{R}^m$, and details are left as 30. Our goal is to show that for general subsets $K \subseteq M$,

$$K \text{ is c-compact} \Leftrightarrow K \text{ is t-compact.} \tag{2.13}$$

"$\Rightarrow$." Suppose $K \subseteq M$ is c-compact. To show K is t-compact, let $\{U_\alpha\}$ be an open cover of K. Choose a decomposition $K = K_1 \cup \cdots \cup K_n$ and coordinate charts (V_i, ψ_i), $i = 1, \ldots, n$ such that $K_i \subseteq V_i$, and each $\psi_i(K_i) \subseteq \mathbb{R}^m$ is closed and bounded. Then each K_i is t-compact by $(*)$, and so is the union $K = K_1 \cup \cdots \cup K_n$ (see 28).

"$\Leftarrow$." Suppose that K is t-compact. For each $x \in K$, choose a coordinate chart (U_x, φ_x) around x, and let $\varepsilon_x > 0$ such that the *closed* ball of radius ε_x around $\varphi_x(x)$ is contained in $\varphi_x(U_x)$. Let B_x be the *open* ball. The preimage of the closed ball,

$$C_x = \varphi_x^{-1}(\overline{B_x}),$$

is t-compact by $(*)$. The collection $\left\{\varphi_x^{-1}(B_x) \,|\, x \in K\right\}$ is an open cover of K; by t-compactness, we may choose a finite subcover. Thus, let $x_1, \ldots, x_n \in K$ such that

$$K \subseteq \varphi_{x_1}^{-1}(B_{x_1}) \cup \cdots \cup \varphi_{x_n}^{-1}(B_{x_n}).$$

By Part (b) of Proposition 2.34,

$$K_i = K \cap C_{x_i}$$

is t-compact. Using $(*)$ again, its image $\varphi_{x_i}(K_i)$ is closed and bounded. Since $K \cap \varphi_{x_i}^{-1}(B_{x_i}) \subseteq K_i$, we see $K = K_1 \cup \cdots \cup K_n$. This shows that K is c-compact. □

30 (answer on page 276). Verify the statement $(*)$ at the beginning of this proof.

2.6 Orientability

In our informal discussion of surfaces 1.6, we considered a surface in $\mathbb{R}^3$ to be orientable if it has two sides and non-orientable if it has only one side. The Möbius strip and the Klein bottle are typical examples of non-orientable surfaces: We can imagine placing a hand on the surface and moving it along the surface until it returns to the starting point, but on the opposite side. It is not obvious, however, how this construction might depend on the choice of realization as a surface in Euclidean space. Recall that knots in $\mathbb{R}^3$ become unknotted once they are placed in $\mathbb{R}^4$; perhaps, two-sided surfaces become one-sided once they are placed in $\mathbb{R}^4$? To see that this is not the case, it is better to place the hand not *on* the surface but *inside* the surface (using a 2-dimensional hand). For a non-orientable surface, we may move the hand inside the surface until it eventually returns to its initial position but as its *mirror image*. This visualization is much better suited for an intrinsic definition.

Recall that an invertible linear transformation $A:\ \mathbb{R}^n \to \mathbb{R}^n$ is called *orientation preserving* if $\det(A) > 0$ and *orientation reversing* if $\det(A) < 0$. (We do not distinguish between a linear transformation of $\mathbb{R}^n$ and its matrix in the standard basis.) One of the classification results of linear algebra is that orientation preserving transformations are compositions of rotations, shears, and dilations, while orientation reversing transformations involve an additional reflection.

To extend this idea to nonlinear transformations, recall that a diffeomorphism $F: U \to U'$ between open subsets of $\mathbb{R}^n$ has non-zero Jacobian determinant everywhere. We say that F is *orientation preserving* if its Jacobian determinant satisfies

$$\det\big(DF(\mathbf{x})\big) > 0$$

for all $\mathbf{x} \in U$. In such a case, the image U' is obtained from U by suitable rotation, translation, and "distortion"—without using reflections.

We now turn to the formal definition of orientation on manifolds, beginning with charts.

Definition 2.37.

(a) Two charts $(U,\varphi),(V,\psi)$ for a set M are oriented-compatible *if the transition functions $\psi\circ\varphi^{-1}:\ \varphi(U\cap V) \to \psi(U\cap V)$ are orientation preserving.*
(b) An oriented atlas *on M is an atlas such that any two of its charts are oriented-compatible; a* maximal oriented atlas *is one containing every chart that is oriented-compatible with all charts in this atlas.*
(c) An oriented manifold *is a set with a maximal oriented atlas, satisfying the Hausdorff and countability conditions as in Definition 2.16.*
(d) A manifold is called orientable *if it admits an oriented atlas.*

The notion of orientation will become crucial later, since integration of differential forms over manifolds is only defined if the manifold is oriented.

Example 2.38. The spheres S^n are orientable. To see this, consider the atlas with the two charts (U_+,φ_+) and (U_-,φ_-), given by stereographic projections (Section 2.3.1). Then $\varphi_-(U_+\cap U_-) = \varphi_+(U_+\cap U_-) = \mathbb{R}^n\backslash\{\mathbf{0}\}$, with transition map $\varphi_-\circ\varphi_+^{-1}(\mathbf{u}) = \mathbf{u}/||\mathbf{u}||^2$. The Jacobian matrix $D(\varphi_-\circ\varphi_+^{-1})(\mathbf{u})$ has entries[†]

[†] Here $\delta_{ij} = \begin{cases} 0 & \text{if } i\neq j \\ 1 & \text{if } i=j \end{cases}$ is the Kronecker delta.

$$\big(D(\varphi_- \circ \varphi_+^{-1})(\mathbf{u})\big)_{ij} = \frac{\partial}{\partial u^j}\Big(\frac{u^i}{||\mathbf{u}||^2}\Big) = \frac{1}{||\mathbf{u}||^2}\delta_{ij} - \frac{2u_i u_j}{||\mathbf{u}||^4}. \tag{2.14}$$

Its determinant is (see 31 below)

$$\det\big(D(\varphi_- \circ \varphi_+^{-1})(\mathbf{u})\big) = -||\mathbf{u}||^{-2n} < 0. \tag{2.15}$$

Hence, the given atlas is *not* an oriented atlas. But this is easily remedied: Simply compose one of the charts, say U_-, with the map $(u_1, u_2, \ldots, u_n) \mapsto (-u_1, u_2, \ldots, u_n)$; then, with the resulting new coordinate map $\widetilde{\varphi_-}$, the atlas $(U_+, \varphi_+), (U_-, \widetilde{\varphi_-})$ will be an oriented atlas.

31 (answer on page 276). Check that the given vector $\mathbf{u}$ (regarded as a vector in $\mathbb{R}^n$) is an eigenvector of the matrix A with entries A_{ij} given by (2.14). Show that any vector orthogonal to $\mathbf{u}$ is an eigenvector as well. Find the corresponding eigenvalues, and use this to compute $\det(A)$.

Example 2.39. The complex projective spaces $\mathbb{C}\mathrm{P}^n$ and the complex Grassmannians $\mathrm{Gr}_{\mathbb{C}}(k,n)$ are all orientable. This follows because they are *complex manifolds* (see Remark 2.22), i.e., the transition maps for their standard charts, as maps between open subsets of $\mathbb{C}^m$, are actually complex-holomorphic. This implies that as real maps, their Jacobian determinant is positive everywhere (32 below).

32 (answer on page 277). Let $A \in \mathrm{Mat}_{\mathbb{C}}(n)$ be a complex square matrix, and $A_{\mathbb{R}} \in \mathrm{Mat}_{\mathbb{R}}(2n)$ the same matrix regarded as a real-linear transformation of $\mathbb{R}^{2n} \cong \mathbb{C}^n$. Show that

$$\det_{\mathbb{R}}(A_{\mathbb{R}}) = |\det_{\mathbb{C}}(A)|^2.$$

(Here $|\cdot|$ signifies the complex modulus. You may want to start with the case $n = 1$.)

We shall see shortly that if a connected manifold is orientable, then there are exactly two orientations. Given one orientation, the other one (called the *opposite orientation*) is obtained by the following procedure.

Definition 2.40. *Let M be an oriented manifold, with oriented atlas $\{(U_\alpha, \varphi_\alpha)\}$. Then the* opposite orientation *on M is obtained by replacing each φ_α by its composition with the map $(u^1, \ldots, u^m) \mapsto (-u^1, u^2, \ldots, u^m)$.*

Proposition 2.41. *Let M be an oriented manifold, and (U, φ) a connected chart compatible with the atlas of M. Then this chart is oriented-compatible either with the given orientation of M or with the opposite orientation.*

Proof. Let $\{(U_\alpha, \varphi_\alpha)\}$ be an oriented atlas for M. Given $p \in U$, with image $\mathbf{x} \in \widetilde{U} = \varphi(U)$, let

$$\varepsilon(p) = \pm 1$$

be the sign of the determinant of $D(\varphi_\alpha \circ \varphi^{-1})(\mathbf{x})$ for any chart $(U_\alpha, \varphi_\alpha)$ containing p. This is well-defined, for if (U_β, φ_β) is another such chart, then the Jacobian matrix $D(\varphi_\beta \circ \varphi^{-1})(\mathbf{x})$ is the product of $D(\varphi_\alpha \circ \varphi^{-1})(\mathbf{x})$ and $D(\varphi_\beta \circ \varphi_\alpha^{-1})(\mathbf{x})$, and the latter has positive determinant since the atlas is oriented by assumption. It is also clear that the function $\varepsilon : U \to \{1, -1\}$ is constant near any given p; hence, it is locally constant and therefore constant (since U is connected). It follows that (U, φ) is either oriented-compatible with the orientation of M (if $\varepsilon = +1$) or with the opposite orientation (if $\varepsilon = -1$). □

Example 2.42. $\mathbb{R}\mathrm{P}^2$ is non-orientable. To see this, consider its standard atlas with charts (U_i, φ_i) for $i = 0, 1, 2$. Suppose $\mathbb{R}\mathrm{P}^2$ has an orientation. By the proposition, each of the charts is compatible either with the given orientation or with the opposite orientation. But the transition map between (U_0, φ_0), (U_1, φ_1) is

$$(\varphi_1 \circ \varphi_0^{-1})(u^1, u^2) = \left(\frac{1}{u^1}, \frac{u^2}{u^1}\right)$$

defined on $\varphi_0(U_0 \cap U_1) = \{(u^1, u^2) :\ u^1 \neq 0\}$. This has Jacobian determinant $-u_1^{-3}$, which changes sign. Thus, the two charts cannot be oriented-compatible, even after composing the coordinate map of one of them with $(u^1, u^2) \mapsto (-u^1, u^2)$. This contradiction shows that $\mathbb{R}\mathrm{P}^2$ is not orientable.

Using similar arguments, one can show that $\mathbb{R}\mathrm{P}^n$ for $n \geq 2$ is orientable if and only if n is odd (see Problem 13 at the end of the chapter).

We conclude this section with the following result:

Proposition 2.43. *Suppose M is a connected, orientable manifold. Then there are exactly two orientations on M. In fact, any connected chart determines a unique orientation for which it is an oriented chart.*

33 (answer on page 278). Give a proof of Proposition 2.43, using Proposition 2.41.

2.7 Building New Manifolds

2.7.1 Disjoint Union

Given manifolds M, M' of the same dimensions m, with atlases $\{(U_\alpha, \varphi_\alpha)\}$ and $\{(U'_\beta, \varphi'_\beta)\}$, the disjoint union $N = M \sqcup M'$ is again an m-dimensional manifold with atlas $\{(U_\alpha, \varphi_\alpha)\} \cup \{(U'_\beta, \varphi'_\beta)\}$. This manifold N is not much more interesting than considering M and M' separately but is the first step toward "gluing" M and M' in an interesting way, which often results in genuinely new manifold (more below).

More generally, given a countable collection of manifolds of the same dimension, their disjoint union is a manifold.

2.7.2 Products

Given manifolds M, M' of dimensions m, m' (not necessarily the same), with atlases $\{(U_\alpha, \varphi_\alpha)\}$ and $\{(U'_\beta, \varphi'_\beta)\}$, the cartesian product $M \times M'$ is a manifold of dimension $m + m'$. An atlas is given by the product charts $U_\alpha \times U'_\beta$ with the product maps $\varphi_\alpha \times \varphi'_\beta : (x, x') \mapsto (\varphi_\alpha(x), \varphi'_\beta(x'))$. For example, the 2-torus $T^2 = S^1 \times S^1$ becomes a manifold in this way, and more generally the n-torus

$$T^n = S^1 \times \cdots \times S^1.$$

2.7.3 Connected Sums*

Let M_1, M_2 be connected, oriented manifolds of the same dimension m. The connected sum $M_1 \# M_2$ is obtained by first removing chosen points $p_1 \in M_1,\ p_2 \in M_2$ and gluing in an open cylinder to connect the two "punctured" manifolds

$$M_1 \backslash \{p_1\}, \quad M_2 \backslash \{p_2\}.$$

In more detail, let (U_i, φ_i) be coordinate charts around chosen points $p_i \in M_i$, with $\varphi_i(p_i) = \mathbf{0}$, and let $\varepsilon > 0$ be such that $B_\varepsilon(\mathbf{0}) \subseteq \varphi_i(U_i)$. We assume that (U_2, φ_2) is oriented-compatible and (U_1, φ_1) is oriented-compatible with M_1^{op}, the manifold with *opposite* orientation (!) to that of M_1. Denote

$$Z = S^{m-1} \times (-\varepsilon, \varepsilon)$$

the "open cylinder"; elements of this cylinder will be denoted as pairs $(\mathbf{v}, t)$. We define

$$M_1 \# M_2 = \Big((M_1 \backslash \{p_1\}) \sqcup Z \sqcup (M_2 \backslash \{p_2\}) \Big) / \sim,$$

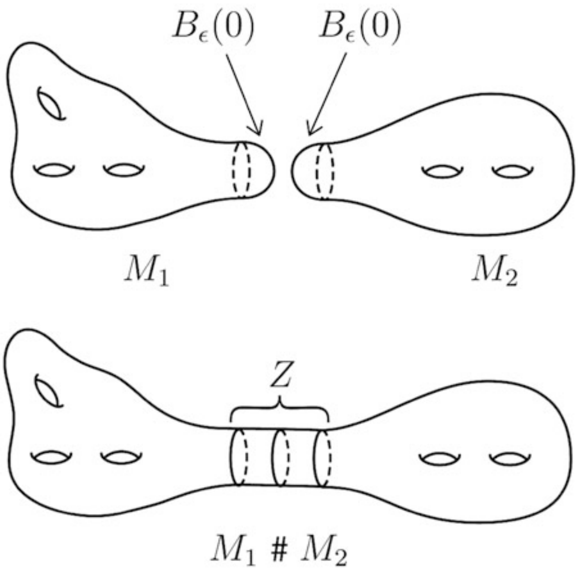
where the equivalence relation identifies $(\mathbf{v}, t) \in S^{m-1} \times (0, \varepsilon)$ with $\varphi_2^{-1}(t\mathbf{v}) \in M_2$, and $(\mathbf{v}, t) \in S^{m-1} \times (-\varepsilon, 0)$ with $\varphi_1^{-1}(-t\mathbf{v}) \in M_1$. Note that both maps

$$S^{m-1} \times (0, \varepsilon) \to M_2,\ (\mathbf{v}, t) \to \varphi_2^{-1}(t\mathbf{v}) \tag{2.16}$$

and

$$S^{m-1} \times (-\varepsilon, 0) \to M_1,\ (\mathbf{v}, t) \to \varphi_1^{-1}(-t\mathbf{v}) \tag{2.17}$$

are orientation preserving; for (2.17), this follows since it is a composition of two orientation reversing maps. By construction, $M_1 \# M_2$ comes with inclusions of $M_1 \backslash \{p_1\}$, $M_2 \backslash \{p_2\}$, and Z; an atlas is obtained by taking charts of $M_i \backslash \{p_i\}$ together with charts for Z. The result is an oriented manifold.



It is a non-trivial fact that the connected sum $M_1 \# M_2$ does not depend on the choices made. (Here, two oriented manifolds are considered "the same" if there is a bijection taking one oriented atlas to the other—that is, if there exists an oriented diffeomorphism, using terminology introduced later.)

One can apply a similar construction to a pair of points $p_1 \neq p_2$ of the same connected, oriented manifold M—the higher-dimensional version of "attaching a handle."

2.7.4 Quotients*

We have seen various examples of manifolds defined as quotient spaces for equivalence relations on given manifolds:

(a) The real projective space $\mathbb{R}\mathrm{P}^n$ can be defined as the quotient $(\mathbb{R}^{n+1}\backslash\{\mathbf{0}\})/\sim$ under the equivalence relation $\mathbf{x} \sim \mathbf{x}' \Leftrightarrow \mathbb{R}\mathbf{x} = \mathbb{R}\mathbf{x}'$, or also as a quotient $S^n/\sim$ for the equivalence relation $\mathbf{x} \sim -\mathbf{x}$ (antipodal identification).
(b) The non-Hausdorff manifold 2.15 was also defined as the quotient under an equivalence relation, $(\mathbb{R} \sqcup \mathbb{R})/\sim$. This non-example illustrates a typical problem for such constructions: Even if the quotient inherits an atlas, it may fail to be Hausdorff.
(c) Our construction of the connected sum of oriented manifolds in Section 2.7.3 also involved a quotient by an equivalence relation.
(d) Let M be a manifold with a given countable atlas $\mathscr{A} = \{(U_\alpha, \varphi_\alpha)\}$. Let

$$Q = \bigsqcup_\alpha U_\alpha$$

be the manifold given as the disjoint union of the chart domains, and let $\pi : Q \to M$ be the map whose restriction to U_α is the obvious inclusion into M. Since π is surjective, this realizes M as a quotient $Q/\sim$, formalizing the idea that every manifold is obtained by gluing charts.

There is a general criterion for deciding when the quotient of a manifold M under an equivalence relation determines a manifold structure on the quotient space $M/\sim$. However, we shall need more tools to formulate this result (see Theorem 4.36).

2.8 Problems

1. There is a different version of stereographic projection for the 2-sphere $S^2 \subseteq \mathbb{R}^3$, as follows. Let $n = (0,0,1)$ and $s = (0,0,-1)$ be the north and south poles, and put $U = S^2 \setminus \{s\}$, $V = S^2 \setminus \{n\}$.
   Let $\varphi : U \to \mathbb{R}^2$ be the map taking $p = (x,y,z)$ to the unique (u,v) such that $p' = (u,v,1)$ is on the line through p and s.
   Let $\psi : V \to \mathbb{R}^2$ be the map taking $p = (x,y,z)$ to the unique (u,v) such that $p' = (u,v,-1)$ is on the line through p and n.
   (a) Explain (by drawing a picture) what the map φ is geometrically.
   (b) Give explicit formulas for φ and ψ.
   (c) Compute $\varphi(U)$ and $\psi(V)$, and show they are open. Then show that φ and ψ are injective.
   (d) Compute the transition map $\varphi \circ \psi^{-1}$, its domain, and range.

2. Let $M \subseteq \mathbb{R}^2$ be the boundary of a square with vertices at $(\pm 1, \pm 1)$:
$$M = \left\{(x,y) \in \mathbb{R}^2 \mid |x| \leq 1 \text{ and } |y| \leq 1, \text{ with } |x| = 1 \text{ or } |y| = 1\right\}.$$
   Decide whether or not the charts (U,φ), (V,ψ) given as
$$U := \{(x,y) \in M \mid y > -1\},\ \varphi : U \to \mathbb{R},\ (x,y) \mapsto \frac{x}{1+y}$$
$$V := \{(x,y) \in M \mid y < 1\},\ \psi : V \to \mathbb{R},\ (x,y) \mapsto \frac{x}{1-y}$$
   define an atlas on M. Prove your answer.

3. On the real projective plane $\mathbb{RP}^2$, with coordinates $(x:y:z)$, consider the following two charts:
$$U = \{(x:y:z) \mid x+y \neq 0\},\ \varphi(x:y:z) = \left(\frac{y}{x+y}, \frac{z}{x+y}\right) \tag{2.18}$$
$$V = \{(x:y:z) \mid x-y \neq 0\},\ \varphi(x:y:z) = \left(\frac{x+y}{x-y}, \frac{z}{x-y}\right). \tag{2.19}$$
   (a) Sketch the images $\varphi(U \cap V)$, $\psi(U \cap V)$, and compute the transition map $\psi \circ \varphi^{-1}$.

(b) Consider the curve $(x-z)^2+(y-z)^2=z^2$ in $\mathbb{R}\mathrm{P}^2$. What kind of conic (ellipse, hyperbola, parabola) is this curve in each of the two charts?

4. Consider the subset of $\mathbb{R}\mathrm{P}^2$ given as

$$S=\{(x:y:z)|\, x^2-xy<z^2\}.$$

Sketch S in each of the standard charts U_0,U_1,U_2 of $\mathbb{R}\mathrm{P}^2$.

5. Every $(k+1)$-dimensional subspace $E\subseteq\mathbb{R}^{n+1}$ determines a subset of $\mathbb{R}\mathrm{P}^n$, consisting of all lines in $\mathbb{R}^{n+1}$ that are contained in E. Such a subset is called a *k-dimensional projective subspace* of the projective space $\mathbb{R}\mathrm{P}^n$. For $k=1$, it is called a *projective line*.
(a) Describe the image of a k-dimensional projective subspace of $\mathbb{R}\mathrm{P}^n$ in a standard chart for the projective space.
(b) Give a bijection between k-dimensional and $(n-k-1)$-dimensional projective subspaces of $\mathbb{R}\mathrm{P}^n$.
(c) Show that any two distinct projective lines in the projective plane $\mathbb{R}\mathrm{P}^2$ have a unique point of intersection.

6. The definition of the projective spaces $\mathbb{R}\mathrm{P}^n$ and $\mathbb{C}\mathrm{P}^n$ generalizes to $\mathbb{F}\mathrm{P}^n$ for any field $\mathbb{F}$. (Here $\mathbb{F}\mathrm{P}^n$ is simply regarded as a set.) In particular, we may take $\mathbb{F}$ to be a finite field.
(a) For $\mathbb{F}$ any field, describe a generalization of the decomposition (2.7).
(b) If $\mathbb{F}$ is a finite field, give a formula for the cardinality $|\mathbb{F}\mathrm{P}^n|$.
(c) Projective lines in $\mathbb{F}\mathrm{P}^n$ are defined similarly to the case $\mathbb{F}=\mathbb{R}$. (See the previous problem.) For $\mathbb{F}$ a finite field, give a formula for the number of projective lines in $\mathbb{F}\mathrm{P}^2$.

7. The party game "*Spot it!*" contains a deck of 55 cards, each of which has 7 different symbols (pictures) printed on its front. The deck is such that any two cards have a unique symbol in common; the various versions of playing the game involve spotting the common symbol as quickly as possible. Explain the mathematics behind this collection of cards, using the geometry of the projective plane over a field with 7 elements (see Problem 6). In particular, find that one could actually make a similar game with 57 cards.

8. Let M be the non-Hausdorff manifold from Example 2.15. Find a subset $A\subseteq M$ that is compact (in the sense of Definition 2.33) but not closed.

9. On $\mathbb{R}^2\backslash\{\mathbf{0}\}$, introduce an equivalence relation by declaring that

$$(x,y)\sim(x',y')\ \Leftrightarrow\ \text{there exists } t>0:\ x'=tx,\quad y'=y/t.$$

Let M be the set of equivalence classes.
(a) Show that M has a natural 1-dimensional atlas, which however is *not* Hausdorff.
(b) Find points $p_1,p_2,p_3\in M$ such that neither p_1,p_2 nor p_2,p_3 have disjoint open neighborhoods, but p_1,p_3 do.

(c) What happens if the equivalence relation is changed to $(x,y) \sim (x',y') \Leftrightarrow x'y' = xy$?

10. Let M be a manifold and $(x_n)_{n=1}^{\infty}$ a sequence of elements of M. We say that $x \in M$ is the *limit* of the sequence $(x_n)_{n=1}^{\infty}$ and write $\lim_{n\to\infty} x_n = x$, if there exists a chart (U,φ) around x such that $x_n \in U$ for all sufficiently large n and $\lim_{n\to\infty} \varphi(x_n) = \varphi(x)$.
    (a) Explain how the Hausdorff property implies the uniqueness of limits: If $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} x_n = y$, then $x = y$.
    (b) Prove that $A \subseteq M$ is closed if and only if for every convergent sequence of elements of A, the limit is also in A.
    (c) Prove that $K \subseteq M$ is compact if and only if every sequence of elements of K has a convergent subsequence with limit in K. (For $M = \mathbb{R}^m$, this is the *Bolzano–Weierstrass Theorem* from multivariable calculus, which you may assume.)

11. For the following problem, recall that a square matrix B is a *projection* if $A^2 = A$, and is an *orthogonal projection* if furthermore $A^\top = A$. A square matrix B is a *rotation* if $B^\top = B^{-1}$ and $\det(B) = 1$.
    (a) Let
    $$f:\ \mathbb{R}\mathrm{P}^2 \to \mathrm{Mat}_{\mathbb{R}}(3)$$
    be the map taking a line ℓ in $\mathbb{R}^3$ to the matrix of orthogonal projection onto ℓ. Show that this map is injective, describe its image, and give a formula for $P = f([\mathbf{x}])$ for any given $[\mathbf{x}] = (x^0 : x^1 : x^2)$.
    (b) Let
    $$g:\ \mathbb{R}\mathrm{P}^2 \to \mathrm{Mat}_{\mathbb{R}}(3)$$
    be the map taking a line ℓ in $\mathbb{R}^3$ to the matrix of rotation by π around that line. Show that this map is injective, describe its image, and give a formula for $Q = g([\mathbf{x}])$, for example, in terms of $P = f([\mathbf{x}])$.

12. In this problem, we describe an alternative approach to the standard atlas of the Grassmannians $\mathrm{Gr}(k,n)$. We will only consider the special case $k = 2, n = 4$. The *Stiefel manifold* $\mathrm{St}(2,4)$ is the set of (4×2)-matrices A of rank 2; it is an open subset of the space of all (4×2)-matrices. The range ("column space") of any such A is a 2-dimensional subspace of $\mathbb{R}^4$, which may be regarded as a point of the Grassmannian $\mathrm{Gr}(2,4)$. This defines a map
    $$\pi:\ \mathrm{St}(2,4) \to \mathrm{Gr}(2,4).$$
    (a) Show that the map π is surjective.
    (b) Define an equivalence relation, by declaring $A \sim A'$ if and only if $\pi(A) = \pi(A')$ (so that A, A' define the same point of the Grassmannian). Show that $A \sim A'$ if and only if $A' = AC$ for an invertible (2×2)-matrix C.
    (c) For indices i,j with $1 \le i < j \le 4$, let $\widetilde{U}_{ij} \subseteq \mathrm{St}(2,4)$ be the subset of (4×2)-matrices A for which the i-th and j-th rows are linearly independent. Show that if A lies in this set, then so does every A' in its equivalence class.

(d) Let $U_{ij} = \pi(\widetilde{U}_{ij}) \subseteq \mathrm{Gr}(2,4)$. Show that for any $p \in U_{ij}$, the fiber $\pi^{-1}(p)$ contains a unique matrix $A(p)$ for which the i-th row is $(\,1\ 0\,)$ and the j-th row is $(\,0\ 1\,)$. For example, for points in U_{13}, this is a matrix of the form

$$A(p) = \begin{pmatrix} 1 & 0 \\ a & b \\ 0 & 1 \\ c & d \end{pmatrix}.$$

(e) Define maps $\varphi_{ij} : U_{ij} \to \mathrm{Mat}_{\mathbb{R}}(2) \cong \mathbb{R}^4$, where $\varphi_{ij}(p)$ is obtained from $A(p)$ by removing the i-th and j-th row. (For example, φ_{13} takes p to (a,b,c,d), using the notation above.) Explicitly compute the transition map between (U_{12}, φ_{12}) and (U_{23}, φ_{23}).

13. Let $M = \mathbb{R}\mathrm{P}^n$ with its standard atlas (U_i, φ_i) for $0 \le i \le n$ (see Section 2.3.2).
    (a) Calculate the determinant of the Jacobian matrix of the transition functions

$$\varphi_i \circ \varphi_j^{-1}$$

    for all $0 \le i < j \le n$.
    (b) Use this result to prove that $\mathbb{R}\mathrm{P}^n$ for $n \ge 1$ is orientable for n odd, but non-orientable for n even.

14. Use a strategy similar to that of Problem 13 to investigate the orientability of the real Grassmannians $\mathrm{Gr}(k,n)$ for $1 \le k < n$.

15. Prove that each of the following sets is a manifold, and find its dimension:
    (a) The set $\mathrm{GL}(n,\mathbb{R})$ of real invertible $n \times n$-matrices
    (b) The set $\mathrm{GL}(n,\mathbb{C})$ of complex invertible $n \times n$-matrices
    (c) The set of real $n \times n$-matrices that have n distinct eigenvalues
    (d) The set of vectors $\mathbf{x} \in \mathbb{R}^n$ with pairwise distinct coordinates
    (Hint: There is a short solution based on one of the theorems in this chapter.)

3

Smooth Maps

3.1 Smooth Functions on Manifolds

A real-valued function on an open subset $U \subseteq \mathbb{R}^m$ is called *smooth at* $\mathbf{x} \in U$ if it is infinitely differentiable on an open neighborhood of $\mathbf{x}$. It is called *smooth on* U if it is smooth at all points of U. The notion of smooth functions on open subsets of Euclidean spaces carries over to manifolds: A function is smooth if its expression in local coordinates is smooth.

Definition 3.1. *Let M be a manifold. A function $f: M \to \mathbb{R}$ is called* smooth at $p \in M$ *if there exists a chart (U, φ) around p such that the function*

$$f \circ \varphi^{-1}: \varphi(U) \to \mathbb{R}$$

is smooth at $\varphi(p)$; it is called smooth *if it is smooth at all points of M. The set of smooth functions on M is denoted $C^\infty(M)$.*

The condition for smoothness at p does not depend on the choice of chart: If (V, ψ) is another chart containing p, then the two maps

$$(f \circ \psi^{-1})|_{\psi(U \cap V)}, \quad (f \circ (\psi)^{-1})|_{\psi(U \cap V)}$$

are related by the transition map $\varphi \circ (\psi)^{-1}$, which is a diffeomorphism. It follows that the first map is smooth at $\varphi(p)$ if and only if the second map is smooth at $\psi(p)$. Consequently, to check if f is smooth on M, it suffices to take *any* atlas $\{(U_\alpha, \varphi_\alpha)\}$ for M (not necessarily the maximal atlas) and verify that for all charts from this atlas, the maps $f \circ \varphi_\alpha^{-1}: \varphi(U_\alpha) \to \mathbb{R}$ are smooth.

G. Gross, E. Meinrenken, *Manifolds, Vector Fields, and Differential Forms*,
Springer Undergraduate Mathematics Series,
https://doi.org/10.1007/978-3-031-25409-3_3

Example 3.2. The "height function"

$$f:\ S^2 \to \mathbb{R},\ \ (x,y,z) \mapsto z$$

is smooth. This may be checked, for example, by using the 6-chart atlas given by projection onto the coordinate planes: For example, in the chart $U = \{(x,y,z) |\ z > 0\}$ with $\varphi(x,y,z) = (x,y)$, we have that

$$(f \circ \varphi^{-1})(x,y) = \sqrt{1-(x^2+y^2)},$$

which is smooth on $\varphi(U) = \{(x,y) |\ x^2+y^2 < 1\}$. (The argument for the other charts in this atlas is similar.) Alternatively, we could also use the atlas with two charts given by stereographic projection.

A similar argument shows, more generally, that for any smooth function $h \in C^\infty(\mathbb{R}^3)$ (for example, the coordinate functions), the restriction $f = h|_{S^2}$ is again smooth.

34 (answer on page 279). Decide whether or not the map

$$f:\ S^2 \to \mathbb{R},\ (x,y,z) \mapsto \sqrt{1-z^2}$$

is smooth.

35 (answer on page 279). Check that the map

$$f : \mathbb{R}\mathrm{P}^2 \to \mathbb{R},\ \ (x:y:z) \mapsto \frac{yz+xz+xy}{x^2+y^2+z^2}$$

is well-defined and use charts to show that it is smooth.

From the properties of smooth functions on $\mathbb{R}^m$, one gets the following properties of smooth $\mathbb{R}$-valued functions on manifolds M:

- If $f,g \in C^\infty(M)$ and $\lambda,\mu \in \mathbb{R}$, then $\lambda f + \mu g \in C^\infty(M)$.
- If $f,g \in C^\infty(M)$, then $fg \in C^\infty(M)$.
- $1 \in C^\infty(M)$ (where "1" denotes the constant function taking on the value 1).

36 (answer on page 279). Prove the assertion that $f,g \in C^\infty(M) \implies fg \in C^\infty(M)$.

These properties say that $C^\infty(M)$ is an *algebra*, with unit the constant function 1. (See Appendix B.2 for some background information on algebras.) Below, we will develop many of the concepts of manifolds in terms of the algebra of smooth functions. In fact, M itself may be recovered from this algebra (see Problem 16).

The following property of smooth functions is often used.

Proposition 3.3. *Let $f \in C^\infty(M)$, and let $J \subseteq \mathbb{R}$ be open. Then the preimage $f^{-1}(J) \subseteq M$ is open.*

Proof. Let $\{(U_\alpha, \varphi_\alpha)\}$ be an atlas of M. For all α, the function $f \circ \varphi_\alpha^{-1} \colon \varphi_\alpha(U_\alpha) \to \mathbb{R}$ is smooth, hence continuous. By multivariable calculus, this implies that the preimage

$$(f \circ \varphi_\alpha^{-1})^{-1}(J) = \varphi_\alpha\big(f^{-1}(J) \cap U_\alpha\big)$$

is open. By Definition 2.25, this means that $f^{-1}(J)$ is open. □

The property in this proposition is the topologist's definition of continuity: A function is continuous if and only if preimages of open sets are open. Continuous functions on M form an algebra $C(M)$, containing $C^\infty(M)$ as a subalgebra.

We may also describe the continuity of functions on manifolds in charts, parallel to Definition 3.1:

37 (answer on page 279). Show that $f \colon M \to \mathbb{R}$ is continuous (in the sense that preimages of open sets are open) if and only if for all charts (U, φ) the function $f \circ \varphi^{-1}$ is continuous.

Definition 3.4. *The* support *of a function $f \colon M \to \mathbb{R}$ is the smallest closed subset*

$$\mathrm{supp}(f) \subseteq M$$

with the property that f is zero outside of $\mathrm{supp}(f)$.

In other words, $p \in M \setminus \mathrm{supp}(f)$ if and only if every open neighborhood of p contains some point where f is non-zero.

38 (answer on page 280). What is the support of the function $f \colon \mathbb{R} \to \mathbb{R}$, $x \mapsto x^2$?

The following result will be needed below.

Lemma 3.5 (Extension by Zero). *Suppose U is an open subset of a manifold M, and let $g \in C^\infty(U)$ be such that* $\mathrm{supp}(g) \subseteq U$ *is closed as a subset of M. Then the function $f \colon M \to \mathbb{R}$, given by*

$$f|_U = g, \quad f|_{M \setminus U} = 0,$$

is smooth.

Proof. By assumption, $V = M \setminus \mathrm{supp}(g)$ is open and contains $M \setminus U$. Thus, U and V are an open cover of M. Since both $f|_U = g$ and $f|_V = 0$ are smooth, it follows that f is smooth. □

39 (answer on page 280). Give examples of an open subset $U \subseteq M$ and a smooth function $g \in C^\infty(U)$ such that:

(a) g does not extend to a smooth function $f \in C^\infty(M)$.

(b) g extends by zero to a smooth function $f \in C^\infty(M)$, even though $\mathrm{supp}(g) \subseteq U$ is not closed in M.

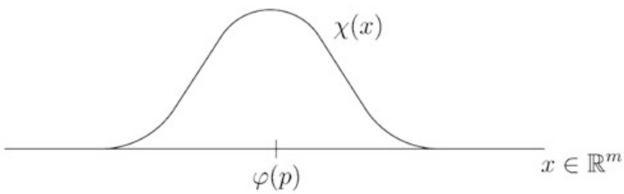
3.2 The Hausdorff Property via Smooth Functions

Suppose M is a possibly non-Hausdorff manifold—that is, it satisfies Definition 2.16 except possibly the Hausdorff condition. The definition of the algebra of smooth functions $C^\infty(M)$ does not use the Hausdorff property; hence, it makes sense in this more general context. In fact, we may use smooth functions to give a simple criterion for the Hausdorff property.

Proposition 3.6 (Criterion for Hausdorff Condition). *Suppose M is a possibly non-Hausdorff manifold. Then M is Hausdorff if and only if it has the* smooth separation property*: For all points $p,q \in M$ with $p \neq q$, there exists a smooth function $f\colon M \to \mathbb{R}$ with $f(p) \neq f(q)$.*

Proof. "$\Leftarrow$." Suppose M satisfies the smooth separation property. To check the Hausdorff property, let $p,q \in M$ with $p \neq q$. Pick $f \in C^\infty(M)$ with $f(p) \neq f(q)$, and choose $\varepsilon > 0$ with $\varepsilon < |f(p) - f(q)|/2$. Then the open ε-intervals around $f(p)$, $f(q)$ are disjoint, and their preimages U,V under f are disjoint open neighborhoods of p,q (see Proposition 3.3).

"$\Rightarrow$." Conversely, suppose M is Hausdorff. To check the smooth separation property, let $p,q \in M$ with $p \neq q$. By the Hausdorff property, there exist disjoint open neighborhoods U,V of p,q. As explained in Section 2.4, we may take these to be the domains of coordinate charts (U,φ) around p and (V,ψ) around q. Choose $\chi \in C^\infty(\mathbb{R}^m)$ with compact support $\mathrm{supp}(\chi)$ contained in $\varphi(U)$, and such that $\chi(\varphi(p)) = 1$. (For example, we may take χ to be a "bump function" on a small ball centered at $\varphi(p)$, see Lemma C.8 in the Appendix.)



Then

$$\mathrm{supp}(\chi \circ \varphi) = \varphi^{-1}(\mathrm{supp}(\chi))$$

is a compact subset of M by 30 and hence is closed by Proposition 2.34 (since M is Hausdorff). Lemma 3.5 shows that $\chi \circ \varphi$ extends by zero to a smooth function $f \in C^\infty(M)$. This function satisfies $f(p) = \chi(\varphi(p)) = 1$, while $f(q) = 0$ since $q \in V \subseteq M \backslash \operatorname{supp}(\chi \circ \varphi)$. $\square$

Corollary 3.7. *Suppose M is a possibly non-Hausdorff manifold. If there exists a smooth injective map $F : M \to \mathbb{R}^N$ for some $N \in \mathbb{N}$, then M is Hausdorff.*

Here, a map $F : M \to \mathbb{R}^N$ is called smooth if its component functions are smooth.

40 (answer on page 280). Explain how this corollary follows from the proposition.

Example 3.8 (Projective Spaces). Write vectors $\mathbf{x} \in \mathbb{R}^{n+1}$ as column vectors; hence, $\mathbf{x}^\top$ is the corresponding row vector. The matrix product $\mathbf{x}\mathbf{x}^\top$ is a square matrix with entries $x^j x^k$. The map

$$F : \mathbb{R}\mathrm{P}^n \to \mathrm{Mat}_{\mathbb{R}}(n+1) \cong \mathbb{R}^{(n+1)^2}, \ \ (x^0 : \ldots : x^n) \mapsto \frac{\mathbf{x}\mathbf{x}^\top}{||\mathbf{x}||^2} \tag{3.1}$$

is well-defined, since the right-hand side does not change when $\mathbf{x}$ is replaced with $\lambda\mathbf{x}$ for a non-zero scalar λ. It is also smooth, as one may check by considering the map in local charts, similarly to 35. Finally, it is injective: Given $F(\mathbf{x})$, one recovers the 1-dimensional subspace $\mathbb{R}\mathbf{x} \subseteq \mathbb{R}^{n+1}$ as the range of the rank 1 orthogonal projection $F(\mathbf{x})$. Hence, the criterion applies, and the Hausdorff condition follows.

41 (answer on page 280). Use a similar argument to verify the Hausdorff condition for $\mathbb{C}\mathrm{P}^n$.

The criterion may also be used for the real and complex Grassmannians (see Problem 12) and many other examples. In the opposite direction, the criterion tells us that if the Hausdorff condition does *not* hold, then no smooth injective map into $\mathbb{R}^N$ exists. This is one reason why it is often difficult to "visualize" non-Hausdorff manifolds.

Example 3.9. Consider the non-Hausdorff manifold M from Example 2.15. Here, there are two points p, q that do not admit disjoint open neighborhoods, and we see directly that every $f \in C^\infty(M)$ must take on the same values at p and q: With the coordinate charts (U, φ), (V, ψ) in that example,

$$f(p) = f(\varphi^{-1}(0)) = \lim_{t \to 0^-} f(\varphi^{-1}(t)) = \lim_{t \to 0^-} f(\psi^{-1}(t)) = f(\psi^{-1}(0)) = f(q),$$

since $\varphi^{-1}(t) = \psi^{-1}(t)$ for $t < 0$.

Remark 3.10. In our criterion for the Hausdorff condition (Proposition 3.6 and Corollary 3.7) we may replace smooth functions with continuous functions, with essentially the same proof. On the other hand, the analogous result does not hold for arbitrary topological spaces: Separation of points by continuous functions implies separation by disjoint open neighborhoods, but not conversely.

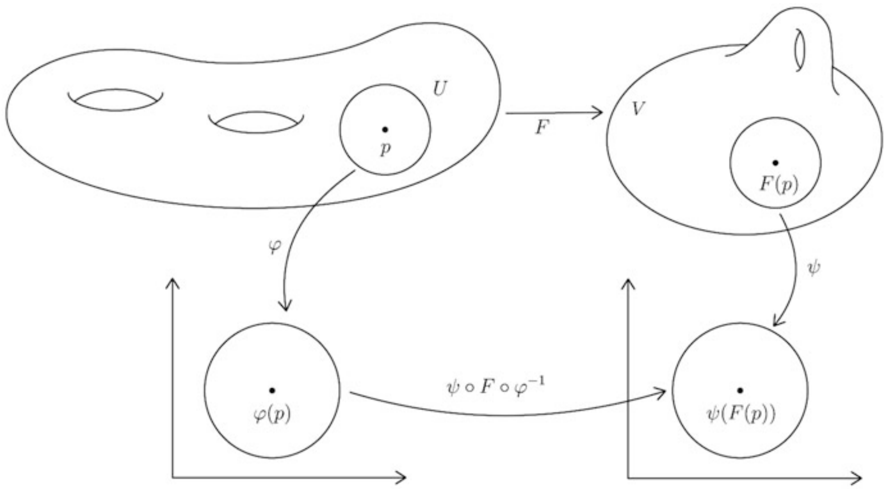
3.3 Smooth Maps Between Manifolds

The notion of smooth maps from M to $\mathbb{R}$ generalizes to smooth maps between manifolds.

Definition 3.11. *A map $F: M \to N$ between manifolds is* smooth at $p \in M$ *if there are coordinate charts (U,φ) around p and (V,ψ) around $F(p)$, such that $F(U) \subseteq V$ and the composition*

$$\psi \circ F \circ \varphi^{-1}: \varphi(U) \to \psi(V)$$

is smooth. The function F is called a smooth map from M to N *if it is smooth at all $p \in M$. The collection of smooth maps $F: M \to N$ is denoted $C^\infty(M,N)$.*



As before, to check smoothness of F, suffice it to take *any* atlas $\{(U_\alpha,\varphi_\alpha)\}$ of M with the property that $F(U_\alpha) \subseteq V_\alpha$ for *some* chart (V_α,ψ_α) of N, and then check smoothness of the maps

$$\psi_\alpha \circ F \circ \varphi_\alpha^{-1}: \varphi_\alpha(U_\alpha) \to \psi_\alpha(V_\alpha).$$

This is because the condition for smoothness at p does not depend on the choice of charts (compare to the remark following Definition 3.1): Given a different choice of charts (U',φ') and (V',ψ') with $F(U') \subseteq V'$, we have

$$\psi' \circ F \circ (\varphi')^{-1} = (\psi' \circ \psi^{-1}) \circ (\psi \circ F \circ \varphi^{-1}) \circ (\varphi \circ (\varphi')^{-1})$$

on $\varphi'(U \cap U')$. Since $(\psi' \circ \psi^{-1})$ and $(\varphi \circ (\varphi')^{-1})$ are smooth, we see that $\psi' \circ F \circ (\varphi')^{-1}$ is smooth at $\varphi'(p)$ if and only if $(\psi \circ F \circ \varphi^{-1})$ is smooth at $\varphi(p)$.

This proof illustrates the motivation behind the requirement that transition charts be diffeomorphisms (i.e., that an atlas is comprised of compatible charts): If some smoothness property holds "locally" in a chart around a point, the compatibility condition is used to verify that the choice of chart is irrelevant.

42 (answer on page 280). Show that $C^\infty(M, \mathbb{R}) = C^\infty(M)$. (Part of the task is understanding the question!)

43 (answer on page 280). Show that the quotient maps

$$\pi : \mathbb{R}^{n+1} \setminus \{\mathbf{0}\} \to \mathbb{R}\mathrm{P}^n$$
$$\pi : \mathbb{C}^{n+1} \setminus \{\mathbf{0}\} \to \mathbb{C}\mathrm{P}^n$$

are smooth.

44 (answer on page 281).

(a) Show that the map $F : \mathbb{R}\mathrm{P}^1 \to \mathbb{R}\mathrm{P}^1$ given by

$$\begin{cases} (1:0) \mapsto (1:0), \\ (t:1) \mapsto (\exp(t^2):1) \end{cases}$$

is smooth. (Why is it well-defined?)

(b) Show that the map $F : \mathbb{C}\mathrm{P}^1 \to \mathbb{C}\mathrm{P}^1$ given by the same formula is *not* smooth.

Proposition 3.3 generalizes to $C^\infty(M,N)$.

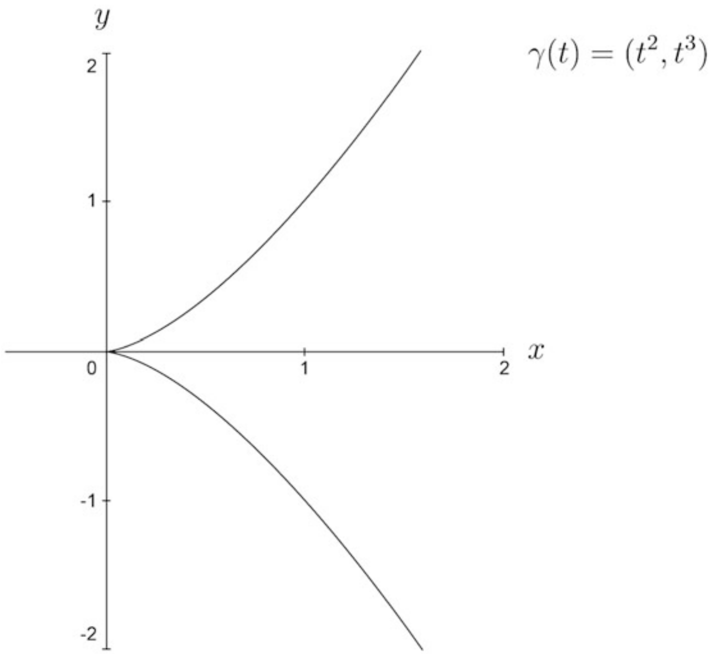
Proposition 3.12. *Smooth functions $F \in C^\infty(M,N)$ are continuous: For any open subset $V \subseteq N$, the preimage $F^{-1}(V)$ is open.*

We leave the proof as an exercise (see Problem 13). Furthermore, a map $F : M \to N$ is continuous at $p \in M$ if and only if its local coordinate expressions $\psi \circ F \circ \varphi^{-1}$ are continuous at $\varphi(p)$.

45 (answer on page 281). Let $F : M \to N$ be a continuous map of manifolds, in the sense of Proposition 3.12. Let $C \subseteq M$ be compact. Prove that $F(C) \subseteq N$ is compact.

Smooth functions $\gamma : J \to M$ from an open interval $J \subseteq \mathbb{R}$ to M are called *(smooth) curves in M*. Note that the image of a smooth curve need not look smooth.

Example 3.13. The image of $\gamma : \mathbb{R} \to \mathbb{R}^2,\ t \mapsto (t^2, t^3)$ has a "cusp singularity" at $(0,0)$.



3.4 Composition of Smooth Maps

Just as for smooth maps between open subsets of Euclidean spaces, the composition of smooth maps between manifolds is again smooth.

Proposition 3.14. *Suppose $F_1 : M_1 \to M_2$ is smooth at $p \in M_1$ and $F_2 : M_2 \to M_3$ is smooth at $F_1(p)$. Then the composition*

$$F_2 \circ F_1 : M_1 \to M_3$$

is smooth at p. Hence, if $F_1 \in C^\infty(M_1, M_2)$ and $F_2 \in C^\infty(M_2, M_3)$, then $F_2 \circ F_1 \in C^\infty(M_1, M_3)$.

Proof. Let (U_3, φ_3) be a chart around $F_2(F_1(p))$. Choose a chart (U_2, φ_2) around $F_1(p)$ with $F_2(U_2) \subseteq U_3$, as well as a chart (U_1, φ_1) around p with $F_1(U_1) \subseteq U_2$. Such charts always exist; see ✎ 46 below. Then $F_2(F_1(U_1)) \subseteq U_3$, and we have

$$\varphi_3 \circ (F_2 \circ F_1) \circ \varphi_1^{-1} = (\varphi_3 \circ F_2 \circ \varphi_2^{-1}) \circ (\varphi_2 \circ F_1 \circ \varphi_1^{-1}).$$

Since $\varphi_2 \circ F_1 \circ \varphi_1^{-1} : \varphi_1(U_1) \to \varphi_2(U_2)$ is smooth at $\varphi_1(p)$ and $\varphi_3 \circ F_2 \circ \varphi_2^{-1} : \varphi_2(U_2) \to \varphi_3(U_3)$ is smooth at $\varphi_2(F_1(p))$, the composition is smooth at $\varphi_1(p)$. $\square$

46 (answer on page 281). Let $F \in C^\infty(M,N)$. Given $p \in M$ and *any* open neighborhood $V \subseteq N$ of $F(p)$, show that there exists a chart (U,φ) around p such that $F(U) \subseteq V$.

Example 3.15. As a simple application, once we know that the inclusion $i:\ S^2 \to \mathbb{R}^3$ of the 2-sphere is smooth, we see that for any open neighborhood $U \subseteq \mathbb{R}^3$ and any $h \in C^\infty(U)$, the restriction $h|_{S^2}:\ S^2 \to \mathbb{R}$ is smooth (cf. Example 3.2). Indeed, this is immediate from Proposition 3.14 since

$$h|_{S^2} = h \circ i$$

(where we think of i as a map $S^2 \to U$). This simple observation applies to many similar examples.

47 (answer on page 281). Let $f \in C^\infty(M)$ be a function with $f > 0$ everywhere on M. Show (without using charts) that the function

$$\frac{1}{f}:\ M \to \mathbb{R},\ \ p \mapsto \frac{1}{f(p)}$$

is smooth.

3.5 Diffeomorphisms of Manifolds

Definition 3.16. *A smooth map $F : M \to N$ is called a* diffeomorphism *if it is invertible, with a smooth inverse $F^{-1} : N \to M$. Manifolds M,N are called* diffeomorphic *if there exists a diffeomorphism from M to N.*

You should verify that being diffeomorphic is an equivalence relation (transitivity is implied by Proposition 3.14). A diffeomorphism of manifolds is a bijection of the underlying sets that identifies the maximal atlases of the manifolds. Manifolds that are diffeomorphic are therefore considered “the same manifold.”

Example 3.17. By definition, every coordinate chart (U,φ) on a manifold M gives a diffeomorphism $\varphi:\ U \to \varphi(U)$.

Example 3.18. In Section 3.6.2, we will describe explicit diffeomorphisms between $\mathbb{R}\mathrm{P}^1$ and S^1 and between $\mathbb{C}\mathrm{P}^1$ and S^2.

Example 3.19. The claim from Example 2.24 may now be rephrased as the assertion that the quotient of $S^2 \times S^2$ under the equivalence relation $(\mathbf{x},\mathbf{x}') \sim (-\mathbf{x},-\mathbf{x}')$ is diffeomorphic to $\mathrm{Gr}(2,4)$.

A continuous map $F: M \to N$ between manifolds (or, more generally, topological spaces) is called a *homeomorphism* if it is invertible, with a continuous inverse. Manifolds that are homeomorphic are considered "the same topologically." Since every smooth map is continuous, every diffeomorphism is a homeomorphism.

Example 3.20. The standard example of a homeomorphism of smooth manifolds that is not a diffeomorphism is the map

$$\mathbb{R} \to \mathbb{R}, \quad x \mapsto x^3.$$

Indeed, this map is smooth and invertible, but the inverse map $y \mapsto y^{\frac{1}{3}}$ is not smooth.

The following 48 gives another way of looking at this example: We get two *distinct* manifold structures on $\mathbb{R}$, with the same collection of open sets.

48 (answer on page 282). Consider $M = \mathbb{R}$ with the trivial atlas $\mathscr{A} = \{(\mathbb{R}, \mathrm{id})\}$, and let $M' = \mathbb{R}$ with the atlas $\mathscr{A}' = \{(\mathbb{R}, \varphi)\}$ where $\varphi(x) = x^3$.

(a) Show that $\mathbb{R}$ equipped with the atlas $\mathscr{A}'$ is a 1-dimensional manifold, whose open sets are just the usual open subsets of $\mathbb{R}$.
(b) Show that the maximal atlases generated by $\mathscr{A}$ and $\mathscr{A}'$ are different.
(c) Show that the map $f: M \to M'$ given by $f(x) = x^{1/3}$ is a diffeomorphism.

While the two manifold structures on $\mathbb{R}$ are not equal "on the nose" (the identity map $\mathbb{R} \to \mathbb{R}$ is a homeomorphism $M \to M'$ but not a diffeomorphism), they are still diffeomorphic.

Remark 3.21. In the introduction (Section 1.6), we presented the classification of 2-dimensional compact, connected manifolds (i.e., surfaces) up to diffeomorphism. This classification coincides with their classification up to homeomorphism. That is, homeomorphic 2-manifolds Σ, Σ' are also diffeomorphic. In higher dimensions, the situation is much more complicated: It is possible for two manifolds to be homeomorphic but not diffeomorphic. The first example of "exotic" manifold structures was discovered by John Milnor [14] in 1956, who found that the 7-sphere S^7 admits manifold structures that are not diffeomorphic to the standard manifold structure, even though they induce the standard topology. Kervaire and Milnor [11] proved in 1963 that up to diffeomorphism, there are exactly 28 distinct manifold structures on S^7, and in fact classified all manifold structures on all spheres S^n with the exception of the case $n = 4$. For example, they showed that S^3, S^5, S^6 do not admit exotic (i.e., non-standard) manifold structures, while S^{15} has 16,256 different manifold structures. For S^4, the existence of exotic manifold structures is an open problem; this is known as the *smooth Poincaré conjecture*. Around 1982, Michael Freedman [8] (using results of Simon Donaldson [7]) discovered the existence of exotic manifold structures on $\mathbb{R}^4$; in 1987 Clifford Taubes [21] showed that there are uncountably many such structures. For $\mathbb{R}^n$ with $n \neq 4$, it is known that there are no exotic manifold structures on $\mathbb{R}^n$.

3.6 Examples of Smooth Maps

3.6.1 Products, Diagonal Maps

(a) If M,N are manifolds, then the projection maps

$$\mathrm{pr}_M:\ M\times N\to M,\ \ \mathrm{pr}_N:\ M\times N\to N$$

are smooth. (This follows immediately by taking product charts $U_\alpha\times V_\beta$.)

(b) The diagonal inclusion

$$\mathrm{diag}_M:\ M\to M\times M$$

is smooth. (In a coordinate chart (U,φ) around p and the chart $(U\times U,\varphi\times\varphi)$ around (p,p), the map is the restriction to $\varphi(U)\subseteq\mathbb{R}^n$ of the diagonal inclusion $\mathbb{R}^n\to\mathbb{R}^n\times\mathbb{R}^n$.)

(c) Suppose $F:\ M\to N$ and $F':\ M'\to N'$ are smooth maps. Then the direct product

$$F\times F':\ M\times M'\to N\times N'$$

is smooth. This follows from the analogous statement for smooth maps on open subsets of Euclidean spaces.

3.6.2 The Diffeomorphisms $\mathbb{R}\mathrm{P}^1\cong S^1$ and $\mathbb{C}\mathrm{P}^1\cong S^2$ *

We have stated before that $\mathbb{R}\mathrm{P}^1\cong S^1$. We will now describe an explicit diffeomorphism $F:\ \mathbb{R}\mathrm{P}^1\to S^1$, using the homogeneous coordinates $(w^0:w^1)$ for $\mathbb{R}\mathrm{P}^1$ and regarding S^1 as the unit circle in $\mathbb{R}^2$. Consider the standard atlas $\{(U_0,\varphi_0),\ (U_1,\varphi_1)\}$ for $\mathbb{R}\mathrm{P}^1$, as described in Section 2.3.2. Thus U_i consists of all $(w^0:w^1)$ such that $w^i\neq 0$; the coordinate maps are

$$\varphi_0(w^0:w^1)=\frac{w^1}{w^0},\quad \varphi_1(w^0:w^1)=\frac{w^0}{w^1}$$

and have as range the entire real line. The image of $U_0\cap U_1$ under each of the coordinate maps is $\mathbb{R}\backslash\{0\}$, and the transition map is

$$\varphi_1\circ\varphi_0^{-1}:\ \mathbb{R}\backslash\{0\}\to\mathbb{R}\backslash\{0\},\ u\mapsto\frac{1}{u}.$$

Under the desired identification $\mathbb{R}\mathrm{P}^1\cong S^1$, this atlas should correspond to an atlas with two charts for the circle, S^1, with the same chart images (namely, $\mathbb{R}$) and the same transition functions. A natural candidate is the atlas $\{(U_+,\varphi_+),(U_-,\varphi_-)\}$ given by stereographic projection, see Section 2.3.1: Thus $U_+=S^1\backslash\{(0,-1)\}$, $U_-=S^1\backslash\{(0,1)\}$, with the coordinate maps

$$\varphi_+(x,y)=\frac{x}{1+y},\ \ \varphi_-(x,y)=\frac{x}{1-y}.$$

Again, the range of each coordinate map is the real line $\mathbb{R}$, the image of $U_+\cap U_-$ is $\mathbb{R}\backslash\{0\}$, and by (2.6) the transition map is $\varphi_-\circ\varphi_+^{-1}(u)=1/u$. Hence, there is a unique diffeomorphism $F:\ \mathbb{R}\mathrm{P}^1 \to S^1$ identifying the coordinate charts, in the sense that $F(U_0) = U_+,\ F(U_1) = U_-$, and such that

$$\varphi_+ \circ F|_{U_0} = \varphi_0, \quad \varphi_- \circ F|_{U_1} = \varphi_1.$$

For $(w^0 : w^1) \in U_0$, we obtain

$$F(w^0 : w^1) = \varphi_+^{-1}(\varphi_0(w^0 : w^1)) = \varphi_+^{-1}\big(\frac{w^1}{w^0}\big).$$

Using the formula for the inverse map of stereographic projection,

$$\varphi_+^{-1}(u) = \frac{1}{1+u^2}\big(2u,\ (1-u^2)\big),$$

we arrive at

$$F(w^0 : w^1) = \frac{1}{||\mathbf{w}||^2}\big(2w^1w^0,\ \ (w^0)^2 - (w^1)^2\big). \tag{3.2}$$

(See Problem 5 for an explanation of (3.2) in terms of the squaring map $\mathbb{C} \to \mathbb{C},\ z \mapsto z^2$.)

49 (answer on page 282). Our derivation of (3.2) used the assumption $(w^0 : w^1) \in U_0$, but one obtains the same result for $(w^0 : w^1) \in U_1$. In fact, this is clear without repeating the calculation—why?

50 (answer on page 282). Use a similar strategy to compute the inverse map $G:\ S^1 \to \mathbb{R}\mathrm{P}^1$. (You may want to consider $(w^0 : w^1) = G(x,y)$ for the two cases $(x,y) \in U_+$ and $(x,y) \in U_-$.)

A similar strategy works for the complex projective line. Again, we compare the standard atlas for $\mathbb{C}\mathrm{P}^1$ with the atlas for the 2-sphere S^2, given by stereographic projection. This results in the following:

Proposition 3.22. *There is a unique diffeomorphism $F:\ \mathbb{C}\mathrm{P}^1 \to S^2$ with the property $F|_{U_0} = \varphi_+^{-1} \circ \varphi_0$. In homogeneous coordinates, it is given by the formula*

$$F(w^0 : w^1) = \frac{1}{|w^0|^2 + |w^1|^2}\Big(2\,\mathrm{Re}(w^1\overline{w^0}),\ 2\,\mathrm{Im}(w^1\overline{w^0}),\ |w^0|^2 - |w^1|^2\Big), \tag{3.3}$$

where $\mathrm{Re}(z)$ *and* $\mathrm{Im}(z)$ *are the real and imaginary parts of a complex number z.*

51 (answer on page 282). Prove Proposition 3.22. (You will discover that the transition map for the standard atlas of $\mathbb{C}\mathrm{P}^1$ is not quite the same as for the stereographic atlas of S^2, and a small adjustment is needed.) Also find expressions for the restrictions of the inverse map $G = F^{-1}:\ S^2 \to \mathbb{C}\mathrm{P}^1$ to $U_\pm$.

3.6.3 Maps to and from Projective Space*

In 43, you have verified that the quotient map

$$\pi: \ \mathbb{R}^{n+1} \setminus \{\mathbf{0}\} \to \mathbb{RP}^n, \ x = (x^0, \ldots, x^n) \mapsto (x^0 : \ldots : x^n)$$

is smooth. Given a map $F: \ \mathbb{RP}^n \to N$, we take its *lift* to be the composition $\widetilde{F} = F \circ \pi: \ \mathbb{R}^{n+1} \setminus \{\mathbf{0}\} \to N$. That is,

$$\widetilde{F}(x^0, \ldots, x^n) = F(x^0 : \ldots : x^n).$$

Note that $\widetilde{F}(\lambda x^0, \ldots, \lambda x^n) = \widetilde{F}(x^0, \ldots, x^n)$ for all non-zero $\lambda \in \mathbb{R} \setminus \{0\}$; conversely, every map $\widetilde{F}$ with this property descends to a map F on projective space. Similarly, maps $F: \ \mathbb{CP}^n \to N$ are in 1-1 correspondence with maps $\widetilde{F}: \ \mathbb{C}^{n+1} \setminus \{\mathbf{0}\} \to N$ that are invariant under scalar multiplication.

Lemma 3.23. *A map $F: \ \mathbb{RP}^n \to N$ is smooth if and only the lifted map $\widetilde{F}: \mathbb{R}^{n+1} \setminus \{\mathbf{0}\} \to N$ is smooth. Similarly, a map $F: \ \mathbb{CP}^n \to N$ is smooth if and only the lifted map $\widetilde{F}: \ \mathbb{C}^{n+1} \setminus \{\mathbf{0}\} \to N$ is smooth.*

Proof. Let $\{(U_i, \varphi_i)\}$ be the standard atlas. If $\widetilde{F}$ is smooth, then

$$(F \circ \varphi_i^{-1})(u^1, \ldots, u^n) = \widetilde{F}(u^1, \ldots, u^i, 1, u^{i+1}, \ldots, u^n)$$

are smooth for all i, and hence, F is smooth. Conversely, if F is smooth, then $\widetilde{F} = F \circ \pi$ is smooth. □

52 (answer on page 283). Show that the map

$$\mathbb{CP}^1 \to \mathbb{CP}^2, \ (z^0 : z^1) \mapsto \big((z^0)^2 : (z^1)^2 : z^0 z^1\big)$$

is smooth.

As remarked earlier, the projective space $\mathbb{RP}^n$ may also be regarded as a quotient of the unit sphere $S^n \subseteq \mathbb{R}^{n+1}$, since every point $[\mathbf{x}] = (x^0 : \ldots : x^n) \in \mathbb{RP}^n$ has a representative $\mathbf{x} \in \mathbb{R}^{n+1}$ with $||\mathbf{x}|| = 1$. The quotient map

$$\pi: \ S^n \to \mathbb{RP}^n$$

is again smooth, since it is a composition of the inclusion $\iota: \ S^n \to \mathbb{R}^{n+1} \setminus \{\mathbf{0}\}$ with the quotient map $\mathbb{R}^{n+1} \setminus \{\mathbf{0}\} \to \mathbb{RP}^n$.

Similarly, the complex projective space $\mathbb{CP}^n$ may be regarded as a quotient of the unit sphere S^{2n+1} inside $\mathbb{C}^{n+1} = \mathbb{R}^{2n+2}$, with a smooth quotient map

$$\pi: \ S^{2n+1} \to \mathbb{CP}^n. \tag{3.4}$$

Note that for any point $p \in \mathbb{CP}^n$, the fiber $\pi^{-1}(p) \subseteq S^{2n+1}$ is diffeomorphic to a circle S^1, regarded as the collection of complex numbers of modulus 1. Indeed, given any point $(z^0, \ldots, z^n) \in \pi^{-1}(p)$ in the fiber, the other points are obtained as $(\lambda z^0, \ldots, \lambda z^n)$, where $\lambda \in \mathbb{C}$ with $|\lambda| = 1$. This defines a decomposition of the odd dimensional sphere

$$S^{2n+1} = \bigsqcup_{p \in \mathbb{C}\mathrm{P}^n} \pi^{-1}(p)$$

as a disjoint union of circles, parametrized by the points of $\mathbb{C}\mathrm{P}^n$. This is an example of what differential geometers call a *fiber bundle* or *fibration*. We will not give a formal definition here, but remark that this fibration is "non-trivial" since S^{2n+1} is *not* diffeomorphic to a product $\mathbb{C}\mathrm{P}^n \times S^1$, as we will see later.

3.7 The Hopf Fibration*

The case $n = 1$ of the fibration (3.4) is of particular importance. Let us describe some of the properties of this fibration; our discussion will be somewhat informal, with details deferred to homework problems.

Identifying $\mathbb{C}\mathrm{P}^1 \cong S^2$ as in Proposition 3.22, the map (3.4) becomes a smooth map

$$\pi : \; S^3 \to S^2$$

with fibers diffeomorphic to S^1. Explicitly, by Equation (3.3),

$$\pi(z, w) = \Big(2\mathrm{Re}(w\bar{z}),\; 2\mathrm{Im}(w\bar{z}),\; |z|^2 - |w|^2 \Big) \tag{3.5}$$

for $(z, w) \in S^3 \subseteq \mathbb{R}^4 \cong \mathbb{C}^2$, i.e., $|z|^2 + |w|^2 = 1$. This map appears in many contexts; it is called the *Hopf fibration* (after *Heinz Hopf* (1894–1971)).

To get a picture of the Hopf fibration, recall that stereographic projection through a given point p identifies the complement of p in S^3 with $\mathbb{R}^3$; hence, we obtain a decomposition of $\mathbb{R}^3$ into a collection of circles together with one line (corresponding to the circle in S^3 containing p); the line may be thought of as the circle through the "point at infinity."

To be specific, take p to be the "south pole" $(0, i) \in S^3$. Stereographic projection from this point is the map

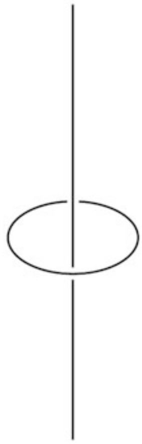
$$F : \; S^3 \backslash \{(0, i)\} \xrightarrow{\cong} \mathbb{R}^3, \quad (z, w) \mapsto \frac{1}{1 - \mathrm{Im}(w)} (z, \mathrm{Re}(w)). \tag{3.6}$$

Denote by $n = (0, 0, 1)$ the north pole and by $s = (0, 0, -1)$ the south pole of S^2. The fiber $\pi^{-1}(s)$ consists of elements of the form $(0, w) \in \mathbb{C}^2$ with $|w| = 1$, and after removing $(0, i)$ from this circle, the image under F is the x^3-axis in $\mathbb{R}^3$:

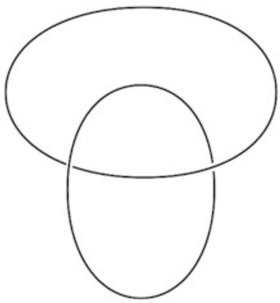
$$F\big(\pi^{-1}(s) \backslash (0, i)\big) = \{\mathbf{x} \in \mathbb{R}^3 |\; x^1 = x^2 = 0\}.$$

Similarly, $\pi^{-1}(n)$ consists of elements $(z, 0) \in \mathbb{C}^2$ with $|z|^2 = 1$; its image under stereographic projection is the unit circle in the x^1x^2-coordinate plane,

$$F\left(\pi^{-1}(n)\right) = \{\mathbf{x} \in \mathbb{R}^3 \mid \|\mathbf{x}\| = 1,\ x^3 = 0\}.$$



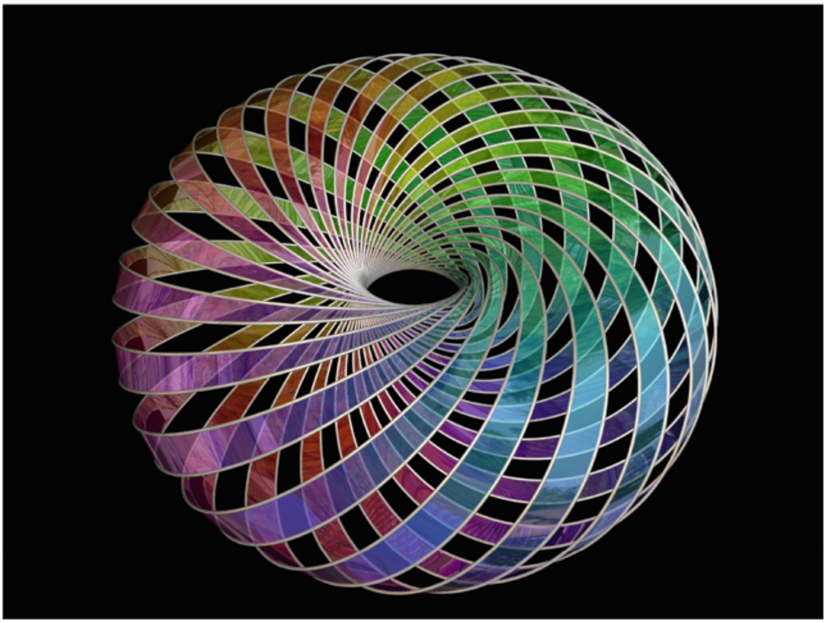
Note that this circle winds around the vertical line exactly once and cannot be continuously shrunk to a point without intersecting the vertical line. This illustrates that the two circles $\pi^{-1}(s)$ and $\pi^{-1}(n)$ are *linked*. In particular, these two circles cannot be separated in S^3 through continuous movements of the circles, without intersecting the circles. When n, s are replaced with a different pair of distinct points $p, q \in S^2$, the corresponding circles $\pi^{-1}(p), \pi^{-1}(q)$ will also be linked. (Indeed, we can continuously move n, s to the new pair p, q, and the corresponding circles will move continuously also.) That is, *any two distinct circles of the Hopf fibration are linked.*



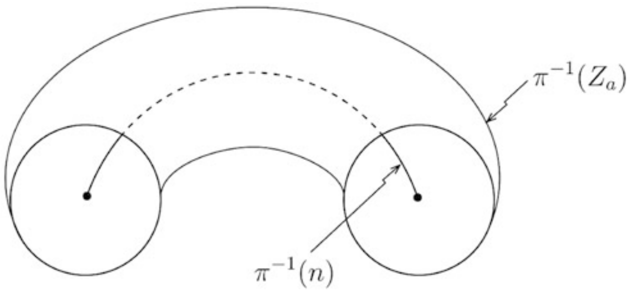
To get a more complete picture, consider the preimage $\pi^{-1}(Z_a)$ of a circle of latitude $a \in (-1, 1)$, i.e.,

$$Z_a = \{\mathbf{x} = (x^0, x^1, x^2) \in S^2 \mid x^2 = a\}.$$

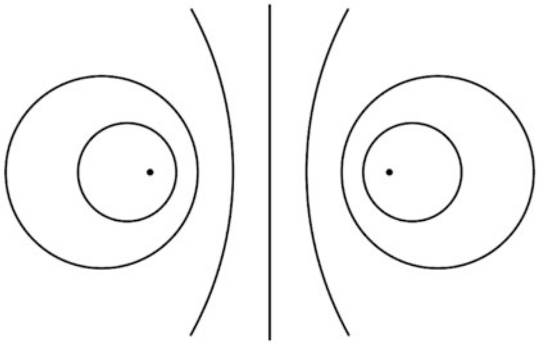
The fiber $\pi^{-1}(p)$ of any $p \in Z_a$ is a circle; since Z_a itself is a circle, we expect that $\pi^{-1}(Z_a)$ is a 2-torus and so is its image $F(\pi^{-1}(Z_a))$. This is confirmed by explicit calculation (see Problem 14 or Problem 15 at the end of the chapter). Each of these 2-tori $\pi^{-1}(Z_a)$ is a union of circles $\pi^{-1}(p),\ p \in Z_a$. Here is a nice picture:



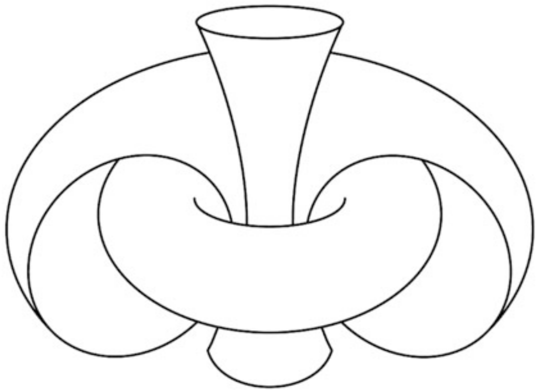
For $a \to 1$, the circle of latitude Z_a approaches the north pole n. Hence, this torus is rather "thin" and surrounds the circle $\pi^{-1}(n)$. The circle $F(\pi^{-1}(p))$ for $p \in Z_a$ lies inside this torus winding around it slowly, in such a way that it is linked with the circle $F(\pi^{-1}(n))$. Note that for p close to n, this circle will be just a small perturbation of the circle $F(\pi^{-1}(n))$.



As the value of a moves toward -1, the tori get "fatter" and larger. The intersection of this collection of 2-tori with the coordinate plane $\{\mathbf{x} \in \mathbb{R}^3 \,|\, x^2 = 0\}$ is sketched below.



A 3-dimensional sketch of this collection of nested 2-tori is as follows.



Here is another interesting feature of the Hopf fibration. Let $U_+ = S^2 \backslash \{s\}$ and $U_- = S^2 \backslash \{n\}$. A calculation (see Problem 15) gives diffeomorphisms

$$\pi^{-1}(U_\pm) \xrightarrow{\cong} U_\pm \times S^1,$$

in such a way that π becomes simply projection onto the first factor, $U_\pm$. In particular, the preimage of the closed upper hemisphere of S^2 is a *solid 2-torus*

$$D^2 \times S^1$$

(with $D^2 = \{z \in \mathbb{C} | \, |z| \leq 1\}$ the unit disk), geometrically depicted as a 2-torus in $\mathbb{R}^3$ together with its interior.* Likewise, the preimage of the closed lower hemisphere is a solid 2-torus $D^2 \times S^1$. The preimage of the equator is a 2-torus $S^1 \times S^1$. We hence see that the S^3 may be obtained by gluing two solid 2-tori along their boundaries $\partial(D^2 \times S^1) = S^1 \times S^1$. More precisely, the gluing identifies $(z, w) \in S^1 \times S^1$ in the boundary of the first solid torus with (w, z) in the boundary of the second solid torus.

* A solid torus is an example of a "manifold with boundary," a concept we have not properly discussed yet (see Example 8.4).

Note that one can also glue two copies of $D^2 \times S^1$ to produce $S^2 \times S^1$. However, here one uses a different gluing map, and indeed, S^3 is *not* diffeomorphic to $S^2 \times S^1$. (We will prove this fact later.)

Remark 3.24. (For those who are familiar with quaternions—see Example B.9 in the appendix for a brief discussion.) Let $\mathbb{H} = \mathbb{C}^2 = \mathbb{R}^4$ be the quaternion numbers. The unit quaternions are a 3-sphere S^3. Generalizing the definition of $\mathbb{R}\mathrm{P}^n$ and $\mathbb{C}\mathrm{P}^n$, there are also quaternionic projective spaces, $\mathbb{H}\mathrm{P}^n$. These are quotients of the unit sphere inside $\mathbb{H}^{n+1} = \mathbb{R}^{4n+4}$. Hence, one obtains a quotient map

$$S^{4n+3} \to \mathbb{H}\mathrm{P}^n;$$

its fibers are diffeomorphic to S^3. For $n = 1$, one can show that $\mathbb{H}\mathrm{P}^1 = S^4$; hence, one obtains a smooth map

$$\pi : \; S^7 \to S^4$$

with fibers diffeomorphic to S^3. This map plays a role in the construction of exotic spheres (cf. Remark 3.21).

3.8 Problems

1. Consider the map $F : \; \mathbb{C}\mathrm{P}^1 \to \mathbb{C}\mathrm{P}^1$, described in homogeneous coordinates as

$$F(z:1) = (z+1:1), \quad F(1:0) = (1:0).$$

   Is this map smooth? Justify your answer.

2. Consider the function $f : \; \mathbb{R}\mathrm{P}^2 \to \mathbb{R}, \; (x:y:z) \mapsto \frac{x^2-y^2-z^2}{x^2+y^2+z^2}$.
   (a) Using the standard charts for the projective plane, show that f is smooth.
   (b) Describe the subsets of $\mathbb{R}\mathrm{P}^2$, where $f < 0$, $f = 0$, or $f > 0$.

3. Show that the following maps are smooth:
   (a) The determinant as a map $\det : \mathrm{Mat}_{\mathbb{R}}(n) \to \mathbb{R}$.
   (b) The map $\mathrm{Inv} : \mathrm{GL}(n,\mathbb{R}) \to \mathrm{GL}(n,\mathbb{R})$ taking each invertible matrix to its inverse $A \mapsto A^{-1}$.

4. Let M be a manifold, and $f \in C^\infty(M)$ a smooth function. Show that if $K \subseteq M$ is compact, then the restriction $f|_K$ takes on a maximum and minimum on K.

5. Identify $\mathbb{R}^2 = \mathbb{C}$; thus $S^1 = \{z : \; |z| = 1\}$.
   (a) Show that the map

$$\mathbb{C} \backslash \{0\} \to S^1, \; z \mapsto \frac{z^2}{|z|^2}$$

   descends to a bijection $f : \; \mathbb{R}\mathrm{P}^1 \to S^1$.
   (b) Writing complex numbers as $z = x + iy$, express $f(x:y)$ in terms of x, y, and compare the result with Equation (3.2).

6. In a similar fashion to Problem 5, give an explanation of Equation (3.3) in terms of quaternions. (See Example B.9 in the appendix for a brief discussion of quaternions.)

7. Show that the map $\mathrm{Gr}(k,n) \to \mathrm{Gr}(n-k,n)$, taking a subspace E to the orthogonal subspace $E^{\perp}$, is a diffeomorphism (cf. ✎ 19).

8. Let M, N be manifolds, and $F: M \to N$ a bijection (of the underlying sets). Show that F is a diffeomorphism if and only if for every coordinate chart (V, ψ) of N, the pair $(F^{-1}(V), \psi \circ F)$ is a chart of M.

9. (a) (Smooth invariance of domain.) Let $U \subseteq \mathbb{R}^m$, and $V \subseteq \mathbb{R}^n$ be open sets, and suppose that $F: U \to V$ is a diffeomorphism. Prove that $m = n$.
   (b) (Invariance of dimension.) Let M be an m-dimensional manifold, and N an n-dimensional manifold. Suppose M and N are diffeomorphic. Prove that $m = n$.

   (This completes the remark made in Example 2.14.)

10. In this problem, we will consider the complex Grassmannians $\mathrm{Gr}_{\mathbb{C}}(k,n)$.
    (a) A complex number $z = x + iy$ defines a complex linear map
    $$\mathbb{C} \to \mathbb{C},\ w \mapsto zw.$$
    Identifying $\mathbb{C} = \mathbb{R}^2$, this can be viewed as a linear map $\mathbb{R}^2 \to \mathbb{R}^2$. What is the corresponding matrix?
    (b) Show that the map
    $$\mathbb{C}\mathrm{P}^1 \to \mathrm{Gr}(2,4),$$
    sending a complex line ℓ in $\mathbb{C}^2$ to the subspace E that is just ℓ itself, regarded as a 2-dimensional real subspace of $\mathbb{C}^2 = \mathbb{R}^4$, is smooth. (Hint: Use the standard charts, viewing the projective space as $\mathrm{Gr}_{\mathbb{C}}(1,2)$, together with the previous part.)
    (c) Indicate how this generalizes to define a smooth map $\mathrm{Gr}_{\mathbb{C}}(k,n) \to \mathrm{Gr}(2k,2n)$.

11. For $k \leq n$, let
    $$\mathrm{St}(k,n) = \{C \in \mathrm{Hom}(\mathbb{R}^k, \mathbb{R}^n) |\ \mathrm{rank}(C) = k\}$$
    be the *Stiefel manifold* of linear maps of rank k (see also Problem 12 at the end of Chapter 1).
    (a) Prove that $\mathrm{St}(k,n)$ is an open subset of $\mathrm{Hom}(\mathbb{R}^k, \mathbb{R}^n) \cong \mathbb{R}^{kn}$ and hence is a smooth manifold. (Hint: Show first that C has rank k if and only if $C^{\top}C$ is invertible.)
    (b) Let
    $$F:\ \mathrm{St}(k,n) \to \mathrm{Gr}(k,n)$$
    be the map taking the rank k map $C:\ \mathbb{R}^k \to \mathbb{R}^n$ to its range: $F(C) = C(\mathbb{R}^k) \subseteq \mathbb{R}^n$. Decide under what condition $F(C) \in U_I$, and in that case, find the resulting map
    $$A_I = \varphi_I(F(C)):\ \mathbb{R}^I \to \mathbb{R}^{I'}.$$

(c) Use these calculations to show that F is smooth.
(d) Explain briefly in which way the map from part (b) is a generalization of the quotient map

$$\mathbb{R}^{m+1} \setminus \{\mathbf{0}\} \to \mathbb{R}\mathrm{P}^m.$$

12. Use Corollary 3.6 to prove that $\mathrm{Gr}(k,n)$ is Hausdorff, by verifying that the map

$$\mathrm{Gr}(k,n) \to \mathrm{Mat}_{\mathbb{R}}(n), \quad E \mapsto P_E, \tag{3.7}$$

taking a subspace E to the matrix of the orthogonal projection onto E is smooth and injective. Discuss a similar map for the complex Grassmannian $\mathrm{Gr}_{\mathbb{C}}(k,n)$. (Hint: Begin by showing that the orthogonal projection onto the subspace S of a vector space V is given by $M(M^\top M)^{-1}M^\top$, where M is a matrix whose columns are basis vectors for S.)

13. Suppose $F \in C^\infty(M,N)$.
(a) Let (U,φ) be a coordinate chart for M and (V,ψ) a coordinate chart for N, with $F(U) \subseteq V$. Show that for all open subsets $W \subseteq N$ the set $U \cap F^{-1}(W)$ is open.
(b) Prove that smooth maps between manifolds are continuous.

14. Let $\pi: S^3 \to S^2$ be the Hopf map. Using the notation from Section 3.7, show that for all $a \in (-1,1)$ the preimage $\pi^{-1}(Z_a) \subseteq S^3$ is the common solution set of the equations $|z|^2 + |w|^2 = 1,\ |z|^2 - |w|^2 = a$. Use an explicit parametrization of the solution set to conclude that $\pi^{-1}(Z_a)$ is a 2-torus.

15. Let $\pi: S^3 \to S^2$ be the Hopf map. Viewing S^3 as the unit sphere inside $\mathbb{C}^2 = \mathbb{R}^4$, and S^2 as $\mathbb{C}\mathrm{P}^1$, it is given by the map $(z,w) \mapsto (z:w)$. Let $(U_0,\varphi_0),\ (U_1,\varphi_1)$ be the standard atlas for $\mathbb{C}\mathrm{P}^1$. We suggest writing elements of U_0 in the form $(1:u)$, and those of U_1 in the form $(v:1)$.
(a) Give explicit diffeomorphisms

$$F_i:\ U_i \times S^1 \to \pi^{-1}(U_i),$$

such that $(\pi \circ F_i)(p, e^{i\theta}) = p$ for all $p \in U_i$. (We regard S^1 as the set of complex numbers $e^{i\theta}$ of absolute value 1.)
(b) Each of the two maps F_0, F_1 restricts to a diffeomorphism

$$(U_0 \cap U_1) \times S^1 \to \pi^{-1}(U_0 \cap U_1).$$

Compute the diffeomorphism

$$K:\ (U_0 \cap U_1) \times S^1 \to (U_0 \cap U_1) \times S^1$$

such that $F_1 = F_0 \circ K$.

16. For any real algebra (with unit) $\mathscr{A}$ (see Appendix B.2), one defines the *spectrum* $\mathrm{Spec}(\mathscr{A})$ to be the set of non-zero algebra homomorphisms $\mathscr{A} \to \mathbb{R}$. Let us consider this notion for $\mathscr{A} = C^\infty(M)$, the algebra of smooth functions on a manifold M. Every $p \in M$ defines an algebra homomorphism

$$\mathrm{ev}_p : C^\infty(M) \to \mathbb{R}, \;\; f \mapsto f(p).$$

We hence obtain a map

$$M \to \mathrm{Spec}(C^\infty(M)), \;\; p \mapsto \mathrm{ev}_p.$$

Assuming that M is compact, prove that this map is a bijection, so that M may be recovered from its algebra of smooth functions. (This problem is somewhat challenging. The conclusion is also true for non-compact M, but the proof is even trickier.)

4

Submanifolds

4.1 Submanifolds

Let M be a manifold of dimension m. We will define a k-dimensional submanifold $S \subseteq M$ to be a subset that looks locally like $\mathbb{R}^k \subseteq \mathbb{R}^m$, regarded as the coordinate subspace defined by $x^{k+1} = \cdots = x^m = 0$.

Definition 4.1. *A subset $S \subseteq M$ is called a* submanifold *of dimension $k \leq m$, if it has the following property: For every $p \in S$, there is a coordinate chart (U, φ) around p such that*

$$\varphi(U \cap S) = \varphi(U) \cap \mathbb{R}^k. \tag{4.1}$$

Charts (U, φ) of M with this property are called submanifold charts *for S.*

Remark 4.2.

(a) A chart (U, φ) such that $U \cap S = \emptyset$ and $\varphi(U) \cap \mathbb{R}^k = \emptyset$ is considered a submanifold chart.
(b) We stress that the existence of submanifold charts is only required for points p that lie in S. For example, the half-open line $S = (0, \infty)$ is a submanifold of $\mathbb{R}$ (of dimension 1). There does not exist a submanifold chart around $p = 0$, but this is not a problem since $0 \notin S$.

Strictly speaking, a submanifold chart for S is not a chart for S, but rather a chart for M that is adapted to S. On the other hand, submanifold charts *restrict* to charts for S, and this may be used to construct an atlas for S.

Proposition 4.3. *Suppose S is a submanifold of M. Then S is a k-dimensional manifold in its own right, with atlas Consisting of all charts $(U \cap S, \varphi')$ such that (U, φ) is a submanifold chart, and $\varphi' = \pi \circ \varphi|_{U \cap S}$ where $\pi : \mathbb{R}^m \to \mathbb{R}^k$ is the projection onto the first k coordinates. The inclusion map*

G. Gross, E. Meinrenken, *Manifolds, Vector Fields, and Differential Forms*,
Springer Undergraduate Mathematics Series,
https://doi.org/10.1007/978-3-031-25409-3_4

$$i: S \to M$$

(taking $p \in S$ to the same point p viewed as an element of M) is smooth.

Proof. Let (U, φ) and (V, ψ) be two submanifold charts for S. We have to show that the charts $(U \cap S, \varphi')$ and $(V \cap S, \psi')$ are compatible. The map

$$\psi' \circ (\varphi')^{-1}: \varphi'(U \cap V \cap S) \to \psi'(U \cap V \cap S)$$

is smooth because it is the restriction of $\psi \circ \varphi^{-1}: \varphi(U \cap V) \to \psi(U \cap V)$ to $\varphi(U \cap V \cap S) = \varphi(U \cap V) \cap \mathbb{R}^k$. Likewise its inverse map is smooth. The Hausdorff condition follows because for any two distinct points $p, q \in S$, one can take disjoint submanifold charts around p, q. (Just take any submanifold charts around p, q, and restrict the chart domains to the intersection with disjoint open neighborhoods.)

To show that S admits a *countable* atlas, we argue that M admits a countable atlas consisting of submanifold charts for S. (The atlas for S itself is then obtained by restriction of the submanifold charts.) By the technical Lemma 4.4 below, there exists a countable collection $\{V_i\}$ of open subsets $V_i \subseteq M$, with the property that every open subset of M can be written as a union of V_i's. Since every $m \in M$ is contained in *some* submanifold chart for S, we see that the collection of *all* submanifold charts is an atlas of M. The domain of a submanifold chart is a union of V_i's. It follows that the collection of V_i's that are contained in the domain of some submanifold chart is an open cover of M. Given such V_i, we can turn V_i itself into the domain of a submanifold chart: Pick a submanifold chart (U, φ) with $V_i \subseteq U$, and let $\psi_i = \varphi|_{V_i}$. The collection of all $\{(V_i, \psi_i)\}$ obtained in this way is the desired countable atlas.

Finally, to check that the inclusion map i is smooth, we verify smoothness near any given $p \in S$. Let (U, φ) be a submanifold chart of S around p, and let $(U \cap S, \varphi')$ be the corresponding chart of S around $p \in S$, so that $\varphi' = \pi \circ \varphi|_{U \cap S}$. The composition

$$\varphi \circ i \circ (\varphi')^{-1}: \varphi'(U \cap S) \to \varphi(U), \quad (x^1, \ldots, x^k) \mapsto (x^1, \ldots, x^k, 0, \ldots, 0),$$

is smooth, as required. □

The proof used the following result.

Lemma 4.4. *For every manifold M, there exists a countable collection $\{V_i\}$ of open subsets $V_i \subseteq M$, with the property that every open subset $U \subseteq M$ is the union of all V_i such that $V_i \subseteq U$.*

Proof. For $M = \mathbb{R}^m$, we may take $\{V_i\}$ to be the collection of all rational ε-balls $B_\varepsilon(\mathbf{x})$, $\varepsilon > 0$. Here, "rational" means that both the center of the ball and its radius are rational: $\mathbf{x} \in \mathbb{Q}^n$, $\varepsilon \in \mathbb{Q}$.

53 (answer on page 284). Prove that every open subset of $\mathbb{R}^m$ is a union of rational ε-balls.

For a general M, start with a countable atlas $\{(U_\alpha, \varphi_\alpha)\}$, and take $\{V_i\}$ to be the collection of all open subsets of the form

$$\varphi_\alpha^{-1}\big(B_\varepsilon(\mathbf{x})\big), \tag{4.2}$$

where $B_\varepsilon(\mathbf{x})$ is a rational ε-ball contained in $\varphi_\alpha(U_\alpha)$. This is countable, due to the fact that countable unions of countable sets are countable. (See Appendix A.1.) It has the desired property: Given an open subset $U \subseteq M$, each $\varphi_\alpha(U \cap U_\alpha)$ is a union of rational ε-balls (by ✎ 53). Hence, U is the union of all subsets of the form (4.2) that are contained in U.

Example 4.5 (Open Subsets). The m-dimensional submanifolds of an m-dimensional manifold are exactly the open subsets.

Example 4.6 (Euclidean Space). For $k \le n$, we may regard $\mathbb{R}^k \subseteq \mathbb{R}^n$ as the subset where the last $n-k$ coordinates are zero. These are submanifolds, with any chart also being a submanifold chart.

Example 4.7 (Spheres). Let $S^n = \{\mathbf{x} \in \mathbb{R}^{n+1} \,|\, ||\mathbf{x}||^2 = 1\}$. Write $\mathbf{x} = (x^0, \dots, x^n)$, and regard

$$S^k \subseteq S^n$$

for $k < n$ as the subset where the last $n-k$ coordinates are zero. These are submanifolds: The charts $(U_\pm, \varphi_\pm)$ for S^n given by stereographic projection

$$\varphi_\pm(x^0, \dots, x^n) = \frac{1}{1 \pm x^0}(x^1, \dots, x^n)$$

are submanifold charts. Alternatively, the charts $(U_i^\pm, \varphi_i^\pm)$, where $U_i^\pm \subseteq S^n$ is the subset where $\pm x^i > 0$, with $\varphi_i^\pm$ the projection to the remaining coordinates, are submanifold charts as well.

Example 4.8 (Projective Spaces). For $k < n$, regard

$$\mathbb{R}\mathrm{P}^k \subseteq \mathbb{R}\mathrm{P}^n$$

as the subset of all $(x^0 : \ldots : x^n)$ for which $x^{k+1} = \cdots = x^n = 0$. These are submanifolds, with the standard charts (U_i, φ_i) for $\mathbb{R}\mathrm{P}^n$ as submanifold charts. (Note that the charts $U_{k+1}, \dots, U_n$ do not intersect $\mathbb{R}\mathrm{P}^k$, but this does not cause a problem.) In fact, the resulting charts for $\mathbb{R}\mathrm{P}^k$ obtained by restricting these submanifold charts are just the standard charts of $\mathbb{R}\mathrm{P}^k$. Similarly,

$$\mathbb{C}\mathrm{P}^k \subseteq \mathbb{C}\mathrm{P}^n$$

are submanifolds, and for $n < n'$, we have $\mathrm{Gr}(k,n) \subseteq \mathrm{Gr}(k,n')$ as a submanifold. This shows the claim that the decomposition (2.7) is a decomposition into submanifolds.

Proposition 4.9 (Graphs Are Submanifolds). *Let $F: M \to N$ be a smooth map between manifolds of dimensions m and n. Then*

$$\mathrm{graph}(F) = \{(F(p), p) |\; p \in M\} \subseteq N \times M \tag{4.3}$$

is a submanifold of $N \times M$, of dimension equal to the dimension of M.

Remark 4.10. Many authors define the graph as a subset of $M \times N$ rather than $N \times M$. An advantage of the convention above is that it gives

$$\mathrm{graph}(F' \circ F) = \mathrm{graph}(F') \circ \mathrm{graph}(F)$$

under composition of relations.

Proof. Given $p \in M$, choose charts (U, φ) around p and (V, ψ) around $F(p)$, with $F(U) \subseteq V$. We claim that (W, κ) with $W = V \times U$ and

$$\kappa(q, p) = (\varphi(p),\; \psi(q) - \psi(F(p))) \tag{4.4}$$

is a submanifold chart for (4.3). Note that this is indeed a chart of $N \times M$ because it is obtained from the product chart $(V \times U, \psi \times \varphi)$ by composition with the diffeomorphism (see 54 below)

$$\psi(V) \times \varphi(U) \to \kappa(W), \;\; (v, u) \mapsto (u, v - \widetilde{F}(u)), \tag{4.5}$$

where $\widetilde{F} = \psi \circ F \circ \varphi^{-1}$. Furthermore, the second component in (4.4) vanishes if and only if $F(p) = q$. That is,

$$\kappa(W \cap \mathrm{graph}(F)) = \kappa(W) \cap \mathbb{R}^m,$$

as required. □

54 (answer on page 284). Prove that the map (4.5) is a diffeomorphism.

This result has the following consequence: If $S \subseteq M$ is a subset of a manifold, such that S can be *locally* described as the graph of a smooth map, then S is a submanifold. In more detail, suppose that S can be covered by open sets $U \subseteq M$, such that for each U there is a diffeomorphism $U \to P \times Q$ taking $S \cap U$ to the graph of a smooth map $Q \to P$, then S is a submanifold.

Example 4.11. The 2-torus $S = f^{-1}(0) \subseteq \mathbb{R}^3$, where

$$f(x, y, z) = (\sqrt{x^2 + y^2} - R)^2 + z^2 - r^2,$$

is a submanifold of $\mathbb{R}^3$, since it can locally be expressed as the graph of a function of two of the coordinates (see 55 below).

55 (answer on page 284). Show that on the subset where $z > 0$, S is the graph of a smooth function on the annulus

$$\{(x,y) | \ (R-r)^2 < x^2 + y^2 < (R+r)^2\}.$$

How many open subsets of this kind (where S is given as the graph of a function of two of the coordinates) are needed to cover S?

Example 4.12. More generally, suppose $S \subseteq \mathbb{R}^3$ is given as a level set $S = f^{-1}(0)$ of a smooth map $f \in C^\infty(\mathbb{R}^3)$. (Actually, we only need f to be defined on an open neighborhood of S.) Let $p \in S$, and suppose

$$\left.\frac{\partial f}{\partial x}\right|_p \neq 0.$$

By the *implicit function theorem* from multivariable calculus, there is an open neighborhood $U \subseteq \mathbb{R}^3$ of p on which the equation $f(x,y,z) = 0$ can be uniquely solved for x. That is,

$$S \cap U = \{(x,y,z) \in U | \ x = F(y,z)\}$$

for a smooth function F, defined on a suitable open subset of $\mathbb{R}^2$. By Proposition 4.9, this shows that S is a submanifold near p and that we may use y,z as coordinates on S near p. Similar arguments apply for $\left.\frac{\partial f}{\partial y}\right|_p \neq 0$ or $\left.\frac{\partial f}{\partial z}\right|_p \neq 0$. Hence, if the gradient

$$\operatorname{grad} f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$$

is non-vanishing at all points $p \in S = f^{-1}(0)$, then S is a 2-dimensional submanifold. Of course, there is nothing special about 2-dimensional submanifolds of $\mathbb{R}^3$, and below we will put this discussion in a more general framework.

Suppose $S \to M$ is a submanifold, and $F \in C^\infty(M,N)$. Then the restriction

$$F|_S : \ S \to N$$

is again smooth. Indeed, since the inclusion $i : \ S \to M$ is smooth (Proposition 4.3), the restriction may be seen as a composition of smooth maps $F|_S = F \circ i$. This is useful in practice because in such cases there is no need to verify smoothness in the local coordinates of S. For example, the map $S^2 \to \mathbb{R}, \ (x,y,z) \mapsto z$ is smooth since it is the restriction of a smooth map $\mathbb{R}^3 \to \mathbb{R}$ to the submanifold S^2. We now invite you to prove a related result.

56 (answer on page 284). Let $S \subseteq M$ be a submanifold, with inclusion map i, and let $F: Q \to S$ be a map from another manifold Q. Then F is smooth if and only if $i \circ F$ is smooth. (In other words, F is smooth as a map to S if and only if it is smooth as a map to M.)

The following proposition shows that the topology of S as a manifold (i.e., its collection of open subsets) coincides with the "subspace topology" as a subset of the manifold M.

Proposition 4.13. *Suppose S is a submanifold of M. Then the open subsets of S for its manifold structure are exactly those of the form $U \cap S$, where U is an open subset of M.*

Proof. We have to show

$$U' \subseteq S \quad \text{is open} \quad \Leftrightarrow \quad U' = U \cap S \quad \text{where} \quad U \subseteq M \quad \text{is open.}$$

"$\Leftarrow$." Suppose $U \subseteq M$ is open, and let $U' = U \cap S$. For any submanifold chart (V, ψ), with corresponding chart $(V \cap S, \psi')$ for S (where, as before, $\psi' = \pi \circ \psi|_{V \cap S}$), we have that

$$\psi'((V \cap S) \cap U') = \pi \circ \psi(V \cap S \cap U) = \pi(\psi(U) \cap \psi(V) \cap \mathbb{R}^k).$$

Now, $\psi(U) \cap \psi(V) \cap \mathbb{R}^k$ is the intersection of the open set $\psi(U) \cap \psi(V) \subseteq \mathbb{R}^n$ with the subspace $\mathbb{R}^k$ and, hence, is open in $\mathbb{R}^k$. Since charts of the form $(V \cap S, \psi')$ cover all of S, this shows that U' is open.

"$\Rightarrow$." Suppose $U' \subseteq S$ is open in S. Define

$$U = \bigcup_V \psi^{-1}(\psi'(U' \cap V) \times \mathbb{R}^{m-k}) \subseteq M,$$

where the union is over any collection of submanifold charts (V, ψ) that cover all of S. This satisfies

$$U \cap S = U'. \tag{4.6}$$

57 (answer on page 284). Verify (4.6).

To show that U is open, it suffices to show that for all submanifold charts (V, ψ), the set $\psi^{-1}(\psi(U' \cap V) \times \mathbb{R}^{m-k})$ is open. Indeed,

$$\begin{aligned} U' \text{is open in } S &\Rightarrow U' \cap V \text{ is open in } S \\ &\Rightarrow \psi'(U' \cap V) \text{ is open in } \mathbb{R}^k \\ &\Rightarrow \psi'(U' \cap V) \times \mathbb{R}^{m-k} \text{ is open in } \mathbb{R}^m \\ &\Rightarrow \psi^{-1}(\psi(U' \cap V) \times \mathbb{R}^{m-k}) \text{ is open in} \end{aligned}$$

$\square$

Corollary 4.14. *A submanifold $S \subseteq M$ is compact with respect to its manifold topology if and only if it is compact as a subset of M.*

In particular, if a manifold M can be realized as a submanifold $M \subseteq \mathbb{R}^n$, then M is compact with respect to its manifold topology if and only if it is a closed and bounded subset of $\mathbb{R}^n$. This can be used to give quick proofs of the facts that the real and complex projective spaces, as well as the real end complex Grassmannians, are all compact.

4.2 The Rank of a Smooth Map

Let $F \in C^\infty(M,N)$ be a smooth map. Then the fibers (level sets)

$$F^{-1}(q) \subseteq M$$

for $q \in N$ need not be submanifolds in general. Similarly, the image

$$F(M) \subseteq N$$

need not be a submanifold—even if we allow self-intersections. (More precisely, there may be points p such that the image $F(U) \subseteq N$ of any open neighborhood U of p is never a submanifold.) Here are some examples:

(a) The fibers $f^{-1}(c)$ of the map $f(x,y) = xy$ are hyperbolas for $c \neq 0$, but $f^{-1}(0)$ is the union of coordinate axes. What makes this possible is that the gradient of f is zero at the origin.

(b) As we mentioned earlier (cf. Example 3.13), the image of the smooth map

$$\gamma\colon\ \mathbb{R} \to \mathbb{R}^2,\quad \gamma(t) = (t^2, t^3)$$

does not look smooth near $(0,0)$ (and replacing $\mathbb{R}$ by an open interval around 0 does not help). What makes this possible is that the velocity $\dot{\gamma}(t)$ vanishes for $t = 0$: The curve described by γ "comes to a halt" at $t = 0$ and then turns around.

In both cases, the problems arise at points where the map does not have maximal rank. After reviewing the notion of the rank of a map from multivariable calculus, we will generalize to manifolds.

4.2.1 The Rank of the Jacobian Matrix

We shall need some notions from multivariable calculus. Let $U \subseteq \mathbb{R}^m$ and $V \subseteq \mathbb{R}^n$ be open subsets, and $F \in C^\infty(U,V)$ a smooth map. Recall from Definition 2.3 that the *Jacobian matrix* of F at p is the matrix of partial derivatives

$$D_pF = \begin{pmatrix} \frac{\partial F^1}{\partial x^1}\Big|_p & \frac{\partial F^1}{\partial x^2}\Big|_p & \cdots & \frac{\partial F^1}{\partial x^m}\Big|_p \\ \frac{\partial F^2}{\partial x^1}\Big|_p & \frac{\partial F^2}{\partial x^2}\Big|_p & \cdots & \frac{\partial F^2}{\partial x^m}\Big|_p \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial F^n}{\partial x^1}\Big|_p & \frac{\partial F^n}{\partial x^2}\Big|_p & \cdots & \frac{\partial F^n}{\partial x^m}\Big|_p \end{pmatrix}.$$

Definition 4.15. *The* rank *of F at $p \in U$ is the rank of the Jacobian matrix D_pF at p.*

Thus, the rank may be computed as the number of linearly independent rows, or equivalently the number of linearly independent columns. Note that

$$\operatorname{rank}_p(F) \leq \min(m,n). \tag{4.7}$$

We will prefer to think of the Jacobian matrix not as an array of numbers, but as a linear map from $\mathbb{R}^m$ to $\mathbb{R}^n$, more conceptually defined as follows.

Definition 4.16. *The* derivative of F at $p \in U$ *is the linear map*

$$D_pF : \mathbb{R}^m \to \mathbb{R}^n, \;\; \mathbf{v} \mapsto \frac{d}{dt}\Big|_{t=0} F(p+t\mathbf{v}).$$

The rank of F at p is the rank of this linear map, i.e., the dimension of its range. Note that we will use the same notation for this linear map and its matrix.

By the chain rule for differentiation, the derivative of a composition of two smooth maps satisfies

$$D_p(F' \circ F) = D_{F(p)}(F') \circ D_p(F). \tag{4.8}$$

In particular, if F' is a diffeomorphism, then $\operatorname{rank}_p(F' \circ F) = \operatorname{rank}_p(F)$, and if F is a diffeomorphism, then $\operatorname{rank}_p(F' \circ F) = \operatorname{rank}_{F(p)}(F')$.

4.2.2 The Rank of Smooth Maps Between Manifolds

Using charts, we may generalize the notion of rank to smooth maps between manifolds.

Definition 4.17. *Let $F \in C^\infty(M,N)$ be a smooth map between manifolds, and $p \in M$. The* rank *of F at $p \in M$ is defined as*

$$\operatorname{rank}_p(F) = \operatorname{rank}_{\varphi(p)}(\psi \circ F \circ \varphi^{-1}),$$

for any two coordinate charts (U,φ) around p and (V,ψ) around $F(p)$ such that $F(U) \subseteq V$.

By (4.8), this is well-defined: If we use different charts (U', φ') and (V', ψ'), then the rank of

$$\psi' \circ F \circ (\varphi')^{-1} = (\psi' \circ \psi^{-1}) \circ (\psi \circ F \circ \varphi^{-1}) \circ (\varphi \circ (\varphi')^{-1})$$

at $\varphi'(p)$ equals that of $\psi \circ F \circ \varphi^{-1}$ at $\varphi(p)$, since the two maps are related by diffeomorphisms.

By (4.7),

$$\operatorname{rank}_p(F) \le \min(\dim M,\ \dim N)$$

for all $p \in M$.

Definition 4.18. *A smooth map $F \in C^\infty(M,N)$ has* maximal rank *at $p \in M$ if*

$$\operatorname{rank}_p(F) = \min(\dim M,\ \dim N).$$

A point $p \in M$ is called a critical point *for F if does not have maximal rank at p.*

58 (answer on page 285). Consider the lemniscate of Gerono:

$$F:\ \mathbb{R} \to \mathbb{R}^2,\quad \theta \mapsto (\cos\theta, \sin\theta\cos\theta).$$

Find $\operatorname{rank}_p(F)$ for all $p \in \mathbb{R}$, and determine the critical points (if any). How does the graph look like?

59 (answer on page 285). Consider the map

$$F:\ \mathbb{R}^3 \to \mathbb{R}^4,\quad (x,y,z) \mapsto (yz, xy, xz, x^2 + 2y^2 + 3z^2).$$

Find $\operatorname{rank}_p(F)$ for all $p \in \mathbb{R}^3$, and determine the critical points (if any).

4.3 Smooth Maps of Maximal Rank

The following discussion will focus on maps $F \in C^\infty(M,N)$ of maximal rank (Definition 4.18). We will consider the three cases where $\dim M$ is equal to, greater than, or less than $\dim N$.

4.3.1 Local Diffeomorphisms

In this section we consider the case $\dim M = \dim N$. Our "workhorse theorem" from multivariable calculus is going to be the following fact.

Theorem 4.19 (Inverse Function Theorem for $\mathbb{R}^m$). *Let $F \in C^\infty(U,V)$ be a smooth map between open subsets of $\mathbb{R}^m$, and suppose that the derivative D_pF at $p \in U$ is invertible. Then there exists an open neighborhood $U_1 \subseteq U$ of p such that F restricts to a diffeomorphism $U_1 \to F(U_1)$.*

Among other things, the theorem tells us that if $F \in C^\infty(U,V)$ is a bijection from $U \to V$, then the inverse $F^{-1}\colon V \to U$ is smooth provided that the differential (i.e., the *first* derivative) of F is invertible everywhere. It is not necessary to check anything involving higher derivatives.

It is good to see, in just one dimension, how this is possible. Given an invertible smooth function $y = f(x)$, with inverse $x = g(y)$, and using $\frac{\mathrm{d}}{\mathrm{d}y} = \frac{\mathrm{d}x}{\mathrm{d}y}\frac{\mathrm{d}}{\mathrm{d}x}$, we have

$$\begin{aligned} g'(y) &= \frac{1}{f'(x)}, \\ g''(y) &= \frac{-f''(x)}{f'(x)^3}, \\ g'''(y) &= \frac{-f'''(x)}{f'(x)^4} + 3\frac{f''(x)^2}{f'(x)^5}, \end{aligned}$$

and so on; only powers of $f'(x)$ appear in the denominator!

Using charts, we can pass from open subsets of $\mathbb{R}^m$ to manifolds.

Theorem 4.20 (Inverse Function Theorem for Manifolds). *Let $F \in C^\infty(M,N)$ be a smooth map between manifolds of the same dimension $m = n$. If $p \in M$ is such that $\operatorname{rank}_p(F) = m$, then there exists an open neighborhood $U \subseteq M$ of p such that F restricts to a diffeomorphism $U \to F(U)$.*

Proof. Choose charts (U,φ) around p and (V,ψ) around $F(p)$ such that $F(U) \subseteq V$. The map

$$\widetilde{F} = \psi \circ F \circ \varphi^{-1},$$

with domain $\widetilde{U} := \varphi(U)$ and codomain $\widetilde{V} := \psi(V)$, has rank m at $\varphi(p)$. Hence, by the inverse function theorem for $\mathbb{R}^m$ (Theorem 4.19), after replacing $\widetilde{U}$ with a smaller open neighborhood of $\varphi(p)$ in $\mathbb{R}^m$ (equivalently, replacing U with a smaller open neighborhood of p in M), the map $\widetilde{F}$ becomes a diffeomorphism from $\widetilde{U}$ onto $\widetilde{F}(\widetilde{U}) = \psi(F(U))$. It then follows that

$$F = \psi^{-1} \circ \widetilde{F} \circ \varphi\colon U \to V$$

is a diffeomorphism $U \to F(U)$. □

Definition 4.21. *A smooth map $F \in C^\infty(M,N)$ is called a* local diffeomorphism *if for every point $p \in M$ there exists an open neighborhood U of p such that $F(U)$ is open, and F restricts to a diffeomorphism $U \to F(U)$.*

By Theorem 4.20, this is equivalent to the condition that

$$\operatorname{rank}_p(F) = \dim M = \dim N$$

for all $p \in M$. It depends on the map in question which of these two conditions is easier to verify.

> **60 (answer on page 286).** Show that the map $\mathbb{R} \to S^1$, $t \mapsto (\cos(2\pi t), \sin(2\pi t))$, is a local diffeomorphism. How does it fail to be a diffeomorphism?

Example 4.22. The quotient map $\pi : S^n \to \mathbb{R}\mathrm{P}^n$ is a local diffeomorphism. For example, one can see that π restricts to diffeomorphisms from the charts $U_j^{\pm} = \{x \in S^n | \pm x^j > 0\}$ (with coordinate map given by the remaining coordinates) to the standard chart U_j of the projective space. Note that π is not bijective and so cannot be a diffeomorphism.

Example 4.23. Let M be a manifold with a countable open cover $\{U_\alpha\}$. Then the *disjoint* union

$$Q = \bigsqcup_\alpha U_\alpha$$

is a manifold (cf. Section 2.7.4). The map $\pi : Q \to M$, given on $U_\alpha \subseteq Q$ by the inclusion into M, is a local diffeomorphism. Since π is surjective, it determines an equivalence relation on Q, with π as the quotient map and $M = Q/\sim$.

> **61 (answer on page 286).** Why did we make the assumption that the cover is *countable*?

> **62 (answer on page 286).** Show that if the U_α's are the domains of coordinate charts, then Q is diffeomorphic to an open subset of $\mathbb{R}^m$. We hence conclude that any manifold is realized as a quotient of an open subset of $\mathbb{R}^m$, in such a way that the quotient map is a local diffeomorphism.

Remark 4.24 (Local Mapping Degree). Suppose $F \in C^\infty(M,N)$ is a smooth map between manifolds of the same dimension $m = n$, where M,N are *oriented*. If F has maximal rank at $p \in M$, and taking the open subset U in Theorem 4.20 to be connected, the diffeomorphism $F|_U : U \to F(U)$ is either orientation preserving or orientation reversing. Put $\varepsilon_p = +1$ in the first case and $\varepsilon_p = -1$ in the second case. If $q \in N$ is a regular value (see Definition 4.27 below), so that F has maximal rank at all $p \in F^{-1}(q)$, and assuming that $F^{-1}(q)$ is finite, one calls

$$\deg_q(F) = \sum_{p \in F^{-1}(q)} \varepsilon_p \in \mathbb{Z}$$

the *degree* of F at q. If q is not in the range of F, we put $\deg_q(F) = 0$. We will see later (Proposition 8.15) that when N is connected and M is compact, then the degree does not depend on the choice of regular value $q \in N$. In particular, it is zero unless F is surjective.

4.3.2 Submersions

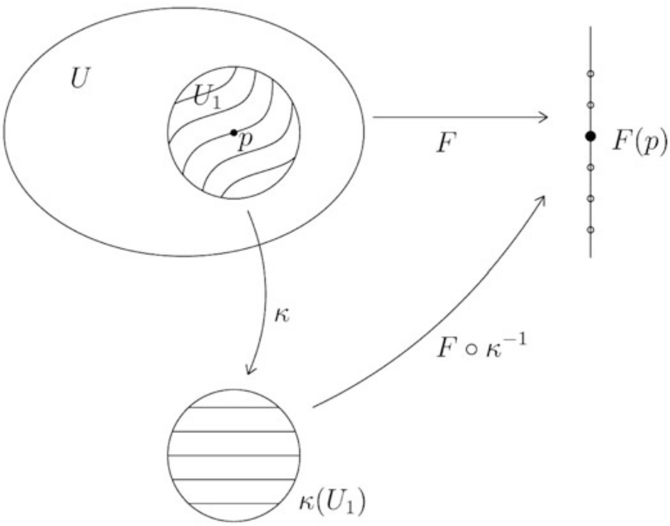
We next consider maps $F: M \to N$ of maximal rank between manifolds of dimensions $m \geq n$. This discussion will rely on the *implicit function theorem* from multivariable calculus.

Theorem 4.25 (Implicit Function Theorem for $\mathbb{R}^m$). *Suppose $F \in C^\infty(U,V)$ is a smooth map between open subsets $U \subseteq \mathbb{R}^m$ and $V \subseteq \mathbb{R}^n$, and suppose $p \in U$ is such that the derivative D_pF is surjective. Then there exist an open neighborhood $U_1 \subseteq U$ of p and a diffeomorphism $\kappa: U_1 \to \kappa(U_1) \subseteq \mathbb{R}^m$ such that*

$$(F \circ \kappa^{-1})(u^1, \ldots, u^m) = (u^{m-n+1}, \ldots, u^m)$$

for all $\mathbf{u} = (u^1, \ldots, u^m) \in \kappa(U_1)$.

Thus, in suitable coordinates, F is given by a projection onto the last n coordinates.



Although it belongs to multivariable calculus, let us recall how to get this result from the inverse function theorem. The idea is to extend F to a map between open subsets of $\mathbb{R}^m$ and then apply the inverse function theorem.

Proof. By assumption, the derivative D_pF has rank equal to n. Hence, it has n linearly independent columns. By re-indexing the coordinates of $\mathbb{R}^m$ (this permutation is itself a change of coordinates, i.e., a diffeomorphism), we may assume that these are the last n columns. That is, writing

$$D_pF = \big(C\ D\big),$$

where C is the $(n\times(m-n))$-matrix formed by the first $m-n$ columns and D the $(n\times n)$-matrix formed by the last n columns, the square matrix D is invertible. Write elements $\mathbf{x}\in\mathbb{R}^m$ in the form $\mathbf{x}=(x',x'')$, where x' are the first $m-n$ coordinates and x'' the last n coordinates. Let

$$G\colon\ U\to\mathbb{R}^m,\ \mathbf{x}=(x',x'')\mapsto(x',F(\mathbf{x})).$$

The derivative D_pG has block form

$$D_pG=\begin{pmatrix} I_{m-n} & 0\\ C & D\end{pmatrix}$$

(where I_{m-n} is the square $(m-n)\times(m-n)$ identity matrix) and is therefore invertible. Hence, by the inverse function theorem, there exists a smaller open neighborhood U_1 of p such that G restricts to a diffeomorphism $\kappa\colon\ U_1\to\kappa(U_1)\subseteq\mathbb{R}^m$. We have

$$G\circ\kappa^{-1}(u',u'')=(u',u'')$$

for all $(u',u'')\in\kappa(U_1)$. Since F is just G followed by projection to the x'' component, we conclude

$$F\circ\kappa^{-1}(u',u'')=u''.$$ □

Again, this result has a version for manifolds.

Theorem 4.26 (Normal Form for Submersions). *Let $F\in C^\infty(M,N)$ be a smooth map between manifolds of dimensions $m\geq n$, and suppose $p\in M$ is such that* $\mathrm{rank}_p(F)=n$*. Then there exist coordinate charts (U,φ) around p and (V,ψ) around $F(p)$, with $F(U)\subseteq V$, such that*

$$(\psi\circ F\circ\varphi^{-1})(u',u'')=u''$$

for all $u=(u',u'')\in\varphi(U)$. In particular, for all $q\in V$, the intersection

$$F^{-1}(q)\cap U$$

is a submanifold of dimension $m-n$.

Proof. Start with coordinate charts (U,φ) around p and (V,ψ) around $F(p)$ such that $F(U)\subseteq V$. Apply Theorem 4.25 to the map $\widetilde{F}=\psi\circ F\circ\varphi^{-1}\colon\ \varphi(U)\to\psi(V)$, to define a smaller neighborhood $\varphi(U_1)\subseteq\varphi(U)$ and change of coordinates κ so that

$$\widetilde{F} \circ \kappa^{-1}(u', u'') = u''.$$

After renaming $(U_1, \kappa \circ \varphi|_{U_1})$ as (U, φ), we have the desired charts for F. The last part of the theorem follows since the map $U \to \mathbb{R}^m$, given as φ followed by translation by $(0, -\psi(q))$, gives a submanifold chart for $F^{-1}(q) \cap U$. $\square$

Definition 4.27. *Let $F \in C^\infty(M,N)$. A point $p \in M$ is called a* regular point *of F, if* $\mathrm{rank}_p(F) = \dim N$*; otherwise, it is called a* critical point *(or* singular point*).*

A point $q \in N$ is called a regular value *of $F \in C^\infty(M,N)$ if for all $p \in F^{-1}(q)$, one has*

$$\mathrm{rank}_p(F) = \dim N.$$

It is called a critical value *(or* singular value*) if it is not a regular value. (Points of N that are not in the image of the map F are considered regular values.)*

We may restate Theorem 4.26 as follows:

Theorem 4.28 (Regular Value Theorem). *For any regular value $q \in N$ of a smooth map $F \in C^\infty(M,N)$, the level set $S = F^{-1}(q)$ is a submanifold of dimension*

$$\dim S = \dim M - \dim N.$$

Example 4.29. The n-sphere S^n may be defined as the level set $F^{-1}(1)$ of the function $F \in C^\infty(\mathbb{R}^{n+1}, \mathbb{R})$ given by

$$F(x^0, \ldots, x^n) = ||\mathbf{x}||^2 = (x^0)^2 + \cdots + (x^n)^2.$$

The derivative of F at $p = \mathbf{x}$ is the $(1 \times (n+1))$-matrix of partial derivatives, that is, the gradient ∇F:

$$D_p F = (2x^0, \ldots, 2x^n).$$

For $\mathbf{x} \neq \mathbf{0}$, this has maximal rank. A real number $q \in \mathbb{R}$ is a regular value of F if and only if $q \neq 0$ (since $\mathbf{0} \notin F^{-1}(q)$ in this case); hence, all the level sets $F^{-1}(q)$ for $q \neq 0$ are submanifolds of dimension $(n+1) - 1 = n$. The number $q = 0$ is a critical value; the level set $F^{-1}(0) = \{\mathbf{0}\}$ is a submanifold but of the "wrong" dimension.

63 (answer on page 286). Let $0 < r < R$. Show that

$$F(x,y,z) = (\sqrt{x^2 + y^2} - R)^2 + z^2$$

has r^2 as a regular value. What is the resulting submanifold?

Example 4.30. The *orthogonal group* $\mathrm{O}(n)$ is the group of matrices $A \in \mathrm{Mat}_{\mathbb{R}}(n)$ satisfying $\langle A\mathbf{x}, A\mathbf{y} \rangle = \langle \mathbf{x}, \mathbf{y} \rangle$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$; here $\langle \cdot, \cdot \rangle$ is the standard inner product (dot product) on $\mathbb{R}^n$. This is equivalent to the property

$$A^\top = A^{-1}$$

of the matrix A, or $A^\top A = I$. We claim that $\mathrm{O}(n)$ is a submanifold of $\mathrm{Mat}_{\mathbb{R}}(n)$. To see this, let us regard $\mathrm{O}(n)$ as the level set $F^{-1}(I)$ of the function

$$F:\ \mathrm{Mat}_{\mathbb{R}}(n) \to \mathrm{Sym}_{\mathbb{R}}(n),\ \ A \mapsto A^\top A,$$

where $\mathrm{Sym}_{\mathbb{R}}(n) \subseteq \mathrm{Mat}_{\mathbb{R}}(n)$ denotes the subspace of symmetric matrices. We want to show that the identity matrix I is a regular value of F. We compute the differential $D_A F:\ \mathrm{Mat}_{\mathbb{R}}(n) \to \mathrm{Sym}_{\mathbb{R}}(n)$ using the definition. (It would be confusing to work with the description of $D_A F$ as a matrix of partial derivatives.)

$$\begin{aligned}(D_A F)(X) &= \frac{d}{dt}\Big|_{t=0} F(A+tX)\\ &= \frac{d}{dt}\Big|_{t=0} ((A^\top + tX^\top)(A+tX))\\ &= A^\top X + X^\top A.\end{aligned}$$

To see that this is surjective for $A \in F^{-1}(I)$, we need to show that for any $Y \in \mathrm{Sym}_{\mathbb{R}}(n)$ there exists a solution $X \in \mathrm{Mat}_{\mathbb{R}}(n)$ for

$$A^\top X + X^\top A = Y.$$

Using $A^\top A = F(A) = I$, we see that $X = \frac{1}{2}AY$ is a solution. We conclude that I is a regular value, and hence that $\mathrm{O}(n) = F^{-1}(I)$ is a submanifold. Its dimension is

$$\dim \mathrm{O}(n) = \dim \mathrm{Mat}_{\mathbb{R}}(n) - \dim \mathrm{Sym}_{\mathbb{R}}(n) = n^2 - \frac{1}{2}n(n+1) = \frac{1}{2}n(n-1).$$

Note that it was important here to regard F as a map to $\mathrm{Sym}_{\mathbb{R}}(n)$; for F viewed as a map to $\mathrm{Mat}_{\mathbb{R}}(n)$, the identity would *not* be a regular value.

Definition 4.31. *A smooth map $F \in C^\infty(M,N)$ is a* submersion *if* $\mathrm{rank}_p(F) = \dim N$ *for all $p \in M$.*

Thus, for a submersion, *all* level sets $F^{-1}(q)$ are submanifolds.

Example 4.32. Local diffeomorphisms are submersions; here the level sets $F^{-1}(q)$ are discrete points, i.e., 0-dimensional manifolds.

Example 4.33. For a product manifold $N \times Q$, the projection to the first factor

$$\mathrm{pr}_N:\ N \times Q \to N$$

is a submersion. The normal form theorem for submersions, Theorem 4.26, shows that locally, *any* submersion $F:\ M \to N$ is of this form. That is, given $p \in M$, there are an open neighborhood $U \subseteq M$ of p and a map $\psi \in C^\infty(U, N \times Q)$ (where Q is a manifold of dimension $m-n$, for example $Q = \mathbb{R}^{m-n}$) such that ψ is a diffeomorphism onto its image and such that the diagram

$$\begin{tikzcd} M \supseteq U \arrow[r, "\psi"] \arrow[dr, "F|_U"'] & N \times Q \arrow[d, "\mathrm{pr}_N"] \\ & N \end{tikzcd}$$

commutes. (That is, $F|_U = \mathrm{pr}_N \circ \psi$.)

4.3.3 Example: The Steiner Surface*

In this section, we give a more detailed example, investigating the smoothness of level sets.

Example 4.34 (Steiner's Surface). Let $S \subseteq \mathbb{R}^3$ be the solution set of

$$y^2z^2 + x^2z^2 + x^2y^2 = xyz.$$

Is this a surface in $\mathbb{R}^3$? (We use *surface* as another term for *2-dimensional manifold*; by a *surface in M*, we mean a 2-dimensional submanifold.) Actually, it is *not*. If we set one of x, y, z to 0, then the equation holds if and only if one of the other two coordinates is also 0. Hence, the intersection of S with the set where $xyz = 0$ (the union of the coordinate hyperplanes) is the union of the three coordinate axes.

Let $U \subseteq \mathbb{R}^3$ be the subset where $xyz \neq 0$; then $S \cap U$ is entirely contained in the set

$$\left\{(x,y,z) \in \mathbb{R}^3 \mid |x| \leq 1, |y| \leq 1, |z| \leq 1\right\}.$$

Let $V \subseteq \mathbb{R}^3$ be an open set around, say, $(2,0,0)$; by replacing it with a possibly smaller open set, we may assume that $V \cap S \subseteq \mathbb{R}^1$. Thus, $V \cap S$ is an open subset of $\mathbb{R}^1$ and thus a 1-dimensional manifold. On the other hand, Proposition 4.13 shows that $V \cap S$ is an open subset of S. Hence, if S were a surface, $V \cap S$ would be a 2-dimensional manifold, contradicting invariance of dimension (cf. Chapter 3, Problem 9).

64 (answer on page 287). Show that $U \cap S$ is entirely contained in the set

$$\left\{(x,y,z) \in \mathbb{R}^3 \,\middle|\, |x| \leq 1, |y| \leq 1, |z| \leq 1\right\}$$

as claimed.

Let us therefore rephrase the question: Is $S \cap U$ a surface? To investigate the problem, consider the function

$$f(x,y,z) = y^2z^2 + x^2z^2 + x^2y^2 - xyz.$$

65 (answer on page 287). Find the critical points of

$$f(x,y,z) = y^2z^2 + x^2z^2 + x^2y^2 - xyz.$$

Conclude that $S \cap U$ is indeed a submanifold.

What does $S \cap U$ look like? It turns out that there is a nice answer. First, let us divide the defining equation by xyz. The equation takes on the form

$$xyz\left(\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2}\right) = 1. \tag{4.9}$$

Since $\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2} > 0$, the solution set of (4.9) is contained in the set of all (x,y,z) such that $xyz > 0$. On this subset, we introduce new variables

$$\alpha = \frac{\sqrt{xyz}}{x}, \;\; \beta = \frac{\sqrt{xyz}}{y}, \;\; \gamma = \frac{\sqrt{xyz}}{z};$$

the old variables x, y, z are recovered as

$$x = \beta\gamma, \;\; y = \alpha\gamma, \;\; z = \alpha\beta.$$

In terms of α, β, γ, Equation (4.9) becomes the equation $\alpha^2 + \beta^2 + \gamma^2 = 1$.

Actually, it is even better to consider the corresponding points

$$(\alpha : \beta : \gamma) = \left(\frac{1}{x} : \frac{1}{y} : \frac{1}{z}\right) \in \mathbb{R}\mathrm{P}^2$$

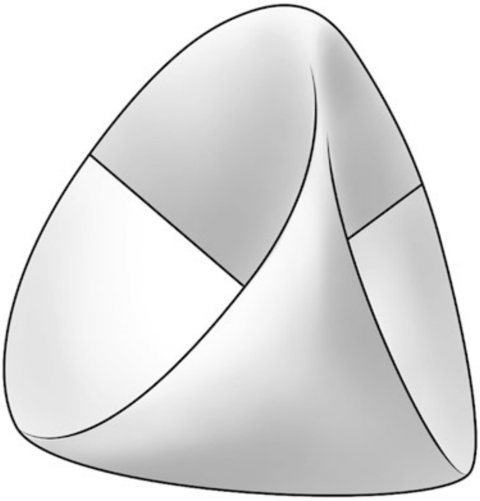
because we could take either sign of the square root of xyz (changing the sign of all α, β, γ does not affect x, y, z). We conclude that the map $U \to \mathbb{R}\mathrm{P}^2$, $(x,y,z) \mapsto (\frac{1}{x} : \frac{1}{y} : \frac{1}{z})$ restricts to a diffeomorphism from $S \cap U$ onto

$$\mathbb{R}\mathrm{P}^2 \backslash \{(\alpha : \beta : \gamma) |\; \alpha\beta\gamma = 0\}.$$

The image of the map

$$\mathbb{R}\mathrm{P}^2 \to \mathbb{R}^3, \;\; (\alpha : \beta : \gamma) \mapsto \frac{1}{|\alpha|^2 + |\beta|^2 + |\gamma|^2}(\beta\gamma, \alpha\gamma, \alpha\beta)$$

is called *Steiner's surface*, even though it is not a submanifold (not even an *immersed* submanifold). Below is a picture.



Note that the subset of $\mathbb{R}\mathrm{P}^2$ defined by $\alpha\beta\gamma = 0$ is a union of three $\mathbb{R}\mathrm{P}^1 \cong S^1$, each of which maps into a coordinate axis (but not the entire coordinate axis). For example, the circle defined by $\alpha = 0$ maps to the set of all $(x,0,0)$ with $-\frac{1}{2} \le x \le \frac{1}{2}$. In any case, S is the union of the Steiner surface with the three coordinate axes.

Example 4.35. Let $S \subseteq \mathbb{R}^4$ be the solution set of

$$y^2z^2 + x^2z^2 + x^2y^2 = xyz, \quad y^2z^2 + 2x^2z^2 + 3x^2y^2 = xyzw.$$

Again, this cannot quite be a surface because it contains the coordinate axes for x,y,z. Closer investigation shows that S is the union of the three coordinate axes, together with the image of an injective map

$$\mathbb{R}\mathrm{P}^2 \to \mathbb{R}^4, \ (\alpha : \beta : \gamma) \mapsto \frac{1}{\alpha^2+\beta^2+\gamma^2}(\beta\gamma, \alpha\gamma, \alpha\beta, \alpha^2 + 2\beta^2 + 3\gamma^2).$$

It turns out (see Example 4.46 and Problem 14 below) that the latter is a submanifold, which realizes $\mathbb{R}\mathrm{P}^2$ as a surface in $\mathbb{R}^4$.

4.3.4 Quotient Maps*

A surjective submersion $F: M \to N$ may be regarded as the quotient map for an equivalence relation on M, where $p \sim p'$ if and only if p, p' are in the same fiber of F. It is natural to ask the converse (see Section 2.7.4): Under what conditions does an equivalence relation $\sim$ on a manifold M determine a manifold structure on the quotient space $N = M/\sim$, in such a way that the quotient map

$$\pi: M \to M/\sim$$

is a submersion. (In Problem 9, you are asked to show that there can be at most one such manifold structure on $M/\sim$.) The answer involves the graph of the equivalence relation

$$R = \{(p,p') \in M \times M |\ p \sim p'\}.$$

Theorem 4.36. *There is a manifold structure on $M/\sim$ with the property that the quotient map $\pi:\ M \to M/\sim$ is a submersion, if and only if the following conditions are satisfied:*

(a) R is a closed submanifold of $M \times M$.
(b) The map $\mathrm{pr}_1:\ M \times M \to M,\ (p,q) \mapsto p$ restricts to a submersion $\mathrm{pr}_1|_R:\ R \to M$.

We will not present the proof of this result, which may be found, for example, in Bourbaki [1, Section 5.9]. One direction is 66 below. The special case that $\mathrm{pr}_1|_R$ is a local diffeomorphism (in particular, $\dim R = \dim M$) is Problem 1 at the end of the chapter.

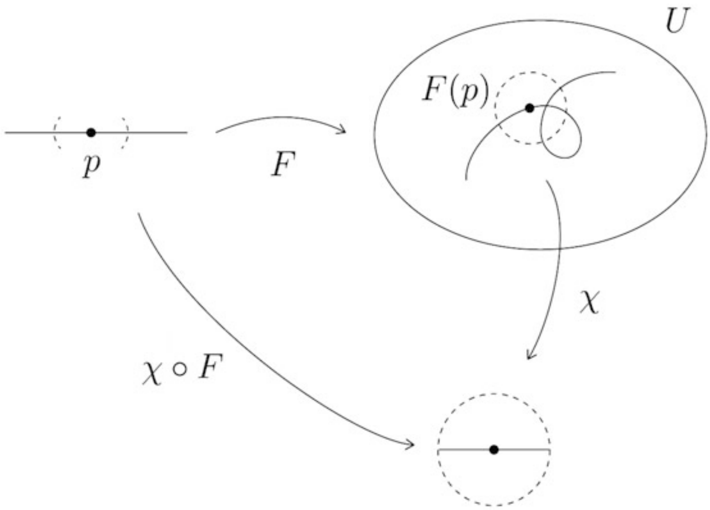
> **66 (answer on page 287).** Suppose $M/\sim$ has the structure of a (possibly non-Hausdorff) manifold, in such a way that π is a submersion. Show that R is a submanifold of $M \times M$, which is closed if and only if $M/\sim$ satisfies the Hausdorff property.

4.3.5 Immersions

We next consider maps $F:\ M \to N$ of maximal rank between manifolds of dimensions $m \le n$. Once again, such a map can be put into a "normal form": By choosing suitable coordinates, it becomes linear.

Proposition 4.37. *Suppose $F \in C^\infty(U,V)$ is a smooth map between open subsets $U \subseteq \mathbb{R}^m$ and $V \subseteq \mathbb{R}^n$, and suppose $p \in U$ is such that the derivative D_pF is injective. Then there exist smaller neighborhoods $U_1 \subseteq U$ of p and $V_1 \subseteq V$ of $F(p)$, with $F(U_1) \subseteq V_1$, and a diffeomorphism $\chi:\ V_1 \to \chi(V_1)$, such that for all $u \in U_1$,*

$$(\chi \circ F)(u) = (u,0) \in \mathbb{R}^m \times \mathbb{R}^{n-m}.$$



Proof. Since D_pF is injective, it has m linearly independent rows. By re-indexing the rows (which amounts to a change of coordinates on V), we may assume that these are the first m rows. That is, writing

$$D_pF = \begin{pmatrix} A \\ C \end{pmatrix},$$

where A is the $(m \times m)$-matrix formed by the first m rows and C is the $((n-m) \times m)$-matrix formed by the last $n-m$ rows, the square matrix A is invertible. We shall write elements $\mathbf{v} \in \mathbb{R}^n$ in the form $\mathbf{v} = (v', v'')$, where v' are the first m coordinates and v'' the last $n-m$ coordinates.

Consider the map

$$H:\ U \times \mathbb{R}^{n-m} \to \mathbb{R}^n,\ (x,y) \mapsto F(x) + (0,y).$$

Its Jacobian at $(p,0)$ is

$$D_{(p,0)}H = \begin{pmatrix} A & 0 \\ C & I_{n-m} \end{pmatrix},$$

which is invertible. Hence, by the inverse function theorem for $\mathbb{R}^n$ (Theorem 4.19), H is a diffeomorphism from some neighborhood of $(p,0)$ in $U \times \mathbb{R}^{n-m}$ onto some neighborhood V_1 of $H(p,0) = F(p)$, which we may take to be contained in V. Let

$$\chi:\ V_1 \to \chi(V_1) \subseteq U \times \mathbb{R}^{n-m}$$

be the inverse; thus

$$(\chi \circ H)(x,y) = (x,y)$$

for all $(x,y) \in \chi(V_1)$. Replace U with the smaller open neighborhood

$$U_1 = F^{-1}(V_1) \cap U$$

of p. Then $F(U_1) \subseteq V_1$, and

$$(\chi \circ F)(u) = (\chi \circ H)(u,0) = (u,0)$$

for all $u \in U_1$. □

The manifolds version reads as follows.

Theorem 4.38 (Normal Form for Immersions). *Let $F \in C^\infty(M,N)$ be a smooth map between manifolds of dimensions $m \le n$, and $p \in M$ a point with*

$$\mathrm{rank}_p(F) = m.$$

Then there are coordinate charts (U,φ) around p and (V,ψ) around $F(p)$ such that $F(U) \subseteq V$ and

$$(\psi \circ F \circ \varphi^{-1})(u) = (u,0).$$

In particular, $F(U) \subseteq N$ is a submanifold of dimension m.

Proof. Once again, this is proved by introducing charts around p and $F(p)$, to reduce to a map between open subsets of $\mathbb{R}^m$, $\mathbb{R}^n$, and then use the multivariable version of the result (Proposition 4.37) to obtain a change of coordinates, putting the map into normal form. We leave the details as an exercise to the reader (see Problem 2 at the end of the chapter). □

Definition 4.39. *A smooth map $F: M \to N$ is an* immersion *if* $\mathrm{rank}_p(F) = \dim M$ *for all $p \in M$.*

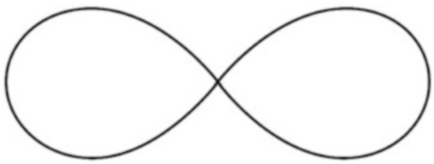
Theorem 4.38 gives a local normal form for immersions.

Example 4.40. Let $J \subseteq \mathbb{R}$ be an open interval, and $\gamma: J \to M$ a smooth map, i.e., a *smooth curve*. We see that γ is an immersion, provided that $\mathrm{rank}_p(\gamma) = 1$ for all $p \in M$. In local coordinates (U,φ), this means that $\frac{d}{dt}(\varphi \circ \gamma)(t) \neq 0$ for all t with $\gamma(t) \in U$. For example, the curve $\gamma(t) = (t^2, t^3)$ from Example 3.13 fails to have this property at $t = 0$.

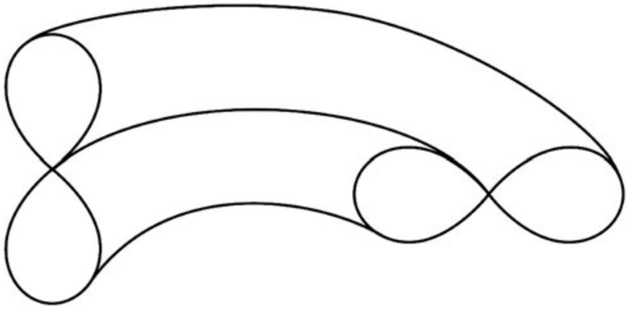
Example 4.41 (Figure Eight). The map (cf. ✐ 58)

$$\gamma: \mathbb{R} \to \mathbb{R}^2, \ t \mapsto \big(\sin(t), \sin(2t)\big)$$

is an immersion; indeed, for all $t \in \mathbb{R}$, we have $D_t\gamma \equiv \dot{\gamma}(t) \neq 0$. The image is a figure eight, as below.



Example 4.42 (A Mystery Immersion). Consider the surface in $\mathbb{R}^3$ obtained by the following procedure. Start with the figure eight in the xz-plane, as in previous example. Shift in the x-direction by an amount $R > 1$, so that the resulting figure lies in the region where $x > 0$. Then rotate the plane containing the figure eight about the z-axis, while at the same time rotating the figure eight about its center, with exactly half the speed of rotation. That is, after a full turn $\varphi \mapsto \varphi + 2\pi$, the figure eight has performed a half-turn. (Think of a single propeller plane flying in a circle, with the figure eight as its propeller.)



The picture suggests that the resulting subset $S \subseteq \mathbb{R}^3$ is the image of an immersion

$$\iota:\ \Sigma \to \mathbb{R}^3$$

of a *compact, connected surface* Σ. To make Σ uniquely defined, we should assume that the map ι is 1-1 on Σ. Since we know the classification of such surfaces, this raises the question: Which surface is it? Let us first try to come up with a good guess, without writing formulas.

67 (answer on page 288). What is the surface Σ, in terms of the classification of compact, connected surfaces? Hint: It may be instructive to investigate the subset $\Sigma_+ \subseteq \Sigma$ generated by half of the figure eight, corresponding to $-\pi/2 < t < \pi/2$ in terms of the parametrization from the previous example.

To get an explicit formula for the immersion, note that the procedure described above is a composition $F = F_3 \circ F_2 \circ F_1$ of the three maps. The map

$$F_1:\ (t,\varphi) \mapsto (\sin(t), \sin(2t), \varphi) = (u,v,\varphi)$$

describes the figure eight in the uv-plane (with φ just a bystander). Next,

$$F_2:\ (u,v,\varphi) \mapsto \Big(u\cos(\frac{\varphi}{2}) + v\sin(\frac{\varphi}{2}),\ v\cos(\frac{\varphi}{2}) - u\sin(\frac{\varphi}{2}), \varphi\Big) = (a,b,\varphi)$$

rotates the uv-plane as it moves in the direction of φ, by an angle of $\varphi/2$; thus $\varphi = 2\pi$ corresponds to a half-turn. Finally,

$$F_3 : \ (a, b, \varphi) \mapsto ((a+R)\cos\varphi, \ (a+R)\sin\varphi, \ b) = (x, y, z)$$

takes this family of rotating uv-planes and wraps it around the circle in the xy-plane of radius R, with φ now playing the role of the angular coordinate. The resulting map $F = F_3 \circ F_2 \circ F_1 : \ \mathbb{R}^2 \to \mathbb{R}^3$ is given by $F(t, \varphi) = (x, y, z)$, where

$$\begin{aligned} x &= \Big(R + \cos(\frac{\varphi}{2})\sin(t) + \sin(\frac{\varphi}{2})\sin(2t)\Big)\cos\varphi, \\ y &= \Big(R + \cos(\frac{\varphi}{2})\sin(t) + \sin(\frac{\varphi}{2})\sin(2t)\Big)\sin\varphi, \\ z &= \cos(\frac{\varphi}{2})\sin(2t) - \sin(\frac{\varphi}{2})\sin(t). \end{aligned}$$

To verify that this is an immersion, it would be cumbersome to work out the Jacobian matrix of F directly. It is much easier to use that F_1 is an immersion, F_2 is a diffeomorphism, and F_3 is a local diffeomorphism from the open subset where $|a| < R$ onto its image.

68 (answer on page 288). With these formulas in place, confirm the result from 67. (Hint: Use the gluing diagram for the Klein bottle to realize it as $\mathbb{R}^2/\sim$ for a suitable equivalence relation.)

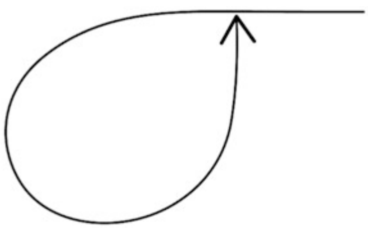
Example 4.43. Let M be a manifold, and $S \subseteq M$ a k-dimensional submanifold. Then the inclusion map $\iota : \ S \to M, \ \ x \mapsto x$ is an immersion. Indeed, if (V, ψ) is a submanifold chart for S, with $p \in U = V \cap S$, and letting $\varphi = \psi|_{V \cap S}$, we have that

$$(\psi \circ \iota \circ \varphi^{-1})(u) = (u, 0),$$

which shows

$$\operatorname{rank}_p(\iota) = \operatorname{rank}_{\varphi(p)}(\psi \circ \iota \circ \varphi^{-1}) = k.$$

By an *embedding*, we will mean an immersion given as the inclusion map for a submanifold. Not every injective immersion is an embedding; the following picture indicates an injective immersion $\mathbb{R} \to \mathbb{R}^2$ whose image is not a submanifold.



In practice, showing that an injective smooth map is an immersion (which amounts to showing that the rank is maximal everywhere) tends to be easier than proving that its image is a submanifold (which amounts to constructing submanifold charts). Fortunately, for compact manifolds, we have the following fact.

Theorem 4.44. *Let $F : M \to N$ be an injective immersion, where the manifold M is compact. Then the image $F(M) \subseteq N$ is an embedded submanifold.*

Proof. We have to show that there exists a submanifold chart for $S = F(M)$ around any given point $F(p) \in N$, for $p \in M$. By Theorem 4.38, we can find charts (U, φ) around p and (V, ψ) around $F(p)$, with $F(U) \subseteq V$, such that the local coordinate expression $\widetilde{F} = \psi \circ F \circ \varphi^{-1}$ is in normal form, i.e.,

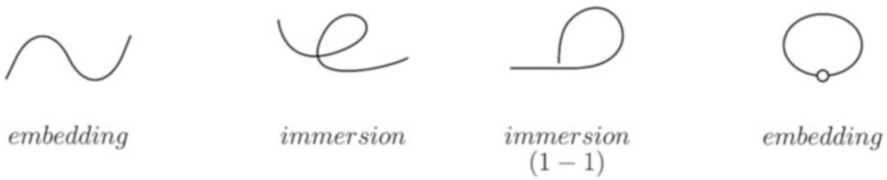
$$\widetilde{F}(u) = (u, 0).$$

We would like to take (V, ψ) as a submanifold chart for $S = F(M)$, but this may not work yet since the normal form above is only given for $F(U) \cap V$, and the set $F(M) \cap V = S \cap V$ may be strictly larger than that. Note however that $A := M \backslash U$ is compact; hence, its image $F(A) \subseteq N$ is compact (cf. ✎ 45) and therefore closed (see Proposition 2.34; note that we are using the fact that N is Hausdorff). Since F is injective, we have that $p \notin F(A)$. Replace V with the smaller open neighborhood

$$V_1 = V \backslash (V \cap F(A)).$$

Then $(V_1, \psi|_{V_1})$ is the desired submanifold chart. □

Remark 4.45. Unfortunately, the terminology for submanifolds used in the literature is not quite uniform. For example, some authors refer to injective immersions $\iota : S \to M$ as submanifolds (thus, a submanifold is taken to be a map rather than a subset). On the other hand, a general immersion (injective or not) is often called an "immersed submanifold," since it is thought of as a generalization of a submanifold where one allows self-intersections. To clarify, "our" submanifolds are sometimes called "embedded submanifolds" or "regular submanifolds" .



Example 4.46. Let A, B, C be distinct real numbers. We will leave it as a homework—Problem 14 below—to verify that the map from Example 4.35

$$F : \mathbb{R}\mathrm{P}^2 \to \mathbb{R}^4, \ \ (\alpha : \beta : \gamma) \mapsto (\beta\gamma, \alpha\gamma, \alpha\beta, \ A\alpha^2 + B\beta^2 + C\gamma^2)$$

(where we use representatives (α, β, γ) such that $\alpha^2 + \beta^2 + \gamma^2 = 1$) is an injective immersion. Hence, by Theorem 4.44, it is an embedding of $\mathbb{R}\mathrm{P}^2$ as a submanifold of $\mathbb{R}^4$.

To summarize the outcome from the last few sections:

> If a smooth map $F \in C^\infty(M,N)$ has maximal rank near a given point $p \in M$, then one can choose local coordinates around p and around $F(p)$ such that the coordinate expression of F becomes a linear map.

In particular, near any given point of m, submersions look like surjective linear maps, while immersions look like injective linear maps.

Remark 4.47. This generalizes further to maps of *constant rank*. That is, if $\mathrm{rank}_p(F)$ is independent of p on some open subset $U \subseteq M$, then for all $p \in U$ one can choose coordinates near p and near $F(p)$ in which F becomes linear. In particular, the image of a sufficiently small open neighborhood of p is a submanifold of N.

4.3.6 Further Remarks on Embeddings and Immersions

Remark 4.48. Let M be a manifold of dimension m. Given $k \in \mathbb{N}$, one may ask if there exists an embedding of M into $\mathbb{R}^k$, or at least an immersion into $\mathbb{R}^k$? For example, one knows that compact 2-manifolds (surfaces) Σ can be embedded into $\mathbb{R}^4$ (even $\mathbb{R}^3$ if Σ is orientable) and immersed into $\mathbb{R}^3$.

These and similar question belong to the realm of *differential topology*, with many deep and difficult results. The *Whitney embedding theorem* states that every m-dimensional manifold M can be realized as an embedded submanifold of $\mathbb{R}^{2m}$. (For a much weaker version of this result, see Theorem C.10 in Appendix C.) This was improved later by various authors to $\mathbb{R}^{2m-1}$, provided that m is not a power of 2, but it is not known what the optimal bound is, in general. The *Whitney immersion theorem* states that every m-dimensional manifold M can be immersed into $\mathbb{R}^{2m-1}$. There had been conjectured optimal bounds $k = 2m - \alpha(m)$ (due to Massey), for a specific function $\alpha(m)$, so that any m-dimensional manifold can be immersed into $\mathbb{R}^{2m-\alpha(m)}$. This conjecture was proved in a 1985 paper of Cohen [6].

Remark 4.49. Another area of differential topology concerns the classification of immersions $M \to N$ up to isotopy, in particular for the case that the target is $N = \mathbb{R}^k$. We say that two immersions $F_0,\ F_1 :\ M \to N$ are *isotopic* if there exists a smooth map

$$F :\ \mathbb{R} \times M \to N$$

such that

$$F_0 = F(0,\cdot),\ F_1 = F(1,\cdot),$$

and such that all $F_t = F(t,\cdot)$ are immersions.

In the late 1950s, Stephen Smale developed criteria for the existence of isotopies and for example gave a striking application to the problem of "sphere eversion" [20]. To explain this result, consider a standard 2-sphere in $\mathbb{R}^3$, with the "outer side" of the sphere painted red, and the "inner side" painted blue. Consider the following question: "Is it possible to turn the sphere inside out—allowing it to pass through itself, but without creating kinks or edges—ending up with the red paint on the inner side and the blue paint on the outer side?"

In mathematical terms, letting $F_0 :\ S^2 \to \mathbb{R}^3$ be the standard inclusion of the sphere, and $F_1 :\ S^2 \to \mathbb{R}^3$ its composition with the map $\mathbf{x} \mapsto -\mathbf{x}$, this is the question whether the immersions F_0 and F_1 are isotopic. Our experience with immersions of the circle in $\mathbb{R}^2$ tends to suggest that this is probably not possible (a circle in $\mathbb{R}^2$ cannot be turned inside out). It hence came as quite a surprise when Stephen Smale proved, in 1957, that such a “sphere eversion” does in fact exist. In subsequent years, various mathematicians developed concrete visualizations for sphere eversions, one of which is the subject of the 1994 short movie “Outside In.”

4.4 Problems

1. Let $\sim$ be an equivalence relation on a manifold M with the property that
$$R = \{(p, p') \in M \times M|\ p \sim p'\}$$
is a closed submanifold of $M \times M$, and such that the map
$$\mathrm{pr}_1 :\ M \times M \to M,\ (p, p') \mapsto p$$
restricts to a local diffeomorphism $\mathrm{pr}_1|_R :\ R \to M$. Prove that $M/\sim$ inherits a unique manifold structure such that the quotient map $\pi :\ M \to M/\sim$ is a local diffeomorphism. (Note: This is a special case of Theorem 4.36, which we had stated without proof.)
2. Complete the proof of Theorem 4.38.
3. Recall (cf. Section 3.6.3) that $\mathbb{C}\mathrm{P}^n$ can be regarded as a quotient of S^{2n+1}. Using charts, show that the quotient map $\pi :\ S^{2n+1} \to \mathbb{C}\mathrm{P}^n$ is a submersion. Hence, its fibers $\pi^{-1}(q)$ are 1-dimensional submanifolds. Indeed, as discussed before, these fibers are circles. As a special case, the Hopf fibration $S^3 \to S^2$ (cf. Section 3.7) is a submersion.
4. Let $\pi :\ M \to N$ be a submersion. Show that for any submanifold $S \subseteq N$, the preimage
$$\pi^{-1}(S) \subseteq M$$
is a submanifold. (Hint: Use the normal form for submersions.)
5. Show that submersions $\pi :\ M \to N$ are open maps: That is, the image $\pi(U)$ of any open subset $U \subseteq M$ is again open.
6. Consider the map $F : \mathbb{R} \to \mathbb{R},\ x \mapsto x - x^3$. For all $p \in \mathbb{R}$, find the quantity $\varepsilon_p = \pm 1$ from Remark 4.24. For any regular value $q \in \mathbb{R}$, find the set of preimages $F^{-1}(q)$, and compute the quantity $\sum_{p \in F^{-1}(q)} \varepsilon_p$ (for a fixed choice of orientation on $\mathbb{R}$).
7. Show that a smooth map $\pi :\ M \to N$ is a submersion if and only if for all $p \in M$, there exist an open neighborhood $V \subseteq N$ of $\pi(p)$ and a smooth map $\iota :\ V \to M$ such that $\iota(\pi(p)) = p$ and $\pi \circ \iota = \mathrm{id}_V$.

8. Let M, N, Q be manifolds, $\pi : M \to N$ a surjective submersion, and $F : N \to Q$ a map. Prove that F is smooth if and only if the lifted map $\widetilde{F} = F \circ \pi : M \to Q$ is smooth. (Hint: Use Problem 7.)

$$\begin{tikzcd} M \arrow[d, "\pi"] \arrow[dr, "\widetilde{F}"] & \\ N \arrow[r, "F"] & Q \end{tikzcd}$$

9. Let $\sim$ be an equivalence relation on a manifold M. Show that there exists *at most one* manifold structure on $M/\sim$ for which the quotient map is a submersion. (Hint: Use Problem 8.)

10. (a) Construct an embedding $S^1 \times S^1 \to S^3$.
    (b) Construct an embedding $S^n \times \mathbb{R} \to \mathbb{R}^{n+1}$.
    (c) Construct an embedding $S^n \times S^m \to S^{n+m+1}$.

11. Let M, N, Q be manifolds, $i : M \to N$ an embedding, and $F : Q \to M$ a map. Prove that F is smooth if and only if $\widetilde{F} = i \circ F : Q \to N$ is smooth.

$$\begin{tikzcd} Q \arrow[r, "F"] \arrow[dr, "F"'] & M \arrow[d, "i"] \\ & N \end{tikzcd}$$

Show that the conclusion fails if "embedding" is replaced with "immersion."

12. Let M be a manifold, and $i : N \to M$ an injective map. Show that there exists *at most one* manifold structure on N for which the inclusion map is an embedding.

13. Show that any smooth map $F : M \to N$ can be written as a composition of two smooth maps
$$i : M \to Q, \quad \pi : Q \to N,$$
where i is an embedding, and π is a submersion. (Hint: Consider the graph of F.)

14. Let A, B, C be *distinct* real numbers.
    (a) Calculate the Jacobian of the map
$$\mathbb{R}^3 \to \mathbb{R}^5, \ (x,y,z) \mapsto \big(yz, xz, xy, Ax^2 + By^2 + Cz^2, \ x^2 + y^2 + z^2 - 1\big),$$
and show that it has maximal rank except at $(0,0,0)$.
(Hint: You may want to consider three cases $x = 0$, $yz \neq 0$; $x = y = 0$, $z \neq 0$; and $xyz \neq 0$. There are more cases, but those follow "by symmetry." Also, you may find it useful to observe that when $Au + Bv + Cw = 0$ and $u + v + w = 0$, then the vanishing of one of u, v, w implies the vanishing of the other two.)

(b) Using the results from part (a), show that the map

$$f:\ S^2 \to \mathbb{R}^4,\ (x,y,z) \mapsto \big(yz, xz, xy, Ax^2+By^2+Cz^2\big)$$

is an immersion. (Here $S^2 = \{(x,y,z) \mid x^2+y^2+z^2=1\}$).

(c) Show that the map

$$g:\ \mathbb{R}\mathrm{P}^2 \to \mathbb{R}^4,\ (x:y:z) \mapsto \frac{1}{x^2+y^2+z^2}\big(yz, xz, xy, Ax^2+By^2+Cz^2\big)$$

is an injective immersion. Since $\mathbb{R}\mathrm{P}^2$ is compact, it is an embedding. (Hint: To show that g is an immersion, use that $f = g\circ\pi$ where $\pi:\ S^2 \to \mathbb{R}\mathrm{P}^2$ is the quotient map.)

15. For a real (2×2)-matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Mat}_{\mathbb{R}}(2),$$

let $||\cdot||$ denote its norm, defined as $||A||^2 = a^2+b^2+c^2+d^2$.

(a) Use the regular value theorem to show that the set

$$S = \{A \in \mathrm{Mat}_{\mathbb{R}}(2) \mid\ ||A||^2 = 1,\ \det(A) = 0\}$$

is a 2-dimensional submanifold of $\mathrm{Mat}_{\mathbb{R}}(2) \cong \mathbb{R}^4$.

(b) Show that the map

$$\pi:\ S \to \mathbb{R}\mathrm{P}^1$$

taking $A \in S$ to its (1-dimensional) range $\mathrm{ran}(A) \subseteq \mathbb{R}^2$ is smooth. Determine the fibers $\pi^{-1}(u:v)$, for $u^2+v^2=1$.

(c) Prove that S is a 2-torus.

(d) Similarly, argue that each of the two regions bounded by this 2-torus (given by $\det(A) \geq 0$ and $\det(A) \leq 0$, respectively) is a solid 2-torus (i.e., a product $D^2 \times S^1$ where $D^2 \subseteq \mathbb{R}^2$ is the closed unit disk).

16. (This may be seen as a continuation of the preceding problem.) Let

$$G = \{A \in \mathrm{Mat}_{\mathbb{R}}(2):\ \det(A) \in \{-1,1\}\}.$$

(a) Show that G is a submanifold of $\mathrm{Mat}_{\mathbb{R}}(2)$, and that it is a subgroup of $\mathrm{GL}(2,\mathbb{R}) = \{A \in \mathrm{Mat}_{\mathbb{R}}(2) \mid \det(A) \neq 0\}$.

(b) Show G has two components, both of which are diffeomorphic to $S^1 \times \mathbb{R}^2$. In particular, G is non-compact.

(c) Let

$$q:\ \mathrm{Mat}_{\mathbb{R}}(2)\backslash\{O\} \to S^3,\quad A \mapsto \frac{A}{||A||}$$

(here O denotes the zero matrix). Show that the restriction of this map to G is an inclusion as an open subset and that the complement is exactly the 2-torus described in the previous problem.

Note: The identity component of G is denoted $\mathrm{SL}(2,\mathbb{R})$. The problem shows, in particular, that this group is diffeomorphic to the interior of a solid 2-torus.

17. Prove that for any smooth map between manifolds $F \in C^\infty(M,N)$, and any given $p \in M$, there exists an open neighborhood $U \subseteq M$ of p with the property that

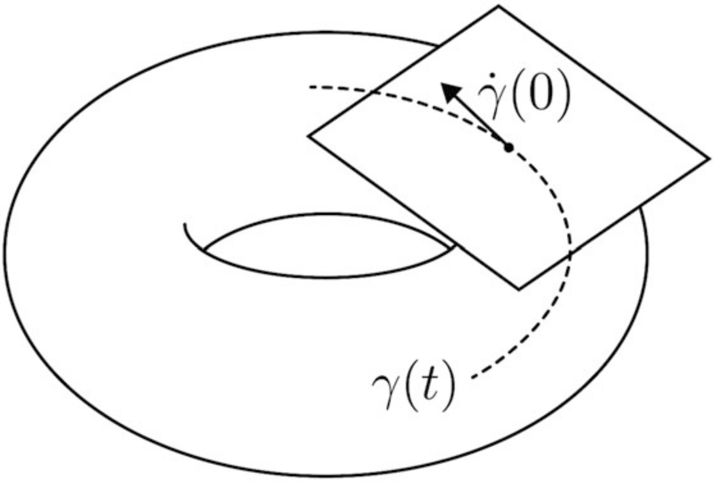
$$\mathrm{rank}_p(F) \leq \mathrm{rank}_{p'}(F)$$

for all $p' \in U$. (One says that the rank is *lower semicontinuous*.)

5 Tangent Spaces

5.1 Intrinsic Definition of Tangent Spaces

For embedded submanifolds $M \subseteq \mathbb{R}^n$, the tangent space T_pM at $p \in M$ can be defined as the set of all *velocity vectors* $v = \dot{\gamma}(0)$, for smooth curves $\gamma\colon J \to M$ with $\gamma(0) = p$; here $J \subseteq \mathbb{R}$ is an open interval around 0.



It turns out (not obvious!) that T_pM becomes a vector subspace of $\mathbb{R}^n$. (Warning: In pictures, we tend to draw the tangent space as an *affine* subspace, where the origin has been moved to p.)

G. Gross, E. Meinrenken, *Manifolds, Vector Fields, and Differential Forms*,
Springer Undergraduate Mathematics Series,
https://doi.org/10.1007/978-3-031-25409-3_5

Example 5.1. Consider the sphere $S^n \subseteq \mathbb{R}^{n+1}$, given as the set of $\mathbf{x}$ such that $||\mathbf{x}|| = 1$. A curve $\gamma(t)$ lies in S^n if and only if $||\gamma(t)|| = 1$. Taking the derivative of the equation $\gamma(t) \cdot \gamma(t) = 1$ at $t = 0$, we obtain (after dividing by 2, and using $\gamma(0) = p$)

$$p \cdot \dot{\gamma}(0) = 0.$$

That is, T_pM consists of vectors $\mathbf{v} \in \mathbb{R}^{n+1}$ that are orthogonal to $p \in S^n$.

Conversely, every such vector $\mathbf{v}$ is of the form $\dot{\gamma}(0)$: Given $\mathbf{v}$, we may take $\gamma(t) = (p + t\mathbf{v})/||p + t\mathbf{v}||$, for example.

Thus,

$$T_pS^n = (\mathbb{R}p)^\perp$$

is the hyperplane orthogonal to the line through p.

For general manifolds M, without a given embedding into a Euclidean space, we would like to make sense of "velocity vectors" of curves, and hence of the tangent space, *intrinsically*. The basic observation is that the curve $t \mapsto \gamma(t)$ defines a "directional derivative" on functions $f \in C^\infty(M)$:

$$f \mapsto \left.\frac{d}{dt}\right|_{t=0} f\big(\gamma(t)\big).$$

69 **(answer on page 290).** Show that if M is a submanifold of $\mathbb{R}^n$, then the map $C^\infty(M) \to \mathbb{R}$, $f \mapsto \frac{d}{dt}\big|_{t=0} f\big(\gamma(t)\big)$ depends only on $p = \gamma(0)$ and the velocity vector $\mathbf{v} = \dot{\gamma}(0)$.

For a general manifold, we think of tangent vectors not as vectors in some ambient Euclidean space, but as the set of directional derivatives.

Definition 5.2 (Tangent Spaces—First Definition). *Let M be a manifold, $p \in M$. The tangent space T_pM is the set of all linear maps $v : C^\infty(M) \to \mathbb{R}$ of the form*

$$v(f) = \left.\frac{d}{dt}\right|_{t=0} f\big(\gamma(t)\big),$$

for smooth curves $\gamma \in C^\infty(J, M)$ with $\gamma(0) = p$, for some open interval $J \subseteq \mathbb{R}$ around 0*. The elements $v \in T_pM$ are called the* tangent vectors *to M at p.*

As it stands, T_pM is defined as a certain subset of the infinite-dimensional vector space

$$\mathrm{Hom}(C^\infty(M), \mathbb{R})$$

of all linear maps $C^\infty(M) \to \mathbb{R}$. The following local coordinate description makes it clear that T_pM is a linear subspace of this vector space, of dimension equal to the dimension of M.

Theorem 5.3. *Let (U,φ) be a coordinate chart around p. A linear map $v : C^\infty(M) \to \mathbb{R}$ is in T_pM if and only if it has the form,*

$$v(f) = \sum_{i=1}^{m} a^i \frac{\partial(f\circ\varphi^{-1})}{\partial u^i}\bigg|_{u=\varphi(p)} \tag{5.1}$$

for some $\mathbf{a} = (a^1,\dots,a^m) \in \mathbb{R}^m$.

Proof. Given a linear map v of this form, let $\tilde{\gamma} : \mathbb{R} \to \varphi(U)$ be a curve with $\tilde{\gamma}(t) = \varphi(p) + t\mathbf{a}$ for $|t|$ sufficiently small. Let $\gamma = \varphi^{-1}\circ\tilde{\gamma}$. Then

$$\begin{aligned}\frac{d}{dt}\bigg|_{t=0} f(\gamma(t)) &= \frac{d}{dt}\bigg|_{t=0} (f\circ\varphi^{-1})(\varphi(p)+t\mathbf{a}) \\ &= \sum_{i=1}^{m} a^i \, \frac{\partial(f\circ\varphi^{-1})}{\partial u^i}\bigg|_{u=\varphi(p)}\end{aligned}$$

by the chain rule. Conversely, given any curve γ with $\gamma(0) = p$, let $\tilde{\gamma} = \varphi\circ\gamma$ be the corresponding curve in $\varphi(U)$ (defined for small $|t|$). Then $\widetilde{\gamma}(0) = \varphi(p)$, and

$$\begin{aligned}\frac{d}{dt}\bigg|_{t=0} f(\gamma(t)) &= \frac{d}{dt}\bigg|_{t=0} (f\circ\varphi^{-1})(\tilde{\gamma}(t)) \\ &= \sum_{i=1}^{m} a^i \, \frac{\partial(f\circ\varphi^{-1})}{\partial u^i}\bigg|_{u=\gamma(p)},\end{aligned}$$

where $\mathbf{a} = \dfrac{d\tilde{\gamma}}{dt}\bigg|_{t=0}$. □

We can use this result as an alternative definition of the tangent space, namely:

Definition 5.4 (Tangent Spaces—Second Definition). *Let (U,φ) be a chart around p. The tangent space T_pM is the set of all linear maps $v : C^\infty(M) \to \mathbb{R}$ of the form*

$$v(f) = \sum_{i=1}^{m} a^i \frac{\partial(f\circ\varphi^{-1})}{\partial u^i}\bigg|_{u=\varphi(p)} \tag{5.2}$$

for some $\mathbf{a} = (a^1,\dots,a^m) \in \mathbb{R}^m$.

Remark 5.5. From this version of the definition, it is immediate that T_pM is an m-dimensional vector space. It is not immediately obvious from this second definition that T_pM is independent of the choice of coordinate chart, but this follows from the equivalence with the first definition. Alternatively, one may check directly that the subspace of $\mathrm{Hom}(C^\infty(M),\mathbb{R})$ characterized by (5.2) does not depend on the chart, by studying the effect of a change of coordinates (see Problem 2 at the end of the chapter).

According to (5.2), any choice of coordinate chart (U,φ) around p defines a vector space isomorphism $T_pM \cong \mathbb{R}^m$, taking v to $\mathbf{a} = (a^1,\dots,a^m)$. In particular, we see that if $U \subseteq \mathbb{R}^m$ is an open subset, and $p \in U$, then T_pU is the subspace of the space of linear maps $C^\infty(U) \to \mathbb{R}$ spanned by the partial derivatives at p. That is, T_pU has a basis

$$\frac{\partial}{\partial x^1}\bigg|_p,\dots,\frac{\partial}{\partial x^m}\bigg|_p$$

identifying $T_pU \cong \mathbb{R}^m$. Given

$$v = \sum a^i \frac{\partial}{\partial x^i}\bigg|_p,$$

the coefficients a^i are obtained by applying v to the coordinate functions $x^1,\dots,x^m : U \to \mathbb{R}$, that is, $a^i = v(x^i)$.

We now describe yet another approach to tangent spaces that again characterizes "directional derivatives" in a coordinate-free way, but without reference to curves γ. Note first that every tangent vector satisfies the *product rule*, also called the *Leibniz rule*.

Lemma 5.6. *Let $v \in T_pM$ be a tangent vector at $p \in M$. Then*

$$v(fg) = f(p)\,v(g) + v(f)\,g(p) \tag{5.3}$$

for all $f,g \in C^\infty(M)$.

Proof. Letting v be represented by a curve γ, this follows from

$$\begin{aligned} v(fg) &= \frac{d}{dt}\bigg|_{t=0}\Big(f\big(\gamma(t)\big)g\big(\gamma(t)\big)\Big) \\ &= f(\gamma(0))\Big(\frac{d}{dt}\bigg|_{t=0} g\big(\gamma(t)\big)\Big) + \Big(\frac{d}{dt}\bigg|_{t=0} f\big(\gamma(t)\big)\Big)g(\gamma(0)) \\ &= f(p)\,v(g) + v(f)\,g(p), \end{aligned}$$

where we used the product rule for functions of $t \in \mathbb{R}$. □

Alternatively, using local coordinates, Equation (5.3) amounts to the product rule for partial derivatives.

Note that there is an abundance of linear functionals $v \in \mathrm{Hom}(C^\infty(M),\mathbb{R})$ that do *not* satisfy the product rule. For example, the evaluation map $\mathrm{ev}_p : f \mapsto f(p)$ is linear but does not satisfy (5.3) with respect to p (or any other point).

70 (answer on page 290). Let $M = \mathbb{R}$. Give several examples of linear maps $v \in \mathrm{Hom}(C^\infty(\mathbb{R}),\mathbb{R})$ that do not satisfy the product rule (5.3) with respect to $p = 0$.

71 (answer on page 290). Suppose that $v : C^\infty(M) \to \mathbb{R}$ is a linear map satisfying the product rule (5.3). Prove the following two facts:

(a) v vanishes on constants. That is, if $f \in C^\infty(M)$ is the constant map, then $v(f) = 0$.

(b) Suppose $f, g \in C^\infty(M)$ with $f(p) = g(p) = 0$. Then $v(fg) = 0$.

It turns out that the product rule completely characterizes tangent vectors.

Theorem 5.7. *A linear map $v : C^\infty(M) \to \mathbb{R}$ defines an element of T_pM if and only if it satisfies the product rule (5.3).*

The proof of Theorem 5.7 will require the following fact from multivariable calculus:

Lemma 5.8 (Hadamard's Lemma). *Let $U = B_R(\mathbf{0}) \subseteq \mathbb{R}^m$ be an open ball of radius $R > 0$ centered at $\mathbf{0}$, and $h \in C^\infty(U)$ a smooth function. Then there exist smooth functions $h_i \in C^\infty(U)$ with*

$$h(\mathbf{u}) = h(\mathbf{0}) + \sum_{i=1}^{m} u^i h_i(\mathbf{u}) \tag{5.4}$$

for all $\mathbf{u} \in U$. For any choice of such functions,

$$h_i(\mathbf{0}) = \frac{\partial h}{\partial u^i}(\mathbf{0}). \tag{5.5}$$

Proof. For fixed $\mathbf{u} \in U$, consider the function $t \mapsto h(t\mathbf{u}) = h(tu^1, \ldots, tu^n)$. We have that

$$h(\mathbf{u}) - h(\mathbf{0}) = \int_0^1 \frac{d}{dt} h(t\mathbf{u})\, dt = \int_0^1 \sum_{i=1}^{n} u^i \frac{\partial h}{\partial u^i}(t\mathbf{u})\, dt = \sum_{i=1}^{n} u^i h_i(\mathbf{u}),$$

where

$$h_i(\mathbf{u}) = \int_0^1 \frac{\partial h}{\partial u^i}(t\mathbf{u})\, dt,$$

are smooth functions of $\mathbf{u}$. Taking the derivative of (5.4)

$$\frac{\partial h}{\partial u^i} = \frac{\partial}{\partial u^i}\left(h(\mathbf{0}) + \sum_{i=1}^{m} u^i h_i(\mathbf{u}) \right) = h_i(\mathbf{u}) + \sum_k u^k \frac{\partial h_k}{\partial u^i}$$

and putting $\mathbf{u} = \mathbf{0}$, we see that $\left.\frac{\partial h}{\partial u^i}\right|_{\mathbf{u}=\mathbf{0}} = h_i(\mathbf{0})$. □

Proof of Theorem 5.7. Let $v : C^\infty(M) \to \mathbb{R}$ be a linear map satisfying the product rule (5.3). The proof consists of the following three steps.

Step 1: *If $f_1 = f_2$ on some open neighborhood U of p, then $v(f_1) = v(f_2)$.* Equivalently, letting $f = f_1 - f_2$, we show that $v(f) = 0$ if $f|_U = 0$. Choose a "bump function" $\chi \in C^\infty(M)$ with $\chi(p) = 1$, and $\chi|_{M\setminus U} = 0$ (cf. Lemma C.8). Then $f\chi = 0$. The product rule tells us that

$$0 = v(f\chi) = v(f)\chi(p) + v(\chi)f(p) = v(f).$$

Step 2: *Let (U, φ) be a chart around p, with image $\widetilde{U} = \varphi(U)$. Then there is unique linear map $\widetilde{v}\colon C^\infty(\widetilde{U}) \to \mathbb{R}$, again satisfying the product rule, such that $\widetilde{v}(\widetilde{f}) = v(f)$ whenever $\widetilde{f}$ agrees with $f \circ \varphi^{-1}$ on some neighborhood of $\widetilde{p} = \varphi(p)$.*
We want to define $\widetilde{v}$ by putting $\widetilde{v}(\widetilde{f}) = v(f)$, for any choice of function f such that $\widetilde{f}$ agrees with $f \circ \varphi^{-1}$ on some neighborhood of $\widetilde{p}$. (Note that such a function f always exists, for any given $\widetilde{f}$.) If g is another function such that $\widetilde{f}$ agrees with $g \circ \varphi^{-1}$ on some neighborhood of $\widetilde{p}$, it follows from Step 1 that $v(f) = v(g)$. This shows that $\widetilde{v}$ is well-defined; the product rule for v implies the product rule for $\widetilde{v}$.

Step 3: *In a chart (U, φ) around p, the map $v\colon C^\infty(M) \to \mathbb{R}$ is of the form (5.2).* Since the condition (5.2) does not depend on the choice of chart around p, we may assume that $\widetilde{p} = \varphi(p) = \mathbf{0}$ and that $\widetilde{U}$ is an open ball around $\mathbf{0}$. Define $\widetilde{v}$ as in Step 2. Given $f \in C^\infty(M)$, let $\tilde{f} = f \circ \varphi^{-1}$. By Lemma 5.8 (Hadamard's Lemma), we have that

$$\tilde{f}(\mathbf{u}) = \tilde{f}(\mathbf{0}) + \sum_{i=1}^{m} u^i h_i(\mathbf{u}),$$

where $h_i \in C^\infty(\widetilde{U})$ with $h_i(\mathbf{0}) = \frac{\partial \widetilde{f}}{\partial u^i}(\mathbf{0})$. Using that $\widetilde{v}$ satisfies the product rule, and in particular that it vanishes on constants, we obtain

$$v(f) = \widetilde{v}(\widetilde{f}) = \sum_{i=1}^{m} \widetilde{v}(u^i)\, h_i(\mathbf{0}) = \sum_{i=1}^{m} a^i \, \frac{\partial \widetilde{f}}{\partial u^i}(\mathbf{0}),$$

where we put $a^i = \widetilde{v}(u^i)$. $\square$

To summarize, we have the following alternative definition of tangent spaces.

Definition 5.9 (Tangent Spaces—Third Definition). *The tangent space T_pM is the space of linear maps $C^\infty(M) \to \mathbb{R}$ satisfying the product rule,*

$$v(fg) = f(p)v(g) + v(f)\,g(p)$$

for all $f, g \in C^\infty(M)$.

At first sight, this characterization may seem a bit less intuitive than the definition as directional derivatives along curves. But it has the advantage of being less redundant—a tangent vector may be represented by many curves (cf. ✎ 69). Furthermore, it is immediate from this third definition (just as for the second definition, in terms of coordinates) that T_pM is a linear subspace of the vector space $\mathrm{Hom}(C^\infty(M), \mathbb{R})$. (The fact that $\dim T_pM = \dim M$ is less obvious, though—it is easiest to see from the second definition.)

The following remark gives yet another characterization of the tangent space. Please read it only if you enjoy abstractions—*otherwise skip this*!

Remark 5.10 (A Fourth Definition). There is a fourth definition of T_pM, as follows. For any $p \in M$, let $C_p^\infty(M)$ denote the subspace of smooth functions vanishing at p, and let $C_p^\infty(M)^2$ consist of finite sums $\sum_i f_i g_i$ where $f_i, g_i \in C_p^\infty(M)$. We have a direct sum decomposition

$$C^\infty(M) = \mathbb{R} \oplus C_p^\infty(M),$$

where $\mathbb{R}$ is regarded as the constant functions.

Since any tangent vector $v: \ C^\infty(M) \to \mathbb{R}$ vanishes on constants, v is effectively a map $v: C_p^\infty(M) \to \mathbb{R}$. By the product rule, v vanishes on the subspace $C_p^\infty(M)^2 \subseteq C_p^\infty(M)$. Thus v descends to a linear map $C_p^\infty(M)/C_p^\infty(M)^2 \to \mathbb{R}$, i.e., an element of the dual space $(C_p^\infty(M)/C_p^\infty(M)^2)^*$.

The map

$$T_pM \to (C_p^\infty(M)/C_p^\infty(M)^2)^*$$

just defined is an *isomorphism* and can therefore be used as a definition of T_pM. This may appear very fancy on first sight, but really it just says that a tangent vector is a linear functional on $C^\infty(M)$ that vanishes on constants and depends only on the first order Taylor expansion of the function at p. Furthermore, this viewpoint lends itself to generalizations that are relevant to algebraic geometry and non-commutative geometry: The "vanishing ideals" $C_p^\infty(M)$ are the maximal ideals in the algebra of smooth functions, with $C_p^\infty(M)^2$ their second power (in the sense of products of ideals). Thus, for any maximal ideal $\mathscr{I}$ in a commutative algebra $\mathscr{A}$, one may regard $(\mathscr{I}/\mathscr{I}^2)^*$ as a "tangent space."

After this lengthy discussion of tangent spaces, observe that the "velocity vectors" of curves are naturally elements of the tangent space. Indeed, let $J \subseteq \mathbb{R}$ be an open interval, and $\gamma \in C^\infty(J,M)$ a smooth curve. Then for any $t_0 \in J$, the tangent (or *velocity*) vector

$$\dot{\gamma}(t_0) \in T_{\gamma(t_0)}M$$

at time t_0 is given in terms of its action on functions by

$$(\dot{\gamma}(t_0))(f) = \frac{d}{dt}\bigg|_{t=t_0} f(\gamma(t)). \tag{5.6}$$

We will also use the notation $\dfrac{d\gamma}{dt}(t_0)$ or $\dfrac{d\gamma}{dt}\bigg|_{t_0}$ to denote this velocity vector.

5.2 Tangent Maps

5.2.1 Definition of the Tangent Map, Basic Properties

For smooth maps $F \in C^\infty(U,V)$ between open subsets $U \subseteq \mathbb{R}^m$ and $V \subseteq \mathbb{R}^n$ of Euclidean spaces, and any given $p \in U$, we considered the derivative to be the linear map

$$D_pF: \mathbb{R}^m \to \mathbb{R}^n, \; \mathbf{a} \mapsto \left.\frac{d}{dt}\right|_{t=0} F(p+t\mathbf{a}).$$

(cf. Definition 4.16). The corresponding matrix is the Jacobian matrix of partial derivatives of F. The following definition generalizes the derivative to smooth maps between manifolds.

Definition 5.11. *Let M,N be manifolds and $F \in C^\infty(M,N)$ a smooth map. For any $p \in M$, we define the* tangent map *to be the linear map*

$$T_pF: T_pM \to T_{F(p)}N$$

given by

$$\big(T_pF(v)\big)(g) = v(g \circ F)$$

for $v \in T_pM$ and $g \in C^\infty(N)$.

One needs to verify that the right-hand side does indeed define a tangent vector.

72 (answer on page 290). Show that for all $v \in T_pM$, the map $g \mapsto v(g \circ F)$ satisfies the product rule at $q = F(p)$ and hence defines an element of T_qN.

Proposition 5.12. *If $v \in T_pM$ is represented by a curve $\gamma: J \to M$, then $(T_pF)(v)$ is represented by the curve $F \circ \gamma$.*

Proof. Let $\gamma: J \to M$ be a smooth curve passing through p at $t = 0$, such that

$$v(g) = \left.\frac{d}{dt}\right|_{t=0} g(\gamma(t))$$

for any $g \in C^\infty(M)$ (i.e., $v = \dot\gamma(0)$ in the notation of (5.6)). Then $F \circ \gamma: J \to N$ is a smooth curve passing through $F(p) = q$ at $t = 0$. By definition, for any $h \in C^\infty(N)$,

$$(T_pF(v))(h) = v(h \circ F) = \left.\frac{d}{dt}\right|_{t=0} (h \circ F)(\gamma(t)) = \left.\frac{d}{dt}\right|_{t=0} h(F \circ \gamma(t)).$$

That is, $T_pF(v)$ is represented by the curve $F \circ \gamma$. □

Remark 5.13 (Pullbacks, Push-Forwards). For smooth maps $F \in C^\infty(M,N)$, one can consider various "pull-backs" of objects on N to objects on M and "push-forwards" of objects on M to objects on N. Pullbacks are generally denoted by F^* and push-forwards by F_*. For example, functions on N can be pulled back to functions on M:

$$g \in C^\infty(N) \;\rightsquigarrow\; F^*g = g \circ F \in C^\infty(M).$$

Curves on M can be pushed forward to curves on N:

$$\gamma\colon J \to M \quad \rightsquigarrow \quad F_*\gamma = F \circ \gamma\colon J \to N.$$

Tangent vectors to M can also be pushed forward to tangent vectors to N:

$$v \in T_pM \quad \rightsquigarrow \quad F_*(v) = (T_pF)(v).$$

In these terms, the defining equation of the tangent map reads

$$(F_*v)(g) = v(F^*g)$$

and Proposition 5.12 states that if v is represented by the curve γ, then F_*v is represented by the curve $F_*\gamma$.

Proposition 5.14 (Chain Rule). *Let M,N,Q be manifolds and $F \in C^\infty(M,N)$ and $G \in C^\infty(N,Q)$ smooth maps. Then,*

$$T_p(G \circ F) = T_{F(p)}G \circ T_pF.$$

Proof. Observe that $T_p(G \circ F)$ is determined by its action on tangent vectors $v \in T_pM$, and the resulting tangent vector $(T_p(G \circ F))(v) \in T_{G(F(p))}Q$ is determined by its action on functions $g \in C^\infty(Q)$. We have, using the definitions,

$$\begin{aligned}
\big(T_p(G\circ F)(v)\big)(g) &= v(g \circ (G \circ F)) \\
&= v((g \circ G) \circ F) \\
&= \big((T_pF)(v)\big)(g \circ G) \\
&= \Big((T_{F(p)}G)\big((T_pF)(v)\big)\Big)(g) \\
&= \big((T_{F(p)}G \circ T_pF)(v)\big)(g).
\end{aligned}$$

□

73 (answer on page 290). Give a second proof of Proposition 5.14, using the characterization of tangent vectors as velocity vectors of curves (cf. Proposition 5.12).

74 (answer on page 290).

(a) Show that the tangent map of the identity map $\mathrm{id}_M : M \to M$ at $p \in M$ is the identity map on the tangent space:

$$T_p\,\mathrm{id}_M = \mathrm{id}_{T_pM}.$$

(b) Show that if $F \in C^\infty(M,N)$ is a diffeomorphism, then T_pF is a linear isomorphism, with inverse

$$(T_pF)^{-1} = (T_{F(p)}F^{-1}).$$

(c) Suppose that $F \in C^\infty(M,N)$ is a *constant map*, that is, $F(M) = \{q\}$ for some element $q \in N$. Show that T_pF is the zero map, for all $p \in M$.

5.2.2 Coordinate Description of the Tangent Map

To get a better understanding of the tangent map, let us first consider the special case where $F \in C^\infty(U,V)$ is a smooth map between open subsets of Euclidean spaces $U \subseteq \mathbb{R}^m$ and $V \subseteq \mathbb{R}^n$. For $p \in U$, the tangent space T_pU is canonically identified with $\mathbb{R}^m$, using the basis

$$\frac{\partial}{\partial x^1}\bigg|_p, \dots, \frac{\partial}{\partial x^m}\bigg|_p \in T_pU$$

of the tangent space (cf. Remark 5.5). Similarly, $T_{F(p)}V \cong \mathbb{R}^n$, using the basis given by partial derivatives $\frac{\partial}{\partial y^j}\Big|_{F(p)}$. Using these identifications, the tangent map becomes a linear map $T_pF : \mathbb{R}^m \to \mathbb{R}^n$, i.e., it is given by an $(n \times m)$-matrix. This matrix is exactly the Jacobian.

Proposition 5.15. *Let $F \in C^\infty(U,V)$ be a smooth map between open subsets $U \subseteq \mathbb{R}^m$ and $V \subseteq \mathbb{R}^n$. For all $p \in M$, the tangent map T_pF coincides with the derivative (i.e., Jacobian matrix) D_pF of F at p.*

Proof. For $g \in C^\infty(V)$, we calculate

$$\begin{aligned}\Big((T_pF)\Big(\frac{\partial}{\partial x^i}\bigg|_p\Big)\Big)(g) &= \frac{\partial}{\partial x^i}\bigg|_p (g \circ F)\\ &= \sum_{j=1}^n \frac{\partial g}{\partial y^j}\bigg|_{F(p)} \frac{\partial F^j}{\partial x^i}\bigg|_p\\ &= \Big(\sum_{j=1}^n \frac{\partial F^j}{\partial x^i}\bigg|_p \frac{\partial}{\partial y^j}\bigg|_{F(p)}\Big)(g).\end{aligned}$$

This shows

$$(T_pF)\Big(\frac{\partial}{\partial x^i}\bigg|_p\Big) = \sum_{j=1}^n \frac{\partial F^j}{\partial x^i}\bigg|_p \frac{\partial}{\partial y^j}\bigg|_{F(p)}.$$

Hence, in terms of the given bases of T_pU and $T_{F(p)}V$, the matrix of the linear map T_pF has entries $\frac{\partial F^j}{\partial x^i}\Big|_p$. □

Remark 5.16. For $F \in C^\infty(U,V)$, it is common to write $y = F(x)$ and accordingly write $\big(\frac{\partial y^j}{\partial x^i}\big)$ for the entries of the Jacobian matrix. In these terms, the derivative reads as

$$T_pF\Big(\frac{\partial}{\partial x^i}\bigg|_p\Big) = \sum_j \frac{\partial y^j}{\partial x^i}\bigg|_p \frac{\partial}{\partial y^j}\bigg|_{F(p)}.$$

This suggestive formula is often used for explicit calculations.

75 (answer on page 291). Consider $\mathbb{R}^2$ with standard coordinates x, y. On the open subset $\mathbb{R}^2 \setminus \{(x,0) :\ x \leq 0\}$ introduce *polar coordinates* r, θ by

$$x = r\cos\theta \ , \ y = r\sin\theta;$$

here $0 < r < \infty$ and $-\pi < \theta < \pi$. Express the tangent vectors

$$\left.\frac{\partial}{\partial r}\right|_p \ , \ \left.\frac{\partial}{\partial \theta}\right|_p$$

as a combination of the tangent vectors

$$\left.\frac{\partial}{\partial x}\right|_p \ , \ \left.\frac{\partial}{\partial y}\right|_p .$$

For a general smooth map $F \in C^\infty(M,N)$, we obtain a similar description once we pick coordinate charts. Given $p \in M$, choose charts (U,φ) around p and (V,ψ) around $F(p)$, with $F(U) \subseteq V$. Let $\widetilde{U} = \varphi(U)$ and $\widetilde{V} = \psi(V)$ and put

$$\widetilde{F} = \psi \circ F \circ \varphi^{-1} :\ \widetilde{U} \to \widetilde{V}.$$

Since the coordinate map $\varphi :\ U \to \mathbb{R}^m$ is a diffeomorphism onto $\widetilde{U}$, it gives an isomorphism (cf. 74)

$$T_p\varphi :\ T_pU \to T_{\varphi(p)}\widetilde{U} = \mathbb{R}^m.$$

Similarly, $T_{F(p)}\psi$ gives an isomorphism of $T_{F(p)}V$ with $\mathbb{R}^n$. Note also that since $U \subseteq M,\ V \subseteq N$ are open, we have that $T_pU = T_pM,\ T_{F(p)}V = T_{F(p)}N$. We obtain

$$T_{\varphi(p)}\widetilde{F} = T_{F(p)}\psi \circ T_pF \circ (T_p\varphi)^{-1},$$

which may be depicted in a commutative diagram

$$\begin{tikzcd}
\mathbb{R}^m \arrow[rr, "D_{\varphi(p)}\widetilde{F}"] & & \mathbb{R}^n \\
T_pM = T_pU \arrow[u, "T_p\varphi", "\cong"'] \arrow[rr, "T_pF"'] & & T_{F(p)}V = T_{F(p)}N \arrow[u, "\cong", "T_{F(p)}\psi"']
\end{tikzcd}$$

Thus, the choice of coordinates identifies the tangent spaces T_pM and $T_{F(p)}N$ with $\mathbb{R}^m$ and $\mathbb{R}^n$, respectively, and the tangent map T_pF with the derivative of the coordinate expression of F (equivalently, the Jacobian matrix).

Now that we have recognized T_pF as the derivative expressed in a coordinate-free way, we may liberate some of our earlier definitions from coordinates.

Definition 5.17. *Let* $F \in C^\infty(M,N)$.

- *The* rank of F at $p \in M$, *denoted* $\mathrm{rank}_p(F)$, *is the rank of the linear map* T_pF.
- F *has* maximal rank at p *if* $\mathrm{rank}_p(F) = \min(\dim M, \dim N)$.
- F *is a* submersion *if* T_pF *is surjective for all* $p \in M$.
- F *is an* immersion *if* T_pF *is injective for all* $p \in M$.
- F *is a* local diffeomorphism *if* T_pF *is an isomorphism for all* $p \in M$.
- $p \in M$ *is a* critical point *of* F *if* T_pF *does not have maximal rank at* p.
- $q \in N$ *is a* regular value *of* F *if* T_pF *is surjective for all* $p \in F^{-1}(q)$ *(in particular, if* $q \notin F(M)$*).*
- $q \in N$ *is a* singular value *(sometimes called* critical value*) if it is not a regular value.*

76 (answer on page 291). To illustrate the merits of the coordinate-free definitions, give simple proofs of the facts that the composition of two submersions is again a submersion and that the composition of two immersions is an immersion.

5.2.3 Tangent Spaces of Submanifolds

Suppose $S \subseteq M$ is a submanifold, and $p \in S$. Then the tangent space T_pS is canonically identified as a subspace of T_pM. Indeed, since the inclusion $i: S \hookrightarrow M$ is an immersion, the tangent map is an injective linear map,

$$T_pi: T_pS \to T_pM,$$

and we identify T_pS with the subspace given as the image of this map[*]. As a special case, we see that whenever M is realized as a submanifold of $\mathbb{R}^n$, then its tangent spaces T_pM may be viewed as subspaces of $T_p\mathbb{R}^n = \mathbb{R}^n$.

Proposition 5.18. *Let* $F \in C^\infty(M,N)$ *be a smooth map, and let* $S = F^{-1}(q)$ *be a submanifold given as the fiber of some regular value* $q \in N$. *For all* $p \in S$,

$$T_pS = \ker(T_pF),$$

as subspaces of T_pM.

Proof. Let $m = \dim M$ and $n = \dim N$. Since T_pF is surjective, its kernel has dimension $m - n$. By the regular value theorem, this is also the dimension of S, hence of T_pS. It is therefore enough to show that $T_pS \subseteq \ker(T_pF)$. Letting $i: S \to M$ be the inclusion, we have to show that

$$T_pF \circ T_pi = T_p(F \circ i)$$

[*] Hopefully, the identifications are not getting too confusing: S gets identified with $i(S) \subseteq M$, hence also $p \in S$ with its image $i(p)$ in M, and T_pS gets identified with $(T_pi)(T_pS) \subseteq T_pM$.

is the zero map. But $F \circ i$ is a *constant map*, taking all points of S to the constant value $q \in N$. The tangent map to a constant map is just zero (✐ 74). Hence $T_p(F \circ i) = 0$. $\square$

As a special case, we can apply this result to smooth maps between open subsets of Euclidean spaces, where the tangent maps are directly given by the derivative (Jacobian matrix). Thus, suppose $V \subseteq \mathbb{R}^n$ is open, and $q \in \mathbb{R}^k$ is a regular value of $F \in C^\infty(V, \mathbb{R}^k)$, defining an embedded submanifold $M = F^{-1}(q)$. Then the tangent spaces $T_pM \subseteq T_p\mathbb{R}^n = \mathbb{R}^n$ are given as

$$T_pM = \ker(T_pF) = \ker(D_pF). \tag{5.7}$$

Example 5.19. For any hypersurface $S \subseteq V \subseteq \mathbb{R}^{n+1}$ given as the level set $F^{-1}(c)$ of a function $F:\ V \to \mathbb{R}$ having c as a regular value, we have that

$$(D_pF)(\mathbf{a}) = \frac{d}{dt}\Big|_{t=0} F(p+t\mathbf{a}) = \nabla_p\, F \cdot \mathbf{a}$$

for all $p \in F^{-1}(c)$, and $\mathbf{a} \in T_pV = \mathbb{R}^{n+1}$. Here $\nabla_p\, F \in \mathbb{R}^{n+1}$ is the gradient of F at p (written as a column vector; it is really just the transpose to the $(1 \times (n+1))$-matrix D_pF). Hence,

$$T_pS = \mathrm{span}(\nabla_p\, F)^\perp.$$

For the sphere S^n, given as the regular level set $F^{-1}(1)$ of the function $F:\ \mathbf{x} \mapsto ||\mathbf{x}||^2$, we have $\nabla_p\, F = 2p$; hence, $T_pS^n = \mathrm{span}(p)^\perp$, which recovers the description of T_pS^n from the beginning of the chapter.

As another typical application, suppose that $S \subseteq M$ is a submanifold, and $f \in C^\infty(S)$ is a smooth function given as the restriction $f = h|_S$ of a smooth function $h \in C^\infty(M)$. Consider the problem of finding the critical points $p \in S$ of f. Since f is a scalar function, T_pf fails to have maximal rank if and only if it is zero:

$$\mathrm{Crit}(f) = \{p \in S|\ T_pf = 0\}.$$

Letting $i\colon\ S \to M$ be the inclusion, we have $f = h|_S = h \circ i$; hence, $T_pf = T_ph \circ T_pi$. It follows that $T_pf = 0$ if and only if T_ph vanishes on the range of T_pi, that is on T_pS:

$$\mathrm{Crit}(f) = \{p \in S|\ T_pS \subseteq \ker(T_ph)\}.$$

If $M = \mathbb{R}^m$, then T_ph is just the Jacobian D_ph, whose kernel is sometimes relatively easy to compute—in any case, this approach tends to be faster than a calculation in charts for S.

Example 5.20. Let $S \subseteq \mathbb{R}^3$ be a surface, and $f \in C^\infty(S)$ the "height function" given by $f(x,y,z) = z$. To find $\mathrm{Crit}(f)$, regard f as the restriction of $h \in C^\infty(\mathbb{R}^3),\ h(x,y,z) = z$. The tangent map is

$$T_ph = D_ph = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}$$

as a linear map $\mathbb{R}^3 \to \mathbb{R}$. Hence, $\ker(T_ph)$ is the xy-plane. On the other hand, T_pS for $p \in S$ is 2-dimensional; hence, the condition $T_pS \subseteq \ker(T_ph)$ is equivalent to $T_pS = \ker(T_ph)$. We conclude that the critical points of f are exactly those points of S where the tangent plane is "horizontal," i.e., equal to the xy-plane.

77 (answer on page 291). Consider $f = h|_{S^2}$, where

$$h:\ \mathbb{R}^3 \to \mathbb{R},\ (x,y,z) \mapsto xy.$$

(a) Find $T_p h = D_p h:\ \mathbb{R}^3 \to \mathbb{R}$, and compute its kernel $\ker(T_p h)$.
(b) Find $\mathrm{Crit}(f)$ as the set of all $p \in S^2$ such that $T_p S^2 \subseteq \ker(T_p h)$. How many critical points are there?

Example 5.21. We have discussed various *matrix Lie groups* G as examples of manifolds (cf. Example 4.30). By definition, these are submanifolds $G \subseteq \mathrm{Mat}_{\mathbb{R}}(n)$ (for some $n \in \mathbb{N}$), consisting of invertible matrices with the properties

$$I \in G;\ \ A,B \in G \Rightarrow AB \in G;\ \ A \in G \Rightarrow A^{-1} \in G.$$

The tangent space to the identity (group unit) for such matrix Lie groups G turns out to be important; it is commonly written in lower case Fraktur font:

$$\mathfrak{g} = T_I G \subseteq \mathrm{Mat}_{\mathbb{R}}(n).$$

One calls $\mathfrak{g}$ the *Lie algebra* of G. It is a key fact (see 79) that $\mathfrak{g}$ is closed under commutation of matrices:

$$X_1, X_2 \in \mathfrak{g} \ \ \Rightarrow \ \ [X_1, X_2] = X_1 X_2 - X_2 X_1 \in \mathfrak{g}.$$

A vector subspace $\mathfrak{g} \subseteq \mathrm{Mat}_{\mathbb{R}}(n)$ that is closed under matrix commutators is called a *matrix Lie algebra*.

Some concrete examples:

(a) The matrix Lie group

$$\mathrm{GL}(n,\mathbb{R}) = \{A \in \mathrm{Mat}_{\mathbb{R}}(n) |\ \det(A) \neq 0\}$$

of *all* invertible matrices is an open subset of $\mathrm{Mat}_{\mathbb{R}}(n)$; hence,

$$\mathfrak{gl}(n,\mathbb{R}) = \mathrm{Mat}_{\mathbb{R}}(n)$$

is the entire space of matrices.

(b) For the group $\mathrm{O}(n)$, consisting of matrices with $F(A) := A^\top A = I$, we found in Example 4.30 that $T_A F(X) = X^\top A + A^\top X$. In particular, $T_I F(X) = X^\top + X$. We read off the Lie algebra $\mathfrak{o}(n)$ as the kernel of this map:

$$\mathfrak{o}(n) = \{X \in \mathrm{Mat}_{\mathbb{R}}(n) |\ X^\top = -X\}.$$

78 (answer on page 292). Show that for every $X \in \mathrm{Mat}_{\mathbb{R}}(n)$,

$$\frac{d}{dt}\bigg|_{t=0} \det(I+tX) = \mathrm{tr}(X).$$

Use this result to compute the Lie algebra $\mathfrak{sl}(n,\mathbb{R})$ of the special linear group

$$\mathrm{SL}(n,\mathbb{R}) = \{A \in \mathrm{Mat}_{\mathbb{R}}(n) |\ \det(A) = 1\}.$$

The following 79 develops some important properties of matrix Lie groups.

79 (answer on page 293).

(a) Show (using, for example, the curves definition of the tangent space) that the tangent space at general elements $A \in G$ can be described by left translation

$$T_A G = \{AX |\ X \in \mathfrak{g}\}$$

or also by right translation $T_A G = \{XA |\ X \in \mathfrak{g}\}$.

(b) Show that $A \in G,\ \ X \in \mathfrak{g} \ \ \Rightarrow \ \ AXA^{-1} \in \mathfrak{g}$.

(c) Show that $X, Y \in \mathfrak{g} \ \ \Rightarrow \ \ XY - YX \in \mathfrak{g}$. (Hint: Choose a curve $\gamma(t)$ in G representing Y.)

5.2.4 Example: Steiner's Surface Revisited*

As we discussed in Section 4.3.3, Steiner's "Roman surface" is the image of the map

$$\mathbb{R}\mathrm{P}^2 \to \mathbb{R}^3,\ \ (x:y:z) \mapsto \frac{1}{x^2+y^2+z^2}(yz,\ xz,\ xy). \tag{5.8}$$

(We changed notation from α, β, γ to x, y, z.) Recall that if $p \in \mathbb{R}\mathrm{P}^2$ is not a critical point, then this map restricts to an immersion on an open neighborhood of p. What are the critical points of this map? To investigate this question, one can express the map in local charts and compute the resulting Jacobian matrix. While this approach is perfectly fine, the resulting expressions will become rather complicated.

A simpler approach is to consider the composition with the local diffeomorphism $\pi:\ S^2 \to \mathbb{R}\mathrm{P}^2$, given as

$$S^2 \to \mathbb{R}^3,\ \ (x,y,z) \mapsto (yz,\ xz,\ xy). \tag{5.9}$$

Since π is a surjective local diffeomorphism, the critical points of (5.8) are the images of the critical points of (5.9). In turn, this map is the restriction $F|_{S^2}$ of the map

$$F:\ \mathbb{R}^3 \to \mathbb{R}^3,\ \ (x,y,z) \mapsto (yz,\ xz,\ xy). \tag{5.10}$$

We have $T_p(F|_{S^2}) = T_pF|_{T_pS^2}$; hence, $\ker(T_p(F|_{S^2})) = \ker(T_pF) \cap T_pS^2$. We are interested in the points $p \in S^2$ where this intersection is non-zero.

80 (answer on page 293).

(a) Compute $T_pF = D_pF$, and find its determinant. Conclude that the kernel is empty except when one of the coordinates is 0.

(b) Suppose $p = (x,y,z)$ with $x = 0$. Find the kernel of the tangent map at p, and $\ker(T_pF) \cap T_pS$. Repeat with the cases $y = 0$ and $z = 0$.

(c) What are the critical points of the map (5.8)?

If you have completed 80, you have found that the map (5.8) has 6 critical points. It is thus an immersion away from those points.

5.3 Problems

1. Let $v: C^\infty(\mathbb{R}) \to \mathbb{R}$ be a linear map of the form
$$v(f) = \sum_{i=0}^{N} a_i f^{(i)}(0),$$
where $f^{(i)}$ stands for the i-th derivative. Find a formula for $v(fg)$, and use it to verify that v satisfies the product rule $v(fg) = v(f)g(0) + f(0)v(g)$ if and only if $a_i = 0$ for all $i \neq 1$.

2. Show that the tangent space T_pM from Definition 5.4 is independent of the choice of coordinate charts, by direct computation (without using any of the other equivalent definitions).

3. For
$$f: \mathbb{R}^2 \to \mathbb{R}, \quad f(x,y) = y^2 + x^3 + ax + b$$
describe the set
$$\Delta = \{(a,b) \mid 0 \text{ is a critical value of } f\}.$$
Sketch typical level sets $f^{-1}(0)$ for parameters $(a,b) \notin \Delta$.

4. Show that the equations
$$x^2 + y = 0\ ,\ x^2 + y^2 + z^3 + w^4 + y = 1$$
define a two-dimensional submanifold S of $\mathbb{R}^4$, and find the equation of the tangent space at the point $(x_0, y_0, z_0, w_0) = (-1,-1,-1,-1)$.

5. Consider the function
$$f: \mathbb{R}^3 \backslash \{\mathbf{0}\} \to \mathbb{R},\ (x,y,z) \to x^2 + y^2 - z^2.$$

(a) Show that all values $a \in \mathbb{R}$ are regular values of f.
(b) Sketch the level sets $f^{-1}(a)$ for typical values of a. (Be sure to include values $a > 0$, $a = 0$, $a < 0$.)
(c) Let

$$X = (\mathbb{R}^3 \backslash \{\mathbf{0}\}) / \sim$$

be the quotient space, where $p \sim p'$ if and only if there exists $a \in \mathbb{R}$ with $f(p) = f(p') = 1$, and such that p, p' are in the same component of $f^{-1}(a)$. Show that X is naturally a non-Hausdorff manifold. What is it?

6. Let $S^2 \subseteq \mathbb{R}^3$ be the 2-sphere, and define

$$TS^2 = \{(p,v) \in \mathbb{R}^3 \times \mathbb{R}^3 |\ p \in S^2,\ v \in T_pS^2\}.$$

It may be regarded as a level set of the function

$$F: \mathbb{R}^6 \to \mathbb{R}^2,\ F(p,v) = (p \cdot p,\ p \cdot v).$$

(a) Find the differential $(a,w) \mapsto (T_{(p,v)}F)(a,w)$.
(b) Show that TS^2 is a 4-dimensional submanifold of $\mathbb{R}^6$.
(c) Show similarly that the set of unit tangent vectors

$$M = \{(p,v) \in TS^2 |\ v \cdot v = 1\}$$

is a 3-dimensional submanifold of $\mathbb{R}^6$.
(d) Show each of the two maps $H_i: M \to S^2$ given by

$$H_1(p,v) = p,\ H_2(p,v) = v$$

is a surjective submersion $M \to S^2$, with fibers diffeomorphic to S^1.

7. An *orientation* on a finite-dimensional vector space V may be defined as an equivalence class of bases, where two bases are equivalent if the change-of-basis matrix has positive determinant. Show that if M is an oriented manifold, then every tangent space T_pM acquires an orientation in this sense.

8. Write $(2n \times 2n)$-matrices in block form

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Mat}_{\mathbb{R}}(2n),$$

where each block a, b, c, d is an $(n \times n)$-matrix. Define a special matrix

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

(where 1 stands for the $n \times n$ identity matrix), and let

$$\mathrm{Sp}(2n) = \{A \in \mathrm{Mat}_{\mathbb{R}}(2n):\ AJA^{\top} = J\}.$$

That is, $\mathrm{Sp}(2n) = F^{-1}(J)$, where

$$F:\ \mathrm{Mat}_{\mathbb{R}}(2n) \to \mathrm{Skew}_{\mathbb{R}}(2n)$$

is the map $F(A) = AJA^{\top}$. Here we denote by $\mathrm{Skew}_{\mathbb{R}}(n)$ the set of real $n \times n$ skew-symmetric matrices, that is, matrices C such that $C^{\top} = -C$.

(a) Find the differential $D_A F:\ \mathrm{Mat}_{\mathbb{R}}(2n) \to \mathrm{Skew}_{\mathbb{R}}(2n)$.
(b) Show that for $A \in F^{-1}(J)$, the differential is surjective, so that $F^{-1}(J)$ is a submanifold. What is its dimension?
(c) Show that $\mathrm{Sp}(2n)$ is a matrix Lie group, and describe its Lie algebra $\mathfrak{sp}(2n)$.

9. Let $\mathrm{U}(n) \subseteq \mathrm{Mat}_{\mathbb{C}}(n)$ be the complex $(n \times n)$-matrices A with the property $A^{\dagger} = A^{-1}$, where $A^{\dagger}$ is the Hermitian adjoint (conjugate transpose).
   (a) Show that $\mathrm{U}(n)$ is a matrix Lie group, and describe its Lie algebra $\mathfrak{u}(n)$.
   (b) Let $\mathrm{SU}(n) = \{A \in \mathrm{U}(n) |\ \det(A) = 1\}$. Show that $\mathrm{SU}(n)$ is a matrix Lie group, and describe its Lie algebra $\mathfrak{su}(n)$.

10. (a) Show that the matrix Lie group $\mathrm{SU}(2)$, as a manifold, is diffeomorphic to S^3.
   (b) Identify $\mathbb{R}^3$ with the space S of skew-adjoint complex (2×2)-matrices of trace 0, by the map

$$(x,y,z) \mapsto Z = \begin{pmatrix} iz & x+iy \\ -x+iy & -iz \end{pmatrix}.$$

   Show that for $A \in \mathrm{SU}(2)$, the map $Z \mapsto AZA^{-1}$ preserves the dot product in $\mathbb{R}^3$.
   (c) Prove that the map $q:\ \mathrm{SU}(2) \to \mathrm{SO}(3)$ taking $A \in \mathrm{SU}(2)$ to the map $f:\ \mathbb{R}^3 \to \mathbb{R}^3$ given by $f(Z) = AZA^{-1}$ (for any skew-adjoint (2×2)-matrix) is a group homomorphism, and that it is surjective.
   (d) Show that q is the quotient map for the equivalence relation $A \sim -A$ on $\mathrm{SU}(2)$, and use this to show that $\mathrm{SO}(3)$ is diffeomorphic to $\mathbb{RP}^3$ (cf. ✎ 15).

6

Vector Fields

6.1 Vector Fields as Derivations

A vector field on a manifold may be regarded as a family of tangent vectors $X_p \in T_pM$ for $p \in M$, depending smoothly on the base points $p \in M$. One way of making precise what is meant by “depending smoothly” is the following.

Definition 6.1 (Vector Fields—First Definition). *A collection of tangent vectors $X = \{X_p\}$ depending on $p \in M$ defines a vector field if and only if for all functions $f \in C^\infty(M)$ the function $p \mapsto X_p(f)$ is smooth. The space of all vector fields on M is denoted $\mathfrak{X}(M)$.*

81 (answer on page 295). Verify the (implicit) claim that the set $\mathfrak{X}(M)$ of all vector fields on M is a vector space.

We hence obtain a linear map, denoted by the same letter,

$$X:\ C^\infty(M) \to C^\infty(M)$$

such that

$$\big(X(f)\big)(p) = X_p(f). \tag{6.1}$$

Since each individual tangent vector X_p satisfies a product rule (5.3), it follows that X itself satisfies a product rule. We can use this as an alternative definition, realizing $\mathfrak{X}(M)$ as a subspace of the space $\mathrm{Hom}(C^\infty(M), C^\infty(M))$.

Definition 6.2 (Vector Fields—Second Definition). *A vector field on M is a linear map*

$$X:\ C^\infty(M) \to C^\infty(M)$$

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satisfying the product rule,

$$X(fg) = X(f)g + fX(g) \tag{6.2}$$

for $f, g \in C^\infty(M)$.

82 (answer on page 295). Explain in more detail why these two definitions of vector fields are equivalent.

Remark 6.3. The condition (6.2) says that X is a *derivation* of the algebra $C^\infty(M)$ of smooth functions. More generally, a derivation of an algebra $\mathscr{A}$ is a linear map $D: \mathscr{A} \to \mathscr{A}$ such that for any $a_1, a_2 \in \mathscr{A}$

$$D(a_1 a_2) = D(a_1)\, a_2 + a_1 D(a_2).$$

(See Appendix B.2.3 for some facts about derivations.)

Vector fields can be multiplied by smooth functions:

$$C^\infty(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M), \quad (h, X) \mapsto hX,$$

where $(hX)(f) = hX(f)$. In algebraic terminology, this makes the space of vector fields into a *module* over the algebra of smooth functions (see Appendix B.2.4).

83 (answer on page 295). Explain why hX is again a vector field.

We can also express the smoothness of the collection of tangent vectors $\{X_p\}$ in terms of coordinate charts (U, φ). Recall that for any $p \in U$, and all $f \in C^\infty(M)$, the tangent vector X_p is expressed as

$$X_p(f) = \sum_{i=1}^m a^i \left.\frac{\partial}{\partial u^i}\right|_{\mathbf{u}=\varphi(p)} (f \circ \varphi^{-1}).$$

The vector $\mathbf{a} = (a^1, \ldots, a^m) \in \mathbb{R}^m$ represents X_p in the chart, i.e., $(T_p\varphi)(X_p) = \mathbf{a}$ under the identification $T_{\varphi(p)}\varphi(U) = \mathbb{R}^m$. As p varies in U, the vector $\mathbf{a}$ becomes a function of $p \in U$, or equivalently of $\mathbf{u} = \varphi(p)$.

Proposition 6.4. *The collection of tangent vectors* $\{X_p \in T_pM,\ p \in M\}$ *defines a vector field* X *such that* $X(f)|_p = X_p(f)$, *if and only if for all charts* (U, φ), *the functions* $a^i: \varphi(U) \to \mathbb{R}$ *defined by*

$$X_{\varphi^{-1}(\mathbf{u})}(f) = \sum_{i=1}^m a^i(\mathbf{u}) \left.\frac{\partial}{\partial u^i}\right|_{\mathbf{u}} (f \circ \varphi^{-1})$$

are smooth.

Proof. "$\Leftarrow$." Suppose the coefficient functions a^i are smooth for all charts (U,φ). Then it follows that for all $f \in C^\infty(M)$, the function

$$X(f)\circ\varphi^{-1}:\ \varphi(U)\to\mathbb{R},\ \ \mathbf{u}\mapsto X_{\varphi^{-1}(\mathbf{u})}(f)$$

is smooth. Consequently, $X(f)|_U$ is smooth, and since this is true for all charts, $X(f)$ is smooth.

"$\Rightarrow$." Let X be a vector field, defining a collection of tangent vectors X_p. Given a chart (U,φ), we want to show that the coefficient functions a^i are smooth near any given point $\widetilde{p}=\varphi(p)\in\widetilde{U}=\varphi(U)$. Let $g\in C^\infty(U)$ be a function whose local coordinate expression $\widetilde{g}=g\circ\varphi^{-1}$ is the coordinate function $\mathbf{u}\mapsto u^i$. This function may not directly extend from U to M, but we may choose $f\in C^\infty(M)$ such that

$$f|_{U_1}=g|_{U_1}$$

over a possibly smaller neighborhood $U_1\subseteq U$ of p (see 84 below). Then $f\circ\varphi^{-1}$ coincides with u^i on $\widetilde{U}_1=\varphi(U_1)$, and hence, $X(f)\circ\varphi^{-1}=a^i$ on $\widetilde{U}_1$. In particular, the a^i are smooth on $\widetilde{U}_1$. $\square$

84 (answer on page 295). Explain how to construct the function f in the second part of the proof.

In particular, we see that vector fields on open subsets $U\subseteq\mathbb{R}^m$ are of the form

$$X=\sum_i a^i\frac{\partial}{\partial x^i},$$

where $a^i\in C^\infty(U)$. Under a diffeomorphism $F:\ U\to V$, $x\mapsto y=F(x)$, the coordinate vector fields transform with the Jacobian

$$TF\left(\frac{\partial}{\partial x^i}\right)=\sum_j\left.\frac{\partial F^j}{\partial x^i}\right|_{x=F^{-1}(y)}\frac{\partial}{\partial y^j}.$$

See Proposition 5.15; as in the remark following that proposition, this "change of coordinates" is often written

$$\frac{\partial}{\partial x^i}=\sum_j\frac{\partial y^j}{\partial x^i}\frac{\partial}{\partial y^j}.$$

Here one thinks of the x^i and y^j as coordinates on the same set (U and V are "identified" via F), and one uses the simplified notation $y=y(x)$ instead of $y=F(x)$.

85 (answer on page 295). Consider $\mathbb{R}^3$ with coordinates x,y,z. Introduce new coordinates u,v,w by setting

$$x=e^u v,\ \ y=e^v,\ \ z=uv^2w$$

valid on the region where $x\geq y>1$.

(a) Express u, v, w in terms of x, y, z.
(b) Express the coordinate vector fields $\frac{\partial}{\partial u}, \frac{\partial}{\partial v}, \frac{\partial}{\partial w}$ as a combination of the coordinate vector fields $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$ with coefficients that are functions of x, y, z.

6.2 Lie Brackets

Let M be a manifold. Given vector fields $X, Y : C^\infty(M) \to C^\infty(M)$, the composition $X \circ Y$ is not a vector field. For example, if $X = Y = \frac{\partial}{\partial x}$ as vector fields on $\mathbb{R}$, then $X \circ Y = \frac{\partial^2}{\partial x^2}$ is a second order derivative, which is not a vector field (it does not satisfy the Leibniz rule). However, the commutator turns out to be a vector field.

Theorem 6.5. *For any two vector fields $X, Y \in \mathfrak{X}(M)$ (regarded as derivations, as in Definition 6.2), the commutator*

$$[X,Y] := X \circ Y - Y \circ X : C^\infty(M) \to C^\infty(M)$$

is again a vector field.

Proof. We need to show that $[X,Y]$ is a derivation of the algebra $C^\infty(M)$. Let $f, g \in C^\infty(M)$. Then

$$\begin{aligned}(X \circ Y)(fg) &= X\big(Y(f)g + fY(g)\big) \\ &= (X \circ Y)(f)\, g + f\, (X \circ Y)(g) + Y(f)X(g) + X(f)Y(g).\end{aligned}$$

Similarly,

$$(Y \circ X)(fg) = (Y \circ X)(f)\, g + f\, (Y \circ X)(g) + X(f)Y(g) + Y(f)X(g).$$

Subtracting the latter equation from the former, several terms cancel and we obtain

$$[X,Y](fg) = [X,Y](f)\, g + f\, [X,Y](g)$$

as desired. □

Remark 6.6. A similar calculation applies to derivations of algebras in general: The commutator of two derivations is again a derivation. (The bracket $[\cdot,\cdot]$ is a common notation for the commutator in an algebra, see Appendix B.2.3.)

Definition 6.7. *The vector field*

$$[X,Y] := X \circ Y - Y \circ X$$

is called the Lie bracket *of $X, Y \in \mathfrak{X}(M)$. If $[X,Y] = 0$, we say that the vector fields X, Y* commute.

The Lie bracket is named after Sophus Lie (1842–1899); the space of vector fields with its Lie bracket is a special case of a *Lie algebra*.

Note that the definition immediately implies that the Lie bracket is *anti-commutative* (or *skew-symmetric*):

$$[X,Y] = -[Y,X].$$

It is instructive to see how the Lie bracket of vector fields works out in local coordinates. For open subsets $U \subseteq \mathbb{R}^m$, if

$$X = \sum_{i=1}^{m} a^i \frac{\partial}{\partial x^i}, \quad Y = \sum_{i=1}^{m} b^i \frac{\partial}{\partial x^i},$$

with coefficient functions $a^i, b^i \in C^\infty(U)$, the composition $X \circ Y$ is a second order differential operator, calculated by the product rule:

$$X \circ Y = \sum_{i=1}^{m} \sum_{j=1}^{m} a^j \frac{\partial b^i}{\partial x^j} \frac{\partial}{\partial x^i} + \sum_{i=1}^{m} \sum_{j=1}^{m} a^i b^j \frac{\partial^2}{\partial x^i \partial x^j}$$

(an equality of operators acting on $C^\infty(U)$). Subtracting a similar expression for $Y \circ X$, the terms involving second derivatives cancel, and we obtain

$$[X,Y] = \sum_{i=1}^{m} \sum_{j=1}^{m} \left(a^j \frac{\partial b^i}{\partial x^j} - b^j \frac{\partial a^i}{\partial x^j} \right) \frac{\partial}{\partial x^i}.$$

This calculation applies to general manifolds, by taking local coordinates.

Note: When calculating Lie brackets $X \circ Y - Y \circ X$ of vector fields X, Y in local coordinates, it is not necessary to work out the second order derivatives—we know in advance that these are going to cancel out!

The following fact is often used in calculations.

86 (answer on page 296). Let $X, Y \in \mathfrak{X}(M)$ and $f, g \in C^\infty(M)$. Show that

$$[X, fY] = (Xf)Y + f[X,Y].$$

Derive a similar formula for $[fX, gY]$.

The geometric significance of the Lie bracket will become clear later. At this stage, let us just note that since the Lie bracket of two vector fields is defined in a coordinate-free way, its vanishing or non-vanishing does not depend on choices of coordinates. For example, since the Lie brackets between coordinate vector fields vanish,

$$\left[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right] = 0, \quad 0 \le i < j \le m,$$

two non-zero vector fields X,Y can only possibly be coordinate vector fields in suitable coordinates if their Lie bracket is zero, i.e., if X,Y commute. It is natural to ask whether this necessary condition is also sufficient. If the vector fields are linearly independent, we shall see below (Lemma 6.48) that the answer is yes, at least locally.

Example 6.8. Consider the vector fields on $\mathbb{R}^2$ given by

$$X = \frac{\partial}{\partial x}, \;\; Y = (1+x^2)\frac{\partial}{\partial y}.$$

Does there exist a change of coordinates $(u,v) = \varphi(x,y)$ (at least locally, near any given point) such that in the new coordinates, these vector fields are the coordinate vector fields $\frac{\partial}{\partial u}, \frac{\partial}{\partial v}$? The answer is no: Since

$$[X,Y] = 2x\frac{\partial}{\partial y}$$

is non-vanishing in x,y coordinates, there cannot be a change of coordinates to make it vanish in u,v coordinates.

Example 6.9. Consider the same problem for the vector fields

$$X = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}, \;\; Y = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}.$$

This time, we can verify that X,Y commute: $[X,Y] = 0$. Can we make a coordinate change so that X,Y become the coordinate vector fields? Note that we will have to remove the origin $p = \mathbf{0}$, since X,Y vanish there. Near points of $\mathbb{R}^2\backslash\{\mathbf{0}\}$, it is convenient to introduce polar coordinates

$$x = r\cos\theta, \;\; y = r\sin\theta$$

(with $r > 0$ and θ varying in an open interval of length at most 2π). We have

$$\frac{\partial}{\partial r} = \frac{\partial x}{\partial r}\frac{\partial}{\partial x} + \frac{\partial y}{\partial r}\frac{\partial}{\partial y} = \frac{1}{r}Y, \quad \frac{\partial}{\partial \theta} = \frac{\partial x}{\partial \theta}\frac{\partial}{\partial x} + \frac{\partial y}{\partial \theta}\frac{\partial}{\partial y} = X.$$

Hence,

$$X = \frac{\partial}{\partial \theta}, \; Y = r\frac{\partial}{\partial r}.$$

To get this into the desired form, we make another change of coordinates $\rho = \rho(r)$ in such a way that Y becomes $\frac{\partial}{\partial \rho}$. Since

$$\frac{\partial}{\partial r} = \frac{\partial \rho}{\partial r}\frac{\partial}{\partial \rho} = \rho'(r)\frac{\partial}{\partial \rho},$$

we want $\rho'(r) = \frac{1}{r}$, thus $\rho = \ln(r)$, or $r = e^\rho$. Hence, the desired change of coordinates is

$$x = e^\rho\cos\theta, \;\; y = e^\rho\sin\theta.$$

87 (answer on page 296). Consider the following two vector fields on the open subset of $\mathbb{R}^2$ where $xy > 0$,

$$X = \frac{x}{y}\frac{\partial}{\partial x} + \frac{\partial}{\partial y}, \quad Y = 2\sqrt{xy}\frac{\partial}{\partial x}.$$

(a) Compute their Lie bracket $[X,Y]$.
(b) Can you find coordinates u, v in which X, Y become the coordinate vector fields?

Definition 6.10. *Let $S \subseteq M$ be a submanifold. A vector field $X \in \mathfrak{X}(M)$ is called* tangent to S *if for all $p \in S$, the tangent vector X_p lies in $T_pS \subseteq T_pM$. (Thus X restricts to a vector field $X|_S \in \mathfrak{X}(S)$.)*

Proposition 6.11. *If two vector fields $X, Y \in \mathfrak{X}(M)$ are tangent to a submanifold $S \subseteq M$, then their Lie bracket $[X,Y]$ is again tangent to S.*

Proposition 6.11 can be proved by using the coordinate expressions of X and Y in submanifold charts. But we will postpone the proof for now since there is a much shorter, coordinate-independent proof (see 92 in the next section).

88 (answer on page 297).
(a) Show that the three vector fields

$$X = y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}, \quad Y = z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}, \quad Z = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$$

on $\mathbb{R}^3$ are tangent to the 2-sphere S^2.
(b) Show that the brackets $[X,Y]$, $[Y,Z]$, $[Z,X]$ are again tangent to the 2-sphere.

As mentioned above, vector fields with the Lie bracket are a special case of *Lie algebras*. More generally, a Lie algebra is a vector space $\mathfrak{g}$ together with a bilinear bracket $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, $(X,Y) \mapsto [X,Y]$, which is skew-symmetric (i.e., $[X,Y] = -[Y,X]$) and satisfies the *Jacobi identity*

$$[X,[Y,Z]] + [Z,[X,Y]] + [Y,[Z,X]] = 0 \tag{6.3}$$

for all $X, Y, Z \in \mathfrak{g}$. Spaces $\mathfrak{X}(M)$ of vector fields on manifolds provide one class of examples. The matrix Lie algebras $\mathfrak{g} \subseteq \mathrm{Mat}_{\mathbb{R}}(n)$ are another class (cf. Example 5.21).

89 (answer on page 297). Verify that the Lie bracket of vector fields on a manifold M satisfies the Jacobi identity; thus $\mathfrak{g} = \mathfrak{X}(M)$ is a Lie algebra. If you enjoy abstractions, verify more generally that the space of derivations of any algebra $\mathscr{A}$ is a Lie algebra $\mathfrak{g} = \mathrm{Der}(\mathscr{A})$.

6.3 Related Vector Fields*

Given a smooth map $F \in C^\infty(M,N)$, the tangent map may be used to map individual tangent vectors at points of M to tangent vectors at points of N. However, this pointwise "push-forward" operation of tangent vectors does not give rise to a push-forward operation of vector *fields* X on M to vector fields Y on N, unless F is a diffeomorphism.

90 (answer on page 297). Let $F \in C^\infty(M,N)$ and $X \in \mathfrak{X}(M)$. We would like to define a vector field $F_*X \in \mathfrak{X}(N)$ such that

$$(F_*X)_{F(p)} = T_pF(X_p)$$

for all $p \in M$. What is the problem with this "definition" when:
(a) F is not surjective?
(b) F is not injective?
(c) F is bijective, but not a diffeomorphism?

The following may be seen as a "workaround," which turns out to be extremely useful.

Definition 6.12. *Let $F \in C^\infty(M,N)$ be a smooth map. Vector fields $X \in \mathfrak{X}(M)$ and $Y \in \mathfrak{X}(N)$ are called F-related, written as*

$$X \sim_F Y,$$

if $T_pF(X_p) = Y_{F(p)}$ for all $p \in M$.

As an immediate consequence of the definition, under composition of smooth maps between manifolds,

$$X \sim_F Y,\ \ Y \sim_G Z \ \ \Rightarrow \ \ X \sim_{G\circ F} Z. \tag{6.4}$$

Example 6.13. Suppose $F:\ M \to N$ is a submersion and $X \in \mathfrak{X}(M)$. Then $X \sim_F 0$ if and only if $T_pF(X_p) = 0$ for all $p \in M$. That is, $X_p \in \ker(T_pF)$. Since F is a submersion, $\ker(T_pF)$ is just the tangent space to the fiber $S = F^{-1}(q)$, where $q = F(p)$ (cf. Proposition 5.18). We conclude that

$$X \sim_F 0 \ \Leftrightarrow\ X \text{ is tangent to the fibers of } F.$$

More generally, $X \sim_F Y$ is the statement that X is a *lift* of the vector field Y; if the submersion F is surjective, it is justified to write this as $Y = F_*X$ since Y is uniquely determined by X.

Example 6.14. Let $\pi:\ S^n \to \mathbb{R}\mathrm{P}^n$ be the usual quotient map for the projective space and $X \in \mathfrak{X}(S^n)$. Then $X \sim_\pi Y$ for some $Y \in \mathfrak{X}(\mathbb{R}\mathrm{P}^n)$ if and only if X is invariant under the transformation $F:\ S^n \to S^n,\ x \mapsto -x$ (that is, $TF \circ X = X \circ F$), and with Y the induced vector field on the quotient.

The following 91 examines the notion of related vector fields in the case of embeddings.

91 (answer on page 298). Suppose $S \subseteq M$ is an embedded submanifold and $i: S \hookrightarrow M$ the inclusion map. For vector fields $X \in \mathfrak{X}(S)$ and $Y \in \mathfrak{X}(M)$, show

$$\begin{aligned} X \sim_i Y &\Leftrightarrow Y \text{ is tangent to } S\text{, with } X \text{ as its restriction} \\ 0 \sim_i Y &\Leftrightarrow Y \text{ vanishes along the submanifold } S. \end{aligned}$$

The F-relation of vector fields has a simple interpretation in terms of the "vector fields as derivations."

Proposition 6.15. *One has $X \sim_F Y$ if and only if for all $g \in C^\infty(N)$,*

$$X(g \circ F) = Y(g) \circ F.$$

Proof. The condition $X(g \circ F) = Y(g) \circ F$ says that

$$(T_pF(X_p))(g) = Y_{F(p)}(g)$$

for all $p \in M$. □

In terms of the pullback notation, with $F^*g = g \circ F$ for $g \in C^\infty(N)$, the proposition amounts to $X \circ F^* = F^* \circ Y$, which can be depicted as the commutative diagram below.

$$\begin{tikzcd} C^\infty(M) \arrow[r, "X"] & C^\infty(M) \\ C^\infty(N) \arrow[u, "F^*"] \arrow[r, "Y"'] & C^\infty(N) \arrow[u, "F^*"'] \end{tikzcd} \tag{6.5}$$

The key fact concerning related vector fields is the following.

Theorem 6.16. *Let $F \in C^\infty(M,N)$, and let vector fields $X_1, X_2 \in \mathfrak{X}(M)$ and $Y_1, Y_2 \in \mathfrak{X}(N)$ be given. Then*

$$X_1 \sim_F Y_1, \quad X_2 \sim_F Y_2 \Rightarrow \quad [X_1, X_2] \sim_F [Y_1, Y_2].$$

Proof. Let $g \in C^\infty(N)$ be arbitrary. Then, since $X_1 \sim_F Y_1$ and $X_2 \sim_F Y_2$, we have

$$\begin{aligned} [X_1, X_2](g \circ F) &= X_1(X_2(g \circ F)) - X_2(X_1(g \circ F)) \\ &= X_1(Y_2(g) \circ F) - X_2(Y_1(g) \circ F) \\ &= Y_1(Y_2(g)) \circ F - Y_2(Y_1(g)) \circ F \\ &= [Y_1, Y_2](g) \circ F. \end{aligned}$$

□

92 (answer on page 298). Prove Proposition 6.11 from page 129: If two vector fields Y_1, Y_2 are tangent to a submanifold $S \subseteq M$, then their Lie bracket $[Y_1, Y_2]$ is again tangent to S, and the Lie bracket of their restriction is the restriction of the Lie brackets.

6.4 Flows of Vector Fields

6.4.1 Solution Curves

Let $\gamma : J \to M$ be a curve, with $J \subseteq \mathbb{R}$ an open interval. In (5.6) we defined the velocity vector at time $t \in J$

$$\dot{\gamma}(t) \in T_{\gamma(t)}M$$

in terms of its action on functions:

$$\big(\dot{\gamma}(t)\big)(f) = \frac{d}{dt} f\big(\gamma(t)\big).$$

The curve representing this tangent vector for a given t, in the sense of Definition 5.2, is the shifted curve $\tau \mapsto \gamma(t+\tau)$. Equivalently, one may think of the velocity vector as the image of the coordinate vector $\frac{\partial}{\partial t}\Big|_t \in T_t J \cong \mathbb{R}$ under the tangent map $T_t\gamma$:

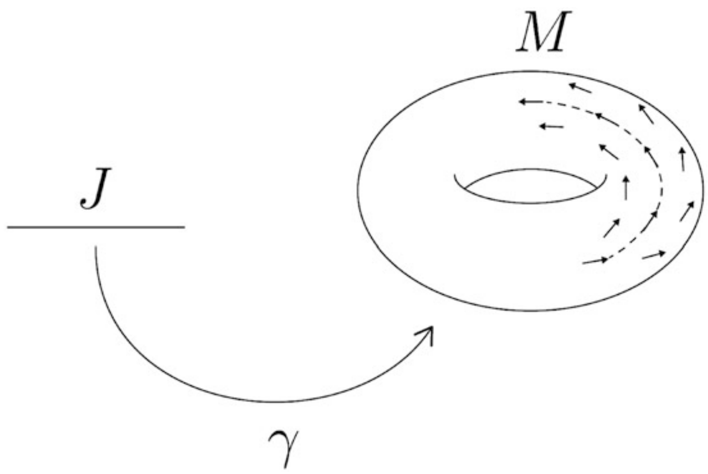
$$\dot{\gamma}(t) = (T_t\gamma)\Big(\frac{\partial}{\partial t}\Big|_t\Big).$$

Definition 6.17. *Suppose $X \in \mathfrak{X}(M)$ is a vector field on a manifold M. A smooth curve $\gamma \in C^\infty(J, M)$, where $J \subseteq \mathbb{R}$ is an open interval, is called a* solution curve *to X if*

$$\dot{\gamma}(t) = X_{\gamma(t)} \tag{6.6}$$

for all $t \in J$.

Geometrically, Equation (6.6) means that the solution curve γ is at all times t tangent to the given vector field, with "speed" as prescribed by the vector field.



We can also restate the definition of solution curves in terms of related vector fields:

$$\frac{\partial}{\partial t} \sim_\gamma X. \tag{6.7}$$

The following proposition is a direct consequence of this characterization.

Proposition 6.18. *Suppose $F \in C^\infty(M,N)$ is a smooth map of manifolds, and let $X \in \mathfrak{X}(M)$ and $Y \in \mathfrak{X}(N)$ be F-related vector fields $X \sim_F Y$. If $\gamma\colon\ J \to M$ is a solution curve for X, then $F \circ \gamma\colon\ J \to N$ is a solution curve for Y.*

Proof. By (6.4),

$$\frac{\partial}{\partial t} \sim_\gamma X,\ \ X \sim_F Y \ \ \Rightarrow\ \ \frac{\partial}{\partial t} \sim_{F\circ\gamma} Y.$$

Alternatively, the claim follows from Proposition 5.12. □

Given a vector field $X \in \mathfrak{X}(M)$ and a point $p \in M$, one may ask about the existence of a solution curve $\gamma\colon\ J \to M$, for some interval J around 0, with the given initial conditions $\gamma(0) = p,\ \dot{\gamma}(0) = X_p$. Furthermore, one may ask whether such a solution is unique, i.e., whether any two solutions agree on their common domain of definition. We shall discuss this problem first for open subsets of Euclidean spaces.

6.4.2 Existence and Uniqueness for Open Subsets of $\mathbb{R}^m$

Consider first the case that $M = U \subseteq \mathbb{R}^m$. Here curves $\gamma(t)$ are of the form

$$\gamma(t) = \mathbf{x}(t) = (x^1(t), \ldots, x^m(t));$$

hence,

$$\dot{\gamma}(t)(f) = \frac{d}{dt} f(\mathbf{x}(t)) = \sum_{i=1}^m \frac{dx^i}{dt} \frac{\partial f}{\partial x^i}(\mathbf{x}(t)).$$

That is,

$$\dot{\gamma}(t) = \sum_{i=1}^{m} \frac{dx^i}{dt} \left. \frac{\partial}{\partial x^i} \right|_{\mathbf{x}(t)} .$$

On the other hand, vector fields have the form $X = \sum_{i=1}^{m} a^i(\mathbf{x}) \frac{\partial}{\partial x^i}$. Hence, (6.6) becomes the system of first-order ordinary differential equations,

$$\frac{dx^i}{dt} = a^i(\mathbf{x}(t)), \quad i = 1, \ldots, m; \tag{6.8}$$

the initial condition $\gamma(0) = p$ takes the form

$$\mathbf{x}(0) = \mathbf{x}_0, \tag{6.9}$$

where $\mathbf{x}_0 = (x_0^1, \ldots, x_0^m) \in U$ is the coordinate vector for p.

Example 6.19. Consider the case of a *constant* vector field $X = \sum a^i \frac{\partial}{\partial x^i}$, with $\mathbf{a} = (a^1, \ldots, a^m) \in \mathbb{R}^m$ regarded as a constant function of $\mathbf{x}$. Then the solutions of (6.8) with initial condition $\mathbf{x}(0) = \mathbf{x}_0 = (x_0^1, \ldots, x_0^m)$ are given by $x^i(t) = x_0^i + a^i t$ for $i = 1, \ldots, m$. That is, the solution curves are affine lines

$$\mathbf{x}(t) = \mathbf{x}_0 + t\mathbf{a}.$$

As a special case, the solution curves for the coordinate vector field $\frac{\partial}{\partial x^j}$, with initial condition $\mathbf{x}(0) = \mathbf{x}_0 = (x_0^1, \ldots, x_0^m)$, are

$$x^i(t) = \begin{cases} x_0^i & i \neq j, \\ x_0^j + t & i = j. \end{cases}$$

Example 6.20. Consider the *Euler vector field* on $\mathbb{R}^m$,

$$X = \sum_{i=1}^{m} x^i \frac{\partial}{\partial x^i}.$$

Here $a^i(x) = x^i$; hence, (6.8) reads as $\frac{dx^i}{dt} = x^i$, with solutions $x^i(t) = c_i e^t$, for arbitrary constants c_i. Such a solution satisfies the initial condition (6.9) if and only if $c^i = x_0^i$; hence, we obtain

$$\mathbf{x}(t) = e^t \, \mathbf{x}_0$$

as the solution of the initial value problem.

93 (answer on page 298). Consider the vector field on $\mathbb{R}^2$,

$$X = -y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y}.$$

Find the solution curve for any given initial condition $(x_0, y_0) \in \mathbb{R}^2$. Draw a picture of the vector field.

Explicit solutions of the initial value problem (6.8), (6.9) can only be found for certain classes of such equations, studied in the theory of ODEs. Even without finding explicit solutions, ODE theory gives a wealth of information on the qualitative behavior of solutions. The first general result says that a solution to the initial value problem always exists and moreover that there is a unique maximal solution.

Theorem 6.21 (Existence and Uniqueness Theorem for ODEs). *Let $U \subseteq \mathbb{R}^m$ be an open subset, and $\mathbf{a} \in C^\infty(U, \mathbb{R}^m)$. For any given $\mathbf{x}_0 \in U$, there is an open interval $J_{\mathbf{x}_0} \subseteq \mathbb{R}$ around 0, and a solution $\mathbf{x}: J_{\mathbf{x}_0} \to U$ of the system of ODEs*

$$\frac{dx^i}{dt} = a^i(\mathbf{x}(t)), \;\; i = 1, \ldots, m$$

with initial condition $\mathbf{x}(0) = \mathbf{x}_0$, and which is maximal in the sense that any other solution to this initial value problem is obtained by restriction to some subinterval of $J_{\mathbf{x}_0}$.

Thus, $J_{\mathbf{x}_0}$ is the maximal open interval on which the solution is defined. The solution depends smoothly on initial conditions, in the following sense. For any given $\mathbf{x}_0$, let $\Phi(t, \mathbf{x}_0)$ be the solution $\mathbf{x}(t)$ of the initial value problem with initial condition $\mathbf{x}_0$. Since we are interested in $\mathbf{x}_0$ as a "variable" rather than a "constant," we will write $\mathbf{x}$ in place of $\mathbf{x}_0$; thus $t \mapsto \Phi(t, \mathbf{x})$ is the solution curve that was earlier denoted $\mathbf{x}(t)$. Then, $\Phi(t, \mathbf{x})$ is a function defined for $t \in J_{\mathbf{x}}$, which is smooth as a function of t, and is characterized by the equations

$$\frac{d}{dt}\Phi(t, \mathbf{x}) = \mathbf{a}\big(\Phi(t, \mathbf{x})\big), \quad \Phi(0, \mathbf{x}) = \mathbf{x}. \tag{6.10}$$

Theorem 6.22 (Dependence on Initial Conditions for ODEs). *For $\mathbf{a} \in C^\infty(U, \mathbb{R}^m)$ as above, the set*

$$\mathscr{J} = \{(t, \mathbf{x}) \in \mathbb{R} \times U \,|\, t \in J_{\mathbf{x}}\}$$

is an open neighborhood of $\{0\} \times U$ in $\mathbb{R} \times U$, and the map

$$\Phi: \; \mathscr{J} \to U, \; (t, \mathbf{x}) \mapsto \Phi(t, \mathbf{x})$$

is smooth.

In general, the interval $J_{\mathbf{x}}$ for given $\mathbf{x}$ may be strictly smaller than $\mathbb{R}$ because a solution might "escape to infinity in finite time," as illustrated in the following example. (This language regards t as a time parameter for the solution curve $\mathbf{x}(t)$.)

Example 6.23. Consider the ODE in one variable,

$$\frac{dx}{dt} = x^2.$$

The initial value problem $x(t_0) = x_0$ for this ODE is solved by the method of *separation of variables*: One formally writes $dt = x^{-2}dx$ and then integrates both sides to obtain $t - t_0 = \int_{x_0}^{x} u^{-2}du = -x^{-1} + x_0^{-1}$. In our case, $t_0 = 0$. Solving for x, we obtain

$$x(t) = \frac{x_0}{1 - tx_0}.$$

(This solution is also correct for $x_0 = 0$, even though the calculation did not apply to this case.)

Note that the solution is only defined for $1 - tx_0 \neq 0$, and since we start at $t_0 = 0$, we must have $1 - tx_0 > 0$. Hence, the domain of definition of the solution curve $x(t)$ with initial condition x_0 is $J_{x_0} = \{t \in \mathbb{R} |\, tx_0 < 1\}$. We read off that

$$\Phi(t, x) = \frac{x}{1 - tx}$$

with domain of definition $\mathscr{J} = \{(t, x) |\, tx < 1\}$.

94 (answer on page 299). For each of the following ODEs: Find the solution curves with initial condition $x(t) = x_0 \in U$; find J_{x_0}, $\mathscr{J}$, and $\Phi(t, x)$.

(a) $\dfrac{dx}{dt} = 1$ on $U = (0, 1) \subseteq \mathbb{R}$.

(b) $\dfrac{dx}{dt} = 1 + x^2$ on $U = \mathbb{R}$.

6.4.3 Existence and Uniqueness for Vector Fields on Manifolds

For general vector fields $X \in \mathfrak{X}(M)$ on manifolds, Equation (6.6) becomes (6.8) after introducing local coordinates. In detail, in a given coordinate chart (U, φ) X becomes the vector field

$$\varphi_*(X) = \sum_{i=1}^{m} a^i(\mathbf{x}) \frac{\partial}{\partial x^i}$$

and $\varphi(\gamma(t)) = \mathbf{x}(t)$ with

$$\frac{dx^i}{dt} = a^i(\mathbf{x}(t)).$$

If $\mathbf{a} = (a^1, \ldots, a^m) :\ \varphi(U) \to \mathbb{R}^m$ corresponds to X in a local chart (U, φ), then any solution curve $\mathbf{x} :\ J \to \varphi(U)$ for $\mathbf{a}$ defines a solution curve $\gamma(t) = \varphi^{-1}(\mathbf{x}(t))$ for X. The existence and uniqueness theorem for ODEs extends to manifolds as follows.

Theorem 6.24 (Solutions of Vector Fields on Manifolds). *Let $X \in \mathfrak{X}(M)$ be a vector field on a manifold M. For any given $p \in M$, there is an open interval $J_p \subseteq \mathbb{R}$ around* 0 *and a unique solution $\gamma :\ J_p \to M$ of the initial value problem*

$$\dot{\gamma}(t) = X_{\gamma(t)}, \quad \gamma(0) = p, \tag{6.11}$$

which is maximal in the sense that any other solution is obtained by restriction to a subinterval. The set

$$\mathscr{J} = \{(t,p) \in \mathbb{R}\times M |\, t \in J_p\}$$

is an open neighborhood of $\{0\}\times M$*, and the map*

$$\Phi : \mathscr{J} \to M, \ (t,p) \mapsto \Phi(t,p)$$

such that $\gamma(t) = \Phi(t,p)$ *solves the initial value problem (6.11) is smooth.*

Proof. Existence and uniqueness of solutions for small times t follows from the existence and uniqueness theorem for ODEs, by considering the vector field in local charts. To prove uniqueness even for large times t, let $\gamma : J \to M$ be a maximal solution of (6.11) (i.e., a solution that cannot be extended to a larger open interval), and let $\gamma_1 : J_1 \to M$ be another solution of the same initial value problem. Suppose that $\gamma_1(t) \neq \gamma(t)$ for some $t \in J \cap J_1$, $t > 0$. Then we can define

$$b = \inf\{t \in J\cap J_1 |\, t > 0, \ \gamma_1(t) \neq \gamma(t)\}.$$

By the uniqueness for small t, we have $b > 0$. We will get a contradiction in each of the following cases.

Case 1: $\gamma_1(b) = \gamma(b) =: q$. Then both $\lambda_1(s) = \gamma_1(b+s)$ and $\lambda(s) = \gamma(b+s)$ are solutions to the initial value problem

$$\lambda(0) = q, \quad \dot{\lambda}(s) = X_{\lambda(s)};$$

hence, they have to agree for small $|s|$, and consequently, $\gamma_1(t), \gamma(t)$ have to agree for t close to b. This contradicts the definition of b.

Case 2: $\gamma_1(b) \neq \gamma(b)$. Using the Hausdorff property of M, we can choose disjoint open neighborhoods U of $\gamma(b)$ and U_1 of $\gamma(b_1)$. For $t = b-\varepsilon$ with $\varepsilon > 0$ sufficiently small, $\gamma(t) \in U$, while $\gamma_1(t) \in U_1$. But this is impossible since $\gamma(t) = \gamma_1(t)$ for $0 \leq t < b$.

These contradictions show that $\gamma_1(t) = \gamma(t)$ for $t \in J_1 \cap J$ and $t > 0$. Similarly, $\gamma_1(t) = \gamma(t)$ for $t \in J_1 \cap J$ and $t < 0$. Hence, $\gamma_1(t) = \gamma(t)$ for all $t \in J_1 \cap J$. Since γ is a maximal solution, it follows that $J_1 \subseteq J$, with $\gamma_1 = \gamma|_{J_1}$.

The result for ODEs about the smooth dependence on initial conditions shows, by working in local coordinate charts, that $\mathscr{J}$ contains an open neighborhood of $\{0\}\times M$, on which Φ is given by a smooth map. The fact that $\mathscr{J}$ itself is open, and the map Φ is smooth everywhere, follows by the "flow property" to be discussed below. (We omit the details of this part of the proof.) $\square$

Note that the uniqueness part uses the Hausdorff property from the definition of manifolds. Indeed, the uniqueness part may fail for non-Hausdorff manifolds.

Example 6.25. Consider is the non-Hausdorff manifold from Example 2.15, $M = \widetilde{M}/\sim$, where

$$\widetilde{M} = (\mathbb{R}\times\{1\}) \sqcup (\mathbb{R}\times\{-1\})$$

is a disjoint union of two copies of the real line (thought of as embedded in $\mathbb{R}^2$), and where $\sim$ glues the two copies along the strictly negative real axis. Let $\pi : \widetilde{M} \to M$

be the quotient map. The vector field $\widetilde{X}$ on $\widetilde{M}$ given as $\frac{\partial}{\partial x}$ on both copies descends to a vector field X on M, i.e.,

$$\widetilde{X} \sim_{\pi} X.$$

The solution curves of the initial value problem for X, with $\gamma(0) = p$ on the negative real axis, are not unique: The solution curves move from the left to the right but may continue on the "upper branch" or on the "lower branch" of M. Concretely, the curves $\widetilde{\gamma}_{+}, \widetilde{\gamma}_{-} : \mathbb{R} \to \widetilde{M}$, given as

$$\widetilde{\gamma}_{\pm}(t) = (t-1, \pm 1),$$

are each a solution curve of $\widetilde{X}$; hence, their images under π are solution curves $\gamma_{\pm}$ of X (cf. Proposition 6.18). These have the same initial condition and coincide for $t < 1$ but are different for $t \geq 1$.

6.4.4 Flows

Given a vector field X, the map $\Phi : \mathscr{J} \to M$ is called the *flow* of X. For any given p, the curve $\gamma(t) = \Phi(t, p)$ is a solution curve. But one can also fix t and consider the time-t flow

$$\Phi_t(p) := \Phi(t, p)$$

as a function of p. It is a smooth map $\Phi_t : U_t \to M$, defined on the open subset

$$U_t = \{p \in M | \ (t, p) \in \mathscr{J}\}.$$

Note that $\Phi_0 = \mathrm{id}_M$, since $t = 0$ describes the initial condition of the initial value problem.

Example 6.26. In Example 6.23, we computed the set $\mathscr{J}$ and the flow for the vector field $X = x^2 \frac{\partial}{\partial x}$ on the real line $\mathbb{R}$. We have found that $\mathscr{J}$ is described by the equation $tx < 1$. Hence, the domain of definition of $\Phi_t(x) = x/(1-tx)$ is

$$U_t = \{x \in \mathbb{R} | \ tx < 1\}.$$

Intuitively, $\Phi_t(p)$ is obtained from the initial point $p \in M$ by flowing for time t along the vector field X. One expects that first flowing for time s, and then flowing for time t, should be the same as flowing for time $t+s$. Indeed, one has the following *flow property*.

Theorem 6.27 (Flow Property). *Let $X \in \mathfrak{X}(M)$, with flow $\Phi : \mathscr{J} \to M$. Let $(s, p) \in \mathscr{J}$ and $t \in \mathbb{R}$. Then*

$$(t, \Phi_s(p)) \in \mathscr{J} \Leftrightarrow (t+s, p) \in \mathscr{J}$$

and in this case

$$\Phi_t(\Phi_s(p)) = \Phi_{t+s}(p).$$

Proof. We claim that, for fixed s, both

$$t \mapsto \Phi_t(\Phi_s(p)) \text{ and } t \mapsto \Phi_{t+s}(p)$$

are maximal solution curves of X, for the same initial condition $q = \Phi_s(p)$. This is clear for the first curve and follows for the second curve by the calculation, for $f \in C^\infty(M)$,

$$\frac{d}{dt} f\big(\Phi_{t+s}(p)\big) = \frac{d}{d\tau}\Big|_{\tau=t+s} f\big(\Phi_\tau(p)\big) = X_{\Phi_\tau(p)}(f)\Big|_{\tau=t+s} = X_{\Phi_{t+s}(p)}(f).$$

Hence, by the uniqueness part of Theorem 6.24, the two curves must coincide. The domain of definition of $t \mapsto \Phi_{t+s}(p)$ is the interval $J_p \subseteq \mathbb{R}$, shifted by s. That is, $t \in J_{\Phi(s,p)}$ if and only if $t+s \in J_p$. □

Corollary 6.28. *For all $t \in \mathbb{R}$, the map Φ_t is a diffeomorphism from its domain U_t onto its image $\Phi_t(U_t)$.*

Proof. Let $p,q \in M$ with $\Phi_t(p) = q$. Thus, $(t,p) \in \mathscr{J}$. Since we always have $(0,p) \in \mathscr{J}$, the theorem shows $(-t,q) \in \mathscr{J}$; furthermore, $\Phi_{-t}(q) = \Phi_{-t}(\Phi_t(p)) = \Phi_0(p) = p$. We conclude that Φ_t takes values in U_{-t}, and Φ_{-t} is an inverse map. □

Example 6.29. Let us illustrate the flow property for various vector fields on $\mathbb{R}$.

(a) The flow property is evident for $\frac{\partial}{\partial x}$ with flow $\Phi_t(x) = x + t$, defined for all t (cf. Example 6.19).
(b) The vector field $x\frac{\partial}{\partial x}$ has flow $\Phi_t(x) = e^t x$, defined for all t (cf. Example 6.20). The flow property holds:

$$\Phi_t(\Phi_s(x)) = e^t\,\Phi_s(x) = e^t e^s x = e^{t+s}x = \Phi_{t+s}(x).$$

(c) The vector field $x^2\frac{\partial}{\partial x}$ has flow $\Phi_t(x) = x/(1-tx)$, defined for $1 - tx > 0$ (cf. Example 6.23). We can explicitly verify the flow property:

$$\Phi_t(\Phi_s(x)) = \frac{\Phi_s(x)}{1 - t\Phi_s(x)} = \frac{\frac{x}{1-sx}}{1 - t\frac{x}{1-sx}} = \frac{x}{1-(t+s)x} = \Phi_{t+s}(x).$$

6.4.5 Complete Vector Fields

Let X be a vector field, and $\mathscr{J} = \mathscr{J}^X \subseteq \mathbb{R} \times M$ be the domain of definition for the flow $\Phi = \Phi^X$.

Definition 6.30. *A vector field $X \in \mathfrak{X}(M)$ is called* complete *if $\mathscr{J}^X = \mathbb{R} \times M$.*

Thus X is complete if and only if all solution curves exist for all time.

Example 6.31. By Example 6.29 above, the vector field $x\frac{\partial}{\partial x}$ on $M = \mathbb{R}$ is complete, but $x^2\frac{\partial}{\partial x}$ is incomplete.

A vector field may fail to be complete if a solution curve escapes to infinity in finite time. This suggests that a vector field X that vanishes outside a compact set must be complete because the solution curves are "trapped" and cannot escape to infinity.

Similarly to the definition of support of a function (cf. Definition 3.4), we define the *support* of a vector field X to be the smallest closed subset

$$\mathrm{supp}(X) \subseteq M$$

with the property that $X_p = 0$ for $p \notin \mathrm{supp}(X)$. We say that X has *compact support* if $\mathrm{supp}(X)$ is compact.

Proposition 6.32. *Every vector field with compact support is complete. In particular, every vector field on a compact manifold is complete.*

Proof. Note that if X vanishes at some point $p \in M$, then the unique solution curve through p is the constant curve $\gamma(t) = p$, defined for all $t \in \mathbb{R}$. Thus, if a solution curve lies in $M \setminus \mathrm{supp}(X)$ for *some* t, then it must be constant and in particular lie in $M \setminus \mathrm{supp}(X)$ for *all* t in its domain. Hence, if a solution curve lies in $\mathrm{supp}(X)$ for some t, then it must lie in $\mathrm{supp}(X)$ for all t in its domain of definition.

Let $U_\varepsilon \subseteq M$ be the set of all p such that the solution curve γ with initial condition $\gamma(0) = p$ exists for $|t| < \varepsilon$ (that is, $(-\varepsilon, \varepsilon) \subseteq J_p$). By smooth dependence on initial conditions (cf. Theorem 6.24), U_ε is open. The collection of all U_ε with $\varepsilon > 0$ covers $\mathrm{supp}(X)$, since every solution curve exists for sufficiently small time. Since $\mathrm{supp}(X)$ is compact, there exists a finite subcover $U_{\varepsilon_1}, \ldots, U_{\varepsilon_k}$ (cf. Definition 2.33). Let ε be the smallest of $\varepsilon_1, \ldots, \varepsilon_k$. Then $U_{\varepsilon_i} \subseteq U_\varepsilon$, for all i, and so $\mathrm{supp}(X) \subseteq U_\varepsilon$. Hence, for any $p \in \mathrm{supp}(X)$, we have

$$(-\varepsilon, \varepsilon) \subseteq J_p,$$

that is, any solution curve $\gamma(t)$ starting in $\mathrm{supp}(X)$ exists for times $|t| < \varepsilon$. For solution curves starting in $M \backslash \mathrm{supp}(X)$, this is true as well. By 95 below (applied to $\delta = \varepsilon/2$, say), we are done. □

95 (answer on page 299). Let $X \in \mathfrak{X}(M)$ be a vector field, and suppose $\delta > 0$ is such that every solution curve exists at least for times t with $|t| \leq \delta$. Use the "flow property" to argue that X is complete.

Theorem 6.33. *If X is a complete vector field, the flow Φ_t defines a 1-parameter group of diffeomorphisms. That is, each Φ_t is a diffeomorphism and*

$$\Phi_0 = \mathrm{id}_M, \quad \Phi_t \circ \Phi_s = \Phi_{t+s}.$$

Conversely, if Φ_t is a 1-parameter group of diffeomorphisms such that the map $(t, p) \mapsto \Phi_t(p)$ is smooth, the equation

$$X_p(f) = \left.\frac{d}{dt}\right|_{t=0} f(\Phi_t(p))$$

defines a complete vector field X on M, with flow Φ_t.

Proof. It remains to show the second statement. Given Φ_t, the linear map

$$C^\infty(M) \to C^\infty(M), \quad f \mapsto \frac{d}{dt}\bigg|_{t=0} f(\Phi_t(p))$$

satisfies the product rule (6.2); hence, it is a vector field X. Given $p \in M$, the curve $\Phi_t(p)$ is an integral curve of X since

$$\frac{d}{dt}\Phi_t(p) = \frac{d}{ds}\bigg|_{s=0} \Phi_{t+s}(p) = \frac{d}{ds}\bigg|_{s=0} \Phi_s(\Phi_t(p)) = X_{\Phi_t(p)}. \qquad \square$$

Remark 6.34. In terms of pullbacks, the relation between the vector field and its flow reads as

$$\frac{d}{dt}\Phi_t^*(f) = \Phi_t^* \frac{d}{ds}\bigg|_{s=0} \Phi_s^*(f) = \Phi_t^* X(f).$$

(To make sense of the derivative, you should think of both sides as evaluated at a point of M.) This identity of linear operators on $C^\infty(M)$

$$\frac{d}{dt}\Phi_t^* = \Phi_t^* \circ X$$

may be viewed as the definition of the flow. (To make sense of the derivative, you should think of both sides as applied to a function and evaluated at a point.)

Example 6.35. Given a square matrix $A \in \mathrm{Mat}_{\mathbb{R}}(m)$, let

$$\Phi_t : \mathbb{R}^m \to \mathbb{R}^m, \quad \mathbf{x} \mapsto e^{tA}\mathbf{x} = \Big(\sum_{N=0}^{\infty} \frac{t^N}{N!} A^N\Big)\mathbf{x}$$

(using the exponential map of matrices). Since $e^{(t+s)A} = e^{tA}e^{sA}$, and since $(t, \mathbf{x}) \mapsto e^{tA}\mathbf{x}$ is a smooth map, Φ_t defines a flow. What is the corresponding vector field X?

For any function $f \in C^\infty(\mathbb{R}^m)$, we calculate

$$X(f)(\mathbf{x}) = \frac{d}{dt}\bigg|_{t=0} f(e^{tA}\mathbf{x}) = \sum_j \frac{\partial f}{\partial x^j}(A\mathbf{x})^j = \sum_{ij} A_i{}^j x^i \frac{\partial f}{\partial x^j}.$$

This shows that

$$X = \sum_{ij} A_i{}^j x^i \frac{\partial}{\partial x^j}. \tag{6.12}$$

As a special case, taking A to be the identity matrix, we get the *Euler vector field* $X = \sum_i x^i \frac{\partial}{\partial x^i}$, with its corresponding flow $\Phi_t(\mathbf{x}) = e^t\mathbf{x}$ (cf. Example 6.20; see also Problem 6 at the end of the chapter).

To conclude this section, we characterize related vector fields in terms of their flows.

Proposition 6.36. *Let $F \in C^\infty(M,N)$, and let $X \in \mathfrak{X}(M)$ and $Y \in \mathfrak{X}(N)$ be complete vector fields, with flows Φ_t^X and Φ_t^Y, respectively. Then*

$$X \sim_F Y \quad \Leftrightarrow \quad F \circ \Phi_t^X = \Phi_t^Y \circ F \quad \textit{for all } t.$$

In short, vector fields are F-related if and only if their flows are F-related.

Proof. Suppose $F \circ \Phi_t^X = \Phi_t^Y \circ F$ for all t. For $g \in C^\infty(N)$, and $p \in M$, taking a t-derivative of

$$g(F(\Phi_t^X(p))) = g(\Phi_t^Y(F(p)))$$

at $t = 0$ on both sides, we get

$$\big(T_pF(X_p)\big)(g) = Y_{F(p)}(g),$$

i.e., $T_pF(X_p) = Y_{F(p)}$. Hence, $X \sim_F Y$.

Conversely, suppose $X \sim_F Y$. By Proposition 6.18, if $\gamma\colon\ J \to M$ is a solution curve for X, with initial condition $\gamma(0) = p$, then $F \circ \gamma\colon\ J \to M$ is a solution curve for Y, with initial condition $F(p)$. That is, $F(\Phi_t^X(p)) = \Phi_t^Y(F(p))$. □

Remark 6.37. The proposition generalizes to possibly incomplete vector fields: The vector fields are F-related if and only if $F \circ \Phi^X = \Phi^Y \circ (\mathrm{id}_\mathbb{R} \times F)$.

Remark 6.38. Flows of incomplete vector fields X can be cumbersome to deal with, since one has to take into account the domain of definition of the flow, $\mathscr{J} \subseteq \mathbb{R} \times M$. However, if one is only interested in the short-time behavior of the flow, near a given point $p \in M$, one can make X complete by multiplying with a compactly supported function $\chi \in C^\infty(M)$ such that $\chi = 1$ on a neighborhood U of p. Indeed,

$$X' = \chi X$$

is compactly supported and therefore complete, and since $X'|_U = X|_U$, its integral curves with initial condition in U coincide with those of X for the time interval where they stay inside U.

6.5 Geometric Interpretation of the Lie Bracket

For any smooth map $F \in C^\infty(M,N)$, we defined the pullback of smooth functions

$$F^*:\ C^\infty(N) \to C^\infty(M), \quad g \mapsto g \circ F.$$

If F is a *diffeomorphism*, then every function in $C^\infty(M)$ can be written as the pullback of a function in $C^\infty(N)$. Hence, we can also pull back vector fields

$$F^* : \mathfrak{X}(N) \to \mathfrak{X}(M), \quad Y \mapsto F^*Y,$$

by requiring that the following diagram commutes (cf. Diagram (6.5)).

$$\begin{tikzcd}
C^\infty(M) \arrow[r, "F^*Y"] & C^\infty(M) \\
C^\infty(N) \arrow[u, "F^*"] \arrow[r, "Y"'] & C^\infty(N) \arrow[u, "F^*"']
\end{tikzcd}$$

That is, by the condition $(F^*Y)(F^*g) = F^*(Y(g))$ for all functions $g \in C^\infty(N)$, i.e., $F^*Y \sim_F Y$, or in more detail

$$(F^*Y)_p = (T_pF)^{-1} Y_{F(p)}$$

for all $p \in M$. By Theorem 6.16, we have $F^*[X,Y] = [F^*X, F^*Y]$.

Now, by Theorem 6.33, any complete vector field $X \in \mathfrak{X}(M)$ with flow Φ_t gives rise to pullback maps

$$\Phi_t^* : C^\infty(M) \to C^\infty(M) \text{ and } \Phi_t^* : \mathfrak{X}(M) \to \mathfrak{X}(M).$$

Definition 6.39. *The* Lie derivative of a function f *with respect to* X *is the function*

$$L_X(f) = \left.\frac{d}{dt}\right|_{t=0} \Phi_t^* f \in C^\infty(M). \tag{6.13}$$

The Lie derivative of a vector field Y *with respect to* X *is the vector field*

$$L_X(Y) = \left.\frac{d}{dt}\right|_{t=0} \Phi_t^* Y \in \mathfrak{X}(M). \tag{6.14}$$

Below are some clarifications for this definition.

(a) In (6.13), the right-hand side is interpreted in terms of evaluation at $p \in M$:

$$\Big(\left.\frac{d}{dt}\right|_{t=0} \Phi_t^* f\Big)(p) = \left.\frac{d}{dt}\right|_{t=0} \big((\Phi_t^* f)(p)\big) = \left.\frac{d}{dt}\right|_{t=0} f\big(\Phi_t(p)\big).$$

Thus, the Lie derivative of a function measures how the function changes along the flow of the vector field. Since $\gamma(t) = \Phi_t(p)$ is just the solution curve of X with initial condition $\gamma(0) = p$, we see that

$$L_X(f) = X(f).$$

(b) Similarly, to interpret the right-hand side of (6.14), one evaluates at $p \in M$ to obtain a family of tangent vectors in T_pM:

$$\left(\frac{d}{dt}\bigg|_{t=0} \Phi_t^* Y \right)\bigg|_p = \frac{d}{dt}\bigg|_{t=0} (\Phi_t^* Y)_p.$$

Here,

$$(\Phi_t^* Y)_p = (T_p \Phi_t)^{-1} \, Y_{\Phi_t(p)};$$

that is, we use the inverse to the tangent map of the flow of X to move $Y_{\Phi_t(p)}$ to p. If Y were invariant under the flow of X, this would agree with Y_p; hence, $(\Phi_t^* Y)_p - Y_p$ measures how Y fails to be Φ_t-invariant. $L_X(Y)$ is the infinitesimal version of this. That is, the Lie derivative measures infinitesimally how Y changes along the flow of X. As we will see below, the infinitesimal version actually implies the global version.

(c) The definition of Lie derivative also works for incomplete vector fields, since it only involves derivatives at $t = 0$.

(d) In (6.13) and (6.14), it was taken for granted that the right-hand side does indeed define a smooth function or, respectively, a vector field. For functions, this follows from $L_X f = X(f)$, and for vector fields from Theorem 6.40 below.

(e) From now on, we will usually drop the parentheses when writing Lie derivatives, e.g., we write $L_X Y$ in place of $L_X(Y)$.

Theorem 6.40. *For any $X, Y \in \mathfrak{X}(M)$, the Lie derivative $L_X Y$ is the Lie bracket:*

$$L_X Y = [X, Y].$$

Proof. Let $\Phi_t = \Phi_t^X$ be the flow of X. For all $f \in C^\infty(M)$, we obtain by taking the t-derivative at $t = 0$ of both sides of

$$\Phi_t^*(Y(f)) = (\Phi_t^* Y)(\Phi_t^* f)$$

that

$$X(Y(f)) = \left(\frac{d}{dt}\bigg|_{t=0} \Phi_t^* Y \right)(f) + Y\left(\frac{d}{dt}\bigg|_{t=0} \Phi_t^* f \right) = (L_X Y)(f) + Y(X(f)).$$

(Again, in this calculation, you may take all the terms as evaluated at a point $p \in M$, so that the calculation really just involves derivatives of $\mathbb{R}$-valued functions of t.) That is, $L_X Y = X \circ Y - Y \circ X = [X, Y]$. $\square$

96 (answer on page 300). Justify the calculation above of the t-derivative at $t = 0$.

We see in particular that $L_X Y$ is skew-symmetric in X and Y—this was not obvious from the definition!

The result $[X, Y] = L_X Y$ gives an interpretation of the Lie bracket, as measuring infinitesimally how Y changes along the flow of X. The following result strengthens this interpretation of the Lie bracket.

Theorem 6.41. *Let X and Y be complete vector fields with flows Φ_t and Ψ_s, respectively. Then*

$$\begin{aligned}
[X,Y] = 0 &\Leftrightarrow \Phi_t^* Y = Y \quad \textit{for all } t \\
&\Leftrightarrow \Psi_s^* X = X \quad \textit{for all } s \\
&\Leftrightarrow \Phi_t \circ \Psi_s = \Psi_s \circ \Phi_t \quad \textit{for all } s,t.
\end{aligned}$$

Proof. The calculation

$$\frac{d}{dt}(\Phi_t)^* Y = (\Phi_t)^* L_X Y = (\Phi_t)^* [X,Y]$$

shows that $\Phi_t^* Y$ is independent of t if and only if $[X,Y] = 0$. Since $[Y,X] = -[X,Y]$, interchanging the roles of X and Y, this is also equivalent to $\Psi_s^* X$ being independent of s. The property $\Phi_t^* Y = Y$ means that Y is Φ_t-related to itself; hence, by Proposition 6.36, it takes the flow of Ψ_s to itself, that is,

$$\Phi_t \circ \Psi_s = \Psi_s \circ \Phi_t.$$

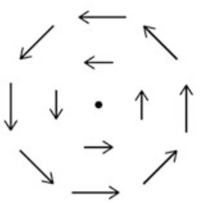
Conversely, if this equation holds, then $\Phi_t^*(\Psi_s^* f) = \Psi_s^*(\Phi_t^* f)$ for all $f \in C^\infty(M)$. Differentiating with respect to s at $s = 0$, we obtain

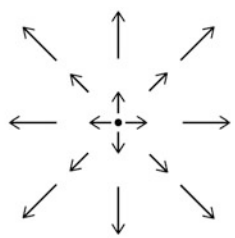
$$\Phi_t^*(L_Y(f)) = L_Y(\Phi_t^* f).$$

Differentiating with respect to t at $t = 0$, we get $L_X(L_Y(f)) = L_Y(L_X(f))$, that is, $L_{[X,Y]} = [L_X, L_Y] = 0$, and therefore, $[X,Y] = 0$. □

Example 6.42. Let $X = \frac{\partial}{\partial y} \in \mathfrak{X}(\mathbb{R}^2)$. Then $[X,Y] = 0$ if and only if Y is invariant under translation in the y-direction.

Example 6.43. The vector fields from Example 6.9, $X = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$ and $Y = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$, commute. This is verified by direct calculation but can also be "seen" in the following picture:

$X =$ 

$Y =$ 

The flow of X is rotations around the origin, but Y is invariant under rotations. Likewise, the flow of Y is by dilations away from the origin, but X is invariant under dilations.

Similar, but less elegant, statements hold when X and Y are possibly incomplete. Let X be a vector field, with flow $\Phi: \mathscr{J}^X \to M$. For $t \in \mathbb{R}$, let $U_t \subseteq M$ be the open subset of all $q \in M$ such that $(t,q) \in \mathscr{J}^X$ (so that Φ_t restricts to a diffeomorphism from U_t onto its image). Given another vector field Y, we have that $[X,Y] = 0$ if and only if

$$Y|_{U_t} \sim_{\Phi_t} Y$$

for all $t \in \mathbb{R}$.

97 (answer on page 300). Give an example of a manifold M and vector fields X, Y with $[X,Y] = 0$, such that there exist $p \in M$ and $s,t \in \mathbb{R}$ with

$$\Phi_t(\Psi_s(p)) \neq \Psi_s(\Phi_t(p))$$

even though both sides are defined (in the sense that (t,p) and $(t, \Psi_s(p))$ are in the domain of X, and (s,p) and $(s, \Phi_t(p))$ are in the domain of Y).

What we can say for $[X,Y] = 0$ is that if $U \subseteq M$ is an open subset and $R \subseteq \mathbb{R}^2$ an open rectangle around $(0,0)$ such that $\Phi_s(\Psi_t(p))$, $\Phi_t(\Psi_s(p))$ are both defined for *all* $(s,t) \in R$ and all $p \in U$, then the two are equal.

Remark 6.44. Theorem 6.41 shows that the Lie bracket of (complete) vector fields X, Y vanishes if and only if their flows commute. More precisely, the Lie bracket measures the *extent* to which the flows fail to commute. Indeed, for a function f, we have that

$$\Phi_t^* f = f + tX(f) + \cdots, \quad \Psi_s^* f = f + sY(f) + \cdots,$$

where the dots indicate higher order terms in the Taylor expansion. Using a short calculation, we find

$$(\Phi_t^* \Psi_s^* - \Psi_s^* \Phi_t^*) f = st\, [X,Y](f) + \cdots,$$

where the dots indicate terms cubic or higher in s,t.

The Jacobi identity (6.3) for vector fields

$$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0$$

also has interpretations in terms of Lie derivatives and flows. Bringing the last two terms to the right-hand side, and using skew-symmetry, the identity is equivalent to $[X,[Y,Z]] = [[X,Y],Z] + [Y,[X,Z]]$. That is,

$$L_X[Y,Z] = [L_X Y, Z] + [Y, L_X Z].$$

This says (by definition) that L_X is a derivation of the Lie bracket. The Jacobi identity is now explained as the derivative at $t = 0$ of the identity

$$\Phi_t^*[Y,Z] = [\Phi_t^* Y, \Phi_t^* Z], \tag{6.15}$$

where Φ_t is the flow of X (which we take to be complete, for simplicity).

98 (answer on page 301). Explain why the identity in (6.15) holds.

6.6 Frobenius Theorem

We saw that for any vector field $X \in \mathfrak{X}(M)$, there are *solution curves* through any given point $p \in M$. The image of such a curve is (an immersed) submanifold to which X is everywhere tangent. One might similarly ask about "integral surfaces" for pairs of vector fields X, Y and more generally "integral submanifolds" for collections of vector fields. But the situation gets more complicated. To see what can go wrong, recall that by Proposition 6.11, if two vector fields are tangent to a submanifold, then so is their Lie bracket.

99 (answer on page 301). On $\mathbb{R}^3$, consider the vector fields

$$X = \frac{\partial}{\partial x}, \; Y = \frac{\partial}{\partial y} + x\frac{\partial}{\partial z}.$$

Show that there does *not* exist a surface $S \subseteq \mathbb{R}^3$ such that both X and Y are everywhere tangent to S.

100 (answer on page 301). On $\mathbb{R}^3$, consider the vector fields

$$X = y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}, \;\; Y = z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}, \;\; Z = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}.$$

Show that for $p \neq (0,0,0)$, the vector fields X, Y, Z span a 2-dimensional subspace of $T_p\mathbb{R}^3$ and find a 2-dimensional surface S passing through p.

To formulate the "integrability problem," we make the following definition.

Definition 6.45. *Suppose $X_1, \ldots, X_r$ are vector fields on the manifold M, such that the tangent vectors*

$$X_1|_p, \ldots, X_r|_p \in T_pM$$

are linearly independent for all $p \in M$. An r-dimensional submanifold $S \subseteq M$ is called an integral submanifold *if the vector fields $X_1, \ldots, X_r$ are all tangent to S.*

Remark 6.46. In practice, one is given these vector fields only locally, on an open neighborhood U of a given point $p \in M$. (In such a case, simply replace M with U in the definition.) For instance, in 100, once it is observed that X, Y, Z span a 2-dimensional subspace of $T_p\mathbb{R}^3$, one would take X_1, X_2 to be two of the vector fields X, Y, Z that are linearly independent on a neighborhood U of p and replace $\mathbb{R}^3$ with that neighborhood.

Let us *suppose* that there we are given $X_1, \dots, X_r$ as above and that there exists an r-dimensional integral submanifold S through every given point $p \in M$. By Proposition 6.11, since the X_i are tangent to S, each of the Lie brackets $[X_i, X_j]$ is tangent to S at p and is hence a linear combination of $X_1|_p, \dots, X_r|_p$. It follows that

$$[X_i, X_j] = \sum_{k=1}^{r} c_{ij}^k X_k \tag{6.16}$$

for certain functions $c_{ij}^k \in C^\infty(M)$, where smoothness holds by 101 below. We refer to (6.16) as the *Frobenius condition*, named after F. G. Frobenius (1849–1917).

101 (answer on page 301). Show that if $X_1, \dots, X_r \in \mathfrak{X}(M)$ are linearly independent everywhere, and $Y \in \mathfrak{X}(M)$ is such that

$$Y|_p = \sum_{k=1}^{r} a_k(p) X_k|_p$$

for all $p \in M$, then the coefficients are smooth functions $a_k \in C^\infty(M)$.

102 (answer on page 302). Suppose that each of the two collections $X_1', \dots, X_r'$ and $X_1, \dots, X_r$ is linearly independent at all points $p \in M$, and spans the same subspace of T_pM everywhere. Show that the first set of vector fields satisfies the Frobenius condition if and only if the second set does.

We shall see that the Frobenius condition is not only necessary but also sufficient for the existence of integral submanifolds. Note that Lemma 6.48 below is a special case of Frobenius theorem; we will reduce the general case to this special case. We first prove a simpler version of Lemma 6.48, dealing with a single vector field.

The *critical set* of a vector field $X \in \mathfrak{X}(M)$ is the set of points $p \in M$ such that $X_p = 0$. The following result gives a local normal form for vector fields, away from their critical set.

Lemma 6.47 (Flow Straightening Lemma). *Let $X \in \mathfrak{X}(M)$ be a vector field, and $p \in M$ such that $X_p \neq 0$. Then there exists a coordinate chart (U, φ) about p, with corresponding local coordinates $x^1, \dots, x^m$, such that*

$$X|_U = \frac{\partial}{\partial x^1}.$$

In this Lemma, we are using the coordinate chart to identify U with the open subset $\varphi(U) \subseteq \mathbb{R}^m$ and thus think of $\frac{\partial}{\partial x^1}$ as a vector field on U. Avoiding the identification, one should write $X|_U \sim_\varphi \frac{\partial}{\partial x^1}$.

Proof. Choose a submanifold N of codimension 1, with $p \in N$, such that X is not tangent to N at p (see 103 below). The idea is to use the time variable t of the flow of X as the x^1-coordinate and complete to a coordinate system near p by choosing coordinates $x^2, \ldots, x^n$ on N. Let us first assume that X is complete, and denote its flow by $\Phi : \mathbb{R} \times M \to M$. (We indicate at the end of the proof how to deal with the incomplete case.)

Claim: The restriction of the flow,

$$\Phi|_{\mathbb{R}\times N} : \mathbb{R} \times N \to M, \tag{6.17}$$

has maximal rank at $(0, p)$.

Proof of Claim: With the standard identification $T_0\mathbb{R} = \mathbb{R}$, the tangent map to Φ at $(0, p)$ is

$$T_{(0,p)}\Phi : \mathbb{R} \times T_pM \to T_pM, \; (s, v) \mapsto v + sX_p. \tag{6.18}$$

Indeed, $(T_{(0,p)}\Phi)(0, v) = v$, since $\Phi|_{0\times M}$ is the identity map of M; on the other hand, $(T_{(0,p)}\Phi)(s, 0) = sX_p$ since $\Phi|_{\mathbb{R}\times\{p\}} : \mathbb{R} \to M$ is the integral curve of X through p. Hence, (6.18) restricts to an isomorphism $\mathbb{R} \times T_pN \to T_pM$, proving the claim.

By the inverse function theorem, there exists a neighborhood of $(0, p)$ in $\mathbb{R} \times N$ on which $\Phi|_{\mathbb{R}\times N}$ restricts to a diffeomorphism. We may take this neighborhood to be of the form $(-\varepsilon, \varepsilon) \times V$, where V is the domain of a chart (V, ψ) around p in N. In conclusion,

$$\Phi|_{(-\varepsilon,\varepsilon)\times V} : (-\varepsilon, \varepsilon) \times V \to M \tag{6.19}$$

is a diffeomorphism onto its image $U \subseteq M$. This diffeomorphism takes $\frac{\partial}{\partial t}$ to $X|_U$, by the property

$$\frac{\partial}{\partial t} \sim_\Phi X, \tag{6.20}$$

of the flow, which is just a restatement of (6.7). On the other hand, the coordinate map

$$(-\varepsilon, \varepsilon) \times V \to \mathbb{R}^m, \; (t, q) \mapsto (t, \psi(q))$$

takes $\frac{\partial}{\partial t}$ to the coordinate vector field $\frac{\partial}{\partial x^1}$. Hence, its composition with the map $U \to (-\varepsilon, \varepsilon) \times V$ inverse to (6.19) is the desired coordinate map $\varphi : U \to \mathbb{R}^m$, taking $X|_U$ to $\frac{\partial}{\partial x^1}$.

The case of possibly incomplete X is essentially the same, but working with the flow domain $\mathscr{J} \subseteq \mathbb{R} \times M$ in place of $\mathbb{R} \times M$. Choosing N as before, the intersection of the flow domain with $\mathbb{R} \times N$ is a codimension 1 submanifold of $\mathscr{J}$; as before, we find a neighborhood of the form $(-\varepsilon, \varepsilon) \times V \subseteq \mathscr{J} \cap (\mathbb{R} \times N)$ over which Φ restricts to a diffeomorphism. The rest of the proof is unchanged. ☐

103 (answer on page 302). Suppose $E \subseteq T_pM$ is a subspace of dimension k. Show that there exists a codimension k submanifold $N \subseteq M$, with $T_pM = T_pN \oplus E$.

The flow straightening lemma generalizes to collections of commuting vector fields. The proof uses the fact (cf. Theorem 6.41) that if vector fields commute, their flows commute. The idea is to use these flows to build a coordinate system.

Lemma 6.48. *Let $p \in M$, and let $X_1, \ldots, X_r \in \mathfrak{X}(M)$ be vector fields, whose values at $p \in M$ are linearly independent, and with*

$$[X_i, X_j] = 0$$

for all $1 \leq i, j \leq r$. Then there exists a coordinate chart (U, φ) near p, with corresponding local coordinates $x^1, \ldots, x^m$, such that

$$X_1|_U = \frac{\partial}{\partial x^1}, \;\ldots,\; X_r|_U = \frac{\partial}{\partial x^r}.$$

Proof. Again, we will first make the (very strong) assumption that the vector fields X_i are all complete. We will explain at the end how to deal with the general case.

Since the X_i commute, their flows Φ_i commute by Theorem 6.41. Consider the map

$$\Phi:\; \mathbb{R}^k \times M \to M,\;\; (t_1, \ldots, t_r, q) \mapsto \big((\Phi_1)_{t_1} \circ \cdots \circ (\Phi_r)_{t_r}\big)(q).$$

Using that the Φ_i commute, we obtain that

$$\frac{\partial}{\partial t_i} \sim_{\Phi} X_i \tag{6.21}$$

for $i = 1, \ldots, k$. As a consequence, the tangent map to Φ at $(0, \ldots, 0, p)$ is

$$T_{(0,\ldots,0,p)}\Phi(s_1, \ldots, s_r, v) = v + s_1 X_1|_p + \cdots + s_r X_r|_p.$$

Choose a codimension k submanifold $N \subseteq M$, with $p \in N$, such that T_pN is the complement to the subspace spanned by $X_1|_p, \ldots, X_r|_p$. Then the restriction of Φ to $\mathbb{R}^k \times N$ has maximal rank at $(0, \ldots, 0, p)$; hence, it is a diffeomorphism on some neighborhood of this point in $\mathbb{R}^k \times N$. We may take this neighborhood to be of the form $(-\varepsilon, \varepsilon)^k \times V$, with V the domain of a coordinate chart (V, ψ) around p in N. It then follows that

$$\Phi|_{(-\varepsilon,\varepsilon)^k \times V}:\; (-\varepsilon, \varepsilon)^k \times V \to M$$

is a diffeomorphism onto some open neighborhood $U \subseteq M$ of p. We take φ to be the inverse of this diffeomorphism followed by the map

$$(-\varepsilon, \varepsilon)^k \times V \to \mathbb{R}^m,\; (t_1, \ldots, t_r, q) \mapsto (t_1, \ldots, t_r, \psi(q)).$$

In the incomplete case, we replace $\mathbb{R}^k \times M$ with the open subset $\mathscr{J} \subseteq \mathbb{R}^k \times M$ on which Φ is defined, in the sense that (t_r, q) is in the flow domain of X_r, $(t_{r-1}, (\Phi_r)_{t_r})$ is in the flow domain of X_{r-1}, and so on. Since the $(\Phi_i)_{t_i}$ commute locally, for t_i sufficiently small, Equation (6.21) holds over a possibly smaller open neighborhood $\mathscr{J}'$ of $\{0\} \times M$ inside $\mathbb{R}^k \times M$. Taking $(-\varepsilon, \varepsilon)^k \times V$ to be contained in $\mathscr{J}'$, the rest of the proof is as before. $\square$

We are now in a position to prove the following result of Frobenius.

Theorem 6.49 (Frobenius Theorem). *Let $X_1,\ldots,X_r \in \mathfrak{X}(M)$ be vector fields such that $X_1|_p,\ldots,X_r|_p \in T_pM$ are linearly independent for all $p \in M$. The following are equivalent:*

(a) There exists an integral submanifold through every $p \in M$.
(b) The Lie brackets $[X_i,X_j]$ satisfy the Frobenius condition (6.16), for suitable functions $c_{ij}^k \in C^\infty(M)$.

In fact, it is then possible to find a coordinate chart (U,φ) near any given point of M, with coordinates denoted $(x^1,\ldots,x^m)$, in such a way that the integral submanifolds are given by $x^{r+1} = \mathrm{const},\ldots,x^m = \mathrm{const}$.

Proof. We have seen that the Frobenius condition (6.16) is necessary for the existence of integral submanifolds; it remains to show that it is also sufficient. Thus, suppose (6.16) holds, and consider $p \in M$.

By choosing a coordinate chart around p, we may assume that M is an open subset U of $\mathbb{R}^m$, with p the origin. Since $X_1|_p,\ldots,X_r|_p$ are linearly independent, they form part of a basis of $\mathbb{R}^m$, and by a linear change of coordinates, we can assume that the $X_i|_p$ coincide with the first r coordinate vectors at the origin. Thus

$$X_i = \sum_{j=1}^{r} a_{ij}(\mathbf{x})\frac{\partial}{\partial x^j} + \sum_{j=r+1}^{m} b_{ij}(\mathbf{x})\frac{\partial}{\partial x^j},$$

where $a_{ij}(\mathbf{0}) = \delta_{ij}$ and $b_{ij}(\mathbf{0}) = 0$. Since the matrix with entries a_{ij} is invertible at the origin, it remains invertible for points near the origin. Hence, we may define new vector fields $X_1',\ldots,X_r'$, on a possibly smaller neighborhood of the origin, such that

$$X_i = \sum_{j=1}^{r} a_{ij}X_j'.$$

By ✎ 102, the vector fields $X_1',\ldots,X_r'$ again satisfy the Frobenius condition, and clearly, S is an integral submanifold for $X_1',\ldots,X_r'$ if and only if it is an integral submanifold for $X_1,\ldots,X_r$. The new vector fields $X_1',\ldots,X_r'$ have the form

$$X_i' = \frac{\partial}{\partial x^i} + \sum_{j=r+1}^{m} b_{ij}'(\mathbf{x})\frac{\partial}{\partial x^j}, \tag{6.22}$$

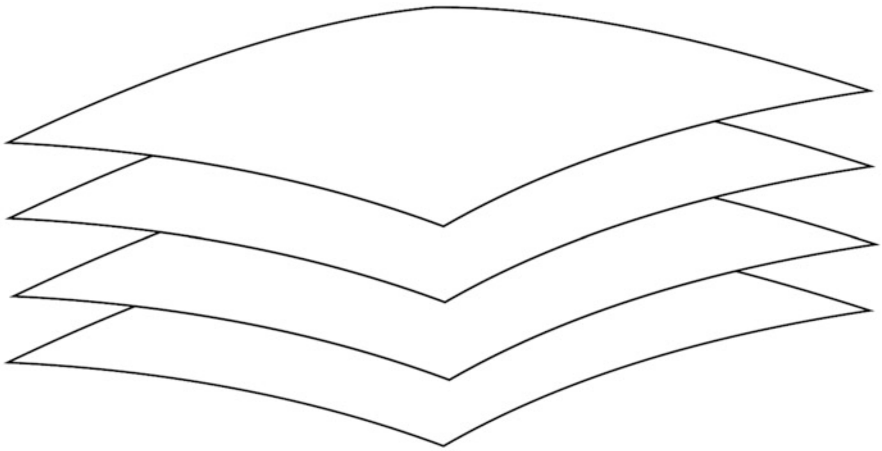
where $b_{ij}'(\mathbf{0}) = 0$.

We claim that the vector fields (6.22) commute. This follows from the Frobenius condition which says that

$$[X_i',X_j'] = \sum_{k=1}^{r} (c')_{ij}^k X_k'$$

for some functions $(c')_{ij}^k$. Let us compare the coefficients of $\frac{\partial}{\partial x^k}$, $k = 1,\ldots,r$ in the pointwise basis given by the coordinate vectors. On the right-hand side, the coefficient is $(c')_{ij}^k$. On the left-hand side, the coefficient is 0, since the Lie bracket between two vector fields of the form (6.22) lies in the pointwise span of $\frac{\partial}{\partial x_{r+1}}, \ldots, \frac{\partial}{\partial x^m}$. This means $(c')^k_{ij} = 0$ and establishes that $[X'_i, X'_j] = 0$, as claimed. By Lemma 6.48, we may change the coordinates to arrange that $X'_1, \ldots, X'_r$ become the first r coordinate vector fields. In such coordinates, it is evident that the level sets for the remaining coordinates are integral submanifolds. $\square$

Thus, if $X_1, \ldots, X_r$ are pointwise linearly independent and satisfy the Frobenius condition, then any $p \in M$ has an open neighborhood with a "nice" decomposition into r-dimensional integral submanifolds.



This is an example of a *foliation*.

There is a more general (and also more elegant) version of Frobenius theorem. Suppose $r \le m$ is given, and

$$\mathscr{E} \subseteq \mathfrak{X}(M)$$

is a subspace of the space of vector fields, with the following properties:

- For all $p \in M$, the subspace $E_p = \{X_p | \, X \in \mathscr{E}\}$ is an r-dimensional subspace of T_pM.
- If $X \in \mathfrak{X}(M)$ is a vector field with the property $X_p \in E_p$ for all $p \in M$, then $X \in \mathscr{E}$.

One calls $\mathscr{E}$ a *rank r distribution*. (Using the terminology of vector bundles, developed in Chapter 9, the definition says that $\mathscr{E}$ is the space of sections of a rank r subbundle $E \subseteq TM$.) Note that if $X_1, \ldots, X_r$ are pointwise linearly independent vector fields, as in the statement of Theorem 6.49, then one obtains a rank r distribution by letting $\mathscr{E}$ be the set of all $\sum_{i=1}^r f_i X_i$ with $f_i \in C^\infty(M)$. However, the new setting is more general, since it also allows for situations such as in ✎ 100.

An r-dimensional submanifold $S \subseteq M$ is called an *integral submanifold* of the rank r distribution $\mathscr{E}$ if all vector fields from $\mathscr{E}$ are tangent to S, or equivalently

$$T_pS = \{X_p | \, X \in \mathscr{E}\}$$

for all $p \in S$. The distribution $\mathscr{E}$ is called *integrable* if there exists an integral submanifold through every $p \in M$.

104 (answer on page 302). Let $\Phi : M \to N$ be a submersion. Show that

$$\mathscr{E} = \{X \in \mathfrak{X}(M) | \, X \sim_{\Phi} 0\}$$

is a rank r distribution, where $r = \dim M - \dim N$. Show that this distribution is integrable.

Using this terminology, we have the following version of Frobenius theorem:

Theorem 6.50 (Frobenius Theorem). *A rank r distribution $\mathscr{E}$ is integrable if and only if $\mathscr{E}$ is a Lie subalgebra of $\mathfrak{X}(M)$: That is, $X, Y \in \mathscr{E} \Rightarrow [X, Y] \in \mathscr{E}$.*

105 (answer on page 302). Explain how this version of Frobenius theorem follows from the earlier version, Theorem 6.49.

6.7 Problems

1. In each part, compute the Lie brackets $[X, Y]$ of the vector fields X and Y.
   (a)
   $$X = e^y \frac{\partial}{\partial x} + \sin(x) \frac{\partial}{\partial z}, \quad Y = x \frac{\partial}{\partial y} + z^2 \frac{\partial}{\partial z}.$$
   (b)
   $$X = e^r \sin(\theta) \frac{\partial}{\partial r}, \quad Y = r \cos(\theta) \frac{\partial}{\partial \theta}.$$
2. Let $X, Y \in \mathfrak{X}(\mathbb{R}^m)$ be the linear vector fields defined by $A, B \in \mathrm{Mat}_{\mathbb{R}}(m)$, as in (6.12). Compute $[X, Y]$.
3. Give an example of two vector fields $X, Y \in \mathfrak{X}(\mathbb{R}^3)$ such that for *all* $p \in \mathbb{R}^3$ the three tangent vectors
   $$X_p, \; Y_p, \; [X, Y]_p$$
   are a basis.
4. For any n, give an example of vector fields X, Y on $\mathbb{R}^n$ such that X, Y together with the iterated Lie brackets $[X, Y], [[X, Y], Y], \ldots$ span $T_p\mathbb{R}^n = \mathbb{R}^n$ for any $p \in \mathbb{R}^n$.
5. Let $\pi : M \to N$ be a surjective submersion. A vector field $X \in \mathfrak{X}(M)$ is called a *lift* of a vector field $Y \in \mathfrak{X}(N)$ if
   $$(T_p\pi)(X_p) = Y_{\pi(p)}$$
   for all $p \in M$.

(a) Show that if X_1 is a lift of Y_1 and X_2 is a lift of Y_2, then $[X_1, X_2]$ is a lift of $[Y_1, Y_2]$.
(b) Show that if X_1 is a lift of Y_1 and X_2 is a lift of Y_2, then $[X_1, X_2]$ is tangent to all fibers of π if and only if the vector fields Y_1 and Y_2 commute, i.e., $[Y_1, Y_2] = 0$.

6. The *Euler vector field* on $\mathbb{R}^m$ is the vector field

$$E = \sum_{i=1}^{m} x^i \frac{\partial}{\partial x^i} \in \mathfrak{X}(\mathbb{R}^m),$$

with flow $\Phi_t^E(\mathbf{x}) = e^t \mathbf{x}$ (cf. Example 6.20).
(a) Show that $f \in C^\infty(\mathbb{R}^n \backslash \{\mathbf{0}\})$ satisfies

$$L_E f = kf$$

for some $k \in \mathbb{R}$ if and only if it is *homogeneous of degree k*, i.e.,

$$f(\lambda \mathbf{x}) = \lambda^k f(\mathbf{x})$$

for all $\lambda > 0$. Similarly, $X = \sum_i a_i \frac{\partial}{\partial x^i} \in \mathfrak{X}(\mathbb{R}^n)$ satisfies $L_E X = kX$ if and only if the coefficient functions a_i are homogeneous of degree $k+1$.
(b) Show that if $f \in C^\infty(\mathbb{R}^m)$ is a non-zero function satisfying $L_E f = kf$ for $k \in \mathbb{R}$, then k is a non-negative integer and f is a homogeneous polynomial of degree k.
(c) Let $U = B_R(\mathbf{0}) \subseteq \mathbb{R}^m$ be an open ball around $\mathbf{0}$, and let $\Psi\colon U \to \Psi(U) \subseteq \mathbb{R}^m$ be a diffeomorphism with the property that $\Psi^* E = E|_U$. Prove that Ψ is the restriction of a linear map.

(Hint: In each case, begin by re-expressing the statement involving E in terms of its flow.)

7. Let $\gamma\colon J \to M$ be a solution curve of $X \in \mathfrak{X}(M)$. Suppose $\dot{\gamma}(t_0) = 0$ for some $t_0 \in J$. Show that $\gamma(t) = \gamma(t_0)$ for all $t \in \mathscr{J}$.

8. Show that

$$\Phi(t, x) = x + tx$$

cannot be the flow of a vector field X on $\mathbb{R}$.

9. Let $\gamma: J \to M$ be a maximal solution curve for $X \in \mathfrak{X}(M)$, and $f \in C^\infty(M)$ with $f > 0$ everywhere. Show that there exists a diffeomorphism $\psi : J \to \widetilde{J}$ onto an open interval such that $\widetilde{\gamma} = \gamma \circ \psi^{-1}$ is a maximal solution curve for $\widetilde{X} = fX$.

10. Let $M = \{x |\, x > 0\} \subseteq \mathbb{R}$.
(a) Show that the function

$$\Phi(t, x) = \left(\sqrt{x} + t\right)^2,$$

with domain of definition

$$\left\{(t,x)\,\middle|\, x>0,\ \sqrt{x}+t>0\right\}$$

is the flow of a vector field on M.

(b) Compute the vector field on M having the flow described in part (a).

11. (a) Let $f \in C^\infty(\mathbb{R})$ be any function such that the vector field $f(x)\frac{\partial}{\partial x}$ is incomplete. Show that the vector fields on $\mathbb{R}^2$,

$$X = f(y)\frac{\partial}{\partial x},\quad Y = f(x)\frac{\partial}{\partial y},$$

are complete, but their sum $X+Y$ is incomplete.

(b) Let $X,Y \in \mathfrak{X}(M)$, where X is complete and Y has compact support. Show that $X+Y$ is complete.

12. Consider the vector fields on $M = \mathbb{R}^3\backslash\{(x,y,z)|y=z\}$,

$$X = (y-z)\frac{\partial}{\partial x},\ \ Y = \frac{\partial}{\partial y}+\frac{\partial}{\partial z}.$$

Show that for all $p \in M$, there is an integral surface for X,Y through p. Find explicit descriptions for these surfaces.

13. Give an example of vector fields $X,Y \in \mathfrak{X}(\mathbb{R}^3)$ such that all of the following conditions hold:
   - X,Y are linearly independent at all points of $\mathbb{R}^3$.
   - $X,Y,Z=[X,Y]$ are linearly dependent at all points of the coordinate hyperplane given by $x=0$.
   - There is no 2-dimensional submanifold $S \subseteq \mathbb{R}^3$ such that X,Y are tangent to S.

14. Let $S \subseteq M$ be a submanifold and $X \in \mathfrak{X}(M)$. Prove that X is tangent to S if and only if for all $f \in C^\infty(M)$,

$$f|_S = 0 \quad \Rightarrow (X(f))|_S = 0.$$

15. Consider

$$S^3 = \{(x,y,z,w) \in \mathbb{R}^4|\ x^2+y^2+z^2+w^2 = 1\}.$$

(a) Show that

$$X = w\frac{\partial}{\partial x}+z\frac{\partial}{\partial y}-y\frac{\partial}{\partial z}-x\frac{\partial}{\partial w}$$

is tangent to S^3.

(b) Find another vector field Y (given by a similar formula) that is also tangent to S^3, and such that X, Y, and $Z := [X,Y]$ span the tangent space T_pS^3 for all $p \in S^3$.

Note: Only for $n = 1,3,7$ is it possible to find n vector fields spanning the tangent space to S^n for all $p \in S^n$. For instance, the 2-sphere S^2 does not even admit a vector field that is non-zero at all points of S^2 (see Corollary 8.40).

16. A *Lie group* is a group G, which is also a manifold, with the properties that the operations of group multiplication and inversion are smooth (e.g., the matrix Lie groups from Example 5.21). The tangent space at the group unit $e \in G$ is denoted $\mathfrak{g} = T_eG$.
(a) For $a \in G$, let $l_a : \; G \to G, \; g \mapsto ag$ be the *left translation*. Prove that l_a is a diffeomorphism of G.
(b) A vector field $X \in \mathfrak{X}(G)$ is called *left-invariant* if it satisfies $(T l_a)(X_g) = X_{ag}$ for all $a, g \in G$. Let $\mathfrak{X}(G)^L$ be the subspace of left-invariant vector fields. Prove that
$$\mathfrak{X}(G)^L \to T_eG = \mathfrak{g}, \;\; X \mapsto X_e$$
is a vector space isomorphism.
(c) Prove that the Lie bracket of two left-invariant vector fields is again left-invariant and use this to define the structure of a Lie algebra on $\mathfrak{g}$.

7

Differential Forms

In multivariable calculus, differential forms appear as a useful computational and organizational tool—unifying, for example, the various div, grad, and curl operations, and providing an elegant reformulation of the classical integration formulas of Green, Kelvin-Stokes, and Gauss. The full power of differential forms appears in their coordinate-free formulation on manifolds, which is the topic of this chapter.

7.1 Review: Differential Forms on $\mathbb{R}^m$

A differential k-form on an open subset $U \subseteq \mathbb{R}^m$ is an expression of the form

$$\omega = \sum_{i_1 \cdots i_k} \omega_{i_1 \dots i_k} \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k},$$

where $\omega_{i_1 \dots i_k} \in C^\infty(U)$ are functions, and the indices are numbers

$$1 \le i_1 < \cdots < i_k \le m.$$

Let $\Omega^k(U)$ be the vector space consisting of such expressions, with the obvious addition and scalar multiplication. It is convenient to introduce a shorthand notation $I = \{i_1, \dots, i_k\}$ for the index set and write $\omega = \sum_I \omega_I \mathrm{d}x^I$ with

$$\omega_I = \omega_{i_1 \dots i_k}, \quad \mathrm{d}x^I = \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k}.$$

Since a k-form is determined by these functions ω_I, and since there are $\binom{m}{k} = \frac{m!}{k!(m-k)!}$ ways of picking k-element subsets from $\{1, \dots, m\}$, the space $\Omega^k(U)$ can be identified with vector-valued smooth functions,

$$\Omega^k(U) = C^\infty(U, \mathbb{R}^{\frac{m!}{k!(m-k)!}}).$$

G. Gross, E. Meinrenken, *Manifolds, Vector Fields, and Differential Forms*,
Springer Undergraduate Mathematics Series,
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The $\mathrm{d}x^I$ are just formal expressions; at this stage they do not have any particular significance or meaning. The motivation for writing the differentials is to suggest an associative product operation

$$\Omega^k(U) \times \Omega^l(U) \to \Omega^{k+l}(U)$$

given by the "rule of computation"

$$\mathrm{d}x^i \wedge \mathrm{d}x^j = -\mathrm{d}x^j \wedge \mathrm{d}x^i$$

for all i, j; in particular, $\mathrm{d}x^i \wedge \mathrm{d}x^i = 0$. In turn, using the product structure we may define the *exterior differential*

$$\mathrm{d}\colon\ \Omega^k(U) \to \Omega^{k+1}(U),\ \ \mathrm{d}\Big(\sum_I \omega_I \mathrm{d}x^I\Big) = \sum_{i=1}^{m}\sum_I \frac{\partial \omega_I}{\partial x^i}\mathrm{d}x^i \wedge \mathrm{d}x^I. \tag{7.1}$$

The key property of the exterior differential is the following fact.

Proposition 7.1. *The exterior differential satisfies*

$$\mathrm{d}\circ\mathrm{d} = 0,$$

i.e., $\mathrm{dd}\omega = 0$ *for all* ω.

Proof. By definition,

$$\mathrm{dd}\omega = \sum_{j=1}^{m}\sum_{i=1}^{m}\sum_I \frac{\partial^2 \omega_I}{\partial x^j \partial x^i}\mathrm{d}x^j \wedge \mathrm{d}x^i \wedge \mathrm{d}x^I,$$

which vanishes by the equality of the mixed partials $\frac{\partial \omega_I}{\partial x^i \partial x^j} = \frac{\partial \omega_I}{\partial x^j \partial x^i}$. (We have $\mathrm{d}x^i \wedge \mathrm{d}x^j = -\mathrm{d}x^j \wedge \mathrm{d}x^i$, but the coefficients in front of $\mathrm{d}x^i \wedge \mathrm{d}x^j$ and $\mathrm{d}x^j \wedge \mathrm{d}x^i$ are the same.) □

106 (answer on page 303). Let $U \subseteq \mathbb{R}^m$ be an open subset.
(a) Show that $\Omega^k(U) = \{0\}$ for $k > m$.
(b) Write an expression for a general m-form $\omega \in \Omega^m(U)$. What is $\mathrm{d}\omega$?

107 (answer on page 303). Find the exterior differential of each of the following forms on $\mathbb{R}^3$ (with coordinates (x, y, z)).
(a) $\alpha = y^2 e^x \mathrm{d}y + 2ye^x \mathrm{d}x$.
(b) $\beta = y^2 e^x \mathrm{d}x + 2ye^x \mathrm{d}y$.
(c) $\rho = e^{x^2 y}\sin z\,\mathrm{d}x \wedge \mathrm{d}y + 2\cos(z^3 y)\mathrm{d}x \wedge \mathrm{d}z$.
(d) $\omega = \dfrac{\sin e^{xy} - \cos\sin z^3 x}{1 + (x+y+z)^4 + (7xy)^6}\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z$.

The exterior differential of forms on $\mathbb{R}^3$ is closely related to the operators div, grad, and curl from multivariable calculus (see Problem 2).

We will proceed to define differential forms on manifolds, beginning with 1-forms. In local charts (U,φ), 1-forms on U are identified with $\mathbb{R}^m$-valued functions, just as for vector fields. However, 1-forms on manifolds are quite different from vector fields, since their transformation properties under coordinate changes are different; in some sense they are "dual" objects. We will therefore begin with a review of dual spaces in general.

7.2 Dual Spaces

For any real vector space E, we denote by $E^* = \mathrm{Hom}(E,\mathbb{R})$ its dual space, consisting of all linear maps $\alpha : E \to \mathbb{R}$. We will assume that E is finite-dimensional. Then the dual space is also finite-dimensional and $\dim E^* = \dim E$.

It is common to write the value of $\alpha \in E^*$ on $v \in E$ as a *pairing*, using the bracket $\langle \cdot, \cdot \rangle$ notation:

$$\langle \alpha, v \rangle := \alpha(v).$$

This pairing notation emphasizes the duality between α and v. In the notation $\alpha(v)$ we think of α as a function acting on elements of E and in particular on v. However, one may just as well think of v as acting on elements of E^* by evaluation: $v(\alpha) = \alpha(v)$ for all $\alpha \in E^*$. This symmetry manifests notationally in the pairing notation.

Let $e_1,\ldots,e_r$ be a basis of E. Any element of E^* is determined by its values on these basis vectors. For $i = 1,\ldots,r$, let $e^i \in E^*$ (with *upper* indices) be the linear functional such that

$$\langle e^i,\, e_j \rangle = \delta^i{}_j = \begin{cases} 0 & \text{if} \quad i \neq j, \\ 1 & \text{if} \quad i = j. \end{cases}$$

The elements $e^1,\ldots,e^r$ are a basis of E^*; this is called the *dual basis*. An element $\alpha \in E^*$ is described in terms of the dual basis as

$$\alpha = \sum_{j=1}^{r} \alpha_j e^j, \quad \alpha_j = \langle \alpha, e_j \rangle. \tag{7.2}$$

Similarly, for vectors $v \in E$ we have

$$v = \sum_{i=1}^{r} v^i e_i, \quad v^i = \langle e^i, v \rangle. \tag{7.3}$$

Notice the placement of indices: Any given summation has a matching occurrence of the index of summation as an upper index (superscript) and a lower index (subscript).

Remark 7.2. As a special case, for $\mathbb{R}^r$ with its standard basis, we have a canonical identification $(\mathbb{R}^r)^* = \mathbb{R}^r$. For more general E with $\dim E < \infty$, there is no *canonical* isomorphism between E and E^* unless more structure is given. (For infinite-dimensional vector spaces E, the algebraic dual space E^* is not isomorphic to E.)

108 (answer on page 303). Let E be a finite-dimensional real vector space equipped with a (positive definite) inner product $(\cdot,\cdot)$. Show that the map

$$E \to E^*,\; v \mapsto A_v = (\cdot, v)$$

is an isomorphism of vector spaces.

Given a linear map $R:\ E \to F$ between vector spaces, one defines the *dual map*

$$R^* :\ F^* \to E^*$$

(note the direction) by setting

$$\langle R^*\beta,\ v\rangle = \langle \beta, R(v)\rangle$$

for $\beta \in F^*$ and $v \in E$. This satisfies $(R^*)^* = R$ and, under composition of linear maps,

$$(R_1 \circ R_2)^* = R_2^* \circ R_1^*.$$

In terms of bases $e_1, \dots, e_r$ of E and $f_1, \dots, f_s$ of F, and the corresponding dual bases (with upper indices), a linear map $R:\ E \to F$ is given by the matrix with entries

$$R_i{}^j = \langle f^j,\ R(e_i)\rangle,$$

so that by (7.3) we have

$$R(e_i) = \sum_{j=1}^{s} R_i{}^j f_j.$$

By definition of the dual map, $R_i{}^j = \langle f^j,\ R(e_i)\rangle = \langle R^* f^j,\ e_i\rangle$ so that by (7.2) we have

$$R^*(f^j) = \sum_{i=1}^{r} R_i{}^j\, e^i.$$

In particular, the (k,ℓ)-entry of the matrix representation of R^* is given by $R_\ell{}^k$, so that the matrix representation of R^* is the *transpose* of the matrix representation of R. This is compactly represented in the upper- and lower-indices notation as

$$(R^*)^j{}_i = R_i{}^j.$$

Remark 7.3. In the physics literature, it is common and convenient to use Dirac's *"bra-ket"* notation. Elements of E are the "kets" $v = |v\rangle$ (signified with the $|\cdot\rangle$ notation), while elements of E^* are the "bras" $\alpha = \langle\alpha|$ (signified with the $\langle\cdot|$ notation). The pairing between elements of E^* and E is then written as bra-kets (where we write $\langle\cdot|\cdot\rangle$ instead of $\langle\cdot||\cdot\rangle$)

$$\langle\alpha|\ v\rangle = \alpha(v).$$

Concatenating in the other direction $|\cdot\rangle\,\langle\cdot|$, a ket-bra $|w\rangle\langle\alpha|$ with $|w\rangle \in F$ and $\langle\alpha| \in E^*$ signifies the linear map

$$|w\rangle\langle\alpha| : E \to F, \ |v\rangle \mapsto |w\rangle\langle\alpha|v\rangle.$$

Note that if $e_1, \ldots, e_n$ is a basis of E, with dual basis $e^1, \ldots, e^n$, then the identity operator of E may be written as

$$I_E = \sum_i |e_i\rangle\langle e^i|.$$

The coefficients of a general linear map $R : E \to F$ with respect to bases $e_1, \ldots, e_n$ of E and $f_1, \ldots, f_m$ of F are $R_i{}^j = \langle f^j\,|R\,|e_i\rangle$, and one has the suggestive formula

$$|Re_i\rangle = R|e_i\rangle = \sum_j |f_j\rangle\langle f^j\,|R\,|e_i\rangle.$$

Denote by $\langle\alpha| = |\alpha\rangle$ the elements of E^*, but now playing the role of the "given" vector space, and $|v\rangle = \langle v|$ the elements of E but now viewed as elements of $(E^*)^* \cong E$ (assuming $\dim E < \infty$). Then $\langle v|\alpha\rangle = \langle\alpha|v\rangle$, and the definition of dual map reads as

$$\langle\alpha|R|v\rangle = \langle v|R^*|\alpha\rangle.$$

(Note: Here we only considered real vector spaces. In quantum mechanics, one mainly deals with *complex* vector spaces and declares $\langle v|\alpha\rangle = \langle\alpha|v\rangle^*$ (where the star denotes complex conjugate). Accordingly, the *conjugate transpose* is defined by $\langle\alpha|R|v\rangle = \langle v|R^*|\alpha\rangle^*$.)

7.3 Cotangent Spaces

After this general discussion of dual spaces, we now consider the duals of tangent spaces.

Definition 7.4. *The dual of the tangent space T_pM of a manifold M is called the* cotangent space *at p, denoted*

$$T_p^*M = (T_pM)^*.$$

Elements of T_p^*M are called cotangent vectors, or simply covectors. *Given a smooth map $F \in C^\infty(M,N)$ and any point $p \in M$, we have the* cotangent map

$$T_p^*F = (T_pF)^* : \ T^*_{F(p)}N \to T_p^*M$$

defined as the dual to the tangent map.

Thus, a covector at p is a linear functional on the tangent space, assigning to each tangent vector a number. The very definition of the tangent space suggests one such functional: Every function $f \in C^\infty(M)$ defines a linear map, $T_pM \to \mathbb{R}$ taking the tangent vector v to $v(f)$. This linear functional is denoted $(\mathrm{d}f)_p \in T_p^*M$.

Definition 7.5. *Let $f \in C^\infty(M)$ and $p \in M$. The covector*

$$(\mathrm{d}f)_p \in T_p^*M, \quad \langle (\mathrm{d}f)_p, v\rangle = v(f)$$

is called the differential of f at p.

109 (answer on page 304). Show that under the identification of tangent spaces $T_a\mathbb{R} \cong \mathbb{R}$ for $a \in \mathbb{R}$, the differential of f at p is the same as the tangent map

$$T_p f : \; T_pM \to T_{f(p)}\mathbb{R} = \mathbb{R}.$$

Lemma 7.6. *For $F \in C^\infty(M,N)$ and $g \in C^\infty(N)$,*

$$\mathrm{d}(F^*g)_p = T_p^*F((\mathrm{d}g)_{F(p)}).$$

Proof. Every element of the dual space is completely determined by its action on vectors; so suffice it to show that the pairing with any $v \in T_pM$ is the same. This is done by unpacking the definitions:

$$\begin{aligned}
\langle d(F^*g)_p, v\rangle &= v(F^*g) && \text{by definition of the differential}\\
&= v(g \circ F) && \text{by definition of the pullback of functions}\\
&= (T_pF(v))(g) && \text{by definition of the tangent map}\\
&= \langle (dg)_{F(p)}, T_pF(v)\rangle && \text{by definition of the differential}\\
&= \langle T_p^*F((dg)_{F(p)}), v\rangle && \text{by definition of the dual map.}
\end{aligned}$$

□

Consider an open subset $U \subseteq \mathbb{R}^m$, with coordinates $x^1, \ldots, x^m$. Here $T_pU \cong \mathbb{R}^m$, with basis

$$\frac{\partial}{\partial x^1}\bigg|_p, \ldots, \frac{\partial}{\partial x^m}\bigg|_p \in T_pU. \tag{7.4}$$

The basis of the dual space T_p^*U, dual to the basis (7.4), is given by the differentials of the coordinate functions:

$$(\mathrm{d}x^1)_p, \ldots, (\mathrm{d}x^m)_p \in T_p^*U.$$

Indeed,

$$\left\langle (\mathrm{d}x^i)_p, \; \frac{\partial}{\partial x^j}\bigg|_p \right\rangle = \frac{\partial}{\partial x^j}\bigg|_p (x^i) = \delta^i{}_j$$

as required. For $f \in C^\infty(M)$, the coefficients of $(\mathrm{d}f)_p = \sum_i \langle (\mathrm{d}f)_p, \; e_i\rangle e^i$ are determined as

$$\left\langle (\mathrm{d}f)_p, \; \frac{\partial}{\partial x^i}\bigg|_p \right\rangle = \frac{\partial}{\partial x^i}\bigg|_p (f) = \frac{\partial f}{\partial x^i}\bigg|_p.$$

Thus,

$$(\mathrm{d}f)_p = \sum_{i=1}^{m} \left.\frac{\partial f}{\partial x^i}\right|_p (\mathrm{d}x^i)_p.$$

Let $U \subseteq \mathbb{R}^m$ and $V \subseteq \mathbb{R}^n$ be open, with coordinates $x^1, \ldots, x^m$ and $y^1, \ldots, y^n$. For $F \in C^\infty(U,V)$, the tangent map is described by the Jacobian matrix, with entries

$$(D_pF)_i{}^j = \frac{\partial F^j}{\partial x^i}(p)$$

for $i = 1, \ldots, m,\ j = 1, \ldots, n$. We have:

$$(T_pF)\Big(\left.\frac{\partial}{\partial x^i}\right|_p\Big) = \sum_{j=1}^{n} (D_pF)_i{}^j \left.\frac{\partial}{\partial y^j}\right|_{F(p)},$$

hence dually

$$(T_pF)^*(\mathrm{d}y^j)_{F(p)} = \sum_{i=1}^{m} (D_pF)_i{}^j\, (\mathrm{d}x^i)_p. \tag{7.5}$$

We see that, as matrices in the given bases, *the coefficients of the cotangent map are the transpose of the coefficients of the tangent map*.

110 **(answer on page 304).** Consider $\mathbb{R}^3$ with standard coordinates denoted x, y, z, and $\mathbb{R}^2$ with standard coordinates denoted u, v. Let $F:\ \mathbb{R}^3 \to \mathbb{R}^2$ be given by

$$(x,y,z) \mapsto (x^2y + e^z, yz - x).$$

Find $T_pF\big(\left.\frac{\partial}{\partial x}\right|_p\big)$ and $(T_p^*F)\big((dv)_{F(p)}\big)$, for $p = (1,1,1)$.

7.4 1-forms

Similarly to the definition of vector fields, one can define *covector fields*, more commonly known as *1-forms*: Collections of covectors $\alpha_p \in T_p^*M$ depending smoothly on the base point. One approach to making precise the smooth dependence on the base point is to observe that in local coordinates 1-forms are given by expressions $\sum_i f_i \mathrm{d}x^i$ and smoothness should mean that the coefficient functions are smooth.

We will use the following (equivalent) approach. (Compare to the definition of a vector field and to Proposition 6.4.)

Definition 7.7. *A 1-form on M is a linear map*

$$\alpha :\ \mathfrak{X}(M) \to C^\infty(M), \quad X \mapsto \alpha(X) = \langle \alpha, X \rangle,$$

which is $C^\infty(M)$-linear *in the sense that*

$$\alpha(X+Y) = \alpha(X) + \alpha(Y),$$
$$\alpha(fX) = f\alpha(X)$$

for all $X, Y \in \mathfrak{X}(M)$ and $f \in C^\infty(M)$. The vector space of 1-forms is denoted $\Omega^1(M)$.

As the following lemma shows, a 1-form can be regarded as a collection of covectors.

Lemma 7.8. *Let $\alpha \in \Omega^1(M)$ be a 1-form and $p \in M$. Then there is a unique covector $\alpha_p \in T_p^*M$ such that*

$$\alpha(X)_p = \alpha_p(X_p)$$

for all $X \in \mathfrak{X}(M)$. In particular, $\alpha(X)_p = 0$ when $X_p = 0$.

(We indicate the value of the function $\alpha(X)$ at p by a subscript, just like we did for vector fields.)

Proof. We have to show that $\alpha(X)_p$ depends only on the value of X at p. By considering the difference of vector fields having the same value at p, it is enough to show that if $X_p = 0$, then $\alpha(X)_p = 0$. But any vector field vanishing at p can be written as a finite sum $X = \sum_i f_i Y_i$ where $f_i \in C^\infty(M)$ vanish at p. (For example, using local coordinates, we can take the Y_i to correspond to $\frac{\partial}{\partial x^i}$ near p, and the f_i to the coefficient functions.) By C^∞-linearity, this implies that

$$\alpha(X) = \alpha(\sum_i f_i Y_i) = \sum_i f_i \alpha(Y_i)$$

vanishes at p. □

The first example of a 1-form is described in the following definition.

Definition 7.9. *The* exterior differential *of a function $f \in C^\infty(M)$ is the 1-form*

$$\mathrm{d}f \in \Omega^1(M),$$

defined in terms of its pairings with vector fields $X \in \mathfrak{X}(M)$ as $\langle \mathrm{d}f,\ X \rangle = X(f)$.

(The reader should verify that this definition conforms to Definition 7.7.)

Clearly, $\mathrm{d}f$ is the 1-form defined by the family of covectors $(\mathrm{d}f)_p$, as in Definition 7.5. Note that the set of critical points of f may be described as the *zero set* of this 1-form: $p \in M$ is a critical point of f if and only if $(\mathrm{d}f)_p = 0$.

Just as for vector fields, 1-forms can be multiplied by functions. (This makes $\Omega^1(M)$ into a module over the algebra $C^\infty(M)$.) Hence one has more general examples of 1-forms as finite sums,

$$\alpha = \sum_i f_i \,\mathrm{d}g_i,$$

where $f_i, g_i \in C^\infty(M)$.

111 (answer on page 304). Prove the product rule

$$\mathrm{d}(fg) = f\mathrm{d}g + g\mathrm{d}f$$

for $f, g \in C^\infty(M)$.

Similarly to vector fields, 1-forms can be *restricted* to open subsets $U \subseteq M$.

Lemma 7.10. *Given an open subsets $U \subseteq M$ and any $\alpha \in \Omega^1(M)$, there is a unique 1-form $\alpha|_U \in \Omega^1(U)$ such that*

$$(\alpha|_U)_p = \alpha_p$$

for all $p \in U$.

112 (answer on page 305). Prove this Lemma. (You will need to define $\alpha|_U(Y)$ for all $Y \in \mathfrak{X}(U)$; here Y need not be a restriction of a vector field on M. Use bump functions to resolve this issue.)

Let us describe the space of 1-forms on open subsets $U \subseteq \mathbb{R}^m$. Given $\alpha \in \Omega^1(U)$, we have

$$\alpha = \sum_{i=1}^{m} \alpha_i \, \mathrm{d}x^i$$

with coefficient functions $\alpha_i = \left\langle \alpha, \, \frac{\partial}{\partial x^i} \right\rangle \in C^\infty(U)$: The right-hand side takes on the correct values at any $p \in U$ and is uniquely determined by those values. General vector fields on U may be written

$$X = \sum_{i=1}^{m} X^i \frac{\partial}{\partial x^i}$$

(to match the notation for 1-forms, we write the coefficients as X^i rather than a^i, as we did in the past), where the coefficient functions are recovered as $X^i = \langle \mathrm{d}x^i, \, X \rangle$. The pairing of the 1-form α with the vector field X is then

$$\langle \alpha, \, X \rangle = \sum_{i=1}^{m} \alpha_i X^i.$$

Lemma 7.11. *Let $\alpha : \; p \mapsto \alpha_p \in T_p^* M$ be a collection of covectors. Then α defines a 1-form, with*

$$\alpha(X)_p = \alpha_p(X_p)$$

for $p \in M$, if and only if for all charts (U, φ), the coefficient functions for α in the chart are smooth.

113 (answer on page 305). Prove Lemma 7.11. (You may want to use Lemma 7.10.)

7.5 Pullbacks of Function and 1-forms

Recall again that for any manifold M, the vector space $C^\infty(M)$ of smooth functions is an algebra, with product given by pointwise multiplication. Any smooth map $F \in C^\infty(M,N)$ between manifolds defines an algebra homomorphism, called the *pullback*

$$F^* : \; C^\infty(N) \to C^\infty(M), \;\; f \mapsto F^*(f) := f \circ F.$$

114 (answer on page 305). Show that the pullback is indeed an algebra homomorphism by showing that it preserves sums and products:

$$F^*(f) + F^*(g) = F^*(f+g) \quad ; \quad F^*(f)F^*(g) = F^*(fg).$$

Next, show that if $F : \; M \to N$ and $G : \; N \to M$ are two smooth maps between manifolds, then

$$(G \circ F)^* = F^* \circ G^*.$$

(Note the order.)

Recall that for vector fields, there are no general "push-forward" or "pullback" operations under smooth maps $F \in C^\infty(M,N)$, unless F is a diffeomorphism. For 1-forms the situation is better. Indeed, for any $p \in M$ one has the dual to the tangent map

$$T_p^* F = (T_p F)^* : \; T^*_{F(p)} N \to T_p^* M.$$

For a 1-form $\beta \in \Omega^1(N)$, we can therefore define

$$(F^*\beta)_p := (T_p^* F)(\beta_{F(p)}). \tag{7.6}$$

The following lemma shows that this collection of covectors on M defines a 1-form.

Lemma 7.12. *There is a unique 1-form $F^*\beta \in \Omega^1(M)$ such that the covectors $(F^*\beta)_p \in T_p^* M$ are given by (7.6).*

Proof. We shall use Lemma 7.11. To check smoothness near a given $p \in M$, choose coordinate charts (V, ψ) around $F(p)$ and (U, φ) around p, with $F(U) \subseteq V$. Using these charts, we may in fact assume that $M = U$ is an open subset of $\mathbb{R}^m$ (with coordinates x^i) and $N = V$ is an open subset of $\mathbb{R}^n$ (with coordinates y^j). Write

$$\beta = \sum_{j=1}^{n} \beta_j(y)\, \mathrm{d}y^j.$$

By (7.5), the pullback of β is given by

$$F^*\beta = \sum_{i=1}^{m} \Big(\sum_{j=1}^{n} \beta_j(F(x)) \frac{\partial F^j}{\partial x^i} \Big) \mathrm{d}x^i. \tag{7.7}$$

In particular, the coefficients are smooth. $\square$

Lemma 7.12 shows that we have a well-defined pullback map

$$F^* : \ \Omega^1(N) \to \Omega^1(M), \ \ \beta \mapsto F^*\beta.$$

With respect to composition of smooth maps F_1 and F_2, this pullback operation on 1-forms satisfies

$$(F_1 \circ F_2)^* = F_2^* \circ F_1^*.$$

On the other hand, if $g \in C^\infty(N)$ is a smooth function and $F^*g = g \circ F$ is its pullback to M, we have

$$F^*(g\beta) = F^*g\, F^*\beta.$$

Moreover, Lemma 7.6 implies that this pullback "commutes" with the exterior derivative:

$$F^*(\mathrm{d}g) = \mathrm{d}(F^*g).$$

(Note that on the left we are pulling back a form, and on the right a function.)

Formula (7.7)—which is a consequence of Formula (7.5)—shows how to compute the pullback of forms in local coordinates. In detail, suppose $F : \ U \to V$ is a smooth map between open subsets $U \subseteq \mathbb{R}^m$ and $V \subseteq \mathbb{R}^n$ with coordinates x^i and y^j, respectively. Given $\beta = \sum_j \beta_j(y)\mathrm{d}y^j$, one computes $F^*\beta$ by *replacing* y *with* $F(x)$:

$$F^*\beta = \sum_j \beta_j(F(x))\mathrm{d}(F(x))^j = \sum_{ij} \beta_j(F(x))\frac{\partial F^j}{\partial x^i}\mathrm{d}x^i.$$

115 (answer on page 306). Consider the map

$$F : \ \mathbb{R}^3 \to \mathbb{R}^2, \quad (x,y,z) \mapsto (x^3 e^{yz}, \sin x).$$

Let u, v be the coordinates on the target space $\mathbb{R}^2$.

(a) Compute the pullback under F of the 1-forms

$$du, \quad v\cos(u)\mathrm{d}v.$$

(b) Let $g \in C^\infty(\mathbb{R}^2)$ be the function $g(u,v) = uv$. Verify Lemma 7.6 by computing $\mathrm{d}g$, $F^*(\mathrm{d}g)$, F^*g, and $\mathrm{d}(F^*g)$.

In the case of vector fields, one has neither pullback nor push-forward in general, but instead works with the notion of related vector fields, $X \sim_F Y$. The proposition below shows that this fits nicely with the pullback of 1-forms.

Proposition 7.13. *Let $F \in C^\infty(M,N)$, and let $\beta \in \Omega^1(N)$. If $X \in \mathfrak{X}(M)$ and $Y \in \mathfrak{X}(N)$ are F-related, i.e., $X \sim_F Y$, then*

$$\langle F^*\beta, X\rangle = F^*\langle \beta, Y\rangle.$$

Proof. We verify this identity pointwise, at any $p \in M$:

$$\begin{aligned}
\langle F^*\beta, X\rangle_p &= \langle (F^*\beta)_p, X_p\rangle \\
&= \langle T_p^* F(\beta_{F(p)}), X_p\rangle \\
&= \langle \beta_{F(p)}, T_p F(X_p)\rangle \\
&= \langle \beta_{F(p)}, Y_{F(p)}\rangle \\
&= \langle \beta, Y\rangle_{F(p)} \\
&= (F^*\langle \beta, Y\rangle)_p.
\end{aligned}$$

□

116 (answer on page 306). One may verify that $\gamma\colon\ \mathbb{R} \to \mathbb{R}^2,\ \ t \mapsto (\cos t, \sin t)$ is a solution curve of the vector field

$$X = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}.$$

That is, $\frac{\partial}{\partial t} \sim_\gamma X$ (cf. Equation (6.7)). Let $\beta = \mathrm{d}x - \mathrm{d}y \in \Omega^1(\mathbb{R}^2)$. Verify the conclusion of Proposition 7.13 by computing each of

$$\langle \beta, X\rangle,\ \ \gamma^*\beta,\ \ \langle \gamma^*\beta, \frac{\partial}{\partial t}\rangle,\ \ \gamma^*\langle \beta, X\rangle.$$

7.6 Integration of 1-forms

Given a curve $\gamma\colon\ J \to M$ in a manifold, and any 1-form $\alpha \in \Omega^1(M)$, we can consider the pullback $\gamma^*\alpha \in \Omega^1(J)$. By the description of 1-forms on $\mathbb{R}$, this is of the form

$$\gamma^*\alpha = f(t)\mathrm{d}t \tag{7.8}$$

for some smooth function $f \in C^\infty(J)$.

To discuss integration, it is convenient to work with closed intervals rather than open intervals. Let $[a,b] \subseteq \mathbb{R}$ be a closed interval. A map $\gamma\colon\ [a,b] \to M$ into a manifold will be called *smooth* if it extends to a smooth map from an open interval containing $[a,b]$. We will call such a map a smooth *path*.

Definition 7.14. *Given a smooth path $\gamma\colon\ [a,b] \to M$, we define the integral of a 1-form $\alpha \in \Omega^1(M)$ along γ as*

$$\int_\gamma \alpha = \int_a^b f(t)\mathrm{d}t,$$

where f is the function defined by $\gamma^*\alpha = f(t)\mathrm{d}t$.

The fundamental theorem of calculus has the following consequence for manifolds. It is a special case of *Stokes' theorem* (Theorem 8.7).

Proposition 7.15. *Let $\gamma : [a,b] \to M$ be a smooth path, with end points $\gamma(a) = p,\ \gamma(b) = q$. For any $f \in C^\infty(M)$, we have*

$$\int_\gamma \mathrm{d}f = f(q) - f(p).$$

In particular, the integral of $\mathrm{d}f$ *depends only on the end points, rather than the path itself.*

Proof. We have

$$\gamma^*(\mathrm{d}f) = \mathrm{d}(\gamma^* f) = \mathrm{d}(f \circ \gamma) = \frac{\partial (f \circ \gamma)}{\partial t} \mathrm{d}t.$$

Integrating from a to b, we obtain, by the fundamental theorem of calculus, $f(\gamma(b)) - f(\gamma(a))$. □

117 (answer on page 306). Let $\gamma : [a,b] \to \mathbb{R}$ be a smooth path in $M = \mathbb{R}$, and let $f \in C^\infty(\mathbb{R})$ be a smooth function, so that $f\mathrm{d}x$ is a 1-form. Let $F \in C^\infty(\mathbb{R})$ be a primitive, i.e., $F'(x) = f(x)$.

(a) Verify that

$$\int_\gamma f \mathrm{d}x = \int_{\gamma(a)}^{\gamma(b)} f(s)\mathrm{d}s,$$

where the right-hand side is a Riemann integral. (Hint: $f\mathrm{d}x = \mathrm{d}F$.)

(b) Use part (a) to prove the "integration by substitution" formula:

$$\int_{\gamma(a)}^{\gamma(b)} f(x)\mathrm{d}x = \int_a^b f(\gamma(t)) \cdot \gamma'(t)\, \mathrm{d}t,$$

where both sides are Riemann integrals.

118 (answer on page 307). Let $\gamma : [a,b] \to M$ be a smooth path, with end points $\gamma(a) = p,\ \gamma(b) = q$, and let $f, g \in C^\infty(\mathbb{R})$ be smooth functions. Prove the "integration by parts" formula

$$\int_\gamma f\, \mathrm{d}g = f(q)g(q) - f(p)g(p) - \int_\gamma g\, \mathrm{d}f.$$

Let M be a manifold, and $\gamma : [a,b] \to M$ a smooth path. A *reparametrization* of the path is a path $\gamma \circ \kappa : [c,d] \to M$, where $\kappa : [c,d] \to [a,b]$ is a diffeomorphism (in the sense that it extends to a diffeomorphism on slightly larger open intervals). The reparametrization is called *orientation preserving* if $\kappa(c) = a,\ \kappa(d) = b$; *orientation reversing* if $\kappa(c) = b,\ \kappa(d) = a$.

Proposition 7.16 (Reparametrization Invariance of the Integral). *Given a reparametrization $\gamma \circ \kappa$ of the path γ as above, and any $\alpha \in \Omega^1(M)$,*

$$\int_\gamma \alpha = \pm \int_{\gamma\circ\kappa} \alpha,$$

with the plus sign if κ preserves orientation and minus sign of it reverses orientation.

Proof. Since $(\gamma \circ \kappa)^* = \kappa^* \circ \gamma^*$ we have

$$\int_{\gamma\circ\kappa} \alpha = \int_c^d (\gamma\circ\kappa)^*\alpha = \int_c^d \kappa^*(\gamma^*\alpha) = \int_{\kappa(c)}^{\kappa(d)} \gamma^*\alpha,$$

where the last equality follows from integration by substitution, as in 117. If κ is orientation preserving, we therefore have

$$\int_{\gamma\circ\kappa} \alpha = \int_{\kappa(c)}^{\kappa(d)} \gamma^*\alpha = \int_a^b \gamma^*\alpha = \int_\gamma \alpha.$$

If κ is orientation reversing, we have

$$\int_{\gamma\circ\kappa} \alpha = \int_{\kappa(c)}^{\kappa(d)} \gamma^*\alpha = \int_b^a \gamma^*\alpha = -\int_a^b \gamma^*\alpha = -\int_\gamma \alpha. \qquad \square$$

Remark 7.17. We did not fully use the fact that κ is a diffeomorphism. The conclusion holds for any smooth path $\kappa\colon\ [c,d] \to \mathbb{R}$ taking values in $[a,b]$, so that $\gamma\circ\kappa$ is defined.

119 (answer on page 307). Consider the 1-form on $\mathbb{R}^2$

$$\alpha = y^2 e^x \mathrm{d}x + 2ye^x \mathrm{d}y.$$

Find the integral of α along the path

$$\gamma\colon\ [0,1] \to M,\ \ t \mapsto (\sin(\pi t/2), t^3).$$

A 1-form $\alpha \in \Omega^1(M)$ such that $\alpha = \mathrm{d}f$ for some function $f \in C^\infty(M)$ is called *exact*. Proposition 7.15 gives a necessary condition for exactness: The integral of α along paths should depend only on the end points.

Remark 7.18. This *path independence* condition is also sufficient: Define f on the connected components of M, by fixing a base point p_0 on each such component, and put $f(p) = \int_\gamma \alpha$ for any path from p_0 to p. With a little work (using charts), one verifies that f defined in this way is smooth.

If M is an open subset $U \subseteq \mathbb{R}^m$, so that $\alpha = \sum_i \alpha_i \mathrm{d}x^i$, then $\alpha = \mathrm{d}f$ means that $\alpha_i = \frac{\partial f}{\partial x^i}$. A necessary condition for exactness is therefore the equality of the mixed partial derivatives,

$$\frac{\partial \alpha_i}{\partial x^j} = \frac{\partial \alpha_j}{\partial x^i}. \tag{7.9}$$

In multivariable calculus one learns that this condition is also sufficient, provided U is simply connected (e.g., convex)—a result known as *Poincaré Lemma* (cf. Problems 11 and 12 at the end of the chapter). Using the exterior differential of forms in $\Omega^1(U)$, this condition becomes $\mathrm{d}\alpha = 0$. Since α is a 1-form, $\mathrm{d}\alpha$ is a 2-form. Thus, to obtain a coordinate-free version of the condition, we need higher order forms.

7.7 k-forms

To get a feeling for higher degree forms, and constructions with higher forms, we first discuss 2-forms.

7.7.1 2-forms

Definition 7.19. *A 2-form on M is a $C^\infty(M)$-bilinear skew-symmetric map*

$$\alpha: \mathfrak{X}(M) \times \mathfrak{X}(M) \to C^\infty(M), \ \ (X,Y) \mapsto \alpha(X,Y).$$

The space of 2-forms is denoted $\Omega^2(M)$.

Here skew-symmetry means that $\alpha(X,Y) = -\alpha(Y,X)$ for all vector fields X,Y, while $C^\infty(M)$-bilinearity means that for any fixed Y, the map $X \mapsto \alpha(X,Y)$ is $C^\infty(M)$-linear, and for any fixed X, the map $Y \mapsto \alpha(X,Y)$ is $C^\infty(M)$-linear. (Actually, by skew-symmetry it suffices to require $C^\infty(M)$-linearity in the first argument.)

By the same argument as for 1-forms, the value $\alpha(X,Y)_p$ depends only on the values X_p, Y_p. Also, if α is a 2-form then so is $f\alpha$ for any smooth function f.

Our first examples of 2-forms are obtained from 1-forms: Let $\alpha, \beta \in \Omega^1(M)$. Then we define a *wedge product* $\alpha \wedge \beta \in \Omega^2(M)$ as follows:

$$(\alpha \wedge \beta)(X,Y) = \alpha(X)\beta(Y) - \alpha(Y)\beta(X). \tag{7.10}$$

Notice that the right-hand side may be written as a determinant

$$\det \begin{pmatrix} \alpha(X) & \alpha(Y) \\ \beta(X) & \beta(Y) \end{pmatrix}.$$

120 (answer on page 307). Show that Equation (7.10) indeed defines a 2-form.

Another example of a 2-form is the exterior differential of a 1-form. We will soon give a general definition of the differential of any k-form; the following will be a special case.

121 (answer on page 307). Show that for any 1-form $\alpha \in \Omega^1(M)$, the following formula defines a 2-form $\mathrm{d}\alpha \in \Omega^2(M)$:

$$(\mathrm{d}\alpha)(X,Y) = L_X(\alpha(Y)) - L_Y(\alpha(X)) - \alpha([X,Y]).$$

Show furthermore that if $\alpha = \mathrm{d}f$ for a function $f \in C^\infty(M)$, then $\mathrm{d}\alpha = 0$.

For an open subset $U \subseteq \mathbb{R}^m$, a 2-form $\omega \in \Omega^2(U)$ is uniquely determined by its values on coordinate vector fields, by $C^\infty(U)$-bilinearity. By skew-symmetry the functions

$$\omega_{ij} = \omega\Big(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\Big)$$

satisfy $\omega_{ij} = -\omega_{ji}$; hence suffice it to know these functions for $i < j$. As a consequence, we see that the most general 2-form on U is

$$\omega = \sum_{i<j} \omega_{ij} \mathrm{d}x^i \wedge \mathrm{d}x^j.$$

122 (answer on page 308). Using 121 as the definition of the exterior differential of a 2-form, show that the differential of $\alpha = \sum_i \alpha_i \mathrm{d}x^i \in \Omega^1(U)$ is

$$\mathrm{d}\alpha = \sum_{i<j} \Big(\frac{\partial \alpha^j}{\partial x^i} - \frac{\partial \alpha^i}{\partial x^j}\Big) \mathrm{d}x^i \wedge \mathrm{d}x^j.$$

Review the discussion around Equation (7.9) from this perspective.

7.7.2 k-forms

We now generalize to forms of arbitrary degree.

Definition 7.20. *Let k be a non-negative integer. A* k-form *on M is a $C^\infty(M)$-multilinear, skew-symmetric map*

$$\alpha : \underbrace{\mathfrak{X}(M) \times \cdots \times \mathfrak{X}(M)}_{k \text{ times}} \to C^\infty(M).$$

The space of k-forms is denoted $\Omega^k(M)$; in particular, $\Omega^0(M) = C^\infty(M)$.

Here, *skew-symmetry* means that $\alpha(X_1,\dots,X_k)$ changes sign under exchange of any two of its arguments. For example,

$$\alpha(X_1,X_2,X_3,\dots) = -\alpha(X_2,X_1,X_3,\dots).$$

More generally, denoting by S_k the group of permutations of $\{1,\dots,k\}$, and by $\mathrm{sign}(s) = \pm 1$ the sign of a permutation $s \in S_k$ ($+1$ for an even permutation, -1 for an odd permutation) we have

$$\alpha(X_{s(1)},\dots,X_{s(k)}) = \mathrm{sign}(s)\alpha(X_1,\dots,X_k). \tag{7.11}$$

(See Appendix B.1 for more on permutations.)

The $C^\infty(M)$*-multilinearity* means that for any index i, and any fixed $k-1$ vector fields $X_1,\dots,X_{i-1},X_{i+1},\dots,X_k \in \mathfrak{X}(M)$, the map

$$\mathfrak{X}(M) \to C^\infty(M),\;\; X \mapsto \alpha(X_1,\dots,X_{i-1},X,X_{i+1},\dots,X_k)$$

is $C^\infty(M)$-linear, i.e., a 1-form. (Given the skew-symmetry, it suffices to check $C^\infty(M)$-linearity in any one of the arguments, for instance, for $i=1$.)

The $C^\infty(M)$-multilinearity implies, in particular, that α is *local* in the sense that the value of $\alpha(X_1,\dots,X_k)$ at any given $p \in M$ depends only on the values $X_1|_p,\dots,X_k|_p \in T_pM$. (This is an application of Lemma 7.8 to any of the arguments.) One thus obtains for any $p \in M$ a skew-symmetric multilinear form

$$\alpha_p :\; T_pM \times \cdots \times T_pM \to \mathbb{R}.$$

For any open subset $U \subseteq M$, one has a *restriction map*

$$\Omega^k(M) \to \Omega^k(U),\;\; \alpha \mapsto \alpha|_U$$

such that $(\alpha|_U)_p = \alpha_p$ for all $p \in M$. The argument for this is essentially the same as for 1-forms, see Lemma 7.10.

Given a $C^\infty(M)$-multilinear map $\eta :\; \mathfrak{X}(M) \times \cdots \times \mathfrak{X}(M) \to C^\infty(M)$ (with k arguments), not necessarily skew-symmetric, one obtains a k-form $\alpha = \mathrm{Sk}\eta \in \Omega^k(M)$ through the process of *skew-symmetrization*:

$$(\mathrm{Sk}\eta)(X_1\dots,X_k) = \sum_{s\in S_k} \mathrm{sign}(s)\,\eta(X_{s(1)},\dots,X_{s(k)}).$$

123 (answer on page 308). Confirm that $\mathrm{Sk}\eta$ does indeed define a k-form. Also show that if $\alpha \in \Omega^k(M)$ (so that α is already skew-symmetric) then $\mathrm{Sk}\alpha = k!\,\alpha$.

This may be applied, for example, to define the wedge product of 1-forms $\alpha_1,\dots,\alpha_k \in \Omega^1(M)$, as the skew-symmetrization of the multilinear form

$$(X_1,\dots,X_k) \mapsto \langle\alpha_1,X_1\rangle\cdots\langle\alpha_k,X_k\rangle.$$

That is, $\alpha_1 \wedge \cdots \wedge \alpha_k \in \Omega^k(M)$ is given by the formula

$$(\alpha_1 \wedge \cdots \wedge \alpha_k)(X_1, \ldots, X_k) = \sum_{s \in S_k} \mathrm{sign}(s) \alpha_1(X_{s(1)}) \cdots \alpha_k(X_{s(k)}).$$

(More general wedge products will be discussed below.)

Let us describe the space of k-forms on open subsets $U \subseteq \mathbb{R}^m$. Using C^∞-multilinearity, a k-form $\alpha \in \Omega^k(U)$ is uniquely determined by its values on coordinate vector fields $\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^m}$, i.e., by the functions

$$\alpha_{i_1 \ldots i_k} = \alpha\Big(\frac{\partial}{\partial x^{i_1}}, \ldots, \frac{\partial}{\partial x^{i_k}}\Big).$$

By skew-symmetry we only need to consider *ordered* index sets $I = \{i_1, \ldots, i_k\} \subseteq \{1, \ldots, m\}$, that is, $i_1 < \cdots < i_k$. Using the wedge product notation, we obtain

$$\alpha = \sum_{i_1 < \cdots < i_k} \alpha_{i_1 \ldots i_k} \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k}. \tag{7.12}$$

For k-forms ω on general manifolds M, this gives a description of $\omega|_U$ in coordinate charts (U, φ). Let us also note the following useful consequence.

Lemma 7.21. *Every differential form $\alpha \in \Omega^k(M)$ is locally, near a given point $p \in M$, a linear combination of k-forms of the type*

$$f_0 \, \mathrm{d}f_1 \wedge \cdots \wedge \mathrm{d}f_k \in \Omega^k(M), \tag{7.13}$$

where $f_0, \ldots, f_k \in C^\infty(M)$.

Proof. Given $p \in M$, choose a coordinate chart (U, φ) around p. In these coordinates, $\alpha|_U$ has the form (7.12). This is not quite the desired form since the coordinate functions and coefficient functions are only defined on U. But this is easily fixed: Choose $h_1, \ldots, h_m \in C^\infty(M)$ so that $h_i \circ \varphi^{-1}$ agrees with the coordinate function x^i near $\varphi(p)$, and choose $g_{i_1,\ldots,i_k} \in C^\infty(M)$ such that $g_{i_1,\ldots,i_k} \circ \varphi^{-1}$ agrees with $\alpha_{i_1 \ldots i_k}$ near p. Then

$$\sum_{i_1 < \cdots < i_k} g_{i_1,\ldots,i_k} \mathrm{d}h_{i_1} \wedge \cdots \wedge \mathrm{d}h_{i_k}$$

agrees with α near p. □

7.7.3 Wedge Product

We next turn to the definition of a *wedge product* of forms of arbitrary degree.

Definition 7.22. *The wedge product of $\alpha \in \Omega^k(M)$ and $\beta \in \Omega^l(M)$ is the element*

$$\alpha \wedge \beta \in \Omega^{k+l}(M)$$

given by the formula

$$(\alpha \wedge \beta)(X_1, \ldots, X_{k+l}) = \\ \frac{1}{k!\,l!} \sum_{s \in S_{k+l}} \mathrm{sign}(s) \alpha(X_{s(1)}, \ldots, X_{s(k)}) \, \beta(X_{s(k+1)}, \ldots, X_{s(k+l)}). \tag{7.14}$$

124 (answer on page 308). Show that Definition 7.22 is consistent with our previous definition of the wedge product of two 1-forms: Equation (7.10).

Thus, up to a factor $1/k!\,l!$, the wedge product of α and β is the skew-symmetrization of the map

$$(X_1,\ldots,X_{k+l}) \mapsto \alpha(X_1,\ldots,X_k)\beta(X_{k+1},\ldots,X_{k+l});$$

in particular, it is indeed a $(k+l)$-form.

Note that many of the $(k+l)!$ terms in the sum over S_{k+l} coincide, since α is skew-symmetric in its arguments to begin with, and likewise for β. Indeed, one can get a simpler expression involving only permutations where

$$s(1) < \cdots < s(k), \quad s(k+1) < \cdots < s(k+l).$$

A permutation $s \in S_{k+l}$ having this property is called a (k,l)-*shuffle*. Denote by $S_{k,l}$ the set of (k,l)-shuffles. Every (k,l)-shuffle is uniquely determined by a k-element subset of $\{1,\ldots,k+l\}$ (by taking this subset to be $s(1),\ldots,s(k)$). In particular, there are

$$\binom{k+l}{k} = \frac{(k+l)!}{k!l!}$$

different (k,l)-shuffles.

Example 7.23. The permutation

$$(1\ 4\ 2\ 3\ 5) \in S_5$$

(meaning $s(1)=1,\ s(2)=4,\ s(3)=2,\ s(4)=3,$ and $s(5)=5$) is a $(3,2)$-shuffle, since the first three elements are in order, and likewise the last two elements.

125 (answer on page 309). List all $(3,2)$-shuffles in S_5. How many elements does $S_{3,2}$ have?

Using the notion of (k,l)-shuffles, the wedge product is also given by the formula

$$(\alpha\wedge\beta)(X_1,\ldots,X_{k+l}) = \sum_{s\in S_{k,l}} \operatorname{sign}(s)\alpha(X_{s(1)},\ldots,X_{s(k)})\,\beta(X_{s(k+1)},\ldots,X_{s(k+l)}). \tag{7.15}$$

We tend to prefer this version since it has fewer terms.

126 (answer on page 309). For $\alpha,\beta \in \Omega^2(M)$, and $X_1,X_2,X_3,X_4 \in \mathfrak{X}(M)$, write all the terms of

$$(\alpha \wedge \beta)(X_1,X_2,X_3,X_4).$$

It will be better to use Formula (7.15) with only 6 terms, rather than (7.14) with $4! = 24$ terms (which coincide in groups of 4).

127 (answer on page 309). Identify $S_k \times S_l$ as the subgroup of S_{k+l} preserving $\{1,\dots,k\}$ and (hence) also $\{k+1,\dots,k+l\}$. Show that every $s \in S_{k+l}$ is uniquely a product

$$s = s's'',$$

where $s' \in S_{k,l}$ and $s'' \in S_k \times S_l$. Use this to prove the second formula (7.15) for the wedge product as a sum over (k,l)-shuffles.

It is clear that the wedge product $\alpha \wedge \beta$ is $C^\infty(M)$-bilinear in α and in β. That is, for a fixed β, the map

$$\Omega^k(M) \to \Omega^{k+l}(M),\ \alpha \mapsto \alpha \wedge \beta$$

is $C^\infty(M)$-linear, and for a fixed α the map $\beta \mapsto \alpha \wedge \beta$ is $C^\infty(M)$-linear. In addition, the wedge product has the following properties.

Proposition 7.24. *(a) The wedge product is* graded commutative*: If $\alpha \in \Omega^k(M)$ and $\beta \in \Omega^l(M)$ then*

$$\alpha \wedge \beta = (-1)^{kl} \beta \wedge \alpha.$$

(b) The wedge product is associative: Given $\alpha_i \in \Omega^{k_i}(M)$ we have

$$(\alpha_1 \wedge \alpha_2) \wedge \alpha_3 = \alpha_1 \wedge (\alpha_2 \wedge \alpha_3).$$

The associativity allows us to drop parentheses when writing wedge products.

Proof. (a) There is a canonical bijection between (k,l)-shuffles and (l,k)-shuffles, obtained by interchanging the first k components and last l components. For example,

$$(1\ 2\ 4\ 3\ 5) \in S_{3,2} \quad \leftrightarrow \quad (3\ 5\ 1\ 2\ 4) \in S_{2,3}.$$

In more detail, let $\sigma \in S_{k+l}$ be the permutation

$$\sigma(1) = k+1, \dots,\ \sigma(l) = k+l,\ \sigma(l+1) = 1, \dots, \sigma(l+k) = k.$$

This has sign

$$\mathrm{sign}(\sigma) = (-1)^{kl} \tag{7.16}$$

and we have that

$$s \in S_{k,l} \Leftrightarrow s' = s \circ \sigma \in S_{l,k}. \tag{7.17}$$

Therefore, for any $X_1, \ldots, X_{k+l} \in \mathfrak{X}(M)$,

$$\begin{aligned}
&(\beta \wedge \alpha)(X_1, \ldots, X_{k+l}) \\
&= \sum_{s' \in S_{l,k}} \operatorname{sign}(s')\beta(X_{s'(1)}, \ldots, X_{s'(l)})\alpha(X_{s'(l+1)}, \ldots, X_{s'(k+l)}) \\
&= \sum_{s \in S_{k,l}} \operatorname{sign}(s \circ \sigma)\beta(X_{(s\circ\sigma)(1)}, \ldots, X_{(s\circ\sigma)(l)})\alpha(X_{(s\circ\sigma)(l+1)}, \ldots, X_{(s\circ\sigma)(k+l)}) \\
&= \operatorname{sign}(\sigma) \sum_{s \in S_{k,l}} \operatorname{sign}(s)\beta(X_{s(k+1)}, \ldots, X_{s(k+l)})\alpha(X_{s(1)}, \ldots, X_{s(k)}) \\
&= (-1)^{kl}(\alpha \wedge \beta)(X_{\sigma(1)}, \ldots, X_{\sigma(k+l)}).
\end{aligned}$$

(b) Define a (k,l,m)-*shuffle* to be a permutation s in S_{k+l+m} such that

$$s(1) < \cdots < s(k), \;\; s(k+1) < \cdots < s(k+l), \;\; s(k+l+1) < \cdots < s(k+l+m).$$

By careful bookkeeping, one finds that each of

$$((\alpha_1 \wedge \alpha_2) \wedge \alpha_3)(X_1, \ldots, X_{k_1+k_2+k_3}), \quad (\alpha_1 \wedge (\alpha_2 \wedge \alpha_3))(X_1, \ldots, X_{k_1+k_2+k_3})$$

is given by the same formula

$$\begin{aligned}
\sum_{s \in S_{k_1,k_2,k_3}} &\operatorname{sign}(s)\alpha_1(X_{s(1)}, \ldots, X_{s(k_1)})\alpha_2(X_{s(k_1+1)}, \ldots, X_{s(k_1+k_2)}) \\
&\alpha_3(X_{s(k_1+k_2+1)}, \ldots, X_{s(k_1+k_2+k_3)}). \qquad \square
\end{aligned}$$

128 (answer on page 310). Give some details of the "careful bookkeeping" in the proof of part (b). (Hint: Mimic 127.)

7.7.4 Exterior Differential

Recall that we have defined the exterior differential of 0-forms (i.e., functions) by the formula

$$(\mathrm{d}f)(X) = X(f). \tag{7.18}$$

(In 121, we also indicated a possible definition of d on 1-forms.) We will now extend this definition to all forms.

Theorem 7.25. *There is a unique collection of linear maps* $\mathrm{d}\colon\ \Omega^k(M) \to \Omega^{k+1}(M)$*, extending the map (7.18) for* $k = 0$*, such that* $\mathrm{d}(\mathrm{d}f) = 0$ *for* $f \in C^\infty(M)$*, and such that the* graded product rule *holds: For* $\alpha \in \Omega^k(M)$ *and* $\beta \in \Omega^l(M)$*,*

$$\mathrm{d}(\alpha \wedge \beta) = \mathrm{d}\alpha \wedge \beta + (-1)^k \alpha \wedge \mathrm{d}\beta. \tag{7.19}$$

This exterior differential *has the property*

$$\mathrm{d}(\mathrm{d}\alpha) = 0$$

for all $\alpha \in \Omega^k(M)$.

Proof. Let us first *assume* that such an exterior differential exists. We will establish uniqueness and along the way give a formula.

Observe first that d is necessarily *local*, in the sense that for any open subset $U \subseteq M$ the restriction $(\mathrm{d}\alpha)|_U$ depends only on $\alpha|_U$. Equivalently, $\alpha|_U = 0 \Rightarrow (\mathrm{d}\alpha)|_U = 0$. Indeed, if $\alpha|_U = 0$, and given any $p \in U$, we may choose $f \in C^\infty(M) = \Omega^0(M)$ such that $\mathrm{supp}(f) \subseteq U$ and $f|_p = 1$. Then $f\alpha = 0$, hence the product rule (7.19) gives

$$0 = \mathrm{d}(f\alpha) = \mathrm{d}f \wedge \alpha + f\mathrm{d}\alpha.$$

Evaluating at p we obtain $(\mathrm{d}\alpha)_p = 0$, as claimed.

Now, locally a k-form is a linear combination of expressions $f_0\mathrm{d}f_1 \wedge \cdots \wedge \mathrm{d}f_k$ (cf. Lemma 7.21). The graded product rule and the property $\mathrm{d}\mathrm{d}f = 0$ force us to define

$$\mathrm{d}(f_0\mathrm{d}f_1 \wedge \cdots \wedge \mathrm{d}f_k) = \mathrm{d}f_0 \wedge \mathrm{d}f_1 \wedge \cdots \wedge \mathrm{d}f_k,$$

which specifies d uniquely. We also see that $\mathrm{d} \circ \mathrm{d} = 0$ on k-forms of any degree, since this is the case for $f_0\mathrm{d}f_1 \wedge \cdots \wedge \mathrm{d}f_k$.

For open subsets of $U \subseteq \mathbb{R}^m$, we are thus forced to define the differential of

$$\alpha = \sum_{i_1 < \cdots < i_k} \alpha_{i_1 \cdots i_k} \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k} \in \Omega^k(U)$$

as

$$\mathrm{d}\alpha = \sum_{i_1 < \cdots < i_k} \mathrm{d}\alpha_{i_1 \cdots i_k} \wedge \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k} \in \Omega^{k+1}(U).$$

Conversely, we may use this explicit formula to *define* $(\mathrm{d}\alpha)|_U = \mathrm{d}(\alpha|_U)$ for any coordinate chart domain U; by uniqueness, the local definitions agree on overlaps of any two coordinate chart domains. □

Definition 7.26. *A k-form* $\omega \in \Omega^k(M)$ *is called* exact *if* $\omega = \mathrm{d}\alpha$ *for some* $\alpha \in \Omega^{k-1}(M)$*. It is called* closed *if* $\mathrm{d}\omega = 0$.

By the $\mathbb{R}$-linearity of d, the collection of closed k-forms is a subspace of $\Omega^k(M)$, as is the collection of exact k-forms. Since $\mathrm{d} \circ \mathrm{d} = 0$, the exact k-forms are a subspace of the space of closed k-forms. For the case of 1-forms, we have seen that the integral $\int_\gamma \alpha$ of an exact 1-form $\alpha = \mathrm{d}f$ along a smooth path $\gamma\colon\ [a,b] \to M$ is given by the difference of the values at the end points $p = \gamma(a)$ and $q = \gamma(b)$; in particular, for an exact 1-form the integral does not depend on the choice of path from p to q. In particular, if γ is a *loop* (that is, $p = q$) the integral is zero. A *necessary* condition for α to be exact is that it is closed. An example of a 1-form that is closed but not exact is

$$\alpha = \frac{x\mathrm{d}y - y\mathrm{d}x}{x^2 + y^2} \in \Omega^1(\mathbb{R}^2 \backslash \{\mathbf{0}\}).$$

129 (answer on page 310). Prove that the 1-form α above is closed but not exact.

The quotient space (closed k-forms modulo exact k-forms) is a vector space called the k-th (de Rham) *cohomology*

$$H^k(M) = \frac{\{\alpha \in \Omega^k(M) |\ \alpha \text{ is closed }\}}{\{\alpha \in \Omega^k(M) |\ \alpha \text{ is exact }\}}. \tag{7.20}$$

(See Appendix B.3 for generalities about quotients of vector spaces.) Note that when M is connected, then $H^0(M) = \mathbb{R}$. This is because a closed 0-form is simply a locally constant function f; but for a connected M, a locally constant function must be (globally) constant (cf. 25). On the other hand, there are no non-zero exact 0-forms.

130 (answer on page 311). Explain why there are no exact 0-forms other than 0.

It turns out that whenever M is compact (and often also if M is non-compact), $H^k(M)$ is a finite-dimensional vector space. The dimension of this vector space

$$b_k(M) = \dim H^k(M)$$

is called the k-th *Betti number* of M; these numbers are important invariants of M which one can use to distinguish non-diffeomorphic manifolds. For example, if $M = \mathbb{C}\mathrm{P}^n$ one can show that

$$b_k(\mathbb{C}\mathrm{P}^n) = 1 \quad \text{for} \quad k = 0, 2, \ldots, 2n,$$

and $b_k(\mathbb{C}\mathrm{P}^n) = 0$ otherwise. For $M = S^n$ the Betti numbers are (see Problem 15 at the end of this chapter and Problem 18 at the end of the next chapter)

$$b_k(S^n) = 1 \quad \text{for} \quad k = 0, n,$$

while $b_k(S^n) = 0$ for all other k. Hence $\mathbb{C}\mathrm{P}^n$ cannot be diffeomorphic to S^{2n} unless $n = 1$.

For an example when M is not compact we have

$$b_k(\mathbb{R}^n) = 1 \quad \text{for} \quad k = 0,$$

while $b_k(\mathbb{R}^n) = 0$ for all other k. Put differently, for $k > 0$, every closed k-form on $\mathbb{R}^n$ is exact. This fact is known as the *Poincaré lemma* whose proof is outlined in Problems 11 and 12 at the end of the chapter.

Remark 7.27. Even though they are defined in terms of differential forms, *de Rham's theorem* shows that the (de Rham) cohomology groups are *topological* invariants. In particular, homeomorphic manifolds have the same cohomology groups, even if they are not diffeomorphic (cf. Remark 3.21).

7.8 Lie Derivatives and Contractions*

Any vector field $X \in \mathfrak{X}(M)$ determines a $C^\infty(M)$-linear map

$$\iota_X : \ \Omega^k(M) \to \Omega^{k-1}(M)$$

called *contraction by* X. Thinking of $\alpha \in \Omega^k(M)$ as a $C^\infty(M)$-multilinear form, one simply puts X into the first slot:

$$(\iota_X\alpha)(X_1, \ldots, X_{k-1}) = \alpha(X, X_1, \ldots, X_{k-1}).$$

(If $k = 0$ so that α is a function, one puts $\iota_X\alpha = 0$.)

Given two vector fields X and Y, we have that

$$\iota_X \circ \iota_Y + \iota_Y \circ \iota_X = 0 \tag{7.21}$$

as operators on forms, due to skew-symmetry

$$\alpha(X, Y, X_1, \ldots, X_{k-2}) = -\alpha(Y, X, X_1, \ldots, X_{k-2}).$$

Contractions have the following compatibility with the wedge product, which is similar to Equation (7.19) for the exterior differential,

$$\iota_X(\alpha \wedge \beta) = \iota_X\alpha \wedge \beta + (-1)^k \alpha \wedge \iota_X\beta, \tag{7.22}$$

for all $\alpha \in \Omega^k(M)$ and $\beta \in \Omega^l(M)$.

131 (answer on page 311). Prove Equation (7.22).

132 (answer on page 312). Compute $\iota_Z\alpha$ for

$$\alpha = \sin(x)\mathrm{d}x \wedge \mathrm{d}y, \ \ Z = e^x\frac{\partial}{\partial x} - \frac{\partial}{\partial y}.$$

The exterior differential d and the contraction operators ι_X are both examples of *odd superderivations*, where "super" refers to the signs appearing in the product rule. In the "super" world, a sign change appears whenever two odd objects move past each other: For example, in (7.22) there is a minus in the second term whenever α is odd, since the "odd" operator ι_X appears to the right of α in the second term.

More formally, a collection of linear maps

$$D:\ \Omega^k(M) \to \Omega^{k+r}(M)$$

is called a *degree r superderivation* if it has the property

$$D(\alpha \wedge \beta) = D(\alpha) \wedge \beta + (-1)^{rk} \alpha \wedge D\beta$$

for $\alpha \in \Omega^k(M)$ and $\beta \in \Omega^l(M)$. Thus, d is a superderivation of degree 1, while ι_X is a superderivation of degree -1.

133 (answer on page 312). Show that if D_1 and D_2 are degree r_1 and degree r_2 superderivations on differential forms, respectively, then their *supercommutator* (using the $[\cdot,\cdot]$ notation)

$$[D_1, D_2] = D_1 \circ D_2 - (-1)^{r_1 r_2} D_2 \circ D_1$$

is a degree $r_1 + r_2$ superderivation.

Note that the identity $\mathrm{d} \circ \mathrm{d} = 0$ may be written using supercommutators, as $[\mathrm{d}, \mathrm{d}] = 0$, while the identity $\iota_X \circ \iota_Y + \iota_Y \circ \iota_X = 0$ now reads as $[\iota_X, \iota_Y] = 0$.

From contractions and differentials, we obtain a superderivation $[\mathrm{d}, \iota_X]$ of degree $1 + (-1) = 0$. We shall temporarily take this to be the definition of the *Lie derivative* $L_X:\ \Omega^k(M) \to \Omega^k(M)$,

$$L_X = \mathrm{d} \circ \iota_X + \iota_X \circ \mathrm{d}. \tag{7.23}$$

Thus, $L_X = [\mathrm{d}, \iota_X]$ in supercommutator notation. A "better" definition will be given in the next section, where the simple formula (7.23) will be realized as the result of a theorem. Note that (7.23) is consistent with the earlier notion of the Lie derivative of functions $f \in C^\infty(M) = \Omega^0(M)$:

$$L_X(f) = (\mathrm{d} \circ \iota_X)(f) + (\iota_X \circ \mathrm{d})(f) = \iota_X(\mathrm{d}f) = X(f).$$

By the general result from 133, L_X satisfies the product rule

$$L_X(\alpha \wedge \beta) = L_X \alpha \wedge \beta + \alpha \wedge L_X \beta, \tag{7.24}$$

for $\alpha \in \Omega^k(M)$ and $\beta \in \Omega^l(M)$, as a consequence of the product rules for ι_X and d. The product rule is frequently used for computations.

134 (answer on page 312). For each of the following vector fields $X \in \mathfrak{X}(\mathbb{R}^3)$ and differential forms $\alpha \in \Omega^k(\mathbb{R}^3)$ on $\mathbb{R}^3$, compute the Lie derivative $L_X \alpha$.

(a) $X = y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}$ and $\alpha = -y\mathrm{d}x - x\mathrm{d}y - z\mathrm{d}z$.

(b) $X = \cos z\frac{\partial}{\partial x} + x^3\frac{\partial}{\partial y} + yz^3\frac{\partial}{\partial z}$ and $\alpha = \sin(xy) + (x+y+z)^2$.

(c) $X = \cos z\frac{\partial}{\partial x} - xyz\frac{\partial}{\partial y}$ and $\alpha = (z+y^2)\mathrm{d}x \wedge \mathrm{d}z$.

To summarize, we have introduced three operators

$$\mathrm{d} : \ \Omega^k(M) \to \Omega^{k+1}(M), \ \ L_X : \ \Omega^k(M) \to \Omega^k(M), \ \ \iota_X : \ \Omega^k(M) \to \Omega^{k-1}(M),$$

respectively called the exterior differential, the Lie derivative with respect to $X \in \mathfrak{X}(M)$, and the contraction by $X \in \mathfrak{X}(M)$. These are superderivations of degrees $1, 0, -1$, respectively. One might expect getting other superderivations by taking further supercommutators, but nothing new is obtained.

Theorem 7.28 (Cartan Calculus). *The operators exterior differentiation, Lie derivation, and contraction satisfy the supercommutator relations*

$$[\mathrm{d},\mathrm{d}] = 0, \tag{7.25}$$
$$[L_X, L_Y] = L_{[X,Y]}, \tag{7.26}$$
$$[\iota_X, \iota_Y] = 0, \tag{7.27}$$
$$[\mathrm{d}, L_X] = 0, \tag{7.28}$$
$$[L_X, \iota_Y] = \iota_{[X,Y]}, \tag{7.29}$$
$$[\mathrm{d}, \iota_X] = L_X. \tag{7.30}$$

This collection of identities is referred to as the *Cartan calculus*, after Élie Cartan (1861–1951), and in particular the last identity is called the *Cartan formula* [3]. Basic contributions to the theory of differential forms were made by his son Henri Cartan (1906–1980), who also wrote a textbook [2] on the subject.

Proof. The identities (7.25), (7.27), (7.30) have already been discussed. The identity (7.28) is proved from the definitions, and using $\mathrm{d} \circ \mathrm{d} = 0$:

$$\begin{aligned} [\mathrm{d}, L_X] &= \mathrm{d} \circ L_X - L_X \circ \mathrm{d} \\ &= \mathrm{d} \circ (\iota_X \circ \mathrm{d} + \mathrm{d} \circ \iota_X) - (\iota_X \circ \mathrm{d} + \mathrm{d} \circ \iota_X) \circ \mathrm{d} \\ &= 0. \end{aligned}$$

Consider the identity (7.29). Both $D = [L_X, \iota_Y]$ and $D' = \iota_{[X,Y]}$ are superderivations of degree -1. The identity $D = D'$ is true for functions $f \in C^\infty(M) = \Omega^0(M)$ since both sides act as zero on functions for degree reasons. The identity also holds for differentials of functions, since

$$
\begin{aligned}
[L_X, \iota_Y]\mathrm{d}f &= (L_X \circ \iota_Y)\mathrm{d}f - (\iota_Y \circ L_X)\mathrm{d}f \\
&= L_X L_Y f - \iota_Y \mathrm{d}L_X f \quad \text{using (7.30), (7.28)} \\
&= L_X L_Y f - L_Y L_X f \quad \text{using (7.30)} \\
&= L_{[X,Y]} f.
\end{aligned}
$$

Here the last line uses $[L_X, L_Y] = L_{[X,Y]}$ *on functions*, by definition of the Lie bracket.

By 135 below, the equality of D and D' on functions and differentials of functions shows $D = D'$. The remaining identity (7.26) is left to the reader (see 136). ☐

135 (answer on page 313).

(a) Show that if D is a degree r superderivation on differential forms and $U \subseteq M$ is open, then $(D\alpha)|_U$ depends only on $\alpha|_U$.

(b) Show that if two degree r superderivations D and D' satisfy $Df = D'f$ and $D(\mathrm{d}f) = D'(\mathrm{d}f)$ for all functions $f \in C^\infty(M)$, then $D = D'$.

136 (answer on page 313). Complete the proof of Theorem 7.28 by proving (7.26).

137 (answer on page 314). Show $\iota_X \circ \iota_X = 0$ as a consequence of the Cartan calculus.

138 (answer on page 314).

(a) Use the Cartan calculus to prove the following formula for the exterior differential of a 1-form $\alpha \in \Omega^1(M)$ given in 121

$$
(\mathrm{d}\alpha)(X,Y) = L_X\big(\alpha(Y)\big) - L_Y\big(\alpha(X)\big) - \alpha([X,Y]).
$$

(b) Prove a similar formula for the exterior differential of a 2-form.

The formulas for the differentials of 1-forms and 2-forms generalize to arbitrary k-forms $\alpha \in \Omega^k(M)$. For $X_1, \ldots, X_{k+1} \in \mathfrak{X}(M)$ we have that

$$
\begin{aligned}
(\mathrm{d}\alpha)(X_1, \ldots, X_{k+1}) = &\sum_{i=1}^{k+1} (-1)^{i+1} L_{X_i}(\alpha(X_1, \ldots, \widehat{X_i}, \ldots X_{k+1}) \\
&+ \sum_{i<j} (-1)^{i+j} \alpha([X_i, X_j], X_1, \ldots, \widehat{X_i}, \ldots, \widehat{X_j}, \ldots, X_{k+1}),
\end{aligned}
$$

where the hat notation $\widehat{X_i}$ indicates that the entry X_i is absent. We leave the proof of this general formula as Problem 5.

7.9 Pullbacks

Let $F \in C^\infty(M,N)$ be a smooth map between manifolds. Similar to the pullback of functions (0-forms) and 1-forms, we have a pullback operation for k-forms.

Proposition 7.29. *There is a well-defined linear map*

$$F^* : \ \Omega^k(N) \to \Omega^k(M)$$

such that for all $\beta \in \Omega^k(N)$ and all $p \in M$,

$$(F^*\beta)_p(v_1,\ldots,v_k) = \beta_{F(p)}\big(T_pF(v_1),\ldots,T_pF(v_k)\big). \tag{7.31}$$

The pullback operation satisfies

$$F^*(\beta_1 \wedge \beta_2) = F^*\beta_1 \wedge F^*\beta_2 \tag{7.32}$$

as well as

$$F^* \circ \mathrm{d} = \mathrm{d} \circ F^*. \tag{7.33}$$

Proof. We have to check that the collection of multilinear forms on T_pM given by (7.31) does indeed define a *smooth* k-form $F^*\beta \in \Omega^k(M)$.

Observe that the pullback operation is *local*: If $U \subseteq M$ and $V \subseteq N$ are open subsets with $F(U) \subseteq V$, then $(F^*\beta)_p$ for $p \in U$ depends only on $\beta|_V$. Note also that for wedge products, $F^*(\beta_1 \wedge \beta_2)_p = (F^*\beta_1)_p \wedge (F^*\beta_2)_p$. Locally, near a given point $F(p) \in N$, every β is a linear combination of forms

$$g_0\, \mathrm{d}g_1 \wedge \cdots \wedge \mathrm{d}g_k \in \Omega^k(N) \tag{7.34}$$

with $g_i \in C^\infty(N)$ (cf. Lemma 7.21). Hence, it is enough to show that $F^*\beta$ is smooth when β is a function g or the differential of such a function. But $F^*g = g \circ F$ is the usual pullback of functions, hence is smooth. We claim $F^*(\mathrm{d}g) = \mathrm{d}F^*g$. which shows that $F^*(\mathrm{d}g)$ is smooth as well. Given $p \in M$ and $v \in T_pM$ we calculate

$$(F^*(\mathrm{d}g))_p(v) = (\mathrm{d}g)_{F(p)}(T_pF(v)) = (T_pF(v))(g) = v(F^*g) = \mathrm{d}(F^*g)_p(v),$$

proving the claim.

Thus, the pullback operation is well-defined. Equation (7.32) follows from the pointwise property, and (7.33) is proved by applying both sides to expressions (7.34), using that (7.33) holds for functions g and differentials of functions $\mathrm{d}g$. $\square$

139 (answer on page 314). Let $F \in C^\infty(M,N)$ and $G \in C^\infty(N,Q)$ be smooth maps between manifolds. Show that the pullback on k-forms satisfies

$$(G \circ F)^* = F^* \circ G^*.$$

Equation (7.33) shows how F^* interacts with the differential. As for contractions and Lie derivatives with respect to vector fields, we have the following statement: If $X \in \mathfrak{X}(M)$ and $Y \in \mathfrak{X}(N)$ are F-related vector fields $X \sim_F Y$, then

$$\iota_X \circ F^* = F^* \circ \iota_Y, \quad L_X \circ F^* = F^* \circ L_Y.$$

We leave the proof as Problem 4 at the end of the chapter.

In local coordinates, if $F: U \to V$ is a smooth map between open subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$, with coordinates $x^1, \ldots, x^m$ and $y^1, \ldots, y^n$, the pullback just amounts to "putting $y = F(x)$."

140 (answer on page 315). Denote the coordinates on $\mathbb{R}^3$ by x, y, z and those on $\mathbb{R}^2$ by u, v. Let $F: \mathbb{R}^3 \to \mathbb{R}^2$, $(x, y, z) \mapsto (y^2 z, x)$. Compute

$$F^*(\mathrm{d}u \wedge \mathrm{d}v).$$

The following proposition is the "key fact" we shall need in order to define integration of differential forms.

Proposition 7.30. *Let $U \subseteq \mathbb{R}^m$ with coordinates x^i, and $V \subseteq \mathbb{R}^n$ with coordinates y^j, and let $F \in C^\infty(U, V)$. Suppose $m = n$. Then*

$$F^*(\mathrm{d}y^1 \wedge \cdots \wedge \mathrm{d}y^n) = J\,\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n,$$

where J is the Jacobian determinant,

$$J(\mathbf{x}) = \det\left(\frac{\partial F^i}{\partial x^j}\right)_{i,j=1}^n.$$

Proof.

$$\begin{aligned}
F^*\beta &= \mathrm{d}F^1 \wedge \cdots \wedge \mathrm{d}F^n \\
&= \sum_{i_1 \ldots i_n} \frac{\partial F^1}{\partial x^{i_1}} \cdots \frac{\partial F^n}{\partial x^{i_n}} \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_n} \\
&= \sum_{s \in S_n} \frac{\partial F^1}{\partial x^{s(1)}} \cdots \frac{\partial F^n}{\partial x^{s(n)}} \mathrm{d}x^{s(1)} \wedge \cdots \wedge \mathrm{d}x^{s(n)} \\
&= \sum_{s \in S_n} \operatorname{sign}(s) \frac{\partial F^1}{\partial x^{s(1)}} \cdots \frac{\partial F^n}{\partial x^{s(n)}} \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n \\
&= J\,\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n.
\end{aligned}$$

Where we have used the characterization of the determinant in terms of the group of permutations (see Appendix B.1). In this calculation, we have used the fact that the wedge product $\mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_n}$ is zero unless the indices $i_1, \ldots, i_n$ are obtained from $1, \ldots, n$ by a permutation s, in which case it is given by $\operatorname{sign}(s)\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n$. $\square$

One may regard this result as giving a new (and in some sense better) definition of the Jacobian determinant.

Recall that our Definition 6.39 of the Lie derivative L_X of functions (and also of vector fields) was stated in terms of the pullback by the flow Φ_t of X; providing a geometric interpretation of this operation. In contrast, we have (provisionally) defined the Lie derivative of forms via the Cartan formula (7.23). Having now introduced the notion of pullback of forms, we may reconsider the Lie derivative $L_X\alpha$ of a differential form with respect to a vector field X. Let us assume for simplicity that X is complete, so that the flow Φ_t is globally defined. The following formula shows that L_X measures (infinitesimally) the extent to which α is invariant under the flow of X.

Theorem 7.31. *For X a complete vector field, and $\alpha \in \Omega^k(M)$,*

$$L_X\alpha = \frac{d}{dt}\Big|_{t=0} \Phi_t^*\alpha. \tag{7.35}$$

Here the derivative on the right-hand side is to be understood pointwise, at any $p \in M$, as the derivative of $(\Phi_t^*\alpha)_p$ (a function of t with values in the finite-dimensional vector space of multilinear forms on T_pM). The theorem also holds for incomplete vector fields; indeed, to define the right-hand side at a given $p \in M$ one only needs Φ_t *near* p, and for *small* $|t|$.

Proof. To prove this identity suffice it to check that the right-hand side satisfies a product rule with respect to the wedge product of forms, and that it has the correct values on functions and on differentials of functions.

In detail, let $D\alpha := \frac{d}{dt}\big|_{t=0} \Phi_t^*\alpha$. Then

$$\begin{aligned} D(\alpha_1 \wedge \alpha_2) &= \frac{d}{dt}\Big|_{t=0} (\Phi_t^*\alpha_1 \wedge \Phi_t^*\alpha_2) \\ &= \frac{d}{dt}\Big|_{t=0} (\Phi_t^*\alpha_1) \wedge \alpha_2 + \alpha_1 \wedge \frac{d}{dt}\Big|_{t=0} (\Phi_t^*\alpha_2) \\ &= D\alpha_1 \wedge \alpha_2 + \alpha_1 \wedge D\alpha_2 \end{aligned}$$

(pointwise, at any $p \in M$). By the usual argument, this implies that $D\alpha|_U$ depends only on $\alpha|_U$.

Next, on functions $f \in C^\infty(M)$ we find

$$Df = \frac{d}{dt}\Big|_{t=0} \Phi_t^* f = L_X f.$$

By the definition of the differentials of functions we have

$$D(\mathrm{d}f) = \frac{d}{dt}\Big|_{t=0} \Phi_t^*\mathrm{d}f = \mathrm{d}\,\frac{d}{dt}\Big|_{t=0} \Phi_t^* f = \mathrm{d}L_X f = L_X \mathrm{d}f.$$

This verifies $L_X = D$ on functions and on differentials of functions, and since any form is locally a sum of expressions $f_0\mathrm{d}f_1 \wedge \cdots \wedge \mathrm{d}f_k$, it shows $L_X = D$ on k-forms. $\square$

7.10 Problems

1. On $\mathbb{R}^3$, let
$$\alpha = x^2 \mathrm{d}y - y\mathrm{d}x, \quad \beta = y\mathrm{d}x \wedge \mathrm{d}z, \quad X = x\frac{\partial}{\partial z} - z\frac{\partial}{\partial x}.$$
Calculate the following forms:
(a) $\alpha \wedge \beta$.
(b) $\iota_X \beta$.
(c) $\mathrm{d}\beta$.
(d) $\iota_X(\alpha \wedge \beta)$.

2. Let $U \subseteq \mathbb{R}^3$ be an open subset.
(a) Relate $\mathrm{d}\colon\ \Omega^0(U) \to \Omega^1(U)$ to the gradient, $\mathrm{grad}(f) = \nabla f$.
(b) Relate $\mathrm{d}\colon\ \Omega^1(U) \to \Omega^2(U)$ to the curl, $\mathrm{curl}(\mathbf{F}) = \nabla \times \mathbf{F}$.
(c) Relate $\mathrm{d}\colon\ \Omega^2(U) \to \Omega^3(U)$ to the divergence, $\mathrm{div}(\mathbf{G}) = \nabla \cdot \mathbf{G}$.
(d) Interpret the properties $\mathrm{curl}(\mathrm{grad}(f)) = 0$ and $\mathrm{div}(\mathrm{curl}(F)) = 0$ in terms of the exterior differential d.

3. Consider the 1-form
$$\alpha = e^{xy}(y\mathrm{d}x + x\mathrm{d}y) \in \Omega^1(\mathbb{R}^2).$$
Let $\gamma\colon\ [0,1] \to \mathbb{R}^2$ be the path
$$\gamma(t) = \Big(\sin^3\big(\frac{\pi t}{2}\big),\ \cos(\pi t)\Big).$$
Find the integral $\int_\gamma \alpha$.

4. Let $F \in C^\infty(M,N)$ be a smooth map, and $X \in \mathfrak{X}(M)$ and $Y \in \mathfrak{X}(N)$ F-related vector fields, $X \sim_F Y$. Show that
$$\iota_X \circ F^* = F^* \circ \iota_Y, \ \ L_X \circ F^* = F^* \circ L_Y$$
as operators on differential forms.

5. Use the Cartan calculus to prove the following formula for the exterior differential of a form $\alpha \in \Omega^k(M)$,
$$\begin{aligned}(\mathrm{d}\alpha)(X_1,\ldots,X_{k+1}) = &\sum_{i=1}^{k+1}(-1)^{i+1}L_{X_i}(\alpha(X_1,\ldots,\widehat{X_i},\ldots X_{k+1})\\ &+\sum_{i<j}(-1)^{i+j}\alpha([X_i,X_j],X_1,\ldots,\widehat{X_i},\ldots,\widehat{X_j},\ldots,X_{k+1}),\end{aligned}$$
for $X_1,\ldots,X_{k+1} \in \mathfrak{X}(M)$.

6. For $\alpha \in \Omega^k(M)$, let the *exterior multiplication* $\varepsilon_\alpha\colon\ \Omega(M) \to \Omega(M)$ be the map given by wedge product, $\varepsilon_\alpha \omega = \alpha \wedge \omega$.

(a) Show that if $D\colon\ \Omega(M)\to\Omega(M)$ is a degree r superderivation, then $\varepsilon_\alpha\circ D$ is a degree $r+k$ superderivation.
(b) Show that

$$[D,\varepsilon_\alpha]=\varepsilon_{D\alpha}.$$

Here the left-hand side is the supercommutator $D\circ\varepsilon_\alpha-(-1)^{kr}\varepsilon_\alpha\circ D$.

7. Let $D\colon\ \Omega(M)\to\Omega(M)$ be a superderivation of degree r. Recall from 135 that D may be described in terms of its restriction to open subsets $U\subseteq M$. For a coordinate chart (U,φ), defining local coordinates $x^1,\ldots,x^m$, show that there are unique differential forms $\alpha^i\in\Omega^r(U)$ and $\beta^i\in\Omega^{r+1}(U)$ such that

$$D|_U=\sum_{i=1}^m\varepsilon_{\alpha^i}\circ L_{\frac{\partial}{\partial x^i}}+\sum_{i=1}^m\varepsilon_{\beta^i}\circ\iota_{\frac{\partial}{\partial x^i}}.$$

8. Prove that if $\alpha\in\Omega^k(M)$ and $\beta\in\Omega^l(M)$ are closed forms, then $\alpha\wedge\beta$ is closed, and if one of them is exact then so is the wedge product. Use this to define a product on cohomology, $H^k(M)\times H^l(M)\to H^{k+l}(M)$.

9. Let $\alpha=\sum_{i=1}^m\alpha_i(\mathbf{x})\mathrm{d}x^i\in\Omega^1(\mathbb{R}^m)$ be a closed 1-form on $\mathbb{R}^m$.
(a) Show that

$$f(\mathbf{x})=\sum_{i=1}^m x^i\int_0^1\alpha^i(u\mathbf{x})\,du$$

defines a primitive: $\mathrm{d}f=\alpha$.
(b) Show that if α has compact support, and $m>1$, then it admits a primitive which also has compact support. Show that this need not be true if $m=1$.

10. Let $\alpha=\frac{1}{2}\sum_{i,j=1}^m\alpha_{ij}(\mathbf{x})\mathrm{d}x^i\wedge\mathrm{d}x^j\in\Omega^2(\mathbb{R}^m)$, with $\alpha_{ij}=-\alpha_{ji}$.
(a) State explicitly under what conditions on α_{ij} the form α is closed.
(b) If α is closed, show that

$$\beta=\sum_{ij}\Big(\int_0^1 u\alpha_{ij}(u\mathbf{x})du\Big)x^i\mathrm{d}x^j$$

is a primitive: $\mathrm{d}\beta=\alpha$.

11. Generalize the results from the previous problem to higher degrees: Let

$$\alpha=\frac{1}{k!}\sum_{i_1\cdots i_k}\alpha_{i_1\ldots i_k}\mathrm{d}x^{i_1}\wedge\cdots\wedge\mathrm{d}x^{i_k}\in\Omega^k(\mathbb{R}^m)$$

with $k>0$, where we take the coefficients to be skew-symmetric in the indices.
(a) State explicitly under what conditions on the coefficients α is closed.
(b) If α is closed, show that

$$\beta=\frac{1}{(k-1)!}\sum_{i_1\ldots i_k}\Big(\int_0^1 u^{k-1}\alpha_{i_1\ldots i_k}(u\mathbf{x})du\Big)x^{i_1}\mathrm{d}x^{i_2}\wedge\cdots\wedge\mathrm{d}x^{i_k}$$

is a primitive: $\mathrm{d}\beta=\alpha$.

12. (a) Let $U \subseteq \mathbb{R}^m$ be an open subset. Let $\pi : \mathbb{R} \times U \to U$ be the projection $\pi(t, \mathbf{x}) = \mathbf{x}$, and $i : U \to \mathbb{R} \times U$ the inclusion $i(\mathbf{x}) = (0, \mathbf{x})$. Prove that the collection of maps
$$\mathsf{h} : \Omega^k(\mathbb{R} \times U) \to \Omega^{k-1}(\mathbb{R} \times U)$$
given by
$$\mathsf{h}\left(\sum_I \varphi_I(t,\mathbf{x})\mathrm{d}x^I + \sum_J \psi_J(t,\mathbf{x})\mathrm{d}t \wedge \mathrm{d}x^J\right) = \sum_J \left(\int_0^t \psi_J(u,\mathbf{x})\mathrm{d}u\right)\mathrm{d}x^J$$
(with the convention that h is the zero map on $\Omega^0(\mathbb{R} \times U)$) has the property
$$\mathsf{h}(\mathrm{d}\alpha) + \mathrm{d}\mathsf{h}(\alpha) = \alpha - \pi^* i^* \alpha.$$
(b) Generalize to $\pi : \mathbb{R} \times M \to M$, $i : M \to \mathbb{R} \times M$ for any manifold M, and use it to show that $\pi^* : \Omega^k(M) \to \Omega^k(\mathbb{R} \times M)$ induces an isomorphism in cohomology, with inverse induced by i^*.
(c) Use these results to prove the fact that $H^k(\mathbb{R}^m) = 0$ for $k > 0$.

13. Let $\alpha \in \Omega^k(M)$ be a closed form on a manifold M, with $k > 0$, and $p \in M$ any given point. Show that there exists a closed form $\alpha' \in \Omega^k(M)$ such that α' vanishes on a neighborhood of p, and $\alpha - \alpha'$ is exact. (Hint: Use that $H^k(\mathbb{R}^m) = 0$ for $k > 0$, and a bump function.)

14. Let $\alpha \in \Omega^1(S^m)$ be a closed 1-form, with $m > 1$. Show that α is exact. (You can—but do not have to—use the result from Problem 9.)

15. The goal of this problem is to show that the Betti numbers $b_k(S^m)$ are zero for $m > k > 0$. You may use an induction on m, and the results from Problem 12. Suppose $\alpha \in \Omega^k(S^m)$ is closed, with $m > k > 0$.
(a) Let $U_+, U_- \subseteq S^m$ be the open covering given by the domains of the stereographic projections, and choose primitives $\beta_\pm \in \Omega^{k-1}(U_\pm)$ of $\alpha|_{U_\pm}$. Prove that
$$\beta_+|_{U_+ \cap U_-} - \beta_-|_{U_+ \cap U_-} = \mathrm{d}\gamma$$
for some $\gamma \in \Omega^{k-2}(U_+ \cap U_-)$.
(b) Using γ and a partition of unity for the cover U_+, U_- (see Appendix C.4), show how to modify the primitives $\beta_\pm$, so that they patch together to a global primitive $\beta \in \Omega^{k-1}(S^m)$.

16. Give an example of a 1-form $\alpha \in \Omega^1(\mathbb{R}^3)$ with the property
$$\alpha \wedge \mathrm{d}\alpha = \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z.$$

17. Let M be a 3-dimensional manifold, and $\alpha \in \Omega^1(M)$ a 1-form such that
$$\alpha \wedge \mathrm{d}\alpha \in \Omega^3(M)$$
is non-zero everywhere on M. Prove that there does *not* exist a 2-dimensional submanifold $S \subseteq M$ with the property that

$$\ker(\alpha_p) = T_pS$$

for all $p \in S$.

18. Let $\alpha_1, \ldots, \alpha_s \in \Omega^1(M)$ be 1-forms such that the covectors $(\alpha_1)_p, \ldots (\alpha_s)_p$ are linearly independent for all $p \in M$, and such that

$$d\alpha_i = \sum_j \beta_{ij} \wedge \alpha_j$$

for suitable $\beta_{ij} \in \Omega^1(M)$. Show that

$$\mathscr{E} = \{X \in \mathfrak{X}(M) |\ \alpha_1(X) = \cdots = \alpha_s(X) = 0\}$$

is an integrable distribution, as discussed at the end of Section 6.6. Show conversely that *locally* (on some open neighborhood of any given point in M), every integrable distribution arises in this way. Given $\mathscr{E}$, are the α_i's unique?

19. Let $\mathscr{E} \subseteq \mathfrak{X}(M)$ be a distribution, as discussed at the end of Section 6.6, and let $\Omega^k_{\mathscr{E}}(M)$ be the space of differential k-forms α with the property $\alpha(X_1, \ldots, X_k) = 0$ for all $X_1, \ldots, X_k \in \mathscr{E}$. Show that the following are equivalent:
    - $\mathscr{E}$ is involutive.
    - The exterior differential takes $\Omega^1_{\mathscr{E}}(M)$ to $\Omega^2_{\mathscr{E}}(M)$.
    - For all $k > 0$, the exterior differential takes $\Omega^k_{\mathscr{E}}(M)$ to $\Omega^{k+1}_{\mathscr{E}}(M)$.

20. Let $\alpha \in \Omega^1(M)$ be a nowhere-vanishing 1-form with the property $\alpha \wedge d\alpha = 0$.
    (a) Let $\ker(\alpha) \subseteq \mathfrak{X}(M)$ be the set of vector fields satisfying $\alpha(X) = 0$. Prove that $\ker(\alpha)$ satisfies the Frobenius condition:

    $$X, Y \in \ker(\alpha) \ \Rightarrow\ [X, Y] \in \ker(\alpha).$$

    (Hint: Locally, one can choose a vector field Z with $\alpha(Z) = 1$.)
    (b) Show that *locally*, near any given point $p \in M$, α is of the form $f_0 df_1$, where $f_0, f_1 \in C^\infty(M)$ with $f_0 > 0$ and $df_1 \neq 0$.
    (c) Show that there exists a 1-form $\beta \in \Omega^1(M)$ such that

    $$d\alpha = \beta \wedge \alpha. \tag{7.36}$$

    (First define β locally, and use a partition of unity to patch the local definitions—see Appendix C.4.) Show furthermore that any other 1-form β' with such a property is of the form

    $$\beta' = \beta + h\alpha$$

    with $h \in C^\infty(M)$.
    (d) Show $\alpha \wedge d\beta = 0$.
    (e) Show that the 3-form $\beta \wedge d\beta \in \Omega^3(M)$ is closed and that its cohomology class does not depend on the choice of β. Let $\mathrm{GV}(\alpha) \in H^3(M)$ denote this cohomology class.

(f) Show that if $\alpha' = f\alpha$ for an everywhere-positive function f, then $d\alpha' \wedge \alpha' = 0$ and $\mathrm{GV}(\alpha') = \mathrm{GV}(\alpha)$.

Note: The first part of the exercise shows that $\ker(\alpha) \subseteq \mathfrak{X}(M)$ defines a codimension 1 foliation of M. A second nowhere-vanishing 1-form α' with $\alpha' \wedge d\alpha'$ defines the same foliation if and only if $\alpha' = f\alpha$ with $f \neq 0$. The equivalence class of such 1-forms, with two α's being equivalent if they differ by a positive function, is called a *co-orientation* of the foliation. The cohomology class $\mathrm{GV}(\alpha) \in H^3(M)$ constructed in this exercise is called the *Godbillon-Vey class* of the co-oriented foliation. It is an example of a *characteristic class* of a foliation.

8 Integration

Differential forms of top degree and of compact support can be integrated over *oriented* manifolds and, more generally, over domains (with boundary) in such manifolds. A key result concerning integration is *Stokes' theorem*, a far-reaching generalization of the fundamental theorem of calculus. Stokes' theorem has numerous important applications, such as to winding numbers and linking numbers, mapping degrees, de Rham cohomology, and many more. Our plan as always is to begin with integration on open subsets of Euclidean spaces and generalize to arbitrary manifolds.

8.1 Integration of Differential Forms

8.1.1 Integration Over Open Subsets of $\mathbb{R}^m$

Suppose $U \subseteq \mathbb{R}^m$ is open and $\omega \in \Omega^m(U)$ is a form of top degree $k = m$. Such a differential form is an expression

$$\omega = f \, \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^m,$$

where $f \in C^\infty(U)$ (cf. ✎ 106). If $\mathrm{supp}(f)$ is compact, one defines the integral of ω to be the usual Riemann integral:

$$\int_U \omega = \int_{\mathbb{R}^m} f(x^1, \ldots, x^m) \, \mathrm{d}x^1 \cdots \mathrm{d}x^m. \tag{8.1}$$

Note that we can regard f as a function on all of $\mathbb{R}^m$, due to the compact support condition. Let us now generalize this to manifolds.

G. Gross, E. Meinrenken, *Manifolds, Vector Fields, and Differential Forms*,
Springer Undergraduate Mathematics Series,
https://doi.org/10.1007/978-3-031-25409-3_8

8.1.2 Integration Over Manifolds

The *support* of a differential form $\omega \in \Omega^k(M)$ is the smallest closed subset $\mathrm{supp}(\omega) \subseteq M$ with the property that ω is zero outside of $\mathrm{supp}(\omega)$ (cf. Definition 3.4). Let M be an oriented manifold of dimension m, and $\omega \in \Omega^m(M)$. If $\mathrm{supp}(\omega)$ is contained in an oriented coordinate chart (U, φ), then one defines

$$\int_M \omega = \int_{\mathbb{R}^m} f(\mathbf{x})\, \mathrm{d}x^1 \cdots \mathrm{d}x^m,$$

where $f \in C^\infty(\mathbb{R}^m)$ is the function, with $\mathrm{supp}(f) \subseteq \varphi(U)$, determined from

$$(\varphi^{-1})^* \omega = f\ \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^m.$$

This definition does not depend on the choice of oriented coordinate chart. Indeed, suppose (V, ψ) is another oriented chart with $\mathrm{supp}(\omega) \subseteq V$ and write

$$(\psi^{-1})^* \omega = g\ \mathrm{d}y^1 \wedge \cdots \wedge \mathrm{d}y^m,$$

where $y^1, \ldots, y^m$ are the coordinates on V. Letting $F = \psi \circ \varphi^{-1}$ be the change of coordinates $\mathbf{y} = F(\mathbf{x})$, Proposition 7.30 shows

$$F^*(g \mathrm{d}y^1 \wedge \cdots \wedge \mathrm{d}y^m) = (g \circ F)\, J\, \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^m,$$

where $J(\mathbf{x}) = \det(DF(\mathbf{x}))$ is the determinant of the Jacobian matrix of F at $\mathbf{x}$. Hence, $f(\mathbf{x}) = g(F(\mathbf{x}))J(\mathbf{x})$, and we obtain

$$\int_{\psi(U)} g(\mathbf{y}) \mathrm{d}y^1 \cdots \mathrm{d}y^m = \int_{\varphi(U)} g(F(\mathbf{x}))J(\mathbf{x}) \mathrm{d}x^1 \cdots \mathrm{d}x^m = \int_{\varphi(U)} f(\mathbf{x}) \mathrm{d}x^1 \cdots \mathrm{d}x^m,$$

as required.

Remark 8.1. Here we used the *change-of-variables formula* from multivariable calculus. It was important to work with *oriented* charts, guaranteeing that $J > 0$ everywhere. Indeed, in general the change-of-variables formula involves $|J|$ rather than J itself.

More generally, if the support of ω is compact but not necessarily contained in a single oriented chart, we proceed as follows. Let $(U_i, \varphi_i),\ \ i = 1, \ldots, r$ be a finite collection of oriented charts covering $\mathrm{supp}(\omega)$. Together with $U_0 = M \backslash \mathrm{supp}(\omega)$ this is an open cover of M. For any such open cover, there exists a *partition of unity subordinate to the cover*, i.e., functions $\chi_i \in C^\infty(M)$ with

$$\mathrm{supp}(\chi_i) \subseteq U_i, \qquad \sum_{i=0}^{r} \chi_i = 1.$$

A proof for the existence of such partitions of unity (for *any* open cover, not only finite ones) is given in in Appendix C.4.

The partition of unity allows us to write ω as a sum

$$\omega = (\sum_{i=0}^{r} \chi_i)\,\omega = \sum_{i=0}^{r} (\chi_i \omega),$$

where each $\chi_i\omega$ has compact support in a coordinate chart. (We may drop the term for $i = 0$, since $\chi_0\omega = 0$.) Accordingly, we define

$$\int_M \omega = \sum_{i=0}^{r} \int_M \chi_i \omega.$$

We have to check that this is well-defined, independent of the various choices we have made. To that end, let (V_j, ψ_j) for $j = 0,\dots,s$ be another collection of oriented coordinate charts covering $\mathrm{supp}(\omega)$, put $V_0 = M \setminus \mathrm{supp}(\omega)$, and let $\sigma_0,\dots,\sigma_s$ be a corresponding partition of unity subordinate to the cover by the V_i's.

Then $\{U_i \cap V_j :\ i = 0,\dots,r, j = 0,\dots,s\}$ is an open cover, with the collection of products $\chi_i\sigma_j$ a partition of unity subordinate to this cover. We obtain

$$\sum_{j=0}^{s} \int_M \sigma_j \omega = \sum_{j=0}^{s} \int_M (\sum_{i=0}^{r} \chi_i)\,\sigma_j \omega = \sum_{j=0}^{s} \sum_{i=0}^{r} \int_M \sigma_j \chi_i \omega = \sum_{i=0}^{r} \sum_{j=0}^{s} \int_M \sigma_j \chi_i \omega = \sum_{i=0}^{r} \int_M \chi_i \omega.$$

8.1.3 Integration Over Oriented Submanifolds

Let M be a manifold (not necessarily oriented) and S a k-dimensional oriented submanifold, with inclusion map $i\colon\ S \to M$. We define the integral over S, of any k-form $\omega \in \Omega^k(M)$ such that $S \cap \mathrm{supp}(\omega)$ is compact, as follows:

$$\int_S \omega = \int_S i^* \omega.$$

This definition works equally well for *any* smooth map from S into M, it does not have to be an embedding as a submanifold. For example, the integral of compactly supported 1-forms along arbitrary curves $\gamma\colon\ \mathbb{R} \to M$ can be thus defined. (Compare with the definition of integrals of 1-forms along paths in Section 7.6, where γ was defined on *closed* intervals.)

8.2 Stokes' Theorem

Let M be an m-dimensional oriented manifold.

Definition 8.2. *A* region with (smooth) boundary *in M is a closed subset $D \subseteq M$ of the form*

$$D = \{p \in M |\ f(p) \le 0\},$$

where $f \in C^\infty(M,\mathbb{R})$ is a smooth function having 0 *as a regular value.*

We do not consider f itself as part of the definition of D, only the existence of f is required.

The interior of a region with boundary, given as the largest open subset contained in D, is

$$\mathrm{int}(D) = f^{-1}((-\infty,0)) = \{p \in M|\ f(p) < 0\},$$

and the boundary is

$$\partial D = f^{-1}(0) = \{p \in M|\ f(p) = 0\},$$

a codimension 1 submanifold (i.e., a hypersurface) in M.

Example 8.3. The unit disk

$$D = \{(x,y) \in \mathbb{R}^2|\ x^2 + y^2 \leq 1\}$$

is a region with boundary, with defining function $f(x,y) = x^2 + y^2 - 1$.

Example 8.4. Recall from Example 4.11 that for $0 < r < R$, the function $f \in C^\infty(\mathbb{R}^3)$ given by

$$f(x,y,z) = z^2 + (\sqrt{x^2+y^2} - R)^2 - r^2$$

has zero as a regular value, with $f^{-1}(0)$ a 2-torus. The corresponding region with boundary $D \subseteq \mathbb{R}^3$ is the *solid torus*.

Recall that we are considering D inside an *oriented* manifold M. The boundary ∂D may be covered by oriented submanifold charts (U,φ), in such a way that ∂D is given in the chart by the condition $x^1 = 0$ and D by the condition $x^1 \leq 0$:

$$\varphi(U \cap D) = \varphi(U) \cap \{x \in \mathbb{R}^m|\ x^1 \leq 0\}.$$

(Indeed, given an oriented submanifold chart for which D lies on the side where $x_1 \geq 0$, one obtains a suitable chart by composing with the orientation preserving coordinate change $(x^1,\dots,x^m) \mapsto (-x^1,-x^2,x^3\dots,x^m)$.) We shall call oriented submanifold charts of this kind “region charts” (this is not a standard name).

Remark 8.5. We originally defined submanifold charts in such a way that the last $m-k$ coordinates are zero on S, here we require that the first coordinate be zero. It does not matter, since one can simply reorder coordinates, but works better for our description of the “induced orientation.”

Lemma 8.6. *The restrictions of the region charts to ∂D form an oriented atlas for ∂D.*

Proof. Let (U,φ) and (V,ψ) be two region charts, defining coordinates $x^1,\dots,x^m$ and $y^1,\dots,y^m$, and let $F = \psi \circ \varphi^{-1}:\ \varphi(U \cap V) \to \psi(U \cap V),\ \mathbf{x} \mapsto \mathbf{y} = F(\mathbf{x})$ be the change of coordinate map. Then F restricts to a map

$$G:\ \{\mathbf{x} \in \varphi(U \cap V)|\ x^1 = 0\} \to \{\mathbf{y} \in \psi(U \cap V)|\ y^1 = 0\}.$$

Since $y^1 > 0$ if and only if $x^1 > 0$, the change of coordinates satisfies

$$\left.\frac{\partial y^1}{\partial x^1}\right|_{x^1=0} > 0, \quad \left.\frac{\partial y^1}{\partial x^j}\right|_{x^1=0} = 0, \quad \text{for } j > 0.$$

Hence, the Jacobian matrix $DF(\mathbf{x})|_{x^1=0}$ has a positive $(1,1)$ entry, and all other entries in the first row equal to zero. Using expansion of the determinant across the first row, it follows that

$$\det\big(DF(0,x^2,\ldots,x^m)\big) = \left.\frac{\partial y^1}{\partial x^1}\right|_{x^1=0} \det\big(DG(x^2,\ldots,x^m)\big),$$

which shows that $\det(DG) > 0$. $\square$

In particular, ∂D is again an oriented manifold. To repeat: If $x^1,\ldots,x^m$ are local coordinates near $p \in \partial D$, compatible with the orientation and such that D lies on the side $x^1 \leq 0$, then $x^2,\ldots,x^m$ are local coordinates on ∂D. This convention of "induced orientation" is arranged in such a way that Stokes' theorem holds without an extra sign.

For a top-degree form $\omega \in \Omega^m(M)$ such that $\mathrm{supp}(\omega) \cap D$ is compact, the integral

$$\int_D \omega$$

is defined similarly to the case of $D = M$: One covers $D \cap \mathrm{supp}(\omega)$ by finitely many submanifold charts (U_i, φ_i) with respect to ∂D (this includes charts that are entirely in the interior of D) and puts

$$\int_D \omega = \sum \int_{D \cap U_i} \chi_i \omega,$$

where the χ_i are supported in U_i and satisfy $\sum_i \chi_i = 1$ over $D \cap \mathrm{supp}(\omega)$. By the same argument as for $D = M$, this definition of the integral is independent of the choices made.

Theorem 8.7 (Stokes' Theorem). *Let M be an oriented manifold of dimension m, and $D \subseteq M$ a region with smooth boundary ∂D. Let $\alpha \in \Omega^{m-1}(M)$ be a form of degree $m-1$, such that* $\mathrm{supp}(\alpha) \cap D$ *is compact. Then*

$$\int_D \mathrm{d}\alpha = \int_{\partial D} \alpha.$$

As in Section 8.1.3, the right-hand side means $\int_{\partial D} i^* \alpha$, where $i:\ \partial D \hookrightarrow M$ is the inclusion map.

Proof. We shall see that Stokes' theorem is just a coordinate-free version of the fundamental theorem of calculus. Let (U_i, φ_i) for $i = 1,\ldots,r$ be a finite collection of region charts covering $\mathrm{supp}(\alpha) \cap D$. Let $\chi_1,\ldots,\chi_r \in C^\infty(M)$ be functions with $\chi_i \geq 0$, $\mathrm{supp}(\chi_i) \subseteq U_i$, and such that $\chi_1 + \cdots + \chi_r$ is equal to 1 on $\mathrm{supp}(\alpha) \cap D$. (For instance, we may take $U_1,\dots,U_r$ together with $U_0 = M\backslash\mathrm{supp}(\alpha)$ as an open cover, and take the $\chi_0,\dots,\chi_r \in C^\infty(M)$ to be a partition of unity subordinate to this cover.) Since

$$\int_D \mathrm{d}\alpha = \sum_{i=1}^r \int_D \mathrm{d}(\chi_i\alpha), \qquad \int_{\partial D} \alpha = \sum_{i=1}^r \int_{\partial D} \chi_i\alpha,$$

suffice it to consider the case that α is supported in a region chart.

Using the corresponding coordinates, it hence suffices to prove Stokes' theorem for the case that $\alpha \in \Omega^{m-1}(\mathbb{R}^m)$ is a compactly supported form in $\mathbb{R}^m$:

$$D = \{x \in \mathbb{R}^m |\, x^1 \le 0\}.$$

That is, α has the form

$$\alpha = \sum_{i=1}^m f_i\, \mathrm{d}x^1 \wedge \cdots \wedge \widehat{\mathrm{d}x^i} \wedge \cdots \wedge \mathrm{d}x^m,$$

with compactly supported $f_i \in C^\infty(\mathbb{R}^m)$, where the hat means that the corresponding factor is to be omitted. Only the $i = 1$ term contributes to the integral over $\partial D = \mathbb{R}^{m-1}$ and

$$\int_{\mathbb{R}^{m-1}} \alpha = \int f_1(0,x^2,\dots,x^m)\, \mathrm{d}x^2 \cdots \mathrm{d}x^m.$$

On the other hand,

$$\mathrm{d}\alpha = \Big(\sum_{i=1}^m (-1)^{i+1} \frac{\partial f_i}{\partial x^i}\Big)\, \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^m.$$

Let us integrate each summand over the region D given by $x^1 \le 0$. For $i > 1$, we have

$$\int_\infty^\infty \cdots \int_{-\infty}^\infty \int_{-\infty}^0 \frac{\partial f_i}{\partial x_i}(x^1,\dots,x^m)\, \mathrm{d}x^1 \cdots \mathrm{d}x^m = 0,$$

where we used Fubini's theorem to carry out the x^i-integration first, and applied the fundamental theorem of calculus to the x^i-integration (keeping the other variables fixed). Since the integrand is the derivative of a compactly supported function, the x^i-integral is zero.

It remains to consider the case $i = 1$. Here we have, again by applying the fundamental theorem of calculus,

$$\begin{aligned}\int_D \mathrm{d}\alpha &= \int_\infty^\infty \cdots \int_{-\infty}^\infty \int_{-\infty}^0 \frac{\partial f_1}{\partial x_1}(x^1,\dots,x^m)\mathrm{d}x^1 \cdots \mathrm{d}x^m \\ &= \int_\infty^\infty \cdots \int_{-\infty}^\infty f_1(0,x^2,\dots,x^m)\mathrm{d}x^2 \cdots \mathrm{d}x^m \\ &= \int_{\partial D} \alpha. \qquad \square\end{aligned}$$

As a special case (where $D = M$ with $\partial D = \emptyset$) we have the following corollary.

Corollary 8.8. *Let $\alpha \in \Omega^{m-1}(M)$ be a compactly supported form on the oriented manifold M. Then*

$$\int_M \mathrm{d}\alpha = 0.$$

Note that it is not enough just for $\mathrm{d}\alpha$ to have compact support. For example, if $f(t)$ is a function with $f(t) = 0$ for $t < 0$ and $f(t) = 1$ for $t > 1$, then $\mathrm{d}f$ has compact support, but $\int_{\mathbb{R}\setminus\{0\}} \mathrm{d}f = 1$.

8.3 Winding Numbers and Mapping Degrees

In this section, we survey some typical applications of Stokes' theorem to the (algebraic) topology of manifolds and maps between manifolds. Another application, to Gauss-Bonnet's theorem, will be presented in Section 8.5.

8.3.1 Invariance of Integrals

The following simple consequence of Stokes' theorem will be frequently used.

Theorem 8.9. *Let $\omega \in \Omega^k(M)$ be a closed form on a manifold M, and S a compact, oriented manifold of dimension k. Let $F \in C^\infty(\mathbb{R} \times S, M)$ be a smooth map, thought of as a smooth family of maps*

$$F_t = F(t, \cdot) : \ S \to M.$$

Then the integrals

$$\int_S F_t^* \omega$$

do not depend on t.

Proof. Let $a < b$, and consider the domain $D = [a,b] \times S \subseteq \mathbb{R} \times S$. The boundary ∂D has two components, both diffeomorphic to S. At $t = b$ the orientation is the given orientation on S, while at $t = a$ we get the opposite orientation. Hence, using Stokes' theorem,

$$0 = \int_D F^* \mathrm{d}\omega = \int_D \mathrm{d}F^* \omega = \int_{\partial D} F^* \omega = \int_S F_b^* \omega - \int_S F_a^* \omega.$$

Hence $\int_S F_b^* \omega = \int_S F_a^* \omega$. $\square$

Note that if the F_t's are embeddings, then $\int_S F_t^* \omega$ can be regarded as the integrals of ω over the time-dependent family of submanifolds $S_t = F_t(S) \subseteq M$.

Remark 8.10. If $\omega \in \Omega^k(M)$, and $\varphi : \ S \to M$ has rank $< k$ at some point $p \in S$, then the pullback $\varphi^* \omega$ vanishes at p. This happens, for example, if φ takes values in a submanifold of dimension less than k. In the setting of Theorem 8.9, this shows that if there exists t_0 with $\mathrm{rank}_p(F_{t_0}) < k$ for all $p \in S$, then $\int_S F_t^* \omega = 0$ for *all* t. A special case of this situation is when one member of the family is the constant map to a point.

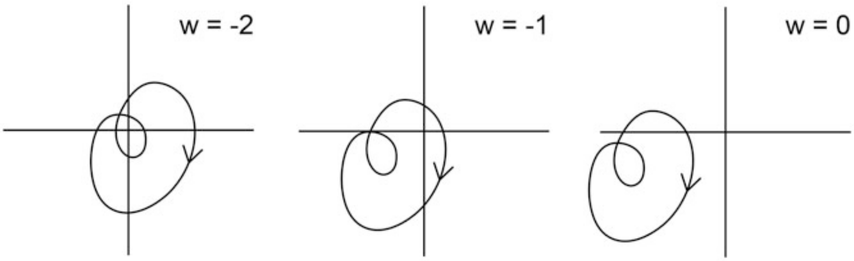
Given a smooth map $\varphi: S \to M$, one refers to a smooth map $F: \mathbb{R} \times S \to M$ with $F_0 = \varphi$ as a *smooth deformation* or *isotopy* of φ. We say that φ can be smoothly deformed into ψ if there exists a smooth isotopy F with $\varphi = F_0$ and $\psi = F_1$. The theorem shows that if S is compact and oriented, and if there is a closed form $\omega \in \Omega^k(M)$ with

$$\int_S \varphi^* \omega \neq \int_S \psi^* \omega$$

then φ cannot be smoothly deformed into ψ. In particular, if $\int_S \varphi^* \omega \neq 0$ then φ cannot be deformed into a constant map (or any map having rank $< k$ everywhere).

8.3.2 Winding Numbers

A *loop* in a manifold M is a smooth map $\gamma: S^1 \to M$. For loops in the punctured plane $\mathbb{R}^2 \setminus \{\mathbf{0}\}$, there is a *winding number* measuring the number of times the loop "winds around" the origin in the positive (counter-clockwise) direction.



In particular, for any $n \in \mathbb{Z}$, the loop

$$\gamma_n: S^1 \to \mathbb{R}^2 \setminus \{\mathbf{0}\}, \quad \gamma_n([t]) = \big(\cos(2\pi n t), \sin(2\pi n t)\big) \tag{8.2}$$

(where we regard S^1 as a quotient $\mathbb{R}/\sim$ under the equivalence relation $t \sim t+1$) should have winding number equal to n. One can show (see Problem 7) that any closed loop γ in $\mathbb{R}^2 \setminus \{\mathbf{0}\}$ can be deformed into a loop of the form (8.2). To use this for a definition for winding numbers in general, one would have to prove that for $n \neq m$, the loop γ_n cannot be deformed into γ_m.

Rather than following this approach (which is perfectly fine), we will use a definition of winding numbers as integrals. Let $\omega \in \Omega^1(\mathbb{R}^2 \setminus \{\mathbf{0}\})$ be the 1-form (cf. ✐ 129)

$$\omega = \frac{1}{2\pi(x^2+y^2)}(x\mathrm{d}y - y\mathrm{d}x).$$

In polar coordinates, $x = r\cos\theta,\ y = r\sin\theta$, one has that

$$\omega = \frac{1}{2\pi}\mathrm{d}\theta,$$

which shows that ω is closed. (It is not exact, since θ is not a globally defined function on $\mathbb{R}^2 \setminus \{\mathbf{0}\}$.)

Definition 8.11. *The* winding number *of a loop* $\gamma : S^1 \to \mathbb{R}^2 \setminus \{\mathbf{0}\}$ *is the integral*

$$w(\gamma) = \int_{S^1} \gamma^* \omega.$$

Since $\mathrm{d}\omega = 0$, the winding number does not change under smooth deformations (isotopies) of the loop.

141 (answer on page 316). Show that the loop (8.2) has winding number $w(\gamma_n) = n$.

Since the winding number is invariant under deformations, we have the following theorem.

Theorem 8.12. *The winding number $w(\gamma)$ of a smooth loop is an integer. Two loops can be deformed into each other if and only if they have the same winding number.*

Problem 9 discusses the invariance of winding numbers under reparametrization of loops (composing γ with a diffeomorphism of S^1). In Problem 10, you are asked to prove the additivity of winding numbers under concatenation:

$$w(\gamma_1 * \gamma_2) = w(\gamma_1) + w(\gamma_2).$$

(Informally, $\gamma_1 * \gamma_2$ is the loop γ_1 followed by the loop γ_2; one may assume "sitting end points" to ensure that the concatenation is smooth.)

A typical application of winding numbers is the non-existence of a smooth retraction of a closed disk onto its boundary, i.e., a smooth map $F : D^2 \to S^1$ whose restriction to $S^1 \subseteq D^2$ is the identity (see Problem 15). Another standard application is to the fundamental theorem of algebra: Every complex polynomial of degree $n > 0$ has a zero (by division, one then concludes the existence of n zeroes, counted with multiplicities), see Problem 16.

8.3.3 Mapping Degree

Let M be a compact, connected, oriented manifold of dimension $m = \dim(M)$. Every top-degree form $\omega \in \Omega^m(M)$ is closed for degree reasons (the space of $(m+1)$-forms on an m-dimensional manifold is trivial). A sufficient condition for this form to not be exact is that its integral is non-zero.

142 (answer on page 316). Show that if $\int_M \omega \neq 0$ then ω is *not* exact.

Top-degree forms having non-zero integral exist: For instance, we may take a local coordinate chart (U, φ) and put

$$\omega = \varphi^*(f\, \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^m), \tag{8.3}$$

where $f \neq 0$ is a non-negative function (e.g., a 'bump function') supported in $\varphi(U) \subseteq \mathbb{R}^m$. Then $\int_M \omega = \int_{\mathbb{R}^m} f\, \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^m > 0$.

In terms of de Rham cohomology (cf. (7.20)), this observation tells us that for a compact, oriented manifold M of dimension m, the top-degree cohomology group $H^m(M)$ is non-trivial. In fact, even more is true.

Theorem 8.13. *Let M be a compact, connected, and oriented manifold of dimension m. Then the integration map $\int_M : \Omega^m(M) \to \mathbb{R}$ induces an isomorphism in cohomology,*

$$H^m(M) \cong \mathbb{R}. \tag{8.4}$$

That is, an m-form on M is exact if and only if its integral vanishes.

Any top-degree form ω with $\int_M \omega = 1$ gives the basis element $[\omega]$ corresponding to 1 under this isomorphism. (The form (8.3) may be arranged to have integral 1, through multiplication by a suitable constant.) The proof of Theorem 8.13 will be left as a guided homework problem (see Problem 20).

Let us now consider two compact oriented manifolds M and N of the same dimension $m = n$, where N is connected.

Definition 8.14 (Brouwer's Mapping Degree). *The degree of a smooth map $F : M \to N$ is the number*

$$\deg(F) = \int_M F^*\omega,$$

for any choice of $\omega \in \Omega^m(N)$ with $\int_N \omega = 1$.

By Theorem 8.9, the degree is invariant under smooth deformations of F. It is also independent of the choice of ω: By Theorem 8.13, any other such form ω' differs from ω by an exact form $\mathrm{d}\alpha$; but $F^*(\omega + d\alpha) = F^*\omega + \mathrm{d}F^*\alpha$, so by Stokes' theorem the integral is unchanged.

Recall that the local mapping degree $\deg_q(F)$ at a regular value q is a signed count of preimages $p \in F^{-1}(q)$, with signs depending on whether F preserves or reverses orientation at p (cf. Remark 4.24).

Theorem 8.15. *Suppose q is a regular value of F. Then*

$$\deg(F) = \deg_q(F),$$

where $\deg_q(F) \in \mathbb{Z}$ is the local mapping degree of F (equal to 0 if q is not in the image of F). In particular, $\deg_q(F)$ does not depend on the choice of regular value q.

Proof. Consider first the case that q is not in the image, hence $\deg_q(F) = 0$. By compactness of M, the image of $F(M) \subseteq N$ is a compact subset, and in particular is a closed subset (cf. Section 2.5). Take ω, with $\int_N \omega = 1$, to be supported in $N \setminus F(M)$. Then $F^*\omega = 0$ and hence $\deg(F) = 0$.

If q is contained in the image of F, the preimage is a finite subset (by compactness)

$$F^{-1}(q) = \{p_1, \ldots, p_r\} \subseteq M.$$

Choose disjoint open neighborhoods U_i around each p_i, so that the map F restricts to a local diffeomorphism $U_i \to F(U_i)$. In particular, each $F(U_i)$ is an open neighborhood of q. Take ω, with $\int_N \omega = 1$, to be supported in the intersection

$$V = \bigcap_{i=1}^{r} F(U_i).$$

Then $F^*\omega$ is supported in the disjoint union of the U_i's. We have that $\int_{U_i} F^*\omega = \varepsilon_{p_i} \int_V \omega = \varepsilon_{p_i}$, where the sign $\varepsilon_{p_i} = \pm 1$ comes from a possible change of orientation. Consequently,

$$\deg(F) = \int_M F^*\omega = \sum_{i=1}^{r} \int_{U_i} F^*\omega = \sum_{i=1}^{r} \varepsilon_{p_i} = \deg_q(F) \in \mathbb{Z},$$

as claimed. □

It is a non-trivial result from differential topology (a consequence of *Sard's theorem*) that regular values *always* exist. Using this fact, the proposition shows that $\deg(F)$ is always an integer.

Remark 8.16. The pullback map $F^* : \Omega^m(N) \to \Omega^m(M)$ induces a map on cohomology, $F^* : H^m(N) \to H^m(M)$. In terms of the identifications $H^m(N) \cong \mathbb{R}$, $H^m(M) \cong \mathbb{R}$, this map is multiplication by the integer $\deg(F)$.

Remark 8.17. Compactness of N is not really needed for the definition of $\deg(F)$. If N is non-compact, take ω to be a top-degree form with support in a compact subset of N, so that the condition $\int_N \omega = 1$ makes sense. Any two such forms differ by the differential of a $(k-1)$-form with support in a compact subset.

143 (answer on page 316). The winding number of a path in $\mathbb{R}^2 \setminus \{\mathbf{0}\}$ can be regarded as the degree of a map. Explain how.

144 (answer on page 316). Let $M, N,$ and Q be compact, connected, oriented manifolds of the same dimension n. Let $F \in C^\infty(M,N)$ and $G \in C^\infty(N,Q)$ be smooth maps. Prove

$$\deg(G \circ F) = \deg(G) \cdot \deg(F).$$

As an application of mapping degree, we can define linking numbers of loops in $\mathbb{R}^3$. Let $\gamma_1, \gamma_2 : S^1 \to \mathbb{R}^3$ be two smooth maps whose images are disjoint, that is, with $\gamma_1([s]) \neq \gamma_2([t])$ for all $[s], [t] \in S^1$.

Definition 8.18. *The* linking number *of the non-intersecting loops* $\gamma_1, \gamma_2 : S^1 \to \mathbb{R}^3$ *is the integer*

$$L(\gamma_1, \gamma_2) = \deg(F),$$

where

$$F : S^1 \times S^1 \to S^2, \ ([s],[t]) \mapsto \frac{\gamma_1([s]) - \gamma_2([t])}{\| \gamma_1([s]) - \gamma_2([t]) \|}.$$

Thus, $L(\gamma_1, \gamma_2) = \int_{S^1 \times S^1} F^* \omega$, for any choice of 2-form $\omega \in \Omega^2(S^2)$ having integral equal to 1.

Intuitively, the linking number describes how many times the loops given by γ_1 and γ_2 "intertwine" around each other in $\mathbb{R}^3$, with the sign distinguishing left-handed and right-handed intertwining. Note that if either γ_1 or γ_2 is a constant loop then the map F is not surjective, hence $L(\gamma_1, \gamma_2) = 0$. More generally, if it is possible to smoothly deform one of the loops—all the while remaining disjoint from the other loop—into a constant loop then the linking number must be zero. In his case, we consider γ_1 and γ_2 "unlinked." Conversely, if the linking number is *not* zero, such a deformation is not possible; it is in this sense that the circles in the Hopf fibration are linked (cf. Section 3.7).

Remark 8.19. The linking number is an important invariant in knot theory, and has applications in physics. It was first defined by Gauss in an 1833 unpublished note, in the context of electromagnetism; thirty years later it was rediscovered by Maxwell in a similar context. For more on the history of linking numbers, including proofs of equivalence between the modern and historical definitions, see [16].

Remark 8.20. Using a similar definition, one can define linking numbers for non-intersecting embeddings $f : M \to \mathbb{R}^{k+1}$, $g : N \to \mathbb{R}^{k+1}$, where M and N are compact oriented manifolds with $\dim M + \dim N = k$.

8.4 Volume Forms

A top-degree differential form $\Gamma \in \Omega^m(M)$ is called a *volume form* if it is non-vanishing everywhere, i.e., $\Gamma_p \neq 0$ for all $p \in M$. In a local coordinate chart (U, φ), this means that

$$(\varphi^{-1})^* \Gamma = f \, \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^m$$

with $f(\mathbf{x}) \neq 0$ for all $\mathbf{x} \in \varphi(U)$.

Suppose $S \subseteq M$ is a submanifold of codimension 1 (a hypersurface), and $X \in \mathfrak{X}(M)$ a vector field that is *nowhere tangent* to S. Let $i : S \to M$ be the inclusion. Given a volume form Γ on M, the form

$$i^* \big(\iota_X \Gamma \big) \in \Omega^{m-1}(S)$$

is a volume form on S.

145 (answer on page 316). Verify the claim that $i^*\left(\iota_X \Gamma\right)$ is a volume form on S.

Example 8.21. The Euclidean space $\mathbb{R}^m$ has a *standard volume form*

$$\Gamma = \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^m.$$

If S is a hypersurface given as a level set $f^{-1}(0)$, where 0 is a regular value of f, then the gradient vector field

$$X = \nabla f = \sum_{i=1}^{m} \frac{\partial f}{\partial x^i} \frac{\partial}{\partial x^i}$$

has the property that X is nowhere tangent to S. By the above, it follows that S inherits a volume form $i^*(\iota_X \Gamma)$.

Example 8.22. As a special case, let $i:\ S^n \to \mathbb{R}^{n+1}$ be the inclusion of the standard n-sphere. Let $X = \sum_{i=0}^{n} x^i \frac{\partial}{\partial x^i}$. Then

$$\iota_X(\mathrm{d}x^0 \wedge \cdots \wedge \mathrm{d}x^n) = \sum_{i=0}^{n} (-1)^i x^i \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^{i-1} \wedge \mathrm{d}x^{i+1} \wedge \cdots \wedge \mathrm{d}x^n$$

pulls back to a volume form on S^n.

Proposition 8.23. *A volume form $\Gamma \in \Omega^m(M)$ determines an orientation on M, by taking as the oriented charts those charts (U, φ) such that*

$$(\varphi^{-1})^* \Gamma = f\ \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^m$$

with $f > 0$ everywhere on $\Phi(U)$.

Proof. Given any chart (U, φ), where U is connected, the function f defined as above cannot change sign, and composing φ with an orientation reversing map if necessary we can arrange $f > 0$. In particular the charts of this type cover M. We need to check that any two charts of this type are oriented compatible.

Suppose (U, φ) and (V, ψ) are two charts, where $(\varphi^{-1})^* \Gamma = f\ \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^m$ and $(\psi^{-1})^* \Gamma = g\ \mathrm{d}y^1 \wedge \cdots \wedge \mathrm{d}y^m$ with $f > 0$ and $g > 0$. If $U \cap V$ is non-empty, let $F = \psi \circ \varphi^{-1}:\ \varphi(U) \to \psi(V)$ be the transition function. Then

$$F^*(\psi^{-1})^* \Gamma|_{U \cap V} = (\varphi^{-1})^* \Gamma|_{U \cap V},$$

hence

$$g(F(\mathbf{x}))\ J(\mathbf{x})\ \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^m = f(\mathbf{x})\ \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^m,$$

where J is the Jacobian determinant of the transition map $F = \psi \circ \varphi^{-1}$. Since $f, g > 0$ it follows that $J > 0$, as required. $\square$

Theorem 8.24. *A manifold M is orientable if and only if it admits a volume form. In this case, any two volume forms Γ, Γ' compatible with the orientation differ by an everywhere positive smooth function:*

$$\Gamma' = f\Gamma, \quad f > 0.$$

Proof. As we saw above, any volume form determines an orientation. Conversely, if M is an oriented manifold, there exists a volume form compatible with the orientation: Let $\{(U_\alpha, \varphi_\alpha)\}$ be an oriented atlas on M. Then each

$$\Gamma_\alpha = \varphi_\alpha^*(\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^m) \in \Omega^m(U_\alpha)$$

is a volume form on U_α; on overlaps $U_\alpha \cap U_\beta$ these are related by the Jacobian determinants of the transition functions, which are *strictly positive* functions. Let $\{\chi_\alpha\}$ be a locally finite partition of unity subordinate to the cover $\{U_\alpha\}$ (see Appendix C.4). The forms $\chi_\alpha \Gamma_\alpha$ have compact support in U_α, hence they extend by zero to global forms on M (somewhat imprecisely, we use the same notation for this extension). The sum

$$\Gamma = \sum_\alpha \chi_\alpha \Gamma_\alpha \in \Omega^m(M)$$

is a well-defined volume form. Indeed, near any point p at least one of the summands is non-zero; and if other summands in this sum are non-zero, they differ by a positive function. $\square$

For a compact manifold M with a given volume form $\Gamma \in \Omega^m(M)$, one can define the *volume of M*,

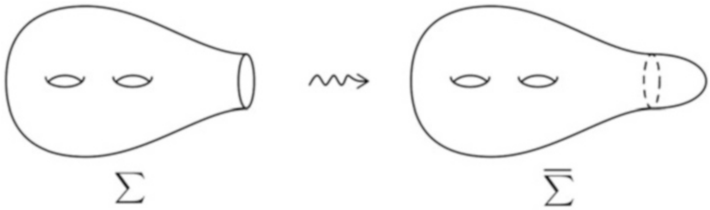
$$\mathrm{vol}(M) = \int_M \Gamma.$$

Here the orientation used in the definition of the integral is taken to be the orientation given by Γ, so $\mathrm{vol}(M) > 0$. By the discussion around Theorem 8.13, this means that Γ cannot be exact, and so represents a non-trivial cohomology class. The compactness of M is essential here: For instance, $\mathrm{d}x$ is an exact volume form on the real line $\mathbb{R}$.

8.5 Applications to Differential Geometry of Surfaces

In this section, we apply the techniques developed in this chapter to the differential geometry of surfaces. In order to keep the discussion at a reasonable length, we will skip many details, and occasionally omit some technical arguments. The two big results proved in this section are the Poincaré theorem and the Gauss-Bonnet theorem. Their generalizations to manifolds of higher dimensions are known as the Poincaré-Hopf theorem and Gauss-Bonnet-Chern theorem, respectively; but their proof in the 2-dimensional case is considerably simpler.

Throughout, we shall restrict our attention to compact surfaces Σ with a possibly non-empty boundary. By this, we mean that Σ can be realized as a domain with boundary inside a surface $\overline{\Sigma}$ without boundary. The particular choice of $\overline{\Sigma}$ is unimportant; one such choice is obtained from "capping" off the boundary components by attaching disks.



By a vector field X on Σ, we mean the restriction of a vector field $\overline{X}$ on $\overline{\Sigma}$; this notion does not depend on the choice of $\overline{\Sigma}$. Similarly, we interpret smooth functions and differential forms on Σ in terms of such extensions.

8.5.1 Euler Characteristic of Surfaces

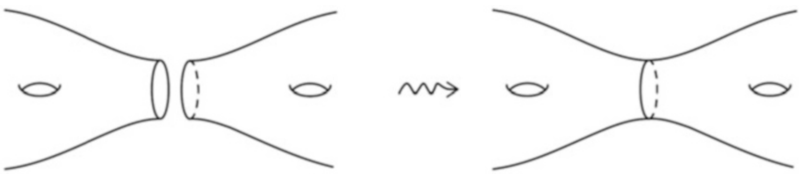
The *Euler characteristic* $\chi(\Sigma) \in \mathbb{Z}$ of a compact surface Σ with boundary (not necessarily connected or oriented) may be defined in various equivalent ways, For example, through the following axioms.

(E1) If Σ is the disjoint union of two surfaces $\Sigma = \Sigma' \sqcup \Sigma''$, then

$$\chi(\Sigma) = \chi(\Sigma') + \chi(\Sigma'').$$

(E2) If Σ is obtained from a surface Σ' by gluing two boundary circles, then

$$\chi(\Sigma) = \chi(\Sigma').$$

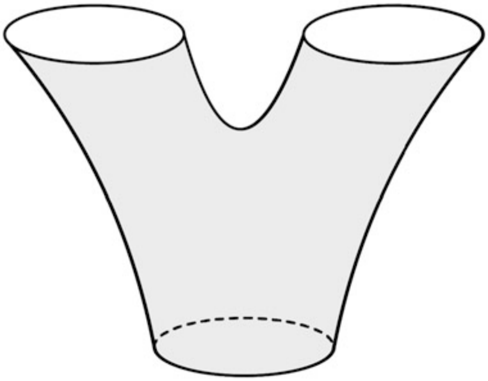


(E3) For a 2-disk D^2,

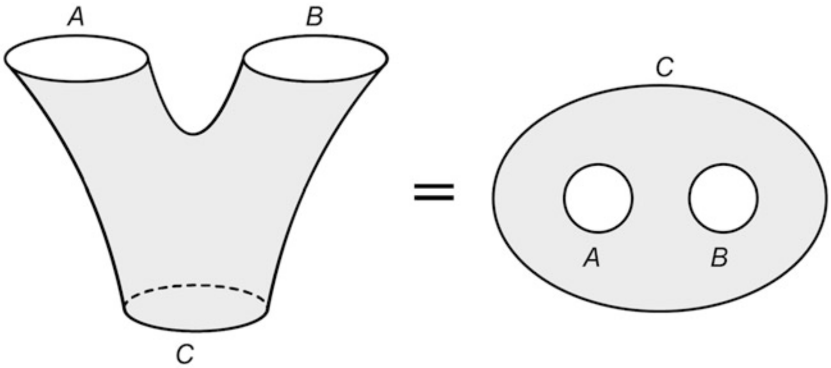
$$\chi(D^2) = 1.$$

Note that in (E2), Σ' may have more connected components than Σ.

Example 8.25. An oriented surface Σ of genus 0 with three boundary components is called a "pair of pants," illustrated in the picture below.



Alternatively, we may think of Σ as being obtained from a disk in $\mathbb{R}^2$ by removing two smaller disks.



By the axioms, removing a disk from a surface decreases the Euler characteristic by 1. Hence,

$$\chi(\Sigma) = 1 - 1 - 1 = -1.$$

More generally, the Euler characteristic of every surface may be computed by cutting the surface into simpler pieces. In the orientable case, the result is as described by the following lemma.

Lemma 8.26. *If Σ is an oriented and connected surface of genus g, with r boundary components, then*

$$\chi(\Sigma) = 2 - 2g - r.$$

146 (answer on page 317). Explain how to obtain this result from the axioms.

For instance, the Euler characteristics of a 2-sphere and of a 2-torus are

$$\chi(S^2) = 2, \ \ \chi(T^2) = 0.$$

147 (answer on page 317). Using the axioms, show that the Möbius strip and the Klein bottle have Euler characteristic 0, while

$$\chi(\mathbb{R}\mathrm{P}^2) = 1.$$

Remark 8.27. Another definition of the Euler characteristic of surfaces uses a choice of a *triangulation* of the surface. One puts

$$\chi(\Sigma) = \#\{\text{vertices}\} - \#\{\text{edges}\} + \#\{\text{faces}\}$$

as an alternating sum of the number of vertices, edges, and faces (i.e., the triangles) appearing in the triangulation.

Remark 8.28. For compact manifolds M without boundary (but of arbitrary dimensions), the Euler characteristic may be defined as the alternating sum of Betti numbers $b^i = \dim H^i(M,\mathbb{R})$ (cf. (7.20) and the surrounding discussion)

$$\chi(M) = \sum_{i=0}^{\dim M} (-1)^i b_i.$$

It is a non-trivial fact that this coincides with our definition in case $\dim M = 2$.

8.5.2 Rotation Numbers for Vector Fields

Let $D \subseteq \mathbb{R}^2$ be a region with boundary, and $X \in \mathfrak{X}(D)$ a vector field. Using the coordinates to write $X = a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y}$ for some smooth functions $a,b: \ D \to \mathbb{R}^2$, we may think of this vector field as a smooth map

$$f_X = (a,b): \ D \to \mathbb{R}^2.$$

Suppose $\gamma: \ S^1 \to D$ is a loop not meeting the set of zeroes of X, i.e., with $X_{\gamma([t])} \neq 0$ for all t. Then we may consider the number of rotations of X along the loop. More formally, $\mathrm{rot}_\gamma(X)$ is the winding number (cf. Section 8.3.2)

$$\mathrm{rot}_\gamma(X) = w(f_X \circ \gamma)$$

of the loop $f_X \circ \gamma: \ S^1 \to \mathbb{R}^2 \setminus \{\mathbf{0}\}$. Since winding numbers are invariant under deformations, the same is true of these rotation numbers, as long as X remains non-vanishing along γ during the deformation. In particular, $\mathrm{rot}_\gamma(X)$ vanishes if γ can be deformed into a constant path, or if X can be deformed into a constant vector field (while remaining non-vanishing along γ). Note also that if γ^- is obtained from the loop γ by traveling in the opposite direction, then

$$\mathrm{rot}_{\gamma^-}(X) = -\mathrm{rot}_\gamma(X).$$

148 (answer on page 317). If we replace X with $-X$, how would the rotation number change? What about if we replace X with cX for some non-zero constant $c \neq 0$?

149 (answer on page 317). Consider the loop $\gamma\colon [t] \mapsto (\cos(2\pi t), \sin(2\pi t))$, and let

$$X = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}, \quad Y = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}.$$

Directly from the definition (in terms of winding numbers), show that $\mathrm{rot}_\gamma(X) = \mathrm{rot}_\gamma(Y) = 1$.

150 (answer on page 318). Continuing with the notation of 149 above. Show that for all $s_1, s_2 \in \mathbb{R}$, not both equal to 0, the vector field

$$W_{s_1,s_2} = s_1 X + s_2 Y$$

is non-vanishing away from the origin $(0,0) \in \mathbb{R}^2$. Use this to argue that the vector field

$$Z = (y-x)\frac{\partial}{\partial x} - (x+y)\frac{\partial}{\partial y}$$

has rotation number $\mathrm{rot}_\gamma(Z) = 1$.

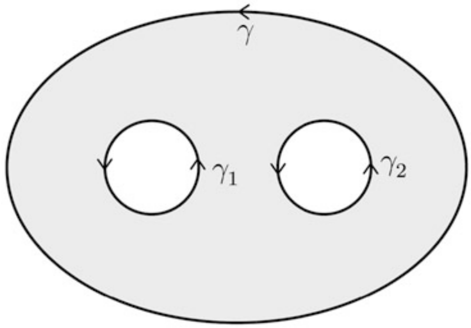
We shall need the following additivity property of rotation numbers. Suppose $\gamma_1, \gamma_2 :\ S^1 \to D$ are two smooth loops, with the same end-points $\gamma_1([0]) = \gamma_2([0])$, and let

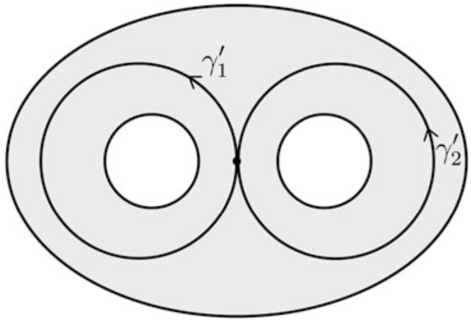
$$\gamma = \gamma_1 * \gamma_2 :\ S^1 \to D$$

be their concatenation, given by the loop γ_1 followed by the loop γ_2 (see Problem 10). Assuming "sitting instances" (i.e., $\gamma_i([t])$ is constant for t close to 0), the concatenation is again smooth. Then

$$\mathrm{rot}_\gamma(X) = \mathrm{rot}_{\gamma_1}(X) + \mathrm{rot}_{\gamma_2}(X), \tag{8.5}$$

by the analogous property for winding numbers (Problem 10). More generally, this identity holds so long as γ_1, γ_2 can be *deformed* within D to loops γ_1', γ_2' for which the concatenation is defined, and γ can be deformed into $\gamma_1' * \gamma_2'$, as illustrated in the following picture. As before, we only allow deformations of paths that do not meet the set of zeroes of X.





151 (answer on page 318). Use the additivity property of rotation numbers to show that there does not exist a nowhere-vanishing vector field X on the pair-of-pants Σ ($g = 0, r = 3$) with the property that X is tangent to all three boundary components. (It is convenient to realize Σ as a domain with boundary in $\mathbb{R}^2$.)

So far, we have worked with the identification of vector fields $X \in \mathfrak{X}(D)$ with $\mathbb{R}^2$-valued functions, expressing $X = a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y}$ for some smooth functions a, b. This relies on the fact that the coordinate vector fields $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}$ define smoothly varying ordered bases for the tangent spaces. This is an example of a *frame*.

Definition 8.29. *Let Σ be a surface, and $U \subseteq \Sigma$ an open subset. A* frame *over U is a pair of pointwise linearly independent vector fields $X_1, X_2 \in \mathfrak{X}(U)$. That is, for all $p \in U$ the tangent vectors $X_1|_p$, $X_2|_p$ are a basis of $T_p\Sigma$.*

If Σ is oriented, then we say that the frame X_1, X_2 is *oriented* provided that these vector fields define oriented bases for the tangent spaces T_pU. Using terminology to be introduced in the next chapter, the frame amounts to a *trivialization of the tangent bundle*, $T\Sigma|_U \to U \times \mathbb{R}^2$.

Given a frame X_1, X_2 over U, every vector field $X \in \mathfrak{X}(U)$ may be uniquely expressed as $X = a_1 X_1 + a_2 X_2$, for a smooth function

$$f_X = (a_1, a_2) : U \to \mathbb{R}^2.$$

If $\gamma : S^1 \to U$ is a loop such that X is non-vanishing at all points of $\gamma(S^1) \subseteq U$, then f_X takes values in $\mathbb{R}^2 \setminus \{\mathbf{0}\}$, and hence the winding number of

$$f_X \circ \gamma : S^1 \to \mathbb{R}^2 \setminus \{\mathbf{0}\}$$

is defined. It counts the number of rotations of X along γ, relative to the given frame X_1, X_2. This number is unchanged under deformations (of the vector field, or of the path), as long as the vector field remains non-vanishing along the loop. We denote this winding number by $\mathrm{rot}_\gamma(X) = w(f_X \circ \gamma)$ as before. It is important to keep in mind, however, that this definition depends on the choice of frame.

152 (answer on page 318). How does the rotation number change if the given frame X_1, X_2 is replaced with $X_1' = X_2,\ X_2' = X_1$?

We now introduce two constructions using frames: Indices of vector fields, and rotation numbers along embedded curves.

Index of a Vector Field

Let $X \in \mathfrak{X}(\Sigma)$ be a vector field with an isolated zero at $p \in \Sigma$ (we assume p is not a boundary point). This means that X vanishes at p, but is non-vanishing on $U \setminus \{p\}$ for some open neighborhood U of p. See the pictures below for some examples. The *index of X at p* signifies the number of rotations of X along a "small" loop γ in $\Sigma \setminus \{p\}$, winding once around p.

To make this precise, pick a coordinate chart (U, φ) centered at p, not containing zeroes of X other than p, and with $\varphi(U) = \mathbb{R}^2$. The choice of chart identifies $U \cong \mathbb{R}^2$, and in particular gives an orientation. Let $X_1, X_2 \in \mathfrak{X}(U)$ be an oriented frame (e.g., given by the coordinate vector fields), and let $\gamma : S^1 \to U \setminus \{p\} \cong \mathbb{R}^2 \setminus \{\mathbf{0}\}$ be a loop of winding number $+1$ (e.g., the standard loop γ_1, in the notation of (8.2)). Since X is non-vanishing on $U \setminus \{p\}$, the rotation number along γ, relative to the frame X_1, X_2, is defined.

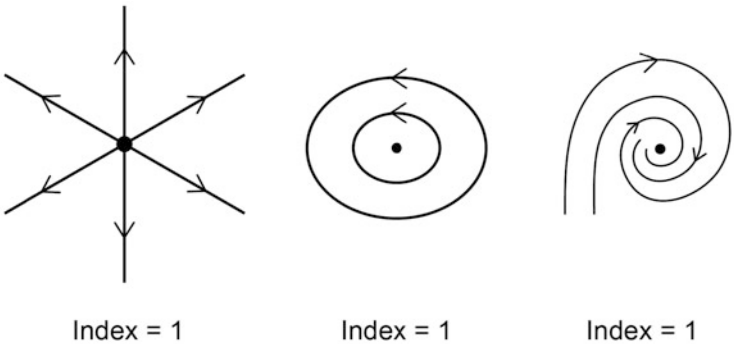
Definition 8.30. *For a vector field $X \in \mathfrak{X}(\Sigma)$ with an isolated zero at $p \in \Sigma$, the rotation number* $\mathrm{rot}_\gamma(X)$ *(for the choice of frame and path γ described above) is called the* index of X at p *and is denoted*

$$\mathrm{index}_p(X) \in \mathbb{Z}.$$

Using the invariance of rotation numbers under deformations, one may prove that the index is well-defined: It does not depend on the choices made. In particular, it does not involve a choice of orientation near p.

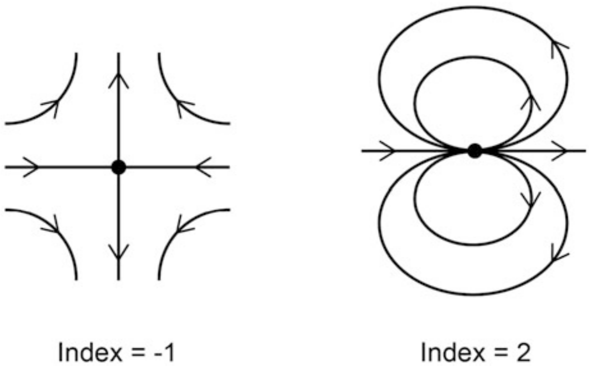
153 (answer on page 318). Show that $\text{index}_p(X)$ does not depend on the choices made.

Example 8.31. In the pictures below, vector fields are indicated through their "phase portraits," by plotting some typical integral curves. For each of the following three types of zeroes of vector fields, the index is $+1$.



Concrete vector fields in $\mathbb{R}^2$ exhibiting this behavior are the vector fields X, Y, Z discussed in 149 and 150 (where you have also computed the rotation numbers).

On the other hand, a zero corresponding to a "saddle point" has index -1, while the vector field on the right has index 2.



Remark 8.32. A change of direction of a vector field, replacing X with $-X$, does not affect its indices at isolated zeroes. In fact, we have already observed earlier that $\mathrm{rot}_\gamma(-X) = \mathrm{rot}_\gamma(X)$.

154 (answer on page 318). Give an example of a vector field on $\mathbb{R}^2$ with an isolated zero at $p = (0,0)$ and with $\mathrm{index}_p(X) = 0$.

155 (answer on page 319). Consider S^2 as the surface $x^2 + y^2 + z^2 = 1$ embedded in $\mathbb{R}^3$.

(a) Show that the vector field

$$X = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$$

on $\mathbb{R}^3$ restricts to a vector field Y on S^2.

(b) Find the zeroes of Y and compute its index at each point.

Rotation Numbers Along Embedded Circles

Using a similar construction, we can define a rotation number of vector fields X along embedded closed curves (circles) $C \subseteq \Sigma$, provided that X is non-vanishing along all points of C. For simplicity, let us assume that both Σ and the curve C come equipped with an *orientation*. On some open neighborhood U of C, we may choose an oriented frame X_1, X_2, with the property that X_1 is tangent to C and defines the given orientation of C.

Pick any orientation preserving diffeomorphism $\gamma\colon\ S^1 \to C$; in other words, γ provides a *parametrization* of C.

Definition 8.33. *Let $C \subseteq \Sigma$ be an embedded circle, and suppose the vector field X is non-vanishing at all points of C. The rotation number* $\mathrm{rot}_\gamma(X)$ *(with respect to the frame X_1, X_2 and loop γ, as above) is called the* rotation number of X with respect to *C and is denoted*

$$\mathrm{rot}_C(X) \in \mathbb{Z}.$$

Again, this definition does not depend on the choices made, since any two choices may be deformed into each other. Intuitively, the rotation number counts the number of rotations of X with respect to the tangent direction of C.

Example 8.34. If the vector field X is everywhere tangent to C, or if it is *transverse* to C in the sense that it is nowhere tangent, we have that $\mathrm{rot}_C(X) = 0$. In the first case, we may take $X_1 = X$, which makes the map $f \circ \gamma$ the constant map. In the second case, take X_1 tangent to C and $X_2 = \pm X$ (on a neighborhood of U where X is non-zero); again the map $f \circ \gamma$ is the constant map.

Example 8.35. Let $D \subseteq \mathbb{R}^2$ be a domain with boundary. Any embedded circle $C \subseteq D$ has an orientation, by declaring the counterclockwise direction to be positive. Suppose $X \in \mathfrak{X}(D)$ is non-vanishing at all points of C. Then

$$\mathrm{rot}_C(X) = \mathrm{rot}_\gamma(X) - 1, \tag{8.6}$$

where $\mathrm{rot}_\gamma(X)$ is the rotation number with respect to the coordinate frame $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}$. The extra -1 appears in (8.6) because the frame X_1, X_2 used for the definition of $\mathrm{rot}_C(X)$, itself makes a full rotation with respect to the coordinate frame $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}$.

Similarly, if X has an isolated zero at $p \in D$ (where p is not a boundary point), and $C \subseteq D$ is a small embedded circle around p (where "small" means that the region inside C is contained in D, and contains no zeroes other than p), then

$$\mathrm{index}_p(X) = 1 + \mathrm{rot}_C(X) \tag{8.7}$$

because $\mathrm{index}_p(X)$ is *by definition* equal to the rotation number $\mathrm{rot}_\gamma(X)$ relative to the coordinate frame.

Remark 8.36. We may define rotation numbers along embedded circles in a possibly non-orientable surface Σ, provided that C comes with a *co-orientation*. Roughly speaking, a choice of co-orientation amounts to declaring a "negative" side of C. Concretely, a co-orientation may be described by the choice of a function ρ having C has a regular level set $C = \rho^{-1}(0)$, where two such functions give the same co-orientation if $\rho' = h\rho$ with $h > 0$. For example, if C is a boundary component of Σ, then C has a natural co-orientation, with $\rho < 0$ on the interior of Σ. On the other hand, the central circle of a Möbius strip does not admit a co-orientation.

8.5.3 Poincaré Theorem

Using the notions introduced above, we may formulate the following result due to Poincaré. Recall that $\chi(\Sigma)$ denotes the Euler characteristic of a surface Σ.

Theorem 8.37 (Poincaré Theorem). *Let Σ be a compact, oriented surface with boundary, and let X be a vector field on Σ with isolated zeroes, all of which are in the interior. Then*

$$\sum_p \mathrm{index}_p(X) - \sum_C \mathrm{rot}_C(X) = \chi(\Sigma).$$

Here the first sum is over all zeroes of X, and the second sum is over all boundary components, with their natural boundary orientation.

(Note that the number of zeroes of X is finite, since Σ is compact.)

Proof. Denote the left-hand side by $\chi_X(\Sigma)$. Our goal is to prove $\chi_X(\Sigma) = \chi(\Sigma)$. The idea is to cut Σ into simpler pieces, where the formula can be verified by hand.

Specifically, our axioms for the Euler characteristic define it via the cutting and gluing of disks along boundary circles, and our plan is to reverse this process and reduce to the case of disks.

Suppose Σ' is obtained from Σ by cutting along an embedded circle $C_* \subseteq \Sigma$, not meeting the boundary and also not meeting the zeroes of X. The vector field X on Σ gives rise to a vector field X' on Σ'. We claim that

$$\chi_{X'}(\Sigma') = \chi_X(\Sigma), \tag{8.8}$$

mirroring Axiom (E2) for the Euler characteristic. Since clearly

$$\sum_{p\in\Sigma'} \mathrm{index}_p(X') = \sum_{p\in\Sigma} \mathrm{index}_p(X),$$

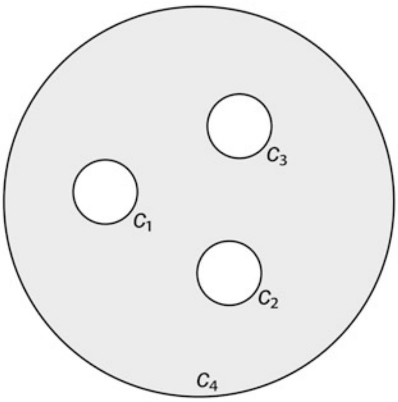
we have to show

$$\sum_{C\subseteq\partial\Sigma'} \mathrm{rot}_C(X') = \sum_{C\subseteq\partial\Sigma} \mathrm{rot}_C(X).$$

Note that the sum on the left-hand side contains two extra terms, given by the two boundary circles $C_{*,1}$, $C_{*,2}$ which are identified to C_*. But since the orientations induced from Σ' are opposite, these terms cancel. This proves (8.8).

Using (8.8), we begin the process of cutting Σ into pieces. Given an isolated zero p of X, choose a disk $D \subseteq \Sigma$ around p, not meeting the boundary or other zeroes of X. For the disk, we have already verified the Poincaré theorem; see Equation (8.7). Hence, by iteratively cutting out disks around the zeroes of X, and using (8.8), we have reduced the problem to the case that the set of zeroes of X is empty.

Using additional cuts through the "handles" of Σ (if any) we may reduce the problem to the case that Σ is a compact oriented surface of genus $g = 0$, with $r > 1$ boundary components, and X is a non-vanishing vector field on Σ. But any such Σ can be realized as a region with boundary $D \subseteq \mathbb{R}^2$. That is, Σ is obtained from a disk in $\mathbb{R}^2$ by removing $r-1$ disjoint disks from its interior. Let $C_1,\ldots,C_{r-1}$ be the inner boundary components and C_r the outer boundary.



Let $\gamma_i : S^1 \to C_i$ be diffeomorphisms inducing the counterclockwise orientation, and let $\mathrm{rot}_{\gamma_i}(X)$ be the corresponding rotation numbers. The counterclockwise orientation of C_i agrees with the boundary orientation (induced from Σ) if $i = r$, but is *opposite* for the inner boundary components. Therefore, using (8.6),

$$\mathrm{rot}_{\gamma_i}(X) = \begin{cases} -\mathrm{rot}_{C_i}(X) + 1 & \text{If } 1 \le i < r, \\ \mathrm{rot}_{C_i}(X) + 1 & \text{If } i = r. \end{cases}$$

On the other hand, we have that $\mathrm{rot}_{\gamma_r}(X) = \sum_{i=1}^{r-1} \mathrm{rot}_{\gamma_i}(X)$ by (8.5). Putting all of this together, we obtain

$$\sum_{i=1}^{r} \mathrm{rot}_{C_i}(X) = \sum_{i=1}^{r-1} (1 - \mathrm{rot}_{\gamma_i}(X)) + (\mathrm{rot}_{\gamma_r}(X) - 1) = r - 2 = -\chi(\Sigma)$$

as desired. □

Remark 8.38. Poincaré theorem also holds for not necessarily oriented surfaces Σ, using the natural co-orientations of the boundary components (cf. Remark 8.36). This may be shown by lifting the vector field to the “oriented double cover,” or directly by using a similar argument as in the oriented case.

Example 8.39. In 155, we found a vector field on S^2 with exactly two zeroes, each of index 1. It follows that $\chi(S^2) = 2$, as we have also found from the axioms. Similarly, 156 agrees with $\chi(T^2) = 0$.

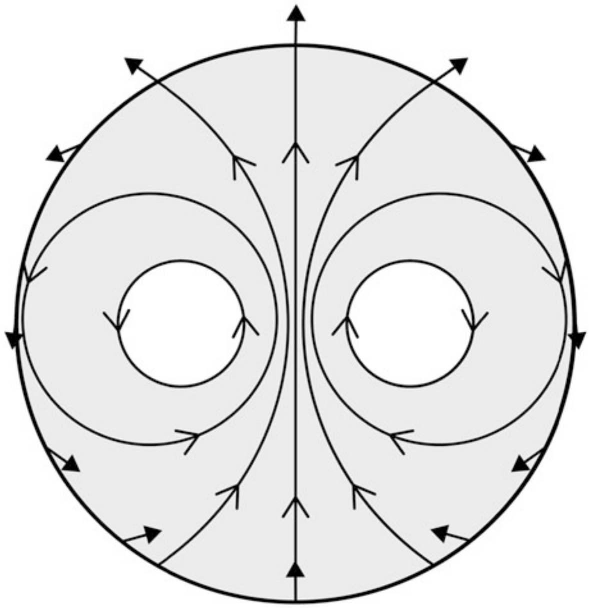
Corollary 8.40. *A compact surface without boundary Σ does not admit a nowhere-vanishing vector field X, unless $\chi(\Sigma) = 0$. More generally, a compact surface with boundary does not admit a nowhere-vanishing vector field X that is everywhere tangent to the boundary, unless $\chi(\Sigma) = 0$.*

Proof. Given a vector field with this property, the left-hand side of the Poincaré theorem is zero, hence $\chi(\Sigma) = 0$. □

For a compact connected surface without boundary Σ, the classification of surfaces tells us that if $\chi(\Sigma) = 0$ then Σ is either a 2-torus or a Klein bottle. The fact that there is no nowhere-vanishing vector field on S^2 is also known as the *Hairy Ball Theorem*.

156 (answer on page 319). Construct a nowhere-vanishing vector field on the 2-torus.

Example 8.41. The following picture shows a nowhere-vanishing vector field X on a pair-of-pants Σ (realized as a domain with boundary in $\mathbb{R}^2$).



Note that X is tangent to two of the three boundary components; the corresponding rotation numbers are 0. The rotation number around the third boundary component is 1. Since X has no zeroes, $\sum_p \operatorname{index}_p(X) = 0$ (an empty sum). Hence

$$\sum_p \operatorname{index}_p(X) - \sum_C \operatorname{rot}_C(X) = -1 = \chi(\Sigma).$$

as expected.

8.5.4 Gauss-Bonnet Theorem

The formulation of Gauss-Bonnet's theorem requires some Riemannian geometry. A *Riemannian metric* on a manifold M is given by inner products

$$\mathrm{g}_p : \; T_pM \times T_pM \to \mathbb{R}$$

on the tangent spaces, depending smoothly on $p \in M$, in the sense that for all vector fields X, Y, the map $p \mapsto \mathrm{g}_p(X_p, Y_p)$ is smooth. The presence of a Riemannian metric allows one to define notions such as distance and curvature. It would take us too far afield to give a proper introduction to the subject; instead, we will limit the discussion to two dimensions, and only cover aspects that are needed to state and prove the Gauss-Bonnet theorem.

The basic plan is to describe Riemannian metrics in terms of 1-forms, using Cartan's *coframe formalism*, and eventually obtain the Gauss-Bonnet theorem as a consequence of Stokes' theorem. Throughout, we take Σ to be a compact, oriented surface, possibly with boundary.

Definition 8.42. *A* coframe *over an open subset $U \subseteq \Sigma$ is a pair of pointwise linearly independent 1-forms $\alpha_1, \alpha_2 \in \Omega^1(U)$.*

For every frame there is a dual coframe, and vice versa, obtained by taking the pointwise dual basis:

$$\alpha_i(X_j) = \delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

Given an orientation on Σ, we call the coframe *oriented* if its dual frame is oriented; equivalently, if the orientation defined by the volume form $\alpha_1 \wedge \alpha_2$ agrees with the given orientation. Given a Riemannian metric on Σ, we call the coframe *orthonormal* if the dual frame is orthonormal, i.e.,

$$\mathrm{g}(X_1, X_1) = \mathrm{g}(X_2, X_2) = 1, \ \mathrm{g}(X_1, X_2) = 0.$$

Locally, the metric is then described in terms of the orthonormal coframe α_1, α_2 by

$$\mathrm{g}(X,Y) = \alpha_1(X)\alpha_1(Y) + \alpha_2(X)\alpha_2(Y). \tag{8.9}$$

(This is true when $X, Y \in \{X_1, X_2\}$, hence it is true in general by the C^∞-bilinearity of both sides.) At each point $p \in U$, the oriented orthonormal frame gives an isometry $T_pM \to \mathbb{R}^2$. Orientation preserving isometries of $\mathbb{R}^2$ fixing the origin are given by rotation matrices

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

Hence, on overlaps $U \cap U'$ of open sets with given oriented orthonormal (co)frames, we have

$$X_1' = \cos(\theta)X_1 - \sin(\theta)X_2, \ \ X_2' = \sin(\theta)X_1 + \cos(\theta)X_2$$

and dually

$$\alpha_1' = \cos(\theta)\alpha_1 + \sin(\theta)\alpha_2, \ \ \alpha_2' = -\sin(\theta)\alpha_1 + \cos(\theta)\alpha_2, \tag{8.10}$$

where the angle function θ on $U \cap U'$ is defined modulo $2\pi\mathbb{Z}$.

Lemma 8.43. *Given a Riemannian metric on the oriented surface Σ, there is a unique volume form*

$$\mathrm{d}A \in \Omega^2(\Sigma)$$

such that for every oriented orthonormal coframe $\alpha_1, \alpha_2 \in \Omega^1(U)$, $dA|_U = \alpha_1 \wedge \alpha_2$.

One calls dA the *area form* for the Riemannian metric; despite the notation it is *not* an exact form in general.

157 (answer on page 319). Use Formulas (8.10) above to show that for a change of oriented orthonormal coframe, $\alpha_1' \wedge \alpha_2'$ agrees with $\alpha_1 \wedge \alpha_2$ on $U \cap U'$.

We therefore have the following proposition.

Proposition 8.44. *For any oriented orthonormal coframe α_1, α_2 over $U \subseteq \Sigma$, there is a unique 1-form $\omega \in \Omega^1(U)$ satisfying the* structure equations

$$\mathrm{d}\alpha_1 = -\omega \wedge \alpha_2, \;\; \mathrm{d}\alpha_2 = \omega \wedge \alpha_1.$$

Proof. Writing $\mathrm{d}\alpha_i = -f_i \, \mathrm{d}A|_U$, we have $\omega = f_1\alpha_1 + f_2\alpha_2$. □

The 1-form ω is called the *spin connection* in Cartan's formalism.

158 (answer on page 319). The Euclidean metric on $\mathbb{R}^2$ is given by

$$\mathrm{g}(\frac{\partial}{\partial x}, \frac{\partial}{\partial x}) = \mathrm{g}(\frac{\partial}{\partial y}, \frac{\partial}{\partial y}) = 1, \quad \mathrm{g}(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}) = 0.$$

What coframe α_1, α_2 describes this metric via (8.9)? What is the area form $\mathrm{d}A$? What is the spin connection ω?

159 (answer on page 319). The hyperbolic metric on the upper half-plane $\{(x,y) | \, y > 0\}$ is given by

$$\mathrm{g}(\frac{\partial}{\partial x}, \frac{\partial}{\partial x}) = \mathrm{g}(\frac{\partial}{\partial y}, \frac{\partial}{\partial y}) = \frac{1}{y^2}, \quad \mathrm{g}(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}) = 0.$$

What coframe α_1, α_2 describes this metric via (8.9)? What is the area form $\mathrm{d}A$? What is the spin connection ω?

Under a coframe rotation (8.10), the spin connection changes to

$$\omega'|_{U\cap U'} = \omega|_{U\cap U'} - \mathrm{d}\theta. \tag{8.11}$$

160 (answer on page 319). Verify Equation (8.11).

In particular, $\mathrm{d}\omega$ agrees with $\mathrm{d}\omega'$ on $U \cap U'$. This shows that the Riemannian metric g determines a global 2-form on Σ, given locally by the differentials $\mathrm{d}\omega$ of the spin connections. Expressing this 2-form in terms of the Riemannian volume form one is led to the following definition.

Definition 8.45 (Gauss Curvature). *The* Gauss curvature *of the Riemannian metric* g *on Σ is the function $K \in C^\infty(\Sigma)$ such that for every local oriented orthonormal coframe $\alpha_1, \alpha_2 \in \Omega^1(U)$, with corresponding spin connection $\omega \in \Omega^1(U)$,*

$$(K\mathrm{d}A)|_U = \mathrm{d}\omega.$$

The following exercise states that the Gauss curvature vanishes if and only if the metric is locally the standard flat metric of the Euclidean plane. This is a first indication that K provides a measure of "curvature."

161 (answer on page 320). Show that the Euclidean plane from 158 has constant Gauss curvature $K = 0$. Conversely, if Σ is a surface with a Riemannian metric of Gauss curvature 0, then there exist local coordinates around any given point $p \in \Sigma$ in which the metric is given by the standard Euclidean metric.

162 (answer on page 320). Show that the hyperbolic plane from 159 has constant Gauss curvature $K = -1$.

Remark 8.46. These examples have the special feature that the Gauss curvature is a constant function. It can be shown that every compact, connected, oriented surface Σ admits a Riemannian metric of constant Gauss curvature K. Using the Gauss-Bonnet theorem to be discussed below, one finds that for a surface without boundary, the sign of K is determined by the sign of the Euler characteristic.

Remark 8.47. The definition of Gauss curvature K does not actually require an orientation on Σ. For a possibly non-orientable surface, and any local orthonormal coframe $\alpha_1, \alpha_2 \in \Omega^1(U)$, with spin connection ω, we may use the equation $\mathrm{d}\omega = (K|_U)\,\alpha_1 \wedge \alpha_2$ to define $K|_U$. Replacing α_1, α_2 with α_2, α_1 changes the sign of both ω and of $\alpha_1 \wedge \alpha_2$.

Using the metric, we can also define the curvature of curves. Let $C \subseteq \Sigma$ be an oriented 1-dimensional submanifold, and

$$i\colon\, C \to \Sigma$$

its inclusion. An oriented orthonormal coframe $\alpha_1, \alpha_2 \in \Omega^1(U)$ will be called *adapted to* C if it satisfies $i^*\alpha_2 = 0$ and $i^*\alpha_1 \in \Omega^1(C)$ gives the prescribed orientation. In terms of the dual oriented orthonormal frame X_1, X_2, this means that X_1 is tangent to C and points in the positive direction. In particular, the restriction of X_1, X_2 to C is uniquely determined by this condition, and likewise for the restriction of the coframe to C.

Hence, if $\alpha_1', \alpha_2' \in \Omega^1(U')$ is another adapted orthonormal coframe, then $i^*\alpha_1$ and $i^*\alpha_1'$ agree over $U \cap U' \cap C$. This allows us to define a volume form on C called the *arc element*

$$\mathrm{d}s \in \Omega^1(C)$$

such that

$$\mathrm{d}s|_{U\cap C} = i^*\alpha_1$$

for every adapted orthonormal coframe. Despite the notation, this form is *not* exact in general.

Similarly, the pullbacks of spin connections $i^*\omega \in \Omega^1(U\cap C)$ patch to a global 1-form on C. (Indeed, the angle function θ relating any two adapted coframes by (8.10) restricts to a constant multiple of 2π along $C\cap(U\cap U')$, hence $i^*\omega - i^*\omega' = i^*\mathrm{d}\theta = 0$.)

Definition 8.48 (Geodesic Curvature). *The geodesic curvature of an oriented embedded* 1*-dimensional submanifold* $C\subseteq\Sigma$ *is the function*

$$k_g \in C^\infty(C)$$

defined by

$$(k_g\,\mathrm{d}s)\big|_{U\cap C} = -i^*\omega,$$

for any adapted coframe $\alpha_1,\alpha_2\in\Omega^1(U)$ *with spin connection* ω*.*

Remark 8.49. The geodesic curvature changes sign under a change of orientation of C, as well as under a change of orientation of Σ. On the other hand, one may define k_g for curves C in possibly non-oriented surfaces, provided that C comes with a co-orientation (cf. Remark 8.36).

Example 8.50. Let Σ be $\mathbb{R}^2$ with the Euclidean metric, as in ✐ 158. We shall compute the geodesic curvature of straight lines through the origin, as well as of circles with center at the origin. For this, it is convenient to consider the orthonormal coframe on $\mathbb{R}^2\setminus\{\mathbf{0}\}$, written in polar coordinates r,φ as

$$\alpha_1 = \mathrm{d}r,\ \alpha_2 = r\mathrm{d}\varphi,$$

or in cartesian coordinates $x = r\cos\varphi,\ y = r\sin\varphi$ as

$$\alpha_1 = \frac{1}{\sqrt{x^2+y^2}}(x\mathrm{d}x + y\mathrm{d}y),\ \ \alpha_2 = \frac{1}{\sqrt{x^2+y^2}}(x\mathrm{d}y - y\mathrm{d}x).$$

One readily checks $\mathrm{g}(X,Y) = \alpha_1(X)\alpha_1(Y)+\alpha_2(X)\alpha_2(Y)$ on coordinate vector fields, confirming that this is indeed an orthonormal coframe; since $\alpha_1\wedge\alpha_2 = \mathrm{d}x\wedge\mathrm{d}y$ it is also oriented. We have $\mathrm{d}\alpha_1 = 0,\ \mathrm{d}\alpha_2 = \mathrm{d}r\wedge\mathrm{d}\varphi = r^{-1}\alpha_1\wedge\alpha_2$ which shows

$$\omega = -\frac{1}{r}\alpha_2.$$

Now let C be the circle of radius $R>0$ given by $x^2+y^2 = R^2$, with orientation given by $i^*\alpha_2$ where i is the inclusion. An adapted oriented orthonormal coframe is given by

$$\alpha_1' = \alpha_2,\ \alpha_2' = -\alpha_1.$$

Indeed, $i^*\alpha_1'$ is compatible with the orientation, while $i^*\alpha_2' = 0$. The spin connection for this coframe is $\omega' = -r^{-1}\alpha_1'$. Since

$$i^*\omega' = -\frac{1}{R}i^*\alpha_1' = -\frac{1}{R}\mathrm{d}s,$$

we conclude that

$$k_g = \frac{1}{R}$$

everywhere along the circle.

A similar calculation shows that straight lines through the origin have geodesic curvature equal to 0; here one may directly use the coframe α_1, α_2 (possibly with sign changes depending on the orientation).

163 (answer on page 320). Let Σ be the hyperbolic plane from 159. Find the geodesic curvature of the curve C given as $y = c$ for some constant c, with the orientation given by the tangent vector $\frac{\partial}{\partial x}$. For more calculations in the hyperbolic plane, see Problem 28.

Remark 8.51. A curve whose geodesic curvature is equal to zero is called an (unparametrized) *geodesic*. For example, straight lines in the Euclidean plane are geodesics.

Let us now consider a compact oriented surface Σ with boundary $\partial\Sigma$. Recall (cf. Lemma 8.6) that the orientation on Σ induces an orientation on the boundary.

Theorem 8.52 (Gauss-Bonnet Theorem). *For a compact, oriented surface with boundary Σ, and any choice of Riemannian metric,*

$$\int_\Sigma K\,\mathrm{d}A + \int_{\partial\Sigma} k_g\ \mathrm{d}s = 2\pi\chi(\Sigma).$$

Proof. Similarly to the proof of Poincaré theorem (Theorem 8.37), our plan is to cut along embedded circles to reduce to the case of disks. Therefore, let us first verify the theorem for the case of a disk.

We may take $\Sigma = D$ to be the standard disk $D \subseteq \mathbb{R}^2$, defined by $x^2 + y^2 \leq 1$, with a possibly non-standard metric g. Let $\alpha_1, \alpha_2 \in \Omega^1(D)$ be an orthonormal coframe defining the metric, and ω the corresponding spin connection. For example, we may construct α_1, α_2 from the coframe dx, dy by using the Gram-Schmidt process. To compute the geodesic curvature of ∂D, we use a different coframe α_1', α_2' defined on some neighborhood U of ∂D in D, which is obtained by an orthogonal rotation. Let $\theta:\ U \to \mathbb{R}\ (\mathrm{mod}\ 2\pi)$ be the corresponding angle function. As we have seen, the corresponding spin connection is $\omega' = \omega - \mathrm{d}\theta$.

Using Stokes' theorem,

$$\begin{aligned}\int_D K\,\mathrm{d}A + \int_{\partial D} k_g\ \mathrm{d}s &= \int_D \mathrm{d}\omega - \int_{\partial D} \omega' \\ &= \int_{\partial D} \omega - \int_{\partial D} \omega' \\ &= \int_{\partial D} \mathrm{d}\theta \\ &= 2\pi n\end{aligned}$$

for some $n \in \mathbb{Z}$.

To find the integer n, note that both integrals change continuously under deformations of the metric; hence their sum (being in $2\pi\mathbb{Z}$) must be constant. But any two Riemannian metrics g_0, g_1 can be deformed into each other, by linear interpolation $g_t = (1-t)g_0 + tg_1$. Hence, to compute n we may take g to be the standard flat metric, given by $\alpha_1 = dx,\ \alpha_2 = dy$ with $K = 0$. For the flat metric we have $k_g = 1$ (cf. Example 8.50), hence $\int_D K dA + \int_{\partial D} k_g\, ds = 0 + 2\pi = 2\pi$. We conclude $n = 1 = \chi(D)$, verifying Gauss-Bonnet for this case.

Next, suppose Σ' is obtained from Σ by cutting along an embedded circle C_* (not meeting the boundary). Let $C_{*,1}, C_{*,2} \subseteq \Sigma'$ be the resulting boundary components of Σ'. The Riemannian metric on Σ determines a Riemannian metric on Σ', and

$$\int_{\Sigma'} K\, dA = \int_{\Sigma} K\, dA.$$

(Note that $\Sigma' \setminus (C_{*,1} \cup C_{*,2})$ is diffeomorphic to $\Sigma \setminus C_*$.) Moreover,

$$\int_{\partial\Sigma'} k_g\, ds = \int_{\partial\Sigma} k_g\, ds.$$

On the left-hand side, $\partial\Sigma'$ includes the two new boundary components $C_{*,1}, C_{*,2}$ but their contributions cancel since the orientations induced from Σ' are opposite. This shows that the left-hand side of the Gauss-Bonnet formula does not change under cutting, just as the right-hand side does not change (by Axiom (E2) for the Euler characteristic).

By iteratively cutting along circles, this reduces the problem to the case of a connected surface of genus zero with $r \geq 1$ boundary components.

Every such surface Σ with $g = 0,\ r > 1$ is obtained from a disk D ($g = 0,\ r = 1$) by removing $r-1$ smaller disks $D_1, \ldots, D_{r-1} \subseteq D$ (disjoint, and not meeting the boundary). The metric g on Σ extends to a metric on D. Using again the behavior under cutting, we have

$$\int_D K\, dA + \int_{\partial D} k_g\, ds = \sum_{i=1}^{r-1} \Big(\int_{D_i} K\, dA + \int_{\partial D_i} k_g\, ds \Big) + \int_{\Sigma} K\, dA + \int_{\partial\Sigma} k_g\, ds.$$

Since we have already proved Gauss-Bonnet for the case of a disk, this identity shows

$$2\pi = (r-1)\cdot 2\pi + \int_{\Sigma} K\, dA + \int_{\partial\Sigma} k_g\, ds,$$

hence $\int_{\Sigma} K\, dA + \int_{\partial\Sigma} k_g\, ds = 2\pi(2-r) = 2\pi\chi(\Sigma)$, as required. $\square$

A remarkable aspect of the Gauss-Bonnet theorem is that it relates a *local quantity* (the Gauss curvature of the surface, computable in coordinate charts in terms of the metric and its derivatives) with a *global quantity* (the Euler characteristic of the surface). The intrinsic proof presented here, not relying on a choice of embedding of the surface, is a variation of a proof given by Chern [4]. In that work, Chern obtained a generalization, now known as the Gauss-Bonnet-Chern theorem, to manifolds of higher dimensions. It may be seen as a starting point of a long line of related results, culminating in the famous Atiyah-Singer index theorem for elliptic differential operators on compact manifolds.

8.6 Problems

1. Let M be a 3-dimensional compact oriented manifold. Define a bilinear form on 1-forms, by setting
$$F(\alpha,\beta)=\int_M \alpha\wedge \mathrm{d}\beta$$
for $\alpha,\beta\in\Omega^1(M)$. Prove that this bilinear form is symmetric:
$$F(\alpha,\beta)=F(\beta,\alpha).$$

2. Let $0<m<n$, and M a compact oriented manifold of dimension m.
   (a) Show that for every closed m-form $\alpha\in\Omega^m(\mathbb{R}^n)$, and every embedding $i: M\hookrightarrow\mathbb{R}^n$,
$$\int_M i^*\alpha=0.$$
   (b) Show that for every closed m-form $\alpha\in\Omega^m(S^n)$, and every embedding $j: M\hookrightarrow S^n$,
$$\int_M j^*\alpha=0.$$
   (Hint: This does *not* require any knowledge of the cohomology of S^n. Instead, reduce to part (a).)
   (c) Prove that neither $\mathbb{R}^n$ nor S^n are diffeomorphic to a product $M\times N$ of M with another manifold N.

3. Let M be a compact oriented manifold, and $D\subseteq M$ a region with boundary $S=\partial D$. Prove that there does *not* exist a smooth map $F:\ D\to S$ whose restriction to $S=\partial D$ is the identity map of S. (By F being smooth, we mean that it extends to a smooth map on an open neighborhood of D in M.)
(Hint: Let $\alpha\in\Omega^{m-1}(S)$ be a volume form on S so that $\int_S\alpha\neq 0$. Assuming that F exists, use Stokes' theorem to argue $\int_S\alpha=0$, a contradiction.)

4. Suppose M is a compact oriented manifold, and $\Phi:\ M\to M$ a smooth map that is isotopic to the identity map (cf. Remark 4.49). Prove that Φ is surjective.

5. Compute the linking number $L(f,g)$ of the maps $f,g:\ S^1\to\mathbb{R}^3\backslash\{\mathbf{0}\}$ given by
$$f(e^{i\theta})=(\cos\theta,\sin\theta,0),\quad g(e^{i\varphi})=(0,-1+\cos\varphi,\sin\varphi).$$

6. Let $F\in C^\infty(M,N)$ be a local diffeomorphism, $\gamma\colon\ [0,1]\to N$ a smooth path, and $p\in M$ with $F(p)=\gamma(0)$. Prove that there is a unique smooth path $\lambda:\ [0,1]\to M$ with $\lambda(0)=p$ and $F\circ\lambda=\gamma$.

7. Use the previous problem to show that any smooth loop $\gamma\colon\ S^1\to\mathbb{R}^2\setminus\{\mathbf{0}\}$ can be deformed into a loop of the form
$$\gamma([t])=\big(\cos(2\pi nt),\ \sin(2\pi nt)\big)$$
for some $n\in\mathbb{Z}$.

8. Find the winding number $w(\gamma)$ of the loop $\gamma\colon S^1 \to \mathbb{R}^2\backslash\{\mathbf{0}\}$ given by
$$\gamma([t]) = \big(\cos(8\pi t)+2, \sin(8\pi t)\big).$$

9. Let $\gamma\colon S^1 \to \mathbb{R}^2\backslash\{\mathbf{0}\}$ be a loop in the punctured plane, and $\Phi\colon S^1 \to S^1$ a diffeomorphism. Show that $w(\gamma\circ\Phi) = w(\gamma)$ if Φ preserves orientation, and $w(\gamma\circ\Phi) = -w(\gamma)$ if Φ reverses orientation.

10. Let $\gamma_1,\gamma_2\colon S^1 \to \mathbb{R}^2\setminus\{\mathbf{0}\}$ be two loops with the property $\gamma_1([0]) = \gamma_2([0])$, and with "sitting end-points," i.e., such that $\gamma_i([t])$ are constant for t close to 0. One defines the *concatenation* $\gamma_1 * \gamma_2$ to be the path
$$(\gamma_1 * \gamma_2)([t]) = \begin{cases} \gamma_1(2t) & \text{if } 0 \le t \le \frac{1}{2}, \\ \gamma_2(2t-1) & \text{if } \frac{1}{2} \le t \le 1. \end{cases}$$
(The loop γ_1 followed by the loop γ_2; the sitting end-points condition ensures that this is again smooth.) Prove that the winding number satisfies
$$w(\gamma_1 * \gamma_2) = w(\gamma_1) + w(\gamma_2).$$

11. Let M be a compact oriented manifold, with a given "base point" $p_0 \in M$. Prove that there does not exist a smooth map $\Psi\colon \mathbb{R}\times M \to M\times M$ such that $\Psi(0,p) = (p,p_0)$ and $\Psi(1,p) = (p_0,p)$.

12. Let M be a compact oriented manifold, and $\Phi\colon M \to M$ an orientation reversing diffeomorphism. Prove that Φ is not isotopic to the identity map. That is, there does not exist a smooth map $\Psi\colon \mathbb{R}\times M \to M$ such that $\Psi(0,\cdot) = \mathrm{id}_M$ and $\Psi(1,\cdot) = \Phi$.

13. Let M be a manifold of even dimension $m = 2n$. A 2-form $\omega \in \Omega^2(M)$ is called *symplectic* if it is closed ($\mathrm{d}\omega = 0$) as well as *non-degenerate*, in the sense that $\Gamma = \omega^n = \omega\wedge\cdots\wedge\omega$ is a volume form. Prove that if M is compact, then a symplectic form cannot be exact.

14. It is known that $\mathbb{C}\mathrm{P}^2$ has a symplectic 2-form ω such that $[\omega]$ is a basis element for $H^2(\mathbb{C}\mathrm{P}^2) \cong \mathbb{R}$ and $[\omega\wedge\omega]$ is a basis element for $H^4(\mathbb{C}\mathrm{P}^2) \cong \mathbb{R}$. Using these facts, prove that $\mathbb{C}\mathrm{P}^2$ does not admit an orientation reversing diffeomorphism.

15. (a) Use winding numbers (Section 8.3.2) to prove that there is no smooth retraction of the unit disk D^2 onto its boundary. That is, there does not exist a smooth map $F\colon D^2 \to S^1$ such that $F|_{S^1}$ is the identity.
(Hint: Assuming the existence of F, consider the family of loops
$$\gamma_t\colon S^1 \to S^1 \subseteq \mathbb{R}^2\setminus\{\mathbf{0}\},\ \mathbf{v} \mapsto F(t\mathbf{v}).)$$
(b) Use part (a) to prove *Brouwer's fixed point theorem*: For every smooth map $\varphi\colon D^2 \to D^2$ there exists $x \in D^2$ such that $\varphi(x) = x$.

16. Use winding numbers to prove the fundamental theorem of algebra: Every complex polynomial $z \mapsto p(z)$ of degree $n > 0$ has a zero.
(Hint: Assuming that p has no zeroes, consider the family of loops
$$\gamma_t : \{z : |z| = 1\} \to \mathbb{C} \setminus \{0\},\ z \mapsto p(tz).)$$

17. Find the linking number $L(\gamma_1, \gamma_2)$ (Definition 8.18), of the loops
$$\gamma_1([s]) = \big(\cos(2\pi s), \sin(2\pi s), 0\big),\ \ \gamma_2(t) = \big(1 + \cos(2\pi t), 0, \sin(2\pi t)\big).$$
(Hint: Rather than computing the integral, use Theorem 8.15 for suitable choice of $q \in S^2 \subseteq \mathbb{R}^3$.)

18. This problem is a continuation of Problem 15 from Chapter 7, computing the cohomology of the m-sphere. You may use induction on m, and the results from Problem 12 of Chapter 7. Let $\alpha \in \Omega^m(S^m)$ be a form of top degree. Thus, α is closed for degree reasons. Suppose
$$\int_{S^m} \alpha = 0.$$
    (a) For $m = 1$, use any method to prove that α is exact.
    (b) For $m > 1$, let $U_+, U_- \subseteq S^m$ be the open covering given by the domains of the stereographic projections, and choose primitives $\beta_\pm \in \Omega^{m-1}(U_\pm)$ of $\alpha|_{U_\pm}$. Show that $\int_{S^{m-1}} \beta_+ = \int_{S^{m-1}} \beta_-$, where $S^{m-1} \subseteq S^m$ is considered as the "equator."
    (c) Prove that
$$\beta_+|_{U_+ \cap U_-} - \beta_-|_{U_+ \cap U_-} = \mathrm{d}\gamma$$
    for some $\gamma \in \Omega^{m-2}(U_+ \cap U_-)$.
    (d) Using γ and a partition of unity for the cover U_+, U_-, show how to modify the primitives $\beta_\pm$, so that they patch together to a global primitive β.
    (e) Explain how this shows that the integration map $\Omega^m(S^m) \to \mathbb{R}$ induces an isomorphism $H^m(S^m) \cong \mathbb{R}$.

19. Let $\alpha \in \Omega^k(\mathbb{R}^m)$ be a closed form of compact support.
    (a) If $k < m$, show that α admits a primitive $\beta \in \Omega^{k-1}(\mathbb{R}^m)$ of compact support. (What happens for $k = 0$?)
    (b) If $k = m$, show that a necessary and sufficient condition for α to admit a primitive $\beta \in \Omega^{k-1}(\mathbb{R}^m)$ of compact support is $\int_{\mathbb{R}^m} \alpha = 0$.

    You may us the fact that for $0 < k < m$, every closed k-form on the sphere S^m is exact (Problem 15 from Chapter 7), while an m-form on S^m is exact if and only if its integral over S^m is zero (Problem 18 above).

20. Give a proof of Theorem 8.13 along the following lines:
    (a) Prove that the map $H^m(M) \to \mathbb{R}$ defined by integration is surjective.
    (b) Argue that M admits a finite atlas $\{(U_1, \varphi_1), \ldots, (U_r, \varphi_r)\}$, with the property that $\varphi_i(U_i) = \mathbb{R}^m$ for $i = 1, \ldots, r$. (It is convenient here to work with Definition 2.33 of compactness.)

(c) Show that one can re-order the indices of the U_i to arrange that $U_k \cup \cdots \cup U_r$ is connected for all k.

(d) Let $\omega \in \Omega^m(M)$ be given, with $\int_M \omega = 0$. Show that ω can be written as a sum

$$\omega = \sum_{i=1}^{r} \omega_i,$$

where ω_i has compact support in U_i, with $\int_{U_i} \omega_i = 0$. (Hint: Use partition of unity, and induction. At each stage, modify ω_k over $U_k \cap (U_{k+1} \cup \cdots \cup U_r)$ to arrange that its integral is zero.)

(e) Use the result from Problem 19 to conclude that the map $H^m(M) \to \mathbb{R}$ is injective.

21. Suppose $\pi : \widehat{\Sigma} \to \Sigma$ is a surjective local diffeomorphism between surfaces, which is a double cover (i.e., every fiber $\pi^{-1}(p)$ consists of two elements). Using the properties of the Euler characteristic show that

$$\chi(\widehat{\Sigma}) = 2\chi(\Sigma).$$

Use this to give another computation of the Euler characteristic of the projective plane and of the Klein bottle.

22. Let $\Sigma_1 \# \Sigma_2$ be the connected sum of two oriented surfaces (cf. Section 2.7.3). How is $\chi(\Sigma_1 \# \Sigma_2)$ related to $\chi(\Sigma_1)$, $\chi(\Sigma_2)$?

23. Consider S^2 as the surface $x^2 + y^2 + z^2 = 1$ embedded in $\mathbb{R}^3$.
   (a) Show that the vector field

$$X = xz\frac{\partial}{\partial x} + yz\frac{\partial}{\partial y} + (z^2 - 1)\frac{\partial}{\partial z}$$

   on $\mathbb{R}^3$ restricts to a vector field Y on S^2.
   (b) Find the zeroes of the vector field Y and compute its index at each point.

24. Consider a linear vector field on $\mathbb{R}^2$, given by

$$X = (ax + by)\frac{\partial}{\partial x} + (cx + dy)\frac{\partial}{\partial y}.$$

Under what condition on the coefficient matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is $p = (0,0)$ an isolated zero of X? What is $\mathrm{index}_p(X)$ in that case?

25. Let $n \in \mathbb{Z}$. Construct an explicit vector field X on $\mathbb{R}^2$, with an isolated zero at $p = (0,0)$ and with $\mathrm{index}_p(X) = n$.
(Suggestion: Make use of polar coordinates, but be sure that your vector field is smooth at the origin.)

26. Suppose Σ is a compact connected surface. Show that there exists a vector field on Σ with exactly one zero.

27. Show that the coframe

$$\alpha_1 = 2(1+x^2+y^2)^{-1}\,\mathrm{d}x,\;\; \alpha_2 = 2(1+x^2+y^2)^{-1}\,\mathrm{d}y,$$

on $\mathbb{R}^2$ gives a Riemannian metric g with Gauss curvature $K = +1$.

28. Let $U = \{(x,y) \in \mathbb{R}^2 | y > 0\}$ be the upper half-plane, equipped with the standard hyperbolic metric (see ✎ 159).
    (a) Compute the geodesic curvature of the curve given as the intersection of U with a straight line $cx + dy = e$.
    (b) Compute the geodesic curvature of the curve given as the intersection of U with a circle of radius $R > 0$ with center at $(a,b) \in \mathbb{R}^2$.

    (Choose any orientation on the curve.) You will find that in both cases, the geodesic curvature is *constant*. Which of these curves are geodesics?

29. How do the Gauss curvature K of an oriented surfaces Σ, and the geodesic curvature k_g of an oriented curve $C \subseteq \Sigma$, change if the metric g is replaced with cg, for some $c > 0$?

30. Let $\Sigma = \{(x,y,z)|\; x^2+y^2+z^2 = R^2\} \subseteq \mathbb{R}^3$ be the sphere of radius $R > 0$, with the Riemannian metric g induced from $\mathbb{R}^3$:

$$\mathrm{g}(X,Y) = X \cdot Y,$$

    for vector fields $X, Y \in \mathfrak{X}(\Sigma)$.
    (a) Construct orthonormal coframes α_1, α_2 for Σ, near any given point p. (Use convenient coordinates.)
    (b) Compute the Gauss curvature K of Σ. (The result should be a positive constant.)

31. Given a positive function f, let $\Sigma \subseteq \mathbb{R}^3$ be the resulting surface of revolution

$$\Sigma = \{(x,y,z)|\; x^2+y^2 = f(z)^2\}.$$

    Let g be the Riemannian metric on Σ induced from $\mathbb{R}^3$:

$$\mathrm{g}(X,Y) = X \cdot Y,$$

    for vector fields $X, Y \in \mathfrak{X}(\Sigma)$.
    (a) Construct orthonormal coframes α_1, α_2 for Σ. (It is convenient to use coordinates z, φ on Σ where $x = f(z)\cos\varphi,\; y = f(z)\sin\varphi$.)
    (b) Compute the Gauss curvature K of Σ. (The result should be a function of z alone.)

32. (a) Find all compact, connected, oriented surfaces with boundary, admitting a Riemannian metric with Gauss curvature $K = 0$ and with geodesic boundary.
    (b) Find all compact, connected, oriented surfaces without boundary, admitting a Riemannian metric whose Gauss curvature K is positive everywhere.

9

Vector Bundles

In earlier chapters, we defined tangent spaces at individual points of a manifold and described vector fields X as collections of tangent vectors $X_p,\ p \in M$ depending smoothly on the base point. Similarly, 1-forms α were seen as collections of covectors $\alpha_p,\ p \in M$ depending smoothly on the base point. We will now explain that the *tangent bundle* given as the disjoint union of all tangent spaces and similarly the *cotangent bundle* given as the disjoint union of all cotangent spaces are manifolds in their own right. This then allows us to interpret vector fields and 1-forms as smooth maps from the base manifold into these bundles. In fact, both are special cases of vector bundles—collections of vector spaces labeled by points of the base manifold, with smooth dependence on the base points. All the natural constructions and concepts of linear algebra—such as direct sums, duals, quotient spaces, tensor products, inner product spaces, and so on—extend to vector bundles in a natural way, compatible with the smooth structures.

9.1 The Tangent Bundle

Let M be a manifold of dimension m. The disjoint union over all the tangent spaces

$$TM = \bigsqcup_{p \in M} T_pM$$

is called the *tangent bundle* of M. It comes with a projection map

$$\pi :\ TM \to M$$

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taking a tangent vector $v \in T_pM$ to its base point p, and with an inclusion map

$$i:\ M \to TM$$

taking $p \in M$ to the zero vector in $T_pM \subseteq TM$.

Proposition 9.1. *For any manifold M of dimension m, the tangent bundle TM is naturally a manifold of dimension $2m$, in such a way that the projection $\pi:\ TM \to M$ and the inclusion $i:\ M \to TM$ are smooth maps.*

Proof. The idea for constructing an atlas for TM is to apply tangent maps to the charts of M. Let (U,φ) be a given chart, with image $\widetilde{U} = \varphi(U)$. Since $T_pU = T_pM$ for all $p \in U$, we have that

$$TU = \bigsqcup_{p\in U} T_pM = \pi^{-1}(U).$$

The tangent maps of the diffeomorphism $\varphi:\ U \to \widetilde{U} \subseteq \mathbb{R}^m$ for $p \in U$ give vector space isomorphisms

$$T_p\varphi : T_pM \to T_{\varphi(p)}\widetilde{U} = \mathbb{R}^m,$$

and the collection of all these maps gives a bijection,

$$T\varphi : TU \to \widetilde{U} \times \mathbb{R}^m,\ v \mapsto \Big(\varphi\big(\pi(v)\big), (T_{\pi(v)}\varphi)(v)\Big)$$

(here $v \in TU$ is an element of T_pU, where $p = \pi(v)$ is its base point). The images $T\widetilde{U} = \widetilde{U} \times \mathbb{R}^m$ of these bijections are open subsets of $\mathbb{R}^{2m}$; hence, $(TU, T\varphi)$ is a chart. We take the collection of all such charts as an atlas for TM.

We need to check that the transition maps are smooth. If (V,ψ) is another coordinate chart with $U \cap V \neq \emptyset$, the transition map for $TU \cap TV = T(U \cap V) = \pi^{-1}(U \cap V)$ is given by

$$T\psi \circ (T\varphi)^{-1} :\ \varphi(U \cap V) \times \mathbb{R}^m \to \psi(U \cap V) \times \mathbb{R}^m. \tag{9.1}$$

But $T_p\psi \circ (T_p\varphi)^{-1} = T_{\varphi(p)}(\psi \circ \varphi^{-1})$ is just the derivative (i.e., Jacobian matrix) for the change of coordinates $\psi \circ \varphi^{-1}$; hence, (9.1) is given by

$$(\mathbf{x}, \mathbf{a}) \mapsto \Big((\psi \circ \varphi^{-1})(\mathbf{x}),\ D_{\mathbf{x}}(\psi \circ \varphi^{-1})(\mathbf{a})\Big).$$

Since the Jacobian matrix depends smoothly on $\mathbf{x}$, this is a smooth map. This shows that any atlas $\mathscr{A} = \{(U_\alpha, \varphi_\alpha)\}$ for M defines an atlas $\{(TU_\alpha, T\varphi_\alpha)\}$ for TM. Taking $\mathscr{A}$ to be countable, the resulting atlas for TM is countable (in fact, we have a bijection between the atlases). The Hausdorff property is satisfied as well. $\square$

164 (answer on page 321). Why does the atlas for TM satisfy the Hausdorff property?

The relationship between the charts of M and the associated charts for TM is depicted by the following commutative diagram.

$$\begin{tikzcd} TU \arrow[r, "T\varphi"] \arrow[d, "\pi"'] & \varphi(U)\times\mathbb{R}^m \arrow[d, "(u,v)\mapsto u"] \\ U \arrow[r, "\varphi"'] & \varphi(U) \end{tikzcd} \tag{9.2}$$

By construction, these charts for TM are compatible with the vector space structure on the fibers, in the sense that $T\varphi$ restricts to *linear maps* $T_pU = T_pM \to \mathbb{R}^m$, from the fibers $T_pU = T_pM$ of $\pi:\ TU \to U$, to the fibers $\varphi(p)\times\mathbb{R}^m \cong \mathbb{R}^m$ of the projection onto the first factor $\varphi(U)\times\mathbb{R}^m \to \varphi(U)$.

Proposition 9.2. *For any smooth map $F \in C^\infty(M,N)$, the map*

$$TF:\ TM \to TN \tag{9.3}$$

given on each $T_pM \subseteq TM$ as the tangent map $T_pF:\ T_pM \to T_{F(p)}N$ is smooth.

Proof. Given $p \in M$, choose charts (U,φ) around p and (V,ψ) around $F(p)$, with $F(U) \subseteq V$. Then $(TU, T\varphi)$ and $(TV, T\psi)$ are charts for TM and TN, respectively, with $TF(TU) \subseteq TV$. Let $\widetilde{F} = \psi\circ F\circ\varphi^{-1}:\ \varphi(U) \to \psi(V)$. The map

$$T\widetilde{F} = T\psi\circ TF\circ(T\varphi)^{-1}:\ \varphi(U)\times\mathbb{R}^m \to \psi(V)\times\mathbb{R}^n$$

is given by

$$(\mathbf{x},\mathbf{a}) \mapsto \Big(\widetilde{F}(\mathbf{x}),\ (D_{\mathbf{x}}\widetilde{F})(\mathbf{a})\Big).$$

It is smooth, by smooth dependence of the differential $D_{\mathbf{x}}\widetilde{F}$ on the base point. Consequently, TF is smooth. □

One refers to (9.3) as the *tangent map* to F. (The same name is used for the individual maps T_pM.)

Example 9.3. Suppose M is given as a submanifold of $\mathbb{R}^n$, with inclusion map $i:\ M \to \mathbb{R}^n$. The tangent map $Ti:\ TM \to T\mathbb{R}^n$ is again an embedding; its image is the disjoint union of the tangent spaces T_pM, realized as subspaces of $\mathbb{R}^n$. That is, TM becomes a submanifold of $\mathbb{R}^{2n}$:

$$TM = \{(p,v) \in \mathbb{R}^n\times\mathbb{R}^n \mid p \in M,\ \ v \in T_pM\}.$$

One can verify directly that TM is a submanifold of $\mathbb{R}^{2n}$, for example by an application of the implicit function theorem. For the case of $M = S^2$ as a submanifold of $\mathbb{R}^3$, this was one of the tasks for Problem 6 in Chapter 5.

165 (answer on page 321). As a simpler version of the same problem, consider $M = S^1$ as the unit circle in $\mathbb{R}^2$. Show that

$$TS^1 = \{(x,y,r,s) \in \mathbb{R}^4 |\, x^2+y^2 = 1,\ \ xr+ys = 0\},$$

and verify that this is a submanifold by checking that $(1,0)$ is a regular value of the function $\Phi(x,y,r,s) = (x^2+y^2,\ xr+ys)$.

9.2 Vector Fields Revisited

Let us now revisit our two definitions of vector fields $X \in \mathfrak{X}(M)$. Our first definition considered vector fields as families of tangent vectors $X_p \in T_pM$ depending smoothly on the base point p, in the sense that for any smooth function $f \in C^\infty(M)$, the function $p \mapsto X_p(f)$ should again be smooth. The second definition described vector fields directly as derivations of the algebra of smooth functions; this approach was cleaner but perhaps not as intuitive as viewing vector fields as collections of tangent vectors. Using the tangent bundle, we now have a simpler way of stating the smooth dependence on the base point p: Literally, $p \mapsto X_p$ should be a smooth map $M \to TM$.

Definition 9.4 (Vector Fields—Third Definition). *A vector field on M is a* section of the tangent bundle: A smooth map $X \in C^\infty(M,TM)$ such that $\pi \circ X$ is the identity.

Recall again that $\pi :\ TM \to M$ is the canonical projection map that takes a tangent vector v to its base point p. Hence, the condition $\pi \circ X = \mathrm{id}_M$ simply states that for every $p \in M$, the tangent vector $X_p = X(p)$ should lie in the tangent space T_pM and not in the tangent space of some other point.

Proposition 9.5. *The new Definition 9.4 of vector fields is equivalent to the earlier definitions.*

Proof. The condition $\pi \circ X = \mathrm{id}_M$ is clearly necessary, as explained above. Consider an M chart (U,φ) and the induced TM chart $(TU, T\varphi)$. The condition $\pi \circ X = \mathrm{id}_M$ guarantees that X maps U to TU. Hence, the coordinate representation of X is the map $(T\varphi) \circ X \circ \varphi^{-1}$

$$(u^1,\ldots,u^m) \mapsto (u^1,\ldots,u^m, a^1(\mathbf{u})\frac{\partial}{\partial u^1},\ldots,a^m(\mathbf{u})\frac{\partial}{\partial u^m}),$$

which is smooth if and only if the functions $a^i : \varphi(U) \to \mathbb{R}^m$ are smooth. We are therefore done by Proposition 6.4. $\square$

It is common practice to use the same symbol X both as a linear map from smooth functions to smooth functions and as a map into the tangent bundle. Thus,

$$X:\ M \to TM, \qquad X:\ C^\infty(M) \to C^\infty(M)$$

coexist. But if it gets too confusing, one uses the Lie derivative notation

$$L_X : C^\infty(M) \to C^\infty(M)$$

for the interpretation as a derivation. Both viewpoints are useful and important, and both have their advantages and disadvantages. For instance, from the tangent bundle viewpoint, it is immediate that vector fields $X : M \to TM$ restrict to open subsets $U \subseteq M$; this map

$$\mathfrak{X}(M) \to \mathfrak{X}(U), \;\; X \mapsto X|_U$$

may seem a little awkward from the derivations viewpoint since $C^\infty(U)$ is not a subspace of $C^\infty(M)$. (There is a restriction map $C^\infty(M) \to C^\infty(U)$, but no natural map in the other direction.) On the other hand, the derivations viewpoint gives the *Lie bracket operation*, defined in terms of $L_{[X,Y]} = L_X \circ L_Y - L_Y \circ L_X$, which seems unexpected from the tangent bundle viewpoint.

166 (answer on page 321). Show that in the description of vector fields as sections of the tangent bundle, two vector fields $X, Y \in \mathfrak{X}(M)$ are F-related if and only if the following diagram commutes.

$$\begin{tikzcd}
TM \arrow[r, "TF"] & TN \\
M \arrow[u, "X"] \arrow[r, "F"'] & N \arrow[u, "Y"']
\end{tikzcd}$$

9.3 The Cotangent Bundle

Similarly to the tangent bundle, the cotangent bundle

$$T^*M = \bigsqcup_p T_p^*M$$

has a natural manifold structure. Denote by $\pi : T^*M \to M$ the projection (taking a covector to its base point), and by $i : M \to T^*M$ the inclusion (taking $p \in M$ to the zero covector at p). If $F : M \to N$ is a *diffeomorphism*, then the union of the cotangent maps $T_p^*F : T^*_{F(p)}N \to T_p^*M$ gives a bijection

$$T^*F : T^*N \to T^*M,$$

again called a *cotangent map*. (Since F is a diffeomorphism, every $q \in N$ is of the form $F(p)$ for a unique $p \in M$.) Note that the base map of T^*F is the inverse map $F^{-1} : N \to M$.

Remark 9.6. We stress that tangent maps TF are defined for an arbitrary smooth map F, while we defined the cotangent map T^*F only in the case that F is a diffeomorphism.

Now, let (U, φ) be a chart of M, with image $\widetilde{U} = \varphi(U) \subseteq \mathbb{R}^m$. Then $T\widetilde{U} = \widetilde{U} \times \mathbb{R}^m$ and also

$$T^*\widetilde{U} = \widetilde{U} \times \mathbb{R}^m$$

(since the dual space to $\mathbb{R}^m$ is identified with $\mathbb{R}^m$ itself). Hence, taking the inverse of the cotangent map $T^*\varphi : T^*\widetilde{U} \to T^*U$, we obtain charts $(T^*U, (T^*\varphi)^{-1})$ for T^*M, called *cotangent charts*. They are described by the commutative diagram below.

$$\begin{tikzcd} T^*U \arrow[r, "(T^*\varphi)^{-1}"] \arrow[d, "\pi"'] & \varphi(U) \times \mathbb{R}^m \arrow[d, "(u,v)\mapsto u"] \\ U \arrow[r, "\varphi"'] & \varphi(U) \end{tikzcd} \tag{9.4}$$

As in the case of the tangent bundle, one checks the compatibility of cotangent charts.

167 (answer on page 321). Let (U, φ), (V, ψ) be charts for the manifold M. Show that the transition map for the corresponding cotangent charts $(T^*U, (T^*\varphi)^{-1})$, $(T^*V, (T^*\psi)^{-1})$,

$$\varphi(U \cap V) \times \mathbb{R}^m \to \psi(U \cap V) \times \mathbb{R}^m$$

is given by the inverse transpose of the transition maps (9.1) for the tangent bundle:

$$(\mathbf{x}, \mathbf{a}) \mapsto \Big((\psi \circ \varphi^{-1})(\mathbf{x}),\ \big((D_{\mathbf{x}}(\psi \circ \varphi^{-1}))^{\top} \big)^{-1}(\mathbf{a}) \Big).$$

We conclude that any (countable) atlas for M determines an (countable) atlas for T^*M. (The Hausdorff property is verified by the same argument as for TM, see 164.)

168 (answer on page 321). Show that if $F : M \to N$ is a diffeomorphism, then the cotangent map $T^*F : T^*N \to T^*M$ is again a diffeomorphism.

Recall that we have defined 1-forms (also called covector fields) as $C^\infty(M)$-linear maps $\alpha : \mathfrak{X}(M) \to C^\infty(M)$. We now obtain an alternative definition as collections of covectors $\alpha_p \in T_p^*M$ depending smoothly on the base point.

Definition 9.7 (1-Forms—Second Definition). *A 1-form on M is a* section of the cotangent bundle, i.e., a smooth map

$$\alpha : M \to T^*M$$

such that $\pi \circ \alpha = \mathrm{id}_M$.

Note that the same symbol α is being used in two ways:

$$\alpha : \mathfrak{X}(M) \to C^\infty(M), \quad \alpha : M \to T^*M.$$

If this gets too confusing, one abandons the first notation and writes $\iota_X \alpha$ (contraction with $X \in \mathfrak{X}(M)$) in place of $\alpha(X)$. One advantage of the new definition of 1-forms is that the restriction $\alpha \mapsto \alpha|_U$ to an open subset $U \subseteq M$ becomes a more natural operation.

Although some of this has been mentioned before, let us point out the differences between tangent and cotangent bundles. Any smooth map $F \in C^\infty(M,N)$ induces a tangent map $TF \in C^\infty(TM,TN)$, but there is no naturally induced map from T^*N to T^*M (or the other way), unless F is a diffeomorphism. On the other hand, while there is no natural push-forward operation from vector fields on M to vector fields on N, there is a natural pullback operation for 1-forms! The existence of a Lie bracket of vector fields may be seen as a special structure of the tangent bundle (in contrast to general vector bundles, see below). There is no such bracket on 1-forms in general. On the other hand, the cotangent bundle also has some special features, such as the existence of a *canonical 1-form* $\theta \in \Omega^1(T^*M)$, and a resulting symplectic structure on T^*M; see Problem 12 below.

9.4 Vector Bundles

Tangent bundles and cotangent bundles are two examples of the concept of a *vector bundle*, which we will now define. Consider a collection of r-dimensional vector spaces E_p labeled by the points $p \in M$ of an m-dimensional manifold M, and let

$$E = \bigsqcup_{p \in M} E_p \tag{9.5}$$

be the disjoint union. We denote by $\pi : E \to M$ the projection and $i : M \to E$ the inclusion of zeros.

Definition 9.8. *A* vector bundle chart *for E is a chart $(\widehat{U}, \widehat{\varphi})$ for which there exists a chart (U, φ) for M, with*

$$\widehat{U} = \pi^{-1}(U), \quad \widehat{\varphi}(\widehat{U}) = \varphi(U) \times \mathbb{R}^r,$$

such that the map $\widehat{\varphi}$ restricts to vector space isomorphisms

$$E_p = \pi^{-1}(p) \to \{\varphi(p)\} \times \mathbb{R}^r \cong \mathbb{R}^r$$

for all $p \in M$. An atlas on E consisting of vector bundle charts is called a vector bundle atlas.

Every vector bundle atlas on E determines a *maximal* vector bundle atlas, consisting of all vector bundle charts that are compatible with all vector bundle charts from the given atlas.

Definition 9.9. *A* vector bundle $\pi: E \to M$ *of rank r over a manifold M is a collection of r-dimensional vector spaces* (9.5) *together with a maximal vector bundle atlas, where the base charts (U, φ) belong to the atlas of M. One calls E the* total space *and M the* base space *of the vector bundle.*

For a vector bundle atlas, it is automatic that the corresponding base charts (U, φ) form an atlas for M.

The vector bundle charts of E may be understood in terms of the following commutative diagram.

$$\begin{tikzcd} \widehat{U} \arrow[r, "\widehat{\varphi}"] \arrow[d, "\pi"'] & \varphi(U) \times \mathbb{R}^r \arrow[d, "(\mathbf{x},\mathbf{v}) \mapsto \mathbf{x}"] \\ U \arrow[r, "\varphi"'] & \varphi(U) \end{tikzcd}$$

The key condition is that $\widehat{\varphi}$ restricts to vector space isomorphisms fiberwise. Thus, just as a manifold M looks locally like an open subset of $\mathbb{R}^m$, a vector bundle E over M looks locally like a product of an open subset of $\mathbb{R}^m$ with the vector space $\mathbb{R}^r$. The transition maps between vector bundle charts are *fiberwise linear.*

Proposition 9.10. *For a vector bundle E, the transition maps*

$$\widehat{\psi} \circ \widehat{\varphi}^{-1}: \ \varphi(U \cap V) \times \mathbb{R}^r \to \psi(U \cap V) \times \mathbb{R}^r$$

are of the form

$$(\mathbf{x}, \mathbf{v}) \mapsto ((\psi \circ \varphi^{-1})(\mathbf{x}), \, A(\mathbf{x})\mathbf{v})$$

with a smooth map $A: \ \varphi(U \cap V) \to \mathrm{GL}(r, \mathbb{R})$. (Here, $\mathrm{GL}(r, \mathbb{R})$ *is the group of invertible $r \times r$ real matrices).*

The proof is left as Problem 1. In our definition of vector bundles, we did not worry about the "Hausdorff property" for the total space; it is in fact automatic.

169 (answer on page 322). Show that the Hausdorff property for any vector bundle $E \to M$ follows from the Hausdorff property of the base M.

Smoothness properties of the various structure maps for vector bundles may be verified in the vector bundle charts. For instance, the inclusion $i: \ M \to E$ is an embedding of M as a submanifold—in fact, the vector bundle charts $(\widehat{U}, \widehat{\varphi})$ are submanifold charts for $i(M) \subseteq E$. From now on, we will identify M with its image in E. Similarly, the projection $\pi: \ E \to M$ is a smooth submersion because in a vector bundle chart it is simply the projection of $\widehat{\varphi}(\widehat{U}) = \varphi(U) \times \mathbb{R}^r$ onto the first factor.

The tangent charts for TM and cotangent charts for T^*M are examples of vector bundle charts; hence, both are vector bundles over M. Here are more examples.

Example 9.11 (Trivial Bundles). The *trivial vector bundle* over M is the direct product $M \times \mathbb{R}^r$. Charts for M directly give vector bundle charts for $M \times \mathbb{R}^r$.

Example 9.12. The unique rank 0 vector bundle over M is called the *zero vector bundle*, $0_M \to M$. Its fibers $(0_M)_p$ are the zero vector spaces $\{0\}$.

Example 9.13 (The Infinite Möbius Strip). View $M = S^1$ as a quotient $\mathbb{R}/\sim$ for the equivalence relation $t \sim t+1$. Let $E = (\mathbb{R} \times \mathbb{R})/\sim$ be the quotient under the equivalence relation $(t,\tau) \sim (t+1, -\tau)$, with the natural projection map

$$\pi : \ E \to M, \ [(t,\tau)] \mapsto [t].$$

The fibers $\pi^{-1}([t])$ have vector space structures, given by

$$\lambda_1[(t,\tau_1)] + \lambda_2[(t,\tau_2)] = [(t, \lambda_1\tau_1 + \lambda_2\tau_2)].$$

(Note that this is well-defined.) In this way, E is a rank 1 vector bundle (a *line bundle*) over S^1. Its total space is an infinite Möbius strip.

Example 9.14 (Tautological Line Bundle of Projective Spaces). For any $p \in \mathbb{R}\mathrm{P}^n$, let $E_p \subseteq \mathbb{R}^{n+1}$ be the 1-dimensional subspace that it represents. Then

$$E = \bigsqcup_{p \in \mathbb{R}\mathrm{P}^n} E_p$$

with the natural projection $\pi : \ E \to \mathbb{R}\mathrm{P}^n$ is a line bundle over $\mathbb{R}\mathrm{P}^n$. It is called the *tautological line bundle*.

Vector bundle charts can be constructed as follows. Let (U_i, φ_i) be the standard atlas for $\mathbb{R}\mathrm{P}^n$, and let $\widehat{U}_i = \pi^{-1}(U_i)$. For $p \in U_i$, with coordinates $\varphi_i(p) = (u^1, \ldots, u^n)$, the fiber $E_p = \pi^{-1}(p) \subseteq \widehat{U}_i$ has a distinguished basis vector $(u^1, \ldots, u^{i-1}, 1, u^i, \ldots, u^n)$. Any other $\mathbf{v} \in E_p$ is a non-zero scalar multiple of this basis vector, say

$$\mathbf{v} = \lambda(u^1, \ldots, u^{i-1}, 1, u^i, \ldots, u^n).$$

We define

$$\widehat{\varphi}_i(\mathbf{v}) = (u^1, \ldots, u^n; \lambda),$$

and then $(\widehat{U}_i, \widehat{\varphi}_i)$ are the desired vector bundle charts.

170 (answer on page 322). Verify that the charts $(\widehat{U}_i, \widehat{\varphi}_i)$ form an atlas, i.e., that they are compatible.

Example 9.15. There is another standard vector bundle $E' \to \mathbb{R}\mathrm{P}^n$ over the projective space, with fibers $E'_p = E_p^\perp \subseteq \mathbb{R}^{n+1}$ the hyperplanes orthogonal to the tautological lines. This bundle has rank n; it is called the *hyperplane bundle*.

Example 9.16 (Vector Bundles over the Grassmannians). The construction of the tautological vector bundle over projective spaces generalizes to the Grassmannians. For $p \in \mathrm{Gr}(k,n)$, let $E_p \subseteq \mathbb{R}^n$ be the k-dimensional subspace that it represents, and put

$$E = \bigsqcup_{p \in \mathrm{Gr}(k,n)} E_p.$$

Vector bundle charts of this *tautological vector bundle* are constructed from the standard charts (U_I, φ_I) of the Grassmannian.

Recall that U_I was characterized as the set of all p such that the orthogonal projection $\Pi_I : \mathbb{R}^n \to \mathbb{R}^I$ restricts to an isomorphism $E_p \to \mathbb{R}^I$, and

$$\varphi_I : U_I \to \mathrm{Hom}(\mathbb{R}^I, \mathbb{R}^{I'}) \cong \mathbb{R}^{k(n-k)}$$

takes $p \in U_I$ to the linear map having E_p as its graph. Let

$$\widehat{\varphi}_I : \pi^{-1}(U_I) \to \mathrm{Hom}(\mathbb{R}^I, \mathbb{R}^{I'}) \times \mathbb{R}^I, \quad \mathbf{v} \mapsto (\varphi_I(\pi(\mathbf{v})), \Pi_I(\mathbf{v})).$$

Then the $(\widehat{U}_I, \widehat{\varphi}_I)$ are vector bundle charts for E (once we identify $\mathrm{Hom}(\mathbb{R}^I, \mathbb{R}^{I'}) = \mathbb{R}^{k(n-k)}$ and $\mathbb{R}^I = \mathbb{R}^k$).

There is another natural vector bundle E' over $\mathrm{Gr}(k,n)$, with fibers $E'_p := E_p^{\perp}$ the orthogonal complement of E_p. In terms of the identification $\mathrm{Gr}(k,n) = \mathrm{Gr}(n-k,n)$, the bundle E' is the tautological vector bundle over $\mathrm{Gr}(n-k,n)$.

It is common to denote vector bundles E over M by "$E \to M$," suggesting the base projection without using an explicit symbol for it (such as π). We will adopt this convention from now on.

Definition 9.17. *A* vector bundle map *(also called* vector bundle homomorphism*) from $E \to M$ to $F \to N$ is a smooth map*

$$\widehat{\Phi} : E \to F$$

of the total spaces, which restricts to linear maps from the fibers of E to fibers of F. The induced map

$$\Phi : M \to N$$

between the base manifolds will be called the base map *of $\widehat{\Phi}$. If the map $\widehat{\Phi}$ is furthermore a diffeomorphism of the total spaces, then it is called a* vector bundle isomorphism.

Vector bundle maps are pictured by commutative diagrams like the one below.

$$\begin{tikzcd} E \arrow[r, "\widehat{\Phi}"] \arrow[d] & F \arrow[d] \\ M \arrow[r, "\Phi"'] & N \end{tikzcd}$$

Example 9.18. Let $E \to M$ be a vector bundle. For any open subset $U \subseteq M$, the restriction $E|_U = \pi^{-1}(U)$ is a vector bundle $E|_U \to U$, in such a way that the inclusion $E|_U \hookrightarrow E$ is a vector bundle map.

Definition 9.19. *A vector bundle isomorphism*

$$\widehat{\Phi}: \ E \to M \times \mathbb{R}^r$$

with a trivial vector bundle is called a trivialization *of M.*

Example 9.20. Any vector bundle chart $(\widehat{U}, \widehat{\varphi})$ defines a vector bundle isomorphism

$$\widehat{\varphi}: \ E|_U := \pi^{-1}(U) \to \varphi(U) \times \mathbb{R}^r.$$

That is, vector bundles are *locally trivial.*

However, vector bundles do not in general admit global trivializations.

171 (answer on page 322). Show that the vector bundle in Example 9.13 (infinite Möbius strip) does not admit a global trivialization.

Remark 9.21. A manifold M is said to be *parallelizable* if its tangent bundle admits a trivialization $TM \cong M \times R^m$. A necessary condition for parallelizability is that the manifold be orientable (Problem 7). Corollary 8.40 shows that an oriented surface ($\dim M = 2$) is parallelizable if and only if its Euler characteristic is 0. It is a remarkable and important fact that compact orientable 3-manifolds are always parallelizable. As mentioned in Problem 15 of Chapter 6, the n-sphere S^n is parallelizable if and only if $n \in \{1,3,7\}$. Problem 16 of Chapter 6 implies that every Lie group is parallelizable.

9.5 Some Constructions with Vector Bundles

In this section we describe several natural constructions producing new vector bundles from given ones.

(a) If $E_1 \to M_1$ and $E_2 \to M_2$ are vector bundles of ranks r_1, r_2, then the cartesian product is a vector bundle

$$E_1 \times E_2 \to M_1 \times M_2$$

of rank $r_1 + r_2$, with fiber at (p_1, p_2) the vector space $E_{p_1} \times E_{p_2}$ (also denoted $E_{p_1} \oplus E_{p_2}$).

(b) Let $\pi: E \to M$ be a given vector bundle and $S \subseteq M$ a submanifold. The *restriction*

$$E|_S := \pi^{-1}(S) \to S$$

is a vector bundle, in such a way that the inclusion map $E|_S \to E$ is a vector bundle map (and also an embedding as a submanifold). Vector bundle charts near any given $p \in S$ may be constructed as follows.
Note first that $E|_S$ is a submanifold (since π is a submersion). If $(\widehat{U}, \widehat{\varphi})$ is a vector bundle chart for $E \to M$ that is also a submanifold chart for the submanifold $E|_S$, then its restriction to $E|_S$ is a vector bundle chart for $E|_S$.
To see that such vector bundle charts exist, near any given $p \in S$, start with any vector bundle chart $(\widehat{U}, \widehat{\varphi})$ for E, with base chart (U, φ), and any submanifold chart (U', φ') for S at p. Replacing U, U' with their intersection, we may assume $U' = U$. Let $\widehat{\varphi}'$ be the composition

$$\pi^{-1}(U) \xrightarrow{\widehat{\varphi}} \varphi(U) \times \mathbb{R}^r \xrightarrow{(\varphi' \circ \varphi^{-1}) \times \mathrm{id}} \varphi'(U) \times \mathbb{R}^r.$$

Then $(\widehat{U}, \widehat{\varphi'})$ is a vector bundle chart and also a submanifold chart.

(c) More generally, suppose $\Phi \in C^\infty(M,N)$ is a smooth map between manifolds, and $\pi: F \to N$ is a vector bundle. Then there is a *pullback bundle* $\Phi^* F \to M$ with fibers

$$(\Phi^* F)_p = F_{\Phi(p)}.$$

One way to prove that this is a vector bundle is to reduce to the case of inclusions of submanifolds. Consider the embedding of M as the submanifold of $N \times M$ given as the graph of Φ:

$$M \cong \mathrm{graph}(\Phi) = \{(\Phi(p), p) |\ p \in M\}.$$

The vector bundle $F \to N$ defines a vector bundle $F \times 0_M \to N \times M$, by cartesian product with the zero bundle $0_M \to M$. We have

$$\Phi^* F \cong (F \times 0_M)|_{\mathrm{graph}(\Phi)}.$$

(d) Let E, E' be two vector bundles over M. Then the *direct sum* (also called *Whitney sum*)

$$E \oplus E' := \bigsqcup_{p \in M} E_p \oplus E'_p$$

is again a vector bundle over M. One way to see this is to regard the direct sum as the pullback of the cartesian product under the diagonal inclusion

$$\mathrm{diag}_M: M \to M \times M,\ p \mapsto (p, p).$$

Indeed, by definition, the fibers of

$$E \oplus E' = \mathrm{diag}_M^*(E \times E')$$

are $E_p \times E'_p = E_p \oplus E'_p$.

172 (answer on page 323). Let $E \to \mathrm{Gr}(k,n)$ be the tautological bundle over the Grassmannian $\mathrm{Gr}(k,n)$ and $E' \to \mathrm{Gr}(k,n)$ the bundle with fibers $E'_p = (E_p)^{\perp}$. Show that $E \oplus E' \to \mathrm{Gr}(k,n)$ is a trivial bundle.

(e) Suppose $\pi : E \to M$ is a vector bundle of rank r. A *vector subbundle* of E along a submanifold $N \subseteq M$ is a collection of subspaces $F_p \subseteq E_p$ for $p \in N$ such that $F = \bigsqcup_{p\in N} F_p$ is a submanifold of E, and is itself a vector bundle over N, with the map $F \to N$ given by restriction of π.
For example, if $N \subseteq M$ is any submanifold, the restriction $E|_N$ is a subbundle (of the same rank).
Consider a subbundle $F \subseteq E$ over the same base, $N = M$. Then the *quotient bundle*

$$E/F := \bigsqcup_{p\in M} E_p/F_p$$

is again a vector bundle over M.

Example 9.22. Given a manifold M with a submanifold S, one calls the restriction $TM|_S$ the *tangent bundle of M along S*; it contains the tangent bundle TS of S as a subbundle.
The *normal bundle* of S in M is defined as a quotient bundle

$$\nu_S = TM|_S/TS \to S$$

with fibers $(\nu_S)_p = T_pM/T_pS$.

(f) For any vector bundle $E \to M$, the *dual bundle*

$$E^* = \bigsqcup_{p\in M} E_p^*$$

(where $E_p^* = \mathrm{Hom}(E_p, \mathbb{R})$ is the dual space to E_p) is again a vector bundle (see Problem 9).

Example 9.23. The dual of the tangent bundle TM is the cotangent bundle T^*M. Given a submanifold $S \subseteq M$, one can consider the set of covectors $\alpha \in T_p^*M$ for $p \in S$ that annihilate T_pS, that is, $\langle \alpha, v\rangle = 0$ for all $v \in T_pS$. This is a vector bundle called the *conormal bundle*

$$\nu_S^* \to S.$$

The notation is justified, since it is the dual bundle to the normal bundle ν_S. (For any vector space V and subspace W, the annihilator of W is canonically isomorphic to the dual of the quotient space V/W; see Appendix B.3.)

9.6 Sections of Vector Bundles

Definition 9.24. *A* smooth section *of a vector bundle* $\pi:\ E \to M$ *is a smooth map* $\sigma:\ M \to E$ *with the property* $\pi \circ \sigma = \mathrm{id}_M$*. The space of smooth sections of* E *is denoted* $\Gamma^\infty(M,E)$*, or simply* $\Gamma^\infty(E)$*.*

Thus, a section is a family of vectors $\sigma_p \in E_p$ depending smoothly on p.

Example 9.25.

(a) Every vector bundle has a distinguished section, the *zero section*

$$p \mapsto \sigma_p = 0,$$

where 0 is the zero vector in the fiber E_p. One usually denotes the zero section itself by 0.

(b) For a *trivial bundle* $M \times \mathbb{R}^r$, a section is the same thing as a smooth function from M to $\mathbb{R}^r$,

$$\Gamma^\infty(M, M \times \mathbb{R}^r) = C^\infty(M, \mathbb{R}^r).$$

Indeed, any such function $f:\ M \to \mathbb{R}^r$ defines a section $\sigma(p) = (p, f(p))$; conversely, any section $\sigma:\ M \to E = M \times \mathbb{R}^r$ defines a function by composition with the projection $M \times \mathbb{R}^r \to \mathbb{R}^r$.
In particular, if $\kappa:\ E|_U \to U \times \mathbb{R}^r$ is a local trivialization of a vector bundle E over an open subset U, then a section $\sigma \in \Gamma^\infty(E)$ restricts to a smooth function $\kappa \circ \sigma|_U:\ U \to \mathbb{R}^r$.

(c) Given a smooth map $\Phi:\ N \to M$, the sections of $\Phi^*TM \to N$ are called *vector fields along* Φ. As a special case, for a smooth curve $\gamma:\ J \to M$, one can consider vector fields along γ; one example of which is the *velocity vector field* $\dot\gamma$, with $\dot\gamma|_t \in T_{\gamma(t)}M$ the velocity at time t.

(d) Let $\pi:\ E \to M$ be a rank r vector bundle. A *frame* for E over $U \subseteq M$ is a collection of sections $\sigma_1, \ldots, \sigma_r$ of $E|_U$, such that $(\sigma_j)_p$ are linearly independent at each point $p \in U$. Any frame over U defines a local trivialization

$$\psi:\ E|_U \to U \times \mathbb{R}^r,$$

given in terms of its inverse map $\psi^{-1}(p,a) = \sum_j a_j (\sigma_j)_p$. Conversely, each local trivialization gives rise to a frame. A global trivialization of E is equivalent to a global frame.
A local frame for the tangent bundle TM is given by a collection of pointwise linearly independent vector fields $X_1, \ldots, X_m \in \mathfrak{X}(U)$ (cf. Definition 8.29). Similarly, the coframes $\alpha_1, \ldots, \alpha_m \in \Omega^1(U)$ from Section 8.5 are the frames for the cotangent bundle.

173 (answer on page 323). Explain how a trivialization of $E|_U$ is equivalent to a frame over U.

The space $\Gamma^\infty(M,E)$ is a vector space under pointwise addition

$$(\sigma_1+\sigma_2)_p = (\sigma_1)_p + (\sigma_2)_p.$$

Moreover, it is a module over the algebra $C^\infty(M)$, under multiplication

$$(f\sigma)_p = f_p\sigma_p.$$

Given smooth sections $\sigma \in \Gamma^\infty(M,E)$ and $\tau \in \Gamma^\infty(M,E^*)$, one can take the pairing to define a function

$$\langle \tau,\sigma\rangle \in C^\infty(M), \quad \langle \tau,\sigma\rangle(p) = \langle \tau_p,\ \sigma_p\rangle.$$

This pairing is $C^\infty(M)$-linear in both entries, i.e.,

$$\langle \tau,\ f\sigma\rangle = f\langle \tau,\ \sigma\rangle = \langle f\tau,\ \sigma\rangle$$

for all $\tau \in \Gamma^\infty(M,E^*)$, $\sigma \in \Gamma^\infty(M,E)$, $f \in C^\infty(M)$. We can use pairings with smooth sections of E to characterize the smooth sections of E^*.

Proposition 9.26.

(a) A family of elements $\tau_p \in E_p^*$ for $p \in M$ defines a smooth section of E^* if and only if for all $\sigma \in \Gamma^\infty(M,E)$, the function

$$M \to \mathbb{R}, \ \ p \mapsto \langle \tau_p,\sigma_p\rangle$$

is smooth.

(b) The space of sections of the dual bundle is identified with the space of $C^\infty(M)$-linear maps

$$\tau :\ \Gamma^\infty(M,E) \to C^\infty(M), \quad \sigma \mapsto \langle \tau,\sigma\rangle.$$

Here $C^\infty(M)$-linear means that $\langle \tau,\ f\sigma\rangle = f\langle \tau,\ \sigma\rangle$ for all functions f.

Proof. The proof uses local bundle charts and bump functions. Details are left as an exercise.*

Remark 9.27. A slightly better formulation of the second part of this proposition reads as follows. Regard $\mathscr{E} = \Gamma^\infty(M,E)$ as a module over the algebra $\mathscr{A} = C^\infty(M)$ of smooth functions, and likewise for $\mathscr{E}^* = \Gamma^\infty(M,E^*)$. The space $\mathrm{Hom}_{\mathscr{A}}(\mathscr{E},\mathscr{A})$ of $\mathscr{A} = C^\infty(M)$-linear maps $\mathscr{E} \to \mathscr{A}$ is again an $\mathscr{A}$-module. There is a natural $\mathscr{A}$-module map

$$\mathscr{E}^* \to \mathrm{Hom}_{\mathscr{A}}(\mathscr{E},\mathscr{A})$$

defined by the pairing of sections. Proposition 9.26 asserts that this map is an isomorphism of $\mathscr{A}$-modules.

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* It is similar to the fact, proved in Section 6.1, that a collection of tangent vectors $X_p \in T_pM$ defines a smooth vector field if and only if for any $f \in C^\infty(M)$ the map $p \mapsto X_p(f)$ is smooth.

9.7 Problems

1. Prove Proposition 9.10, stating that the transition maps for a vector bundle are fiberwise linear.
2. Show that the tautological line bundles over $\mathbb{RP}^n$ are non-trivial for all $n \geq 1$.
3. (a) Recall from the end of Section 8.5.3 that every vector field on S^2 must vanish at some point of S^2. Use this to show that the tangent bundle of the 2-sphere TS^2 is non-trivial, i.e., there does not exist a vector bundle isomorphism $TS^2 \cong S^2 \times \mathbb{R}^2$.
   (b) Prove that the direct sum of TS^2 with the trivial vector bundle $S^2 \times \mathbb{R}$ is a trivial bundle,
   $$TS^2 \oplus (S^2 \times \mathbb{R}) \cong S^2 \times \mathbb{R}^3.$$
   (Hint: Viewing S^2 as embedded in $\mathbb{R}^3$, consider its normal bundle.)
4. Generalizing the previous problem, prove that for a compact, connected oriented surface Σ (without boundary), the tangent bundle $T\Sigma$ is non-trivial unless Σ is a 2-torus, but $T\Sigma \oplus (\Sigma \times \mathbb{R})$ is always trivial. (Again, you may want to use an embedding of into $\mathbb{R}^3$.)
5. Construct a vector bundle isomorphism between the tautological line bundle over $\mathbb{RP}^1$ and the infinite Möbius strip (Example 9.13).
6. Prove that the tangent bundle of S^3 is trivial, and likewise for the tangent bundle of $\mathbb{RP}^3$.
7. Show that the tangent bundle of a non-orientable manifold is never trivial.
8. Let $X \in \mathfrak{X}(M)$ be a complete vector field, with flow Φ_t. Show that there is a unique vector field $\widehat{X} \in \mathfrak{X}(TM)$ on the total space of the tangent bundle, with flow $\widehat{\Phi}_t$ given by
   $$\widehat{\Phi}_t = T\Phi_t : \; TM \to TM.$$
   Explain how to generalize to possibly incomplete vector fields. One calls $\widehat{X}$ the *tangent lift* of X.
9. Prove that for any vector bundle $E \to M$, the dual $E^* = \bigsqcup_p E_p^* \to M$ is a vector bundle: Explain how vector bundle charts for E determine vector bundle charts for E^* and show that compatibility of bundle charts for E implies compatibility of bundle charts for E^*.
10. Let $E \to M$ be a vector bundle. A *bundle metric* on E is a family of (positive definite) inner products $\mathrm{g}_p : \; E_p \times E_p \to \mathbb{R}$ such that for all sections $\sigma, \tau \in \Gamma(E)$ the map
    $$M \to \mathbb{R}, \; p \mapsto \mathrm{g}_p(\sigma_p, \tau_p)$$
    is smooth. Using partitions of unity, prove that every vector bundle admits such a bundle metric. Furthermore, show that a bundle metric determines an isomorphism between E and its dual E^*.

11. Let $V \to M$ be a vector bundle over M. Let $\kappa_t : V \to V$ be the map given by fiberwise multiplication by $t \in \mathbb{R}$. Every $v \in V$ determines a curve $t \mapsto \kappa_t(v)$; taking the velocity vector at $t = 0$, we obtain a map

$$j : V \to TV, \quad v \mapsto \frac{d}{dt}\Big|_{t=0} \kappa_t(v).$$

   (a) Show that j is a vector bundle map, realizing $V \to M$ as a subbundle of $TV \to V$. (This *vertical subbundle* should not be confused with the inclusion $V \to TV$ as the zero section.)
   (b) Prove the following result of Grabowski and Rotkiewicz: If V, W are vector bundles, and $\varphi : V \to W$ a smooth map of the underlying manifolds, compatible with the scalar multiplications in the sense that

$$\kappa_t(\varphi(v)) = \varphi(\kappa_t(v))$$

   for all $v \in V$, then φ is a morphism of vector bundles.
   (Hint: Consider the tangent map $T\varphi$, and use part (a) to regard V and W as subbundles of TV and TW, respectively.)

12. Let M be a manifold, and T^*M its cotangent bundle.
   (a) Show that there exists a unique 1-form $\theta \in \Omega^1(T^*M)$ with the property that for all $\alpha \in \Omega^1(M)$, with corresponding section $\sigma_\alpha \in \Gamma(T^*M)$,

$$(\sigma_\alpha)^*\theta = \alpha.$$

   (b) Let $x^1, \ldots, x^n$ be local coordinates in a chart $U \subseteq M$, and $x^1, \ldots, x^n, p_1, \ldots, p_n$ the resulting coordinates on $T^*U \subseteq T^*M$. Express θ in these coordinates.
   (c) Show furthermore that the closed 2-form $\omega = \mathrm{d}\theta$ is non-degenerate, i.e., symplectic: $\iota_v \omega = 0$ for $v \in T(T^*M)$ implies that $v = 0$.

A

Notions from Set Theory

A.1 Countability

A set X is *countable* if it is either finite (possibly empty) or there exists a bijective map $f:\ \mathbb{N} \to X$. We list some basic facts about countable sets:

- $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$ are countable, $\mathbb{R}$ is not countable.
- If X_1, X_2 are countable, then the cartesian product $X_1 \times X_2$ is countable.
- If X is countable, then any subset of X is countable.
- If X is countable, and $f:\ X \to Y$ is surjective, then Y is countable.
- If $(X_i)_{i\in I}$ are countable sets, indexed by a countable set I, then the (disjoint) union $\bigsqcup_{i\in I} X_i$ is countable.

A.2 Equivalence Relations

A *relation* from a set X to a set Y is simply a subset

$$R \subseteq Y \times X.$$

We write $x \sim_R y$ if and only if $(y,x) \in R$. When R is understood, we write $x \sim y$. If $Y = X$ we speak of a *relation on* X.

Example A.1. Any map $f : X \to Y$ defines a relation, given by its *graph*

$$\mathrm{graph}(f) = \{(f(x),x) | x \in X\}.$$

In this sense relations are generalizations of maps; for example, they are often used to describe "multi-valued" maps.

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Remark A.2. Given another relation $S \subseteq Z \times Y$, one defines a composition $S \circ R \subseteq Z \times X$, where

$$S \circ R = \{(z,x) |\ \exists y \in Y :\ (z,y) \in S,\ (y,x) \in R\}.$$

Our conventions are set up in such a way that if $f:\ X \to Y$ and $g:\ Y \to Z$ are two maps, then $\mathrm{graph}(g \circ f) = \mathrm{graph}(g) \circ \mathrm{graph}(f)$.

Example A.3. On the set $X = \mathbb{R}$ we have relations $\geq, >, <,\ \leq, =$. But there is also the relation defined by the condition $x \sim x' \Leftrightarrow x' - x \in \mathbb{Z}$, and many others.

A relation $\sim$ on a set X is called an *equivalence relation* if it has the following properties:

(a) Reflexivity: $x \sim x$ for all $x \in X$.
(b) Symmetry: $x \sim y \Rightarrow y \sim x$.
(c) Transitivity: $x \sim y,\ y \sim z \Rightarrow x \sim z$.

Given an equivalence relation, we define the *equivalence class* of $x \in X$ (using the notation $[\cdot]$) to be the subset

$$[x] = \{y \in X |\ x \sim y\}.$$

Each element of the equivalent class $[x]$ is called a *representative* of the equivalence class. Note that $[x] = [y]$ if and only if $x \sim y$ if and only if x, y are representatives of the same equivalence class. Note also that X is a disjoint union of its equivalence classes. We denote by $X/\sim$ the set of equivalence classes. That is, all the elements of a given equivalence class are lumped together and represent a single element of $X/\sim$. One defines the *quotient map*

$$q:\ X \to X/\sim,\quad x \mapsto [x].$$

By definition, the quotient map is surjective.

Remark A.4. There are two other useful ways to think of equivalence relations:

- An equivalence relation R on X amounts to a decomposition $X = \bigsqcup_{i \in I} X_i$ as a disjoint union of subsets. Given R, one takes X_i to be the equivalence classes; given the decomposition, one defines $R = \{(y,x) \in X \times X | \exists i \in I :\ x, y \in X_i\}$.
- An equivalence relation amounts to a surjective map $q:\ X \to Y$. Indeed, given R one takes $Y := X/\sim$ with q the quotient map; conversely, given q one defines $R = \{(x',x) \in X \times X |\ q(x) = q(x')\}$.

Remark A.5. Often, we will not write out the entire equivalence relation. For example, if we say "*the equivalence relation on S^2 given by $x \sim -x$*", then it is understood that we also have $x \sim x$, since reflexivity holds for any equivalence relation. Similarly, when we say "*the equivalence relation on $\mathbb{R}$ generated by $x \sim x+1$*", it is understood that we also have $x \sim x+2$ (by transitivity from $x \sim x+1 \sim x+2$) as well as $x \sim x-1$ (by symmetry), hence $x \sim x+k$ for all $k \in \mathbb{Z}$. (Any relation $R_0 \subseteq X \times X$ extends to a unique smallest equivalence relation R; one says that R is the equivalence relation *generated by* R_0.)

Example A.6. Consider the equivalence relation on S^2 given by

$$(x,y,z) \sim (-x,-y,-z).$$

The equivalence classes are pairs of antipodal points; they are in 1–1 correspondence with lines in $\mathbb{R}^3$. That is, the quotient space $S^2/\sim$ is naturally identified with $\mathbb{R}\mathrm{P}^2$.

Example A.7. The quotient space $\mathbb{R}/\sim$ for the equivalence relation $x \sim x+1$ on $\mathbb{R}$ is naturally identified with S^1. If we think of S^1 as a subset of $\mathbb{R}^2$, the quotient map is given by $t \mapsto (\cos(2\pi t), \sin(2\pi t))$.

Example A.8. Similarly, the quotient space for the equivalence relation on $\mathbb{R}^2$ given by $(x,y) \sim (x+k, y+l)$ for $k,l \in \mathbb{Z}$ is the 2-torus T^2.

Example A.9. Let E be a k-dimensional real vector space. Given two ordered bases $(e_1,\ldots,e_k)$ and $(e'_1,\ldots,e'_k)$, there is a unique invertible linear transformation $A: E \to E$ with $A(e_i) = e'_i$. The two ordered bases are called *equivalent* if $\det(A) > 0$. One checks that equivalence of bases is an equivalence relation. There are exactly two equivalence classes; the choice of an equivalence class is called an *orientation* on E.

For example, $\mathbb{R}^n$ has a standard orientation defined by the standard basis $(e_1,\ldots,e_n)$. The opposite orientation is defined, for example, by $(-e_1, e_2, \ldots, e_n)$. A permutation of the standard basis vectors defines the standard orientation if and only if the permutation is even (see Appendix B).

When a function $f: X \to Y$ acts on a set X equipped with an equivalence relation $\sim$, one may try to define a function on the equivalence classes $\widetilde{f}: (X/\sim) \to Y$ in a "natural way" by its action on representatives.

$$\widetilde{f}([x]) = f(x).$$

This only makes sense if $f(x) = f(y)$ whenever $[x] = [y]$, i.e., when f is constant on equivalence classes. In such cases, one says that f *descends* to a map $\widetilde{f}$ (which is often also denoted f by abuse of notation). This terminology comes from the commutative diagram below.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "q"'] & Y \\ X/\sim \arrow[ur, dashed, "\widetilde{f}"'] & \end{tikzcd}$$

Example A.10. For the equivalence relation on $\mathbb{R}$ given by $x \sim x+1$ from Example A.7 above, the identity function $\mathrm{id}_{\mathbb{R}}: \mathbb{R} \to \mathbb{R}$ does *not* descend to a well-defined function on the circle S^1. However, the function $f: x \mapsto x - \lfloor x \rfloor$ mapping x to its *fractional part* does descend to a map $\tilde{f}: (\mathbb{R}/\sim) \to \mathbb{R}$; since $[x] = [y]$ if and only if $x - y \in \mathbb{Z}$.

Remark A.11. Similarly, if $G : X \times X \to X$ is an operation, one may try to define $\widetilde{G} : (X/\sim) \times (X/\sim) \to (X/\sim)$ by $\widetilde{G}([x],[y]) \to [G(x,y)]$. This is only well-defined if $[G(x,y)] = [G(x',y')]$ whenever $[x] = [x']$ and $[y] = [y']$. In such cases, it is often said that $X/\sim$ *inherits* the operation G from X. This is exactly how modular arithmetic is defined on $\mathbb{Z}/n\mathbb{Z}$.

B

Notions from Algebra

B.1 Permutations

Let X be a set with $n < \infty$ elements. A *permutation* of X is an invertible map $s: X \to X$. The set of permutations form a group, with product the composition of permutations, and with identity element the trivial permutation. A permutation interchanging two elements of X, while fixing all other elements, is called a *transposition*.

By choosing an enumeration of the elements of X, we may assume

$$X = \{1, \ldots, n\};$$

the corresponding group is denoted S_n. For $i < j$, we denote by t_{ij} the transposition of the indices i, j (leaving all others fixed).

It is standard practice to denote a permutation $s(1) = i_1, \ldots, s(n) = i_n$ by a symbol*

$$(i_1,\ i_2,\ \cdots, i_n).$$

Alternatively, one can simply list where the elements map to, e.g.,

$$1 \to i_1,\ 2 \to i_2, \ldots,\ n \to i_n.$$

Example B.1. The notation

$$(2, 4, 1, 3) \to (3, 2, 4, 1)$$

* Overloading the parenthesis $(\cdot,\cdot)$ notation. One must use the context to distinguish between a permutation and an ordered tuple.

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signifies the permutation $s(2) = 3,\ s(4) = 2,\ s(1) = 4,\ s(3) = 1$. After listing the elements in the proper order $s(1) = 4,\ s(2) = 3,\ s(3) = 1,\ s(4) = 2$, it is thus described by the symbol

$$(4,3,1,2).$$

By induction, one may prove that every permutation is a product of transpositions. In fact, it is enough to consider transpositions of *adjacent* elements, i.e., those of the form $t_{i\ i+1}$.

Example B.2. For $s = (4,3,1,2)$, use the following transpositions to put s back to the original position:

$$(4,3,1,2) \to (4,1,3,2) \to (1,4,3,2) \to (1,4,2,3) \to (1,2,4,3) \to (1,2,3,4).$$

Reversing arrows, this shows how to write s as a product of five transpositions of adjacent elements: $s = t_{13}\, t_{14}\, t_{23}\, t_{24}\, t_{34}$.

A permutation $s \in S_n$ of $\{1, \ldots, n\}$ is called *even* if the number of pairs (i, j) such that $i < j$ but $s(i) > s(j)$ is even and is called *odd* if the number of such "wrong order" pairs is odd. In particular, every transposition is odd.

Example B.3. The permutation $s = (4,3,1,2)$ has five pairs of indices in the wrong order,

$$(4,3), (4,1), (4,2), (3,1), (3,2).$$

Hence, s is odd.

Of course, computing the sign by listing all pairs that are in the wrong order can be cumbersome. Fortunately, there are much simpler ways of finding the parity. Define a map

$$\mathrm{sign}: \ S_n \to \{1, -1\}$$

by setting $\mathrm{sign}(s) = 1$ if the permutation is even, $\mathrm{sign}(s) = -1$ if the permutation is odd. View $\{1, -1\}$ as a group, with product the usual multiplication.

Theorem B.4. *The map* $\mathrm{sign}: \ S_n \to \{1, -1\}$ *is a group homomorphism. That is,* $\mathrm{sign}(s's) = \mathrm{sign}(s')\mathrm{sign}(s)$ *for all* $s, s' \in S_n$.

Proof (Sketch). This may be proved by examining the effect of precomposing a given permutation s with a transposition $t_{i\,i+1}$ of two adjacent elements—let us call the result $\tilde{s}$. If $i, i+1$ were a "right order" pair for s, then they will be a "wrong order" pair for $\tilde{s}$, and the relative order for all other pairs remains unchanged. Consequently, the signs of $\tilde{s}$ and s are opposite. It follows by induction that if s can be written as a product of N transpositions of adjacent elements, then $\mathrm{sign}(s) = (-1)^N$.

A similar reasoning applies to s', so that $\mathrm{sign}(s') = (-1)^{N'}$. The expressions for s, s' as products of transpositions of adjacent elements gives another such expression for $s's$, involving $N + N'$ transpositions. □

A simple consequence is that $\mathrm{sign}(s) = (-1)^N$ whenever s is a product of N transpositions (not necessarily adjacent ones).

Example B.5. We saw that $s = (4,3,1,2)$ is a product of five transpositions of adjacent elements, hence $\mathrm{sign}(s) = (-1)^5 = -1$. But if we use general transpositions, we only need three steps to put $(4,3,1,2)$ in the initial position:

$$(4,3,1,2) \to (1,3,4,2) \to (1,2,4,3) \to (1,2,3,4).$$

We once again see that $\mathrm{sign}(s) = (-1)^3 = -1$.

The permutation group and the sign function appear in the *Leibniz formula* for the determinant of an $(n \times n)$-matrix A with entries A_{ij}:

$$\det(A) = \sum_{s \in S_n} \mathrm{sign}(s) A_{s(1)1} \cdots A_{s(n)n}.$$

B.2 Algebras

B.2.1 Definition and Examples

An *algebra* (over the field $\mathbb{R}$ of real numbers) is a vector space $\mathscr{A}$, together with a *multiplication* (product) $\mathscr{A} \times \mathscr{A} \to \mathscr{A}$, $(a,b) \mapsto ab$ such that

(a) The multiplication is associative: That is, for all $a,b,c \in \mathscr{A}$

$$(ab)c = a(bc).$$

(b) The multiplication map is linear in both arguments: That is,

$$(\lambda_1 a_1 + \lambda_2 a_2)b = \lambda_1(a_1 b) + \lambda_2(a_2 b),$$

$$a(\mu_1 b_1 + \mu_2 b_2) = \mu_1(ab_1) + \mu_2(ab_2),$$

for all $a, a_1, a_2, b, b_1, b_2 \in \mathscr{A}$ and all scalars $\lambda_1, \lambda_2, \mu_1, \mu_2 \in \mathbb{R}$.

The algebra is called *commutative* if $ab = ba$ for all $a,b \in \mathscr{A}$. A *unital algebra* is an algebra $\mathscr{A}$ with a distinguished element $1_{\mathscr{A}} \in \mathscr{A}$ (called the *unit*) such that

$$1_{\mathscr{A}} a = a = a 1_{\mathscr{A}}$$

for all $a \in \mathscr{A}$.

Remark B.6. One can also consider non-associative product operations on vector spaces, most importantly one has the class of *Lie algebras*. If there is risk of confusion with these or other concepts, we may refer to *associative algebras*.

Remark B.7. One can also consider algebras over other fields.

Example B.8. The space $\mathbb{C}$ of complex numbers (regarded as a real vector space $\mathbb{R}^2$) is a unital, commutative algebra, containing $\mathbb{R} \subseteq \mathbb{C}$ as a subalgebra.

Example B.9. A more sophisticated example is the algebra $\mathbb{H} \cong \mathbb{R}^4$ of quaternions, which is a unital non-commutative algebra. Elements of $\mathbb{H}$ are written as

$$x = a + ib + jc + kd$$

with $a, b, c, d \in \mathbb{R}$; here i, j, k are just formal symbols. The multiplication of quaternions is specified by the rules

$$i^2 = j^2 = k^2 = -1, \; ij = k = -ji, \; jk = i = -kj, \; ki = j = -ik.$$

Elements of the form $z = a + ib$ form a subalgebra of $\mathbb{H}$ isomorphic to $\mathbb{C}$. The norm of a quaternion is defined by $|x| = \sqrt{a^2 + b^2 + c^2 + d^2}$; it has the properties $|x_1 + x_2| \leq |x_1| + |x_2|$ and $|x_1 x_2| = |x_1||x_2|$.

The algebra of quaternions may also be described as an algebra of complex (2×2)-matrices of the form

$$\begin{pmatrix} z & w \\ -\overline{w} & \overline{z} \end{pmatrix},$$

where the algebra multiplication is given by matrix multiplication.

Example B.10. For any n, the space $\mathrm{Mat}_{\mathbb{R}}(n)$ of $(n \times n)$-matrices, with matrix multiplication, is a non-commutative unital algebra. One can also consider matrices with coefficients in $\mathbb{C}$, denoted $\mathrm{Mat}_{\mathbb{C}}(n)$, or in fact with coefficients in any given algebra.

Example B.11. For any set X, the space of functions $f: X \to \mathbb{R}$ is a unital commutative algebra, where the product is given by pointwise multiplication. Given a topological space X, one has the unital algebra $C(X)$ of *continuous* $\mathbb{R}$-valued functions. If X is non-compact, this has a (non-unital) subalgebra $C_0(X)$ of continuous functions vanishing outside a compact set.

B.2.2 Homomorphisms of Algebras

A *homomorphism of algebras* $\Phi: \mathscr{A} \to \mathscr{A}'$ is a linear map preserving products:

$$\Phi(ab) = \Phi(a)\Phi(b).$$

(For a homomorphism of unital algebras, one asks in addition that $\Phi(1_{\mathscr{A}}) = 1_{\mathscr{A}'}$.) It is called an *isomorphism of algebras* if Φ is invertible. For the special case $\mathscr{A}' = \mathscr{A}$, these are also called *algebra automorphisms* of $\mathscr{A}$. Note that the algebra automorphisms form a group under composition.

Example B.12. Consider $\mathbb{R}^2$ as an algebra, with product coming from the identification $\mathbb{R}^2 = \mathbb{C}$. The complex conjugation $z \mapsto \overline{z}$ defines an automorphism $\Phi: \mathbb{R}^2 \to \mathbb{R}^2$ of this algebra.

Example B.13. The algebra $\mathbb{H}$ of quaternions has an automorphism given by cyclic permutation of the three imaginary units:

$$\Phi(x + iu + jv + kw) = x + ju + kv + iw.$$

Example B.14. Let $\mathscr{A} = \mathrm{Mat}_{\mathbb{R}}(n)$ the algebra of $(n \times n)$-matrices. If $U \in \mathscr{A}$ is invertible, then $X \mapsto \Phi(X) = UXU^{-1}$ is an algebra automorphism.

Example B.15. Suppose $\mathscr{A}$ is a unital algebra. Let $\mathscr{A}^\times$ be the set of invertible elements, i.e., elements $u \in \mathscr{A}$ for which there exists $v \in \mathscr{A}$ with $uv = vu = 1_{\mathscr{A}}$. Given u, such a v is necessarily unique so we may write $v = u^{-1}$, and the map $\mathscr{A} \to \mathscr{A}$, $a \mapsto uau^{-1}$ is an algebra automorphism. Such automorphisms are called *inner*.

B.2.3 Derivations of Algebras

Definition B.16. *A* derivation *of an algebra $\mathscr{A}$ is a linear map $D\colon \mathscr{A} \to \mathscr{A}$ satisfying the product rule*

$$D(a_1 a_2) = D(a_1)a_2 + a_1 D(a_2).$$

If $\dim \mathscr{A} < \infty$, then a derivation may be regarded as an *infinitesimal automorphism* of the algebra. Indeed, let $U : \mathbb{R} \to \mathrm{End}(\mathscr{A})$, $t \mapsto U_t$ be a smooth curve with $U_0 = I$, such that each U_t is an algebra automorphism. Consider the Taylor expansion,

$$U_t = I + tD + \cdots$$

with

$$D = \frac{d}{dt}\Big|_{t=0} U_t$$

the velocity vector at $t = 0$. By taking the derivative of the condition

$$U_t(a_1 a_2) = U_t(a_1)U_t(a_2)$$

at $t = 0$, we get the derivation property for D.

Conversely, if D is a derivation, then

$$U_t = \exp(tD) = \sum_{n=0}^{\infty} \frac{t^n}{n!} D^n$$

(using the exponential of a matrix) is a well-defined curve of algebra automorphisms. We leave it as an exercise to check the automorphism property; it involves proving the property

$$D^n(a_1 a_2) = \sum_k \binom{n}{k} D^k(a_1)\, D^{n-k}(a_2)$$

for all $a_1, a_2 \in \mathscr{A}$.

If $\mathscr{A}$ is infinite-dimensional, one may still want to think of derivations D as infinitesimal automorphisms, even though the discussion will run into technical problems. (For instance, the exponential map of infinite rank endomorphisms is not well-defined in general.)

A collection of facts about derivations of algebras $\mathscr{A}$:

- Any given $x \in \mathscr{A}$ defines a derivation

$$D(a) = [x,a] := xa - ax.$$

(Exercise: Verify that this is a derivation.) These are called *inner derivations*. If $\mathscr{A}$ is commutative (for example, $\mathscr{A} = C^\infty(M)$) the inner derivations are all trivial. At the other extreme, for the matrix algebra $\mathscr{A} = \mathrm{Mat}_\mathbb{R}(n)$, one may show that every derivation is inner.
- If $\mathscr{A}$ is a unital algebra, with unit $1_\mathscr{A}$, then $D(1_\mathscr{A}) = 0$ for all derivations D. (This follows by applying the defining property of derivations to $1_\mathscr{A} = 1_\mathscr{A} 1_\mathscr{A}$.)
- Given two derivations D_1, D_2 of an algebra $\mathscr{A}$, their commutator (using the $[\cdot,\cdot]$ notation)

$$[D_1, D_2] = D_1 D_2 - D_2 D_1$$

is again a derivation. Indeed, if $a, b \in \mathscr{A}$ then

$$\begin{aligned} D_1 D_2(ab) &= D_1\big(D_2(a)b + aD_2(b)\big) \\ &= (D_1D_2)(a)b + a(D_1D_2)(b) + D_1(a)D_2(b) + D_2(a)D_1(b). \end{aligned}$$

Subtracting a similar expression with indices 1 and 2 interchanged, one obtains the derivation property of $[D_1, D_2]$.

B.2.4 Modules over Algebras

Definition B.17. *A (left)* module *over an algebra $\mathscr{A}$ is a vector space $\mathscr{E}$ together with a map (module action) $\mathscr{A} \times \mathscr{E} \to \mathscr{E}$, $(a,x) \mapsto ax$ such that*

(a) For $a, b \in \mathscr{A}$ and $x \in \mathscr{E}$,

$$(ab)x = a(bx).$$

(b) The module action is linear in both arguments: That is,

$$(\lambda_1 a_1 + \lambda_2 a_2)x = \lambda_1(a_1 x) + \lambda_2(a_2 x),$$

$$a(\mu_1 x_1 + \mu_2 x_2) = \mu_1(ax_1) + \mu_2(ax_2),$$

for all $a, a_1, a_2 \in \mathscr{A}$, $x, x_1, x_2 \in \mathscr{E}$, and all scalars $\lambda_1, \lambda_2, \mu_1, \mu_2 \in \mathbb{R}$.

Example B.18. Every algebra $\mathscr{A}$ is a module over itself, with the module action given by algebra multiplication from the left.

Example B.19. If the algebra $\mathscr{A}$ is commutative, then the space of derivations is a module over $\mathscr{A}$. Indeed, if D is a derivation and $x \in \mathscr{A}$ then $a \mapsto (xD)(a) := xD(a)$ is again a derivation, since

$$(xD)(ab) = x(D(ab)) = x(D(a)\,b + a(D(b)) = (xD)(a)b + a\,(xD)(b)$$

(using $xa = ax$).

B.3 Dual Spaces and Quotient Spaces

Let E be a vector space over a field $\mathbb{F}$. (We mostly have in mind the cases $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$.) The *dual space* is the space of linear functionals $\varphi : E \to \mathbb{F}$,

$$E^* = \mathrm{Hom}(E, \mathbb{F}),$$

which again a vector space over $\mathbb{F}$. If $n = \dim E < \infty$, and $v_1, \ldots, v_n$ is a basis of E, then the dual space E^* has a basis $\varphi^1, \ldots, \varphi^n$ such that $\varphi^i(v_j) = \delta^i_j$ for all i, j. In particular, $\dim E^* = \dim E$, and in fact E and E^* are isomorphic, for example by the map taking v_i to φ^i. Note, however, that there is no *canonical* isomorphism: The isomorphism just described depends on the choice of a basis, and for a general E it is not possible to describe an isomorphism $E \to E^*$ without making extra choices. Furthermore, if $\dim E = \infty$ then E and E^* are in fact *not* isomorphic; intuitively, E^* is "more" infinite-dimensional than E. For instance, if E has a countable (infinite) basis then E^* does not admit a countable basis.

For any vector space E, any $v \in E$ defines a linear functional on the dual space E^*, given by evaluation:

$$\mathrm{ev}_v : E^* \to \mathbb{F}, \ \ \varphi \mapsto \varphi(v).$$

This defines a canonical linear map

$$E \to E^{**} = (E^*)^*, \quad v \mapsto \mathrm{ev}_v.$$

This map is injective; hence, if $\dim E < \infty$ it is an isomorphism.

Suppose that $E' \subseteq E$ is a subspace. Define an equivalence relation on E, where $v_1 \sim v_2 \Leftrightarrow v_1 - v_2 \in E'$. The set of equivalence classes is called the *quotient space*, and is denoted by

$$E/E' = \{[v] \in E\}.$$

It has a unique vector space structure in such a way that the quotient map $E \to E/E'$ is linear; specifically, $[v_1] + [v_2] = [v_1 + v_2]$ for $v_1, v_2 \in E$ and $\lambda [v] = [\lambda v]$ for $v \in E$ and $\lambda \in \mathbb{F}$. Note that the quotient map $E \to E/E'$ is surjective, with kernel (null space) E'; conversely, if $E \to E''$ is any surjective linear map with kernel E', then E'' is canonically isomorphic to E/E'.

The subspace $E' \subseteq E$ also determines a subspace of the dual space, namely its *annihilator*

$$\mathrm{ann}(E') = \{\varphi \in E^* |\ \varphi(v) = 0 \text{ for all } v \in E'\}.$$

(Other common notations include $(E')^0$, or also $\mathrm{ann}_{E^*}(E')$ to indicate the ambient space.) For $\varphi \in \mathrm{ann}(E')$, one obtains a linear functional on E/E' by

$$E/E' \to \mathbb{F}, \ \ [v] \mapsto \varphi(v).$$

This is well-defined exactly because φ vanishes on E'. Conversely, given a linear functional on E/E', its composition with the quotient map $E \to E/E'$ is a linear functional on E vanishing on E'. This defines an isomorphism

$$(E/E')^* \xrightarrow{\cong} \mathrm{ann}(E').$$

Note that E, E' may be infinite-dimensional here; if they are finite-dimensional then we obtain, in particular,

$$\dim \mathrm{ann}(E') = \dim(E/E') = \dim E - \dim E'.$$

Furthermore, in this case, $\mathrm{ann}(\mathrm{ann}(E')) = E'$ under the identification of E^{**} with E.

Consider next the situation that E comes equipped with a symmetric bilinear form $\beta : E \times E \to \mathbb{F}$. That is, β is linear in each argument, leaving the other fixed, and $\beta(v_1, v_2) = \beta(v_2, v_1)$ for all v_1, v_2. Let us assume for the remainder of this subsection that $\dim E < \infty$. Now, β is called *non-degenerate* if it has the property that for all $v \in V$, one has $\beta(v, w) = 0$ for all $w \in E$ only if $v = \mathbf{0}$. Equivalently, the linear map $\beta^\flat : E \to E^*$, $v \mapsto \beta(v, \cdot)$ is an isomorphism. Hence, a non-degenerate bilinear form gives a concrete isomorphism between E and E^*.

For a subspace E' we define the *orthogonal subspace* as

$$(E')^\perp = \{v \in E \mid \beta(v, w) = 0 \text{ for all } w \in E'\}.$$

The isomorphism $\beta^\flat : E \to E^*$ restricts to an isomorphism $(E')^\perp \to \mathrm{ann}(E')$; in particular, $\dim(E')^\perp = \dim E - \dim E'$ and $(E')^{\perp\perp} = E'$.

If $\mathbb{F} = \mathbb{R}$ and if the bilinear form β is positive definite (i.e., $\beta(v, v) > 0$ for $v \neq \mathbf{0}$) then $(E')^\perp \cap E' = \{\mathbf{0}\}$, and one often refers to $(E')^\perp$ as the *orthogonal complement*. However, in more general situations the space $(E')^\perp$ need not be a complement to E'.

As a final remark, note that a similar discussion goes through for non-degenerate *skew-symmetric* bilinear forms ω, i.e., such that $\omega(v_1, v_2) = -\omega(v_2, v_1)$ and the map $\omega^\flat : E \to E^*$ is an isomorphism. In particular, for $E' \subseteq E$ one can define the ω-orthogonal space $(E')^\omega$, and one has $(E')^{\omega\omega} = E'$.

C

Topological Properties of Manifolds

C.1 Topological Spaces

A *topological space* is a set X together with a collection of subsets $U \subseteq X$ called *open subsets*, with the following properties:

- $\emptyset$, X are open.
- If U, U' are open then $U \cap U'$ is open.
- For any collection $\{U_\alpha\}$ of open subsets, the union $\bigcup_\alpha U_\alpha$ is open.

(Note that in the third condition, the index set need not be finite, or even countable.) The collection of open subsets is called the *topology* of X.

The space $\mathbb{R}^n$ has a standard topology given by the usual open subsets. Likewise, the open subsets of a manifold M define a topology on M. For any set X, one has the *trivial topology* where the only open subsets are $\emptyset$ and X, and the *discrete topology* where every subset is considered open.

An *open neighborhood* of a point p is an open subset containing it. A topological space is called *Hausdorff* if any two distinct points have disjoint open neighborhoods.

Let X be a topological space. Then any subset $A \subseteq X$ has the *subspace topology*, with open sets the collection of all intersections $U \cap A$ such that $U \subseteq X$ is open. Given a surjective map $q: X \to Y$, the space Y inherits a *quotient topology*, whose open sets are all $V \subseteq Y$ such that the preimage $q^{-1}(V) = \{x \in X | \, q(x) \in V\}$ is open.

A subset A is *closed* if its complement $X \backslash A$ is open. Dual to the statements for open sets, one has

- $\emptyset$, X are closed.
- If A, A' are closed then $A \cup A'$ is closed.
- For any collection $\{A_\alpha\}$ of closed subsets, the intersection $\bigcap_\alpha A_\alpha$ is closed.

For any subset A, denote by $\overline{A}$ its *closure*, defined as the smallest closed subset containing A, i.e., the intersection of all closed subsets containing A.

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C.2 Manifolds Are Second Countable

A *basis* for the topology on X is a collection $\mathscr{B} = \{U_\alpha\}$ of open subsets of X such that every open set U is a union of sets from $\mathscr{B}$. (Equivalently, for every open subset U and every $p \in U$ there exists α such that $p \in U_\alpha \subseteq U$.) In contrast to the notion of a basis of a vector space, this collection does not have to be minimal in any sense; for instance, the collection of *all* open subsets of a topological space is a basis.

Example C.1. Let $X = \mathbb{R}^n$. Then the collection of all open balls $B_\varepsilon(x)$, with $\varepsilon > 0$ and $x \in \mathbb{R}^n$, is a basis for the topology on $\mathbb{R}^n$.

A topological space is said to be *second countable* if its topology has a countable basis.

Proposition C.2. $\mathbb{R}^n$ *is second countable.*

Proof. A countable basis is given by the collection of all *rational balls* (cf. ✎ 53), by which we mean ε-balls $B_\varepsilon(x)$ such that $x \in \mathbb{Q}^m$ and $\varepsilon \in \mathbb{Q}_{>0}$. To check that it is a basis, let $U \subseteq \mathbb{R}^m$ be open, and $p \in U$. Choose $\varepsilon \in \mathbb{Q}_{>0}$ such that $B_{2\varepsilon}(p) \subseteq U$. There exists a rational point $x \in \mathbb{Q}^n$ with $||x-p|| < \varepsilon$. This then satisfies $p \in B_\varepsilon(x) \subseteq U$. Since p was arbitrary, this proves the claim. $\square$

The same reasoning shows that for any open subset $U \subseteq \mathbb{R}^m$, the rational ε-balls that are contained in U form a basis for the topology of U.

Proposition C.3. *Manifolds are second countable.*

Proof. Given a manifold M, let $\mathscr{A} = \{(U_\alpha, \varphi_\alpha)\}$ be a countable atlas. Then the set of all $\varphi_\alpha^{-1}(B_\varepsilon(x))$, where $B_\varepsilon(x)$ is a rational ball contained in $\varphi_\alpha(U_\alpha)$, is a countable basis for the topology of M. Indeed, any open subset U is a countable union over all $U \cap U_\alpha$, and each of these intersections is a countable union over all $\varphi_\alpha^{-1}(B_\varepsilon(x))$ such that $B_\varepsilon(x)$ is a rational ε-ball contained in $U \cap U_\alpha$. $\square$

C.3 Manifolds Are Paracompact

A collection $\{U_\alpha\}$ of open subsets of X is called an *open cover* of $A \subseteq X$ if $A \subseteq \bigcup_\alpha U_\alpha$. A *subcover* of the given cover $\{U_\alpha\}$ is an open cover $\{V_\beta\}$ where each V_β's is equal to some U_α. (In other words, it is a sub-collection which is still a cover.) The subset A is called *compact* if every open cover of A has a finite subcover (see also Section 2.5). The *Heine-Borel Theorem* from multivariable calculus states that a subset $A \subseteq \mathbb{R}^m$ of Euclidean space is compact in this sense if and only if it is closed and bounded.

A *refinement* of an open cover $\{U_\alpha\}$ of A is an open cover $\{V_\beta\}$ of A such that each V_β is contained in (not necessarily equal to) some U_α. A topological space is called *paracompact* if every open cover $\{U_\alpha\}$ of X has a locally finite refinement $\{V_\beta\}$, i.e., every point has an open neighborhood meeting only finitely many V_β's.

Proposition C.4. *Manifolds are paracompact.*

We will need the following auxiliary result.

Lemma C.5. *For any manifold M, there exists a sequence of open subsets $W_1, W_2, \ldots$ of M such that*

$$\bigcup W_i = M,$$

and such that each W_i has compact closure with $\overline{W_i} \subseteq W_{i+1}$.

Proof. Start with a countable open cover $O_1, O_2, \ldots$ of M such that each O_i has compact closure $\overline{O}_i$. (We saw in the proof of Proposition C.3 how to construct such a cover, by taking preimages of ε-balls in coordinate charts.) Replacing O_i with $O_1 \cup \cdots \cup O_i$ we may assume $O_1 \subseteq O_2 \subseteq \cdots$. For each i, the covering of the compact set $\overline{O}_i$ by the collection of all O_j's admits a finite subcover. Since the sequence of O_j's is nested, this just means $\overline{O}_i$ is contained in O_j for j sufficiently large. We can thus define $W_1, W_2, \ldots$ as a subsequence $W_i = O_{j(i)}$, starting with $W_1 = O_1$, and inductively letting $j(i)$ for $i > 1$ be the smallest index $j(i)$ such that $\overline{W}_{i-1} \subseteq O_{j(i)}$. ☐

Proof (of Proposition C.4). Let $\{U_\alpha\}$ be an open cover of M. Let W_i be a sequence of open sets as in Lemma C.5. For every i, the compact subset $\overline{W}_{i+1} \backslash W_i$ is contained in the open set $W_{i+2} \backslash \overline{W}_{i-1}$, hence it is covered by the collection of open sets

$$(W_{i+2} \backslash \overline{W}_{i-1}) \cap U_\alpha. \tag{C.1}$$

By compactness, $\overline{W}_{i+1} \backslash W_i$ is already covered by finitely many of the subsets (C.1). Let $\mathscr{V}^{(i)}$ be this finite collection, and $\mathscr{V} = \bigcup_{i=1}^{\infty} \mathscr{V}^{(i)}$ the union. Then

$$\mathscr{V} = \{V_\beta\}$$

is the desired countable open cover of M. Indeed, if $V_\beta \in \mathscr{V}^{(i)}$, then $V_\beta \cap W_{i-1} = \emptyset$. That is, a given W_i meets only V_β's from $\mathscr{V}^{(k)}$ with $k \leq i$. Since these are finitely many V_β's, it follows that the cover $\mathscr{V} = \{V_\beta\}$ is locally finite. ☐

Remark C.6. (See Lang [13], page 35.) One can strengthen the result a bit. Given a cover $\{U_\alpha\}$, we can find a refinement to a cover $\{V_\beta\}$ such that each V_β is the domain of a coordinate chart (V_β, ψ_β), with the following extra properties, for some $0 < r < R$:

(i) $\psi_\beta(V_\beta) = B_R(\mathbf{0})$, and
(ii) M is already covered by the smaller subsets $V'_\beta = \psi_\beta^{-1}(B_r(\mathbf{0}))$.

To prove this, we modify the second half of the proof as follows. For each $p \in \overline{W}_{i+1} \backslash W_i$ choose a coordinate chart (V_p, ψ_p) such that $\psi_p(p) = 0$, $\psi_p(V_p) = B_R(\mathbf{0})$, and $V_p \subseteq (W_{i+2} \backslash \overline{W}_{i-1}) \cap U_\alpha$. Let $V'_p \subseteq V_p$ be the preimage of $B_r(\mathbf{0})$. The V'_p cover $\overline{W}_{i+1} \backslash W_i$; let $\mathscr{V}^{(i)}$ be a finite subcover and proceed as before. This remark is useful for the construction of partitions of unity.

C.4 Partitions of Unity

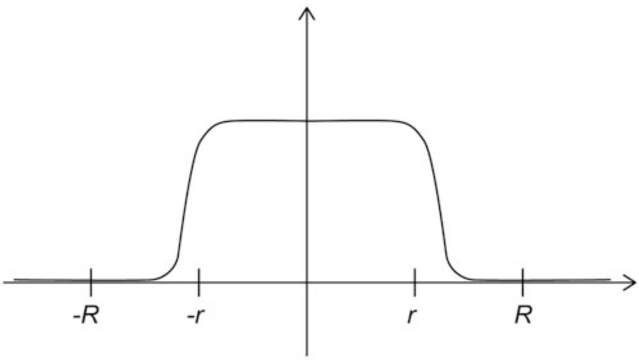
Let M be a manifold. The *support* $\mathrm{supp}(f)$ of a function $f: M \to \mathbb{R}$ is the smallest closed subset such that f vanishes on $M \backslash \mathrm{supp}(f)$. Equivalently, $p \in M \backslash \mathrm{supp}(f)$ if and only if f vanishes on some open neighborhood of p.

Definition C.7. *A* partition of unity subordinate to an open cover $\{U_\alpha\}$ *of a manifold M is a collection of smooth functions $\chi_\alpha \in C^\infty(M)$, with $0 \leq \chi_\alpha \leq 1$, such that* $\mathrm{supp}(\chi_\alpha) \subseteq U_\alpha$*, and*

$$\sum_\alpha \chi_\alpha = 1.$$

Proposition C.9 below states that every open cover admits a partition of unity. To prove it, we will need the following result from multivariable calculus.

Lemma C.8 (Bump Functions). *For all $0 < r < R$, there exists a function $f \in C^\infty(\mathbb{R}^m)$, with* $\mathrm{supp}(f) \subseteq B_R(\mathbf{0})$*, such that $f(\mathbf{x}) = 1$ for $||\mathbf{x}|| \leq r$.*



Proof. Recall that the function

$$h(t) = \begin{cases} 0 & \text{if } t \leq 0, \\ \exp(-1/t) & \text{if } t > 0 \end{cases}$$

is smooth even at $t = 0$. Choose $R_1 \in \mathbb{R}$ with $r < R_1 < R$. We claim that

$$f(\mathbf{x}) = 1 - \frac{h(||\mathbf{x}|| - r)}{h(||\mathbf{x}|| - r) + h(R_1 - ||\mathbf{x}||)}$$

is well-defined, and has the desired properties.

Indeed, for $||\mathbf{x}|| \leq r$ we have that $h(R_1 - ||\mathbf{x}||) > 0$ while $h(||\mathbf{x}|| - r) = 0$, hence $f(\mathbf{x}) = 1$. For $||\mathbf{x}|| > r$ we have that $h(||\mathbf{x}|| - r) > 0$, hence the denominator is > 0 and the expression is well-defined. Finally, if $||\mathbf{x}|| \geq R_1$ we have that $h(R_1 - ||\mathbf{x}||) = 0$, hence the numerator becomes equal to the denominator and $f(\mathbf{x}) = 0$. $\square$

Proposition C.9. *For any open cover $\{U_\alpha\}$ of a manifold, there exists a partition of unity $\{\chi_\alpha\}$ subordinate to that cover. One can take this partition of unity to be locally finite, i.e., for any $p \in M$ there is an open neighborhood U meeting the support of only finitely many χ_α's.*

Proof. Let V_β be a locally finite refinement of the cover U_α, given as domains of coordinate charts (V_β, ψ_β) of the kind described in Remark C.6, and let $V'_\beta \subseteq V_\beta$ be as described there. Since the images of $V'_\beta \subseteq V_\beta$ are $B_r(\mathbf{0}) \subseteq B_R(\mathbf{0})$, we can use Lemma C.8 to define a function $f_\beta \in C^\infty(M)$ with $\mathrm{supp}(f_\beta) \subseteq V_\beta$, and equals to 1 on the closure $\overline{V'_\beta}$. Since the collection of sets V_β is a locally finite cover, the sum $\sum_\beta f_\beta$ is well-defined (near any given point, only finitely many terms are non-zero). Since already the smaller sets V'_β are a cover of M, and the f_β are > 0 on these sets, the sum is strictly positive everywhere.

For each index β, pick an index α such that $V_\beta \subseteq U_\alpha$. This defines a map $d : \beta \mapsto d(\beta)$ between the indexing sets. The functions

$$\chi_\alpha = \frac{\sum_{\beta \in d^{-1}(\alpha)} f_\beta}{\sum_\gamma f_\gamma}$$

give the desired partition of unity: The support is in U_α (since each f_β in the numerator is supported in U_α), and the sum over all χ_α's is equal to 1. Furthermore, the partition of unity is locally finite, since near any given point p only finitely many f_β's are non-zero. $\square$

An important application of partitions of unity is the following result, a weak version of the *Whitney embedding theorem*.

Theorem C.10. *Let M be a manifold admitting a finite atlas with r charts. Then there is an embedding of M as a submanifold of $\mathbb{R}^{r(m+1)}$.*

Proof. Let $\{(U_i, \varphi_i)\}_{i=1}^r$ be a finite atlas for M, and $\chi_1, \ldots, \chi_r$ a partition of unity subordinate to the cover by coordinate charts. Then the products $\chi_i \varphi_i : U_i \to \mathbb{R}^m$ extend by zero to smooth functions $\psi_i : M \to \mathbb{R}^m$. The map

$$F : M \to \mathbb{R}^{r(m+1)},\ p \mapsto (\psi_1(p), \ldots, \psi_r(p), \chi_1(p) \ldots, \chi_r(p))$$

is the desired embedding.

Indeed, F is injective: If $F(p) = F(q)$, choose i with $\chi_i(p) > 0$. Then $\chi_i(q) = \chi_i(p) > 0$, hence both $p, q \in U_i$, and the condition $\psi_i(p) = \psi_i(q)$ gives $\varphi_i(p) = \varphi_i(q)$, hence $p = q$. Similarly T_pF is injective: For $v \in T_pM$ in the kernel of T_pF, choose i such that $\chi_i(p) > 0$, so that $v \in T_pU_i$. Then v, being in the kernel of $T_p\psi_i$ and of $T_p\chi_i$, is in the kernel of $T_p\varphi_i$, hence $v = 0$ (since φ_i is a diffeomorphism). This shows that we get an injective immersion, we leave it as an exercise to verify that the image is a submanifold (e.g., by constructing submanifold charts). $\square$

The theorem applies in particular to all compact manifolds. Actually, one can show that *all* manifolds admit a finite atlas; for a proof see, e.g., the book [9]. Hence, every manifold can be realized as a submanifold of some Euclidean space.

D

Hints and Answers to In-text Questions

Chapter 1

 # 1 (page 9).
For $N = 5$, labeling the vertices as $1,\ldots,5$, these are the five lines $13,14,24,25,35$. Hence, there are 5 bending transformations, whereas $\dim M = 10-6 = 4$. This is not a contradiction; it only shows that the bending transformations are not independent.

 # 2 (page 13).
A punctured $\mathbb{R}\mathrm{P}^2$ is an open Möbius strip (i.e., without its boundary). Removing a point from a surface is equivalent to removing a small disk from the surface. But our discussion showed that $\mathbb{R}\mathrm{P}^2$ can be viewed as a closed Möbius strip glued with a disk along their boundary circles; so, removing that disk leaves the open Möbius strip.

 # 3 (page 13).
The first surface is a *Klein bottle*: Gluing the sides labeled "a" gives a cylinder, and then gluing the sides labeled "b" (with a twist) gives the usual picture of a Klein bottle. For the second surface, start by gluing the "a" parts; the result can be drawn as a disk whose boundary is divided into two segments, both labeled "b." The identification of the two "b" segments amounts to antipodal identification on the boundary of the disk; hence one obtains a *projective plane*.

The third gluing diagram is a bit tricky, since it is not so easy to see directly what the identification of either "a" sides or "b" sides does. One way of finding the surface is to first *cut* the square along the diagonal from the upper left corner to the lower right corner; chose a direction of the diagonal and call this "c." The result is two triangle shaped pieces; flip one of them over and then glue the "b" segments of the two triangles. The result is the standard gluing diagram of the Klein bottle, up to a relabeling of the sides. In summary, the surface is again a *Klein bottle*.

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Chapter 2

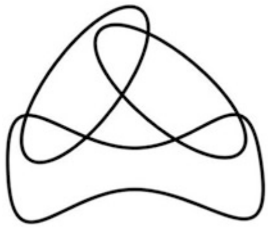
 # 4 (page 20).

(a)
- $A \subseteq A \cup B,\ B \subseteq A \cup B$ implies $f(A) \subseteq f(A \cup B),\ f(B) \subseteq f(A \cup B)$, hence $f(A) \cup f(B) \subseteq f(A \cup B)$. Conversely, suppose that $y \in f(A \cup B)$. Then there exists an $x \in A \cup B$ with $f(x) = y$. If $x \in A$ we have $y \in f(A)$. Otherwise, $x \in B$ and $y \in f(B)$. In either case, $y \in f(A) \cup f(B)$ and so $f(A \cup B) \subseteq f(A) \cup f(B)$.
- $A \cap B \subseteq A,\ A \cap B \subseteq B$ implies $f(A \cap B) \subseteq f(A),\ f(A \cap B) \subseteq f(B)$, hence $f(A \cap B) \subseteq f(A) \cap f(B)$. We prove the reverse inclusion under the assumption that f is injective. Let $y \in f(A) \cap f(B)$. Since f is injective, there is a unique $x \in X$ with $f(x) = y$. Since $y \in f(A)$, we must have $x \in A$. Similarly, $x \in B$. Thus $x \in A \cap B$, and so $y = f(x) \in f(A \cap B)$. For a counterexample if f is *not* injective, let $X = \{0,1\}$ and $Y = \{1\}$. Then we must have $f(x) = 1$, the constant function. Here
$$f(\{0\} \cap \{1\}) = f(\emptyset) = \emptyset, \quad f(\{0\}) \cap f(\{1\}) = \{1\} \cap \{1\} = \{1\}.$$
- If $y \in f(A) \backslash f(B)$, there exists $x \in A$ with $f(x) = y$, but this x cannot be in B since $y \notin f(B)$. This shows
$$f(A) \setminus f(B) \subseteq f(A \setminus B).$$
If f is injective, we prove the reverse inclusion: Let $y \in f(A \setminus B)$, and let $x \in A \setminus B$ be its unique preimage. Since $x \in A$, we have that $y \in f(A)$. But we cannot have $y \in f(B)$, since the unique preimage x is not in B. Thus, $y \in f(A) \backslash f(B)$. If f is *not* injective, we can use the same counterexample $f:\ \{0,1\} \to \{1\}$ as above:
$$f(\{0\}) \setminus f(\{1\}) = \emptyset, \quad f(\{0\} \setminus \{1\}) = f(\{0\}) = \{1\}.$$
- Suppose f is bijective. Then
$$f(A^c) = f(X \setminus A) = f(X) \setminus f(A) = Y \setminus f(A) = \big(f(A)\big)^c.$$
If f is not injective the second equality may fail: An example with $f:\ \{0,1\} \to \{1\}$ as above is $A = \{0\}$, since $f(\{0\}^c) = f(\{1\}) = \{1\}$ while $f(\{0\})^c = \emptyset$. If f is not surjective, then the third equality may fail. For example, let $f:\ \{0,1\} \to \{0,1\}$ be the constant function $f(x) = 0$; and let $A = \{0\}$. Then
$$f(\{0,1\}) \setminus f(\{0\}) = \emptyset, \quad \{0,1\} \setminus f(\{0\}) = \{1\}.$$

(b) We shall only prove the third equality; leaving the other three to the reader (note that the fourth can be proven from the second and third; and the second can be proven from the first and third):
$$x \in f^{-1}(C^c) \Leftrightarrow f(x) \notin C \Leftrightarrow x \notin f^{-1}(C) \Leftrightarrow x \in [f^{-1}(C)]^c.$$

 # 5 (page 21).

While chart compatibility is reflexive and symmetric, it is not transitive and so not an equivalence relation. With chart domains intersecting as in the picture below, it is evident that compatibility in any two regions does not imply compatibility in the third.



 # 6 (page 23).

To find φ_-, note that the affine line passing through $p=(x,y,z)\in S^2\backslash\{n\}$ and $n=(0,0,1)$ is obtained by linear interpolation:

$$t\big(x,y,z\big)+(1-t)(0,0,1),\quad t\in\mathbb{R}. \tag{$\star$}$$

Indeed, for $t=0$ this is the point $n=(0,0,1)$, while for $t=1$ we obtain $p=(x,y,z)$. To find the point of intersection with the xy-place $\mathbb{R}^2$, we need to take the value of t for which the last component is zero. This gives the equation $tz+(1-t)=0$, hence $t=1/(1-z)$. With this value of t, ($\star$) becomes

$$\frac{1}{1-z}(x,y,0),$$

and $\varphi_-(x,y,z)$ is obtained by dropping the 0. The argument for φ_+ is similar.

 # 7 (page 23).

We shall explain the calculation for φ_+; the argument for φ_- is similar. (It is also clear from the geometry that $\varphi_+^{-1},\varphi_-^{-1}$ only differ by the sign of the z-coordinate.) We want to solve $u=x/(1+z),\ v=y/(1+z)$ for (x,y,z) with $x^2+y^2+z^2=1$ and $z+1\neq 0$. We have that

$$u^2+v^2=\frac{x^2+y^2}{(1+z)^2}=\frac{1-z^2}{(1+z)^2}=\frac{(1-z)(1+z)}{(1+z)^2}=\frac{1-z}{1+z},$$

from which one obtains

$$z=\frac{1-(u^2+v^2)}{1+(u^2+v^2)}.$$

Since $x=u(1+z),\ y=v(1+z)$ this results in

$$(x,y,z)=\Big(\frac{2u}{1+(u^2+v^2)},\frac{2v}{1+(u^2+v^2)},\frac{1-(u^2+v^2)}{1+(u^2+v^2)}\Big),$$

as claimed.

 # 8 (page 23).
We shall describe the transition map $\psi\circ\varphi^{-1}:\ \varphi(U\cap V)\to\psi(U\cap V)$, and leave the other map to the reader. We need to express (n,c) in terms of (m,b). Solving $y=mx+b$ for x we obtain

$$x=\frac{1}{m}y-\frac{b}{m}.$$

Thus, $n=\frac{1}{m}$ and $c=-\frac{b}{m}$, which shows that the transition map is

$$(\psi\circ\varphi^{-1})(m,b)=(\frac{1}{m},\ -\frac{b}{m}).$$

Note that this is smooth on $U\cap V$.

 # 9 (page 24).
The surface is an open (i.e., "infinite") Möbius strip. Here are two ways of seeing this.

(1) Note that every affine line is given by an equation $ax+by+c=0$ with $(a,b,c)\in\mathbb{R}^3$. Conversely, (a,b,c) determines an affine line if and only if a,b are not both zero, and two such triples determine the same affine line if and only if they are related by multiplication by a non-zero constant λ. Hence, the set of affine lines is identified with a subset of the projective plane $\mathbb{R}\mathrm{P}^2$, namely $\mathbb{R}\mathrm{P}^2\setminus\{(0:0:1)\}$. This is exactly the "punctured projective plane" from ✐ 2, which is an open Möbius strip.

This proof may be visualized, as follows. First of all, let us identify the space of affine lines in the xy-plane with the space of affine lines in the *affine plane* in $\mathbb{R}^3$, given by $A=\{(x,y,z)|\ z=1\}$. Note that every two-dimensional subspace of $S\subseteq\mathbb{R}^3$, except for the xy-plane, determines an affine line $S\cap A\subseteq A$; conversely, every affine line in A arises in this way. We have thus identified the set of affine lines in the plane with the set of two-dimensional subspaces of $\mathbb{R}^3$, minus the xy-plane. But the map taking such a subspace to the line orthogonal to it identifies the set of two-dimensional subspaces with the set of one-dimensional subspaces, i.e., with $\mathbb{R}\mathrm{P}^2$. We hence see again that the set of affine lines is an $\mathbb{R}\mathrm{P}^2$ minus one point.

(2) Alternatively, consider the map $q:\ S^1\times\mathbb{R}\to M$, taking a pair $(\mathbf{u},t)$ (with $\mathbf{u}$ a unit vector in $\mathbb{R}^2$) to the affine line passing through the point $t\mathbf{u}$ and perpendicular to $\mathbf{u}$. The map q is clearly surjective but not injective, since $(\mathbf{u},t)$ and $(-\mathbf{u},-t)$ describe the same affine line. This, however, is the only ambiguity. We hence see that M is obtained from the infinite cylinder $S^1\times\mathbb{R}$ by "antipodal identification," with q the quotient map. The result is the infinite Möbius strip.

 # 10 (page 24).
We have already found that

$$\varphi_+(x,y,z)=\left(\frac{x}{1+z},\frac{y}{1+z}\right),$$

$$\varphi_+^{-1}(u,v)=\left(\frac{2u}{1+(u^2+v^2)},\frac{2v}{1+(u^2+v^2)},\frac{1-(u^2+v^2)}{1+(u^2+v^2)}\right).$$

One can also calculate that

$$\psi^{-1}(x,z) = \left(x, -\sqrt{1-(x^2+z^2)}, z\right).$$

Thus,

$$\psi \circ \varphi_+^{-1}(u,v) = \left(\frac{2u}{1+(u^2+v^2)}, \frac{1-(u^2+v^2)}{1+(u^2+v^2)}\right)$$

$$\varphi_+ \circ \psi^{-1}(x,z) = \left(\frac{x}{1+z}, \frac{-\sqrt{1-(x^2+z^2)}}{1+z}\right).$$

Both maps are smooth on the relevant domains, proving that (V,ψ) is compatible with (U_+,φ_+).

 # 11 (page 25).
The step

$$\varphi_\alpha(U \cap U_\alpha) \cap \varphi_\alpha(V \cap U_\alpha) = \varphi_\alpha(U \cap V \cap U_\alpha)$$

is justified since the map φ_α is injective. (See ✐ 4.)

 # 12 (page 25).
We are given that $\psi(V) = \varphi(V)$ is open. Since $\varphi : U \to \widetilde{U}$ is a bijection, it restricts to a bijection $\varphi|_V : V \to \varphi|_V(V)$. Thus, (ψ, V) is a chart.

To see that (ψ,V) is compatible with (φ,U), note that

$$\varphi \circ \psi^{-1} : \widetilde{V} \to \widetilde{V}$$

is simply the identity function and so a diffeomorphism.

Finally, suppose that (U,φ) is a chart from an atlas $\mathscr{A}$. Let (W,ρ) be an arbitrary chart from the same atlas. Then, (U,φ) and (W,ρ) are compatible. That is,

$$\rho \circ \varphi^{-1} : \varphi(U \cap W) \to \rho(U \cap W)$$

is a diffeomorphism. Then, its restriction

$$(\rho \circ \varphi^{-1})|_{\varphi(V\cap W)} : \varphi(V \cap W) \to \rho(V \cap W)$$

is also a diffeomorphism. Restricting this map to the open subset $\psi(V \cap W) \subseteq \varphi(U \cap W)$ we obtain another diffeomorphism

$$\rho \circ \psi^{-1} : \psi(V \cap W) \to \rho(V \cap W).$$

This proves that (ψ,V) is compatible with (W,ρ). Since (W,ρ) was an arbitrary chart from the atlas $\mathscr{A}$, we conclude that (ψ,V) is compatible with $\mathscr{A}$, and will be a chart in the maximal atlas generated by it.

 # 13 (page 30).
We shall describe a bijection of sets. In truth, the two are identified as manifolds, and not only as sets; but we do not yet have the tools to demonstrate this.

Consider the (unit) sphere S^n in $\mathbb{R}^{n+1}$. Any line ℓ through the origin in $\mathbb{R}^{n+1}$ intersects the sphere in exactly two antipodal points. Since $\mathbb{R}\mathrm{P}^n$ is the collection of such lines ℓ, we get an equivalent identification as $S^n/\sim$ where $p \sim p'$ if and only if p and p' are antipodal.

Every equivalence class $[p]$ has at least one representative p in the closed upper hemisphere S^n_+, and indeed a unique one unless p lies on the boundary (the equator). Hence we also have $S^n_+/\sim$, where $\sim$ is antipodal identification on the equator. But projection to $\mathbb{R}^n$ (dropping the last coordinate) identifies S^n_+ with the closed unit ball B^n, taking the boundary of S^n_+ to the boundary of B^n (which is the sphere S^{n-1}). So, we get an identification of $\mathbb{R}\mathrm{P}^n$ as $B^n/\sim$, using antipodal identification on its boundary.

An identification $S^1 \cong \mathbb{R}\mathrm{P}^1$ is obtained by mapping the point $\big(\cos(\theta),\sin(\theta)\big) \in S^1$ to the line in $\mathbb{R}^2$ passing through the point $(x,y) = \big(\cos(\theta/2),\sin(\theta/2)\big)$. This is well-defined, since changing θ by $2\pi k$, $k \in \mathbb{Z}$ changes (x,y) by a sign $(-1)^k$, which does not affect the line passing through this point.

 # 14 (page 31).
There are two cases:

Case 1: $0 \le i < j \le n$. We have*

$$\varphi_i \circ \varphi_j^{-1}(u^1,\dots,u^n) = (\frac{u^1}{u^{i+1}},\dots,\frac{u^i}{u^{i+1}},\frac{u^{i+2}}{u^{i+1}},\dots,\frac{u^j}{u^{i+1}},\frac{1}{u^{i+1}},\frac{u^{j+1}}{u^{i+1}},\dots,\frac{u^n}{u^{i+1}}),$$

defined on $\varphi_j(U_i \cap U_j) = \{u \in \mathbb{R}^n \mid u^{i+1} \neq 0\}$.

Case 2: If $0 \le j < i \le n$. We have

$$\varphi_i \circ \varphi_j^{-1}(u^1,\dots,u^n) = (\frac{u^1}{u^i},\dots,\frac{u^j}{u^i},\frac{1}{u^i},\frac{u^{j+1}}{u^i},\dots,\frac{u^{i-1}}{u^i},\frac{u^{i+1}}{u^i},\dots,\frac{u^n}{u^i}),$$

defined on $\varphi_j(U_i \cap U_j) = \{u \in \mathbb{R}^n \mid u^i \neq 0\}$.

In both cases, we see that $\varphi_i \circ \varphi_j^{-1}$ is smooth.

 # 15 (page 32).
Each vector $\mathbf{v} \in \mathbb{R}^3$ determines a rotation, by an angle $||\mathbf{v}||$ about the oriented axis determined by $\mathbf{v}$. Letting $\mathbf{v} = \mathbf{0}$ correspond to the trivial rotation, this gives a surjective map from $\mathbb{R}^3$ onto the set of rotations. Since adding a multiple of 2π to $||\mathbf{v}||$ (while keeping the direction) does not change the rotation, we see that every equivalence class has a representative $\mathbf{v}$ in the solid ball of radius π, denoted $B_\pi(\mathbf{0}) \subseteq \mathbb{R}^4$. The representative is unique unless $\mathbf{v}$ lies on the boundary (rotations by π), since $\mathbf{v}$ and $-\mathbf{v}$ determine the same rotation in that case. We see therefore that the solid ball B^3 with antipodal identification of its boundary S^2 describes the rotation about the origin in $\mathbb{R}^3$. This is the same as our description of $\mathbb{R}\mathrm{P}^3$ from ✎ 13 above.

* If $i = n-1$, then $u^{i+2}/u^{i+1} = 1/u^{i+1}$ is the last entry.

 # 16 (page 34).
We have $\mathbb{R}^n = \mathbb{R}^I \oplus \mathbb{R}^{I'}$. Let $\pi_I : \mathbb{R}^n \to \mathbb{R}^I, \pi_{I'} : \mathbb{R}^n \to \mathbb{R}^{I'}$ denote the (orthogonal) projections onto the two summands. For $E \in U_I$, the restriction $\pi_I|_E : E \to \mathbb{R}^I$ is injective: If $\mathbf{x} \in E$ satisfies $\pi_I(\mathbf{x}) = \mathbf{0}$, then $\mathbf{x} \in E \cap \ker(\pi_I) = E \cap \mathbb{R}^{I'} = \{\mathbf{0}\}$. Hence, for dimension reasons this map $\pi_I|_E$ is a linear isomorphism. We claim that E is the graph of

$$A_I = \pi_{I'} \circ (\pi_I|_E)^{-1} : \mathbb{R}^I \to \mathbb{R}^{I'}.$$

Indeed, every $\mathbf{x} \in E \subseteq \mathbb{R}^n$ has the decomposition $\mathbf{x} = \pi_I(\mathbf{x}) + \pi_{I'}(\mathbf{x})$, hence $\mathbf{x} = \mathbf{y} + A_I(\mathbf{y})$ where $\mathbf{y} = \pi_I(\mathbf{x})$. This shows $E \subseteq \operatorname{graph}(A_I)$; equality holds since both spaces have the same dimension. As for uniqueness, suppose $B_I : \mathbb{R}^I \to \mathbb{R}^{I'}$ is another linear map having E as its graph. For all $\mathbf{x} \in E$, we then have

$$\mathbf{x} = \pi_I(\mathbf{x}) + A_I(\pi_I(\mathbf{x})) = \pi_I(\mathbf{x}) + B_I(\pi_I(\mathbf{x})).$$

Hence $A_I(\pi_I(\mathbf{x})) = B_I(\pi_I(\mathbf{x}))$ for all $\mathbf{x} \in E$. Since $\pi_I|_E$ is an isomorphism, it follows that $A_I = B_I$, completing the proof.

Note: More generally, given finite-dimensional vector spaces V, W, there is a one-to-one correspondence between linear maps $\varphi : V \to W$ on the one hand, and $n = \dim(V)$-dimensional subspaces $E \subseteq W \oplus V$ with the property $E \cap (W \oplus \{\mathbf{0}\}) = \{\mathbf{0}\}$; the correspondence takes a linear map to its graph.

 # 17 (page 36).
Recall that for any two subspaces $U, V \subseteq \mathbb{R}^n$ we have

$$(U \cap V)^\perp = U^\perp + V^\perp.$$

In particular, $E_1 \cap F^\perp = E_2 \cap F^\perp = \{\mathbf{0}\}$ if and only if $E_1^\perp + F = E_2^\perp + F = \mathbb{R}^n$. For convenience, denote $G_1 = E_1^\perp$ and $G_2 = E_2^\perp$.

The set-theoretic union of two proper subspaces of $\mathbb{R}^n$ cannot be all of $\mathbb{R}^n$; hence there exists some $\mathbf{v}_1 \in \mathbb{R}^n \setminus (G_1 \cup G_2)$. If $k = 1$, then we may take $F = \operatorname{span}(\mathbf{v}_1)$ and we are done.

If $k > 1$, note that the set-theoretic union of the subspaces $G_1 + \operatorname{span}(\mathbf{v}_1)$ and $G_2 + \operatorname{span}(\mathbf{v}_1)$ cannot be all of $\mathbb{R}^n$, hence there exists $\mathbf{v}_2 \in \mathbb{R}^n$ in the complement of both subspaces. Proceed recursively to produce $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ and set $F = \operatorname{span}(\mathbf{v}_1, \ldots, \mathbf{v}_k)$.

Note: A similar statement, with a similar proof, holds for any finite (or even just countable) collection $\{E_i\}$ of proper subspaces of a given dimension.

 # 18 (page 36).
Suppose P is the matrix of an orthogonal projection onto $E \subseteq \mathbb{R}^n$. The matrix $I - P$ has the property that for $\mathbf{v} \in E$, $(I - P)\mathbf{v} = \mathbf{v} - \mathbf{v} = \mathbf{0}$, while for $\mathbf{v} \in E^\perp$, $(I - P)\mathbf{v} = \mathbf{v} - \mathbf{0} = \mathbf{v}$. Hence, $I - P$ is the matrix of orthogonal projection to $E^\perp$.

Since P acts trivially on the range $\operatorname{ran} P = E$, we have that

$$P(P\mathbf{v}) = P\mathbf{v}$$

for all $\mathbf{v} \in \mathbb{R}^n$, thus $PP = P$. On the other hand, letting $\langle \cdot, \cdot \rangle$ denote the usual inner product of $\mathbb{R}^n$, we see that for all $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$,

$$\begin{aligned}\langle \mathbf{v}, P\mathbf{w}\rangle &= \langle P\mathbf{v} + (I-P)\mathbf{v}, P\mathbf{w}\rangle = \langle P\mathbf{v}, P\mathbf{w}\rangle = \langle P\mathbf{v}, P\mathbf{w} + (I-P)\mathbf{w}\rangle = \langle P\mathbf{v}, \mathbf{w}\rangle \\ &= \langle \mathbf{v}, P^{\top}\mathbf{w}\rangle;\end{aligned}$$

here we used that the inner product of elements in $\operatorname{ran}(I-P) = E^{\perp}$ and $\operatorname{ran} P = E$ is 0. Hence, $P^{\top} = P$.

In the opposite direction, given $P \in \operatorname{Mat}_{\mathbb{R}}(n)$ with $P^{\top} = P$ and $PP = P$, let $E = \operatorname{ran} P$. For $\mathbf{v} \in E$ we may write $\mathbf{v} = P\mathbf{v}'$, and hence we see (using $PP = P$) that

$$P\mathbf{v} = P(P\mathbf{v}') = P\mathbf{v}' = \mathbf{v}.$$

On the other hand, for $\mathbf{v} \in E^{\perp}$ we have that for all $\mathbf{w} \in \mathbb{R}^n$ (using $P^{\top} = P$)

$$\langle P\mathbf{v}, \mathbf{w}\rangle = \langle \mathbf{v}, P^{\top}\mathbf{w}\rangle = \langle \mathbf{v}, P\mathbf{w}\rangle = 0,$$

hence $P\mathbf{v} = 0$. This shows that P is the matrix of orthogonal projection to E.

 # 19 (page 37).
The bijection takes a k-dimensional subspace E to its orthogonal complement $E^{\perp}$. In terms of orthogonal projections, it takes the rank k orthogonal projection P to the rank $n-k$ orthogonal projection $I-P$.

 # 20 (page 37).
The bijection takes a k-dimensional subspace $E \subseteq V$ to its annihilator $\operatorname{ann}(E) \subseteq V^*$, i.e., the space of all linear functionals that restrict to 0 on E.

 # 21 (page 38).
First note that ✐ 12 implies that $\mathscr{A}_U$ is indeed an atlas for U. To show maximality, let (V, ψ) be a chart for U compatible with the atlas $\mathscr{A}_U$; in particular $V \subseteq U$. Any $\mathscr{A}$-chart $(W, \varphi) \in \mathscr{A}$ with non-empty intersection $W \cap V \neq \emptyset$ is compatible with (V, ψ), since $W \cap V \subseteq W \cap U$ and the restriction of (W, φ) to U is a chart of $\mathscr{A}_U$. Thus, (V, ψ) is compatible with $\mathscr{A}$, and by the maximality of $\mathscr{A}$ we must have $(V, \psi) \in \mathscr{A}$. Therefore, $(V, \psi) \in \mathscr{A}_U$, proving the maximality of $\mathscr{A}_U$. It remains to check countability and Hausdorffness, but reviewing Definition 2.16 we see that $\mathscr{A}_U$ simply inherits these properties from $\mathscr{A}$ when we restrict the charts to U.

 # 22 (page 39).
The "if" direction is obvious. For the "only if" direction, consider a non-empty open subset $U \subseteq M$. Let $(U_\alpha, \varphi_\alpha)$ be a collection of coordinate charts covering M. By ✐ 12 the sets $(U_\alpha \cap U, \varphi_\alpha|_{U_\alpha \cap U})$ are also coordinate charts, and clearly we have $U = \bigcup_\alpha (U_\alpha \cap U)$. Note that in invoking ✐ 12 we have used the fact that U is open.

 # 23 (page 39).
Note that by definition $\varphi^{-1}(V) \subseteq U$. Therefore, $\{(U, \varphi)\}$ is a collection of charts covering $\varphi^{-1}(V)$, and we may use Proposition 2.26. We are now done, since

$$\varphi(U \cap \varphi^{-1}(V)) = \varphi(U) \cap V.$$

 # 24 (page 40).
This follows from the corresponding property for open sets in $\mathbb{R}^m$.

In one direction, suppose A is open, and let $a \in A$. Choose a chart (U, φ) about a. Since $\varphi(U \cap A) \subseteq \mathbb{R}^m$ is open, we may find some open ball B with $\varphi(a) \in B \subseteq \varphi(U \cap A)$. It follows that $\varphi^{-1}(B)$ is an open neighborhood of a with $\varphi^{-1}(B) \subseteq A$.

Conversely, suppose A is not open. That is, there exists some chart (U, φ) such that $\varphi(U \cap A) \subseteq \mathbb{R}^m$ is not open. Thus, there is some $\mathbf{y} \in \varphi(U \cap A)$ such that no open ball about $\mathbf{y}$ is entirely contained in $\varphi(U \cap A)$. We take $a = \varphi^{-1}(\mathbf{y}) \in A$.

 # 25 (page 40).
Suppose M is connected, and let $f: M \to \mathbb{R}$ be a locally constant function. For any $a \in \mathbb{R}$, the level set $f^{-1}(a)$ is an open subset: If $p \in f^{-1}(a)$, then there exists an open neighborhood of p on which f is constant. Since level sets for distinct values of f are disjoint, and M is connected, it follows that there can be only one level set, hence f is constant.

Conversely, if M is a disjoint union of two open sets, then we have the locally constant, non-constant function taking the value 0 on one of the sets and 1 on the other.

 # 26 (page 40).

(a) This follows since the intersection of two closed subsets is again closed, and the intersection of a bounded subset with any subset is bounded.
(b) The union of finitely many closed and bounded subsets is closed and bounded.

 # 27 (page 41).

(a) Write $K = K_1 \cup \cdots \cup K_n$, where each K_i is contained in the domain of some coordinate chart (U_i, φ_i), and $\varphi_i(K_i) \subseteq \mathbb{R}^m$ is closed and bounded. Then,

$$K \cap C = (K_1 \cap C) \cup \cdots \cup (K_n \cap C)$$

and each $K_i \cap C$ is contained in the domain of (U_i, φ_i). The subset

$$\varphi_i(K_i \cap C) \subseteq \mathbb{R}^m$$

is bounded, since it is contained in the bounded subset $\varphi_i(K_i)$. It remains to show that it is also closed. We have $\varphi_i(K_i \cap C) = \varphi_i(K_i) \cap \varphi_i(U \cap C)$, but this does not suffice since $\varphi_i(U \cap C)$ need not be closed in $\mathbb{R}^m$. Instead, we write

$$\varphi_i(K_i \cap C) = \varphi_i(K_i) \cap \Big(\varphi_i(U_i \cap C) \cup \big(\mathbb{R}^m \backslash \varphi_i(U_i)\big)\Big).$$

The set $\varphi_i(U_i \cap C) \cup (\mathbb{R}^m \backslash \varphi_i(U_i))$ is closed in $\mathbb{R}^m$, because its complement

$$\varphi_i(U_i) \setminus \varphi_i(U_i \cap C) = \varphi_i(U_i \cap (M \setminus C))$$

is open.

(b) Write each K_i as a finite union $K_i = K_{i1} \cup \cdots \cup K_{i\ell_i}$, where each K_{ij} is contained in the domain of some coordinate chart (U_{ij}, φ_{ij}) and $\varphi_{ij}(K_{ij})$ is closed and bounded. Then,

$$K = \bigcup_{i=1}^{n} \bigcup_{j=1}^{\ell_i} K_{ij}$$

shows that K is compact.

 # 28 (page 42).
Let $\{U_\alpha\}$ be an open cover of $K = K_1 \cup \cdots \cup K_n$. In particular, it is an open cover of K_i for each $1 \le i \le n$. Since each K_i is compact, there are finitely many indices $\alpha_{i1}, \ldots, \alpha_{i\ell_i}$ such that $K_i \subseteq U_{\alpha_{i1}} \cup \cdots \cup U_{\alpha_{i\ell_i}}$. Then

$$K \subseteq \bigcup_{i=1}^{n} \bigcup_{j=1}^{\ell_i} U_{\alpha_{ij}}$$

is a finite subcover for K.

 # 29 (page 42).
Let $\{U_\alpha\}$ be an open cover of $K \cap C$. Together with the open subset $M \backslash C$, we obtain a cover of K. Since K is compact, there are finitely many indices $\alpha_1, \ldots, \alpha_n$ with

$$K \subseteq (M \backslash C) \cup U_{\alpha_1} \cup \cdots \cup U_{\alpha_n}.$$

Hence $K \cap C \subseteq U_{\alpha_1} \cup \cdots \cup U_{\alpha_n}$.

 # 30 (page 43).
In one direction, suppose $K \subseteq U$ is t-compact. We want to show that $\varphi(K) \subseteq \mathbb{R}^m$ is closed and bounded. By Heine-Borel for $\mathbb{R}^m$, this is equivalent to showing that every open cover $\{\widetilde{U}_\alpha\}$ of $\varphi(K)$ has a finite subcover. Given such an open cover, the preimages $U_\alpha = \varphi^{-1}(\widetilde{U}_\alpha)$ are open (see ✐23), and so define an open cover of K. Letting $\{U_{\alpha_1}, \ldots, U_{\alpha_n}\}$ be a finite subcover of K, it follows that $\{\widetilde{U}_{\alpha_1}, \ldots, \widetilde{U}_{\alpha_n}\}$ is a finite subcover of $\varphi(K)$.

Conversely, suppose $\varphi(K)$ is closed and bounded. To show that K is t-compact, let $\{U_\alpha\}$ be an open cover of K. Then $\{\varphi(U \cap U_\alpha)\}$ is an open cover of $\varphi(K)$. By the Heine-Borel theorem for $\mathbb{R}^m$, there exists a finite subcover; the corresponding U_α's are then an open cover of K.

 # 31 (page 45).
The i-th entry of $A\mathbf{u}$ is

$$(A\mathbf{u})_i = \sum_{j=1}^{n} \left(\frac{1}{||\mathbf{u}||^2} \delta_{ij} - \frac{2u_i u_j}{||\mathbf{u}||^4} \right) u_j = \frac{u_i}{||\mathbf{u}||^2} - \frac{2u_i ||\mathbf{u}||^2}{||\mathbf{u}||^4} = -\frac{u_i}{||\mathbf{u}||^2}.$$

Thus $\mathbf{u}$ is an eigenvector with eigenvalue $-||\mathbf{u}||^{-2}$. On the other hand, if $\mathbf{v} \in \mathbb{R}^n$ is a non-zero vector orthogonal to $\mathbf{u}$, with components v_i, we find that the i-th entry of $A\mathbf{v}$ is

$$(A\mathbf{v})_i = \sum_{j=1}^{n} \left(\frac{1}{||\mathbf{u}||^2} \delta_{ij} - \frac{2u_i u_j}{||\mathbf{u}||^4} \right) v_j = \frac{v_i}{||\mathbf{u}||^2}.$$

Thus, $\mathbf{v}$ is an eigenvector with eigenvalue $||\mathbf{u}||^{-2}$. Since $(\mathrm{span}\{\mathbf{u}\})^\perp$ is $(n-1)$-dimensional, this eigenvalue has multiplicity $n-1$, while the eigenvalue $-||\mathbf{u}||^2$ has multiplicity 1. Since the determinant is the product of eigenvalues (with multiplicities), we obtain

$$\det(A) = (-||\mathbf{u}||^2) \cdot (||\mathbf{u}||^2)^{n-1} = -||\mathbf{u}||^{-2n}.$$

 # 32 (page 45).
We will give two proofs.

(1) Let $A = (a+ib) \in \mathrm{Mat}_{\mathbb{C}}(1)$ be a complex 1×1 matrix; its (complex) determinant is simply $\det_{\mathbb{C}}(A) = a + ib$. To find the real matrix $A_{\mathbb{R}}$ corresponding to A, consider its action standard basis $\{1, i\}$ of $\mathbb{C}$, viewed as a vector space over $\mathbb{R}$:

$$1 \mapsto (a+ib)1 = a+ib, \quad i \mapsto (a+ib)i = -b+ia.$$

The coefficients are the columns of $A_{\mathbb{R}} \in \mathrm{Mat}_{\mathbb{R}}(2)$,

$$A_{\mathbb{R}} := \begin{pmatrix} a & -b \\ b & a \end{pmatrix}.$$

Its real determinant is then

$$\det_{\mathbb{R}}(A_{\mathbb{R}}) = a^2 + b^2 = |a+ib|^2 = |\det_{\mathbb{C}}(A)|^2.$$

For the general case, recall Schur's decomposition theorem: For any $A \in \mathrm{Mat}_{\mathbb{C}}(n)$ there exists an invertible matrix $S \in \mathrm{Mat}_{\mathbb{C}}(n)$ such that $A' = SAS^{-1}$ is upper triangular. Clearly $A'_{\mathbb{R}} = S_{\mathbb{R}} A_{\mathbb{R}} S_{\mathbb{R}}^{-1}$. Since $\det_{\mathbb{C}}(A') = \det_{\mathbb{C}}(A)$ and $\det_{\mathbb{R}}(A'_{\mathbb{R}}) = \det_{\mathbb{R}}(S_{\mathbb{R}} A_{\mathbb{R}} S_{\mathbb{R}}^{-1}) = \det_{\mathbb{R}}(A_{\mathbb{R}})$, it is enough to prove the claim for upper triangular matrices. But if A is upper triangular then $A_{\mathbb{R}}$ is block upper triangular:

$$A = \begin{pmatrix} A_{11} & A_{12} & A_{13} & \cdots & A_{1n} \\ 0 & A_{22} & A_{23} & \cdots & A_{2n} \\ 0 & 0 & A_{33} & \cdots & A_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & A_{nn} \end{pmatrix}, \quad A_{\mathbb{R}} = \begin{pmatrix} (A_{11})_{\mathbb{R}} & (A_{12})_{\mathbb{R}} & (A_{13})_{\mathbb{R}} & \cdots & (A_{1n})_{\mathbb{R}} \\ 0 & (A_{22})_{\mathbb{R}} & (A_{23})_{\mathbb{R}} & \cdots & (A_{2n})_{\mathbb{R}} \\ 0 & 0 & (A_{33})_{\mathbb{R}} & \cdots & (A_{3n})_{\mathbb{R}} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & (A_{nn})_{\mathbb{R}} \end{pmatrix}$$

Here each block is a 2×2 matrix of the form considered above. The determinant of a block upper triangular matrix is the product of the determinants of the blocks; hence

$$\det_{\mathbb{R}}(A_{\mathbb{R}}) = \prod_{j=1}^{n} \det_{\mathbb{R}}((A_{jj})_{\mathbb{R}}) = \prod_{j=1}^{n} |\det_{\mathbb{C}}(A_{jj})|^2 = |\det_{\mathbb{C}}(A)|^2.$$

(2) An alternative argument proceeds by showing that if $A \in \mathrm{Mat}_{\mathbb{C}}(n)$ has eigenvalues $\lambda_1,\ldots,\lambda_n$, then the eigenvalues of $A_{\mathbb{R}} \in \mathrm{Mat}_{\mathbb{R}}(2n)$ are $\lambda_1,\ldots,\lambda_n,\overline{\lambda}_1,\ldots,\overline{\lambda}_n$. This implies the result, since the determinant is the product of eigenvalues.

We will show how to construct eigenvectors of $A_{\mathbb{R}}$ from those of A. (If A has repeated eigenvalues we should also consider generalized eigenvectors; this can be done by a small generalization of the argument given below. Alternatively, use that any matrix A is a limit of matrices with distinct eigenvalues.) Let $J = (iI_n)_{\mathbb{R}} \in \mathrm{Mat}_{\mathbb{R}}(2n)$ be the matrix of the linear map $\mathbb{C}^n \to \mathbb{C}^n$ given as multiplication by i. Since A commutes with iI_n we have that $JA_{\mathbb{R}} = A_{\mathbb{R}}J$. Note also $J^2 = -I_{2n}$. Suppose $\mathbf{z} = \mathbf{x} + i\mathbf{y}$ is an eigenvector with eigenvalue $\lambda = r + is$. In terms of the decomposition of $\mathbf{z},\lambda$ into real and imaginary parts, the eigenvector equation $A\mathbf{z} = \lambda\mathbf{z}$ reads as

$$A(\mathbf{x}+i\mathbf{y}) = s(\mathbf{x}+i\mathbf{y}) + si(\mathbf{x}+i\mathbf{y}).$$

Under the identification $\mathbb{C}^n \cong \mathbb{R}^{2n}$, the complex vector $\mathbf{x}+i\mathbf{y}$ corresponds to $\mathbf{x}+J\mathbf{y}$ (with $\mathbb{R}^n$ regarded as the "real subspace" of $\mathbb{C}^n \cong \mathbb{R}^{2n}$). Hence, the eigenvector equation $A\mathbf{z} = \lambda\mathbf{z}$ is equivalent to the "real" version

$$A_{\mathbb{R}}(\mathbf{x}+J\mathbf{y}) = r(\mathbf{x}+J\mathbf{y}) + sJ(\mathbf{x}+J\mathbf{y}).$$

Using this equation, one verifies that the vectors in $\mathbb{C}^{2n}$,

$$(\mathbf{x}+J\mathbf{y}) - iJ(\mathbf{x}+J\mathbf{y}), \quad (\mathbf{x}+J\mathbf{y}) + iJ(\mathbf{x}+J\mathbf{y}),$$

are eigenvectors of $A_{\mathbb{R}}$ with eigenvalues $\lambda,\overline{\lambda}$, respectively. (Note that in the argument above, to find the eigenvalues of the real matrix $A_{\mathbb{R}}$, one considers it as a transformation of $\mathbb{C}^{2n}$. One should be careful not to confuse this element of $\mathrm{Mat}_{\mathbb{C}}(2n)$ with the original matrix $A \in \mathrm{Mat}_{\mathbb{C}}(n)$.)

 # 33 (page 46).
Suppose $\mathscr{A},\mathscr{A}'$ are two maximal oriented atlases. Given $p \in M$, contained in charts $(U_\alpha,\varphi_\alpha)$ from $\mathscr{A}$ and $(U'_\beta,\varphi'_\beta)$ from $\mathscr{A}'$ let $\varepsilon(p) = \pm 1$ be the sign of the Jacobian determinant of the transition map $\varphi'_\beta \circ \varphi_\alpha^{-1}$. An argument similar to that in the proof of Proposition 2.41 shows that this does not depend on the choice of charts, and that $\varepsilon:\ M \to \pm 1$ is locally constant. Hence, if M is connected the function ε is constant. (See ✐ 25.) If $\varepsilon = +1$ then $\mathscr{A},\mathscr{A}'$ are oriented-compatible, hence are equal (by maximality). If $\varepsilon = -1$ then $\mathscr{A}$ is oriented-compatible (and hence is equal) to the atlas obtained from $\mathscr{A}'$ by passing to the opposite orientation.

Let $\mathscr{A}$ be an oriented maximal atlas, and $\mathscr{A}^{\mathrm{op}}$ the atlas for the opposite orientation. Given a connected chart (U,φ), Proposition 2.41 shows that it must belong either to $\mathscr{A}$ or to $\mathscr{A}^{\mathrm{op}}$.

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Chapter 3

 # 34 (page 54).
The function $h(x,y,z) = \sqrt{1-z^2}$ is defined on the subset of $\mathbb{R}^3$ where $|z| \leq 1$, and is smooth on the open subset where $|z| < 1$. Therefore, we only need to study the smoothness of $f = h|_{S^2}$ at the points of S^2 where $|z| = 1$; these are the points $(0,0,\pm 1)$.

Consider the chart (U, φ) given by

$$U = \{(x,y,z)|\, z > 0\} \quad ; \quad \varphi(x,y,z) = (x,y).$$

In this chart,

$$f \circ \varphi^{-1}(x,y) = f(x,y,\sqrt{1-(x^2+y^2)}) = \sqrt{x^2+y^2}$$

is not smooth at the origin $(0,0)$. The origin corresponds to the point $\varphi^{-1}(0,0) = (0,0,1)$ on the sphere. Similarly, in the chart

$$V = \{(x,y,z)|\, z < 0\} \quad ; \quad \psi(x,y,z) = (x,y)$$

the function is not smooth at the origin, which corresponds to the point $\psi^{-1}(0,0) = (0,0,-1)$. We conclude that f is smooth on $S^2 \setminus \{(0,0,\pm 1)\}$, but is not smooth at the poles $(0,0,\pm 1)$.

 # 35 (page 54).
To see that f is well-defined, it suffices to note that the expression $\frac{yz+xz+xy}{x^2+y^2+z^2}$ is unchanged when x,y,z are all multiplied by the same non-zero scalar $\lambda \neq 0$.

To show it is smooth, we use coordinates. For the standard coordinate chart (U_0, φ_0), we have that

$$\varphi_0^{-1}(u,v) = (1 : u : v).$$

Then

$$f \circ \varphi_0^{-1}(u,v) = \frac{uv+u+v}{1+u^2+v^2}$$

which is a smooth function. The proof for the other charts is analogous.

 # 36 (page 54).
Suppose that $f,g \in C^\infty(M)$, and let (U,φ) be some chart for M. Then, $\varphi^{-1} \circ (fg) = (\varphi^{-1} \circ f)\,(\varphi^{-1} \circ g)$ is smooth as the product of two smooth functions.

 # 37 (page 55).
First, recall that for functions between sets, $f : A \to B$ and $g : B \to C$, and any subset $J \subseteq C$, we have the following equality:

$$(g \circ f)^{-1}(J) = f^{-1}(g^{-1}(J)).$$

Here f^{-1}, g^{-1} refer to the operation of taking preimages and the equality holds regardless of whether the functions are bijective.

Now, in one direction: Suppose that $f : M \to \mathbb{R}^n$ is continuous, and let (U, φ) be a chart for M. We want to show that $f|_U \circ \varphi^{-1}$ is continuous. Let $J \subseteq \mathbb{R}^n$ be an open set. Then, since φ is bijective:

$$(f \circ \varphi^{-1})^{-1}(J) = \varphi(f^{-1}(J) \cap U)$$

which is open by the continuity of f.

The other direction is the same as in the proof of Proposition 3.3.

 # 38 (page 55).
The function $f: \mathbb{R} \to \mathbb{R},\ x \mapsto f(x) = x^2$ has support $\mathrm{supp}(f) = \mathbb{R}$. The function only vanishes at 0, however, every open interval around 0 contains a point where f is non-zero.

 # 39 (page 56).
In each of the examples below, $\mathrm{supp}(g) = U$ is not closed as a subset of M.

(a) $M = \mathbb{R},\ U = \mathbb{R}\backslash\{0\},\ g(x) = \frac{1}{x}$.
(b) $M = \mathbb{R},\ U = \mathbb{R}\backslash\{0\},\ g(x) = x$.

A more interesting example is $g(x) = \exp(-\frac{1}{x^2})$ for $U = \{x\,|\,x > 0\}$.

 # 40 (page 57).
Let $p \neq q \in M$. Then $F(p) \neq F(q) \in \mathbb{R}^N$. Therefore, they must differ along at least one component, say in the i-th component. The i-th component of F is the desired smooth function $f: M \to \mathbb{R}$ such that $f(p) \neq f(q)$.

 # 41 (page 57).
For the complex projective space, we use the map

$$F: \mathbb{C}\mathrm{P}^n \to \mathrm{Mat}_{\mathbb{C}}(n+1) \cong \mathbb{R}^{2(n+1)^2},\ \ (z^0 : \ldots : z^n) \mapsto \frac{\mathbf{z}\mathbf{z}^\dagger}{||\mathbf{z}||^2}$$

(where $\mathbf{z}^\dagger = \overline{\mathbf{z}}^\top$ is the conjugate transpose of the complex column vector $\mathbf{z}$). This map takes a complex one-dimensional subspace $\mathbb{C}\mathbf{z}$, $\mathbf{z} \neq \mathbf{0}$ to the rank 1 orthogonal projection onto that subspace. The map is well-defined, since the right-hand side does not change when $\mathbf{z}$ is replaced with $\lambda\mathbf{z}$ for $\lambda \in \mathbb{C}\backslash\{0\}$. It is injective, since $\mathbb{R}\mathbf{z}$ is recovered as the range of $F(\mathbf{z})$. And it is smooth, as one checks using the standard charts for $\mathbb{C}\mathrm{P}^n$.

 # 42 (page 59).
If $N = \mathbb{R}$, we may simply take $V_\alpha = \mathbb{R},\ \psi_\alpha = \mathrm{id}_{\mathbb{R}}$ in the definition for $C^\infty(M, \mathbb{R})$.

 # 43 (page 59).
Let $\{(U_i, \varphi_i)\}$ be the standard atlas of $\mathbb{R}\mathrm{P}^n$, so that U_i is the set of all $(x^0 : \ldots : x^n)$ with $x^i \neq 0$. For $\mathbb{R}^{n+1} \setminus \{\mathbf{0}\}$, we take the atlas $\{(\widehat{U}_i, \widehat{\varphi}_i)\}$ given by the preimages $\widehat{U}_i = \pi^{-1}(U_i)$ consisting of $(x^0, \ldots, x^n)$ with $x^i \neq 0$, with coordinate maps $\widehat{\varphi}_i$ the obvious inclusions into $\mathbb{R}^{n+1}$. Then the local coordinate expression of π relative to the charts (U_i, φ_i) and $(\widehat{U}_i, \widehat{\varphi}_i)$ is the map

$$\varphi_i \circ \pi \circ \widehat{\varphi}_i^{-1} : \ \widehat{\varphi}_i(\widehat{U}_i) = \{\mathbf{x} \in \mathbb{R}^{n+1} \,|\, x^i \neq 0\} \to \varphi_i(U_i) = \mathbb{R}^n$$

given by

$$(x^0, \ldots, x^n) \mapsto \Big(\frac{x^0}{x^i}, \ldots, \frac{x^{i-1}}{x^i}, \frac{x^{i+1}}{x^i}, \ \ldots, \frac{x^n}{x^i}\Big),$$

which is smooth as required.

The argument for $\pi : \ \mathbb{C}^{n+1} \setminus \{\mathbf{0}\} \to \mathbb{C}\mathrm{P}^n$ is analogous; one obtains the same formula, now defining a smooth (in fact, holomorphic) map from an open subset of $\mathbb{C}^{n+1} = \mathbb{R}^{2(n+1)}$ to $\mathbb{C}^n = \mathbb{R}^{2n}$.

 # 44 (page 59).

(a) In the chart U_1, we have $F(U_1) \subseteq U_1$, and $\varphi_1 \circ F \circ \varphi_1^{-1}(t) = e^{t^2}$, which is smooth. It remains to check smoothness at the point $p = (1:0)$. Since $F(p) = p$, we will verify this using the chart U_0 around p. We have, for $u \neq 0$

$$F(\varphi_0^{-1}(u)) = F(1:u) = F(\frac{1}{u}:1) = (e^{\frac{1}{u^2}}:1) = (1:e^{-\frac{1}{u^2}}).$$

Hence $\varphi_0 \circ F \circ \varphi_0^{-1}$ is the map

$$u \mapsto e^{-\frac{1}{u^2}} \quad \text{for} \quad u \neq 0, \qquad 0 \mapsto 0.$$

As is well-known, this map is smooth even at $u = 0$.

(b) The same calculation applies for the map $F : \ \mathbb{C}\mathrm{P}^1 \to \mathbb{C}\mathrm{P}^1$. However, the conclusion is different: The map

$$z \mapsto e^{-\frac{1}{z^2}} \quad \text{for} \quad u \neq 0, \quad 0 \mapsto 0$$

is *not* smooth (or even continuous) at $z = 0$. (Consider the limit of this function for $z = sa$ as the real number $s \to 0$. If $a = 1$, the limit is 0. If $a = i$, the limit is ∞.)

 # 45 (page 59).

Using Definition 2.33 of compactness: Suppose $\{V_\alpha\}$ is a collection of open sets in N covering $F(C)$; since F is continuous, $\{F^{-1}(V_\alpha)\}$ is a collection of open sets in M covering C. Since C is compact, there exists some finite subcover $F^{-1}(V_1), \ldots, F^{-1}(V_k)$ of C, but then $V_1, \ldots, V_k$ form a finite subcover of $F(C)$. This proves that $F(C)$ is compact.

 # 46 (page 61).

Since F is smooth, it is continuous. Hence the preimage $F^{-1}(V) \subseteq M$ is open. By choosing *any* chart around p, and replacing the chart domain by its intersection with $F^{-1}(V)$, we obtain a chart (U, φ) with $U \subseteq F^{-1}(V)$, i.e., $F(U) \subseteq V$.

 # 47 (page 61).

Since $f > 0$ everywhere on M, we may regard it as a smooth map $f : \ M \to \mathbb{R}_{>0}$. The function $\frac{1}{f}$ is the composition of f with the smooth function with $\mathbb{R}_{>0} \to \mathbb{R}_{>0}, \ x \mapsto \frac{1}{x}$.

 # 48 (page 62).

(a) One just needs to check that $(\mathbb{R}, t \mapsto t^3)$ is a chart, since the entire atlas is composed of a single chart, there are no compatibility conditions to check.
(b) Recall that two atlases belong to the same equivalence class if and only if their union is again an atlas. We already know that $\mathscr{A} \cup \mathscr{A}'$ is not an atlas, since the transition function $\mathrm{id} \circ \sqrt[3]{t}$ is not smooth.
(c) Let us denote the chart of $\mathscr{A}$ by (U, φ) and that of $\mathscr{A}'$ by (V, ψ). We need to check that $\psi \circ f \circ \varphi^{-1}$ is smooth. However,

$$(\psi \circ f \circ \varphi^{-1})(x) = x$$

is the identity map which is obviously smooth. Note that f has an inverse $g : M' \to M$ given by $g(x) = x^3$. It is also smooth since

$$(\varphi \circ g \circ \psi^{-1})(x) = x$$

is again the identity map. We therefore conclude that M and M' are two distinct but diffeomorphic manifolds.

 # 49 (page 64).
The equality of the transition maps $\varphi_- \circ \varphi_+^{-1} = \varphi_1 \circ \varphi_0^{-1}$ shows that $\varphi_+^{-1} \circ \varphi_0$ and $\varphi_-^{-1} \circ \varphi_1$ coincide over $U_0 \cap U_1$. Hence, F agrees with $\varphi_-^{-1} \circ \varphi_1$ over $U_0 \cap U_1$, and therefore over all of U_1. In a bit more detail: $\varphi_- \circ F \circ \varphi_1^{-1}$ is the identity over $\mathbb{R}\backslash\{0\}$, and it is continuous over all of $\mathbb{R}$, hence it must be the identity over all of $\mathbb{R}$.

 # 50 (page 64).
For $(x,y) \in U_+$,

$$G(x,y) = \varphi_0^{-1}(\varphi_+(x,y)) = \varphi_0^{-1}\Big(\frac{x}{1+y}\Big) = \Big(1 : \frac{x}{1+y}\Big) = (1+y : x);$$

a similar calculation gives $G(x,y) = (x : 1-y)$ if $(x,y) \in U_-$. Note that the two expressions agree over $U_+ \cap U_-$.

 # 51 (page 64).
For S^2 we use the atlas given by stereographic projection.

$$U_+ = \{(x,y,z) \in S^2 |\, z \neq -1\} \quad \varphi_+(x,y,z) = \frac{1}{1+z}\,(x,y),$$
$$U_- = \{(x,y,z) \in S^2 |\, z \neq +1\} \quad \varphi_-(x,y,z) = \frac{1}{1-z}\,(x,y).$$

The transition map is $\mathbf{u} \mapsto \frac{\mathbf{u}}{||\mathbf{u}||^2}$, for $\mathbf{u} = (u^1, u^2)$. Regarding $\mathbf{u}$ as a complex number $u = u^1 + iu^2$, the norm $||\mathbf{u}||$ is just the magnitude of u, and the transition map becomes

$$u \mapsto \frac{u}{|u|^2} = \frac{1}{\bar{u}}.$$

Note that it is not quite the same as the transition map for the standard atlas of $\mathbb{C}P^1$, which is given by $u \mapsto u^{-1}$. We obtain a unique diffeomorphism $F: \mathbb{C}P^1 \to S^2$ such that $\varphi_+ \circ F \circ \varphi_0^{-1}$is the identity, while $\varphi_- \circ F \circ \varphi_1^{-1}$ is complex conjugation. In particular, $F|_{U_0} = \varphi_+^{-1} \circ \varphi_0$. For $(w^0 : w^1) \in U_0$, we have

$$\varphi_0(w^0 : w^1) = \frac{w^1}{w^0} = \frac{1}{|w^0|^2} w^1 \overline{w^0} = \frac{1}{|w^0|^2} \big(\mathrm{Re}(w^1 \overline{w^0}), \mathrm{Im}(w^1 \overline{w^0})\big);$$

where the last equality is the standard identification $\mathbb{C} = \mathbb{R}^2$. Applying to this expression the inverse map φ_+^{-1} of the stereographic projection φ^+, we obtain

$$\varphi_+^{-1} \circ \varphi_0(w^0 : w^1) = \frac{1}{|w^0|^2 + |w^1|^2} \Big(2\mathrm{Re}(w^1 \overline{w^0}),\ 2\mathrm{Im}(w^1 \overline{w^0}),\ |w^0|^2 - |w^1|^2\Big).$$

The calculation of the inverse map $G = F^{-1}: S^2 \to \mathbb{C}P^1$ is similar, but easier: For $(x,y,z) \in U_+$, we have that

$$\begin{aligned} G(x,y,z) &= \varphi_0^{-1} \varphi_+(x,y,z) \\ &= \varphi_0^{-1}\Big(\frac{1}{1+z}(x,y)\Big) \\ &= \varphi_0^{-1}\Big(\frac{1}{1+z}(x+iy)\Big) \\ &= \Big(1 : \frac{1}{1+z}(x+iy)\Big) \\ &= (1+z : x+iy). \end{aligned}$$

Similarly for $(x,y,z) \in U_-$ we find that $G(x,y,z) = (x - iy : 1 - z)$. Note that the two expressions agree if $(x,y,z) \in U_+ \cap U_-$.

 # 52 (page 65).

We observe that the map $G: \mathbb{C}^2 \setminus \{\mathbf{0}\} \to \mathbb{C}^3 \setminus \{\mathbf{0}\}$ given by

$$G(z^0, z^1) = ((z^0)^2, (z^1)^2, z^0 z^1)$$

is smooth. Since the quotient map $\mathbb{C}^3 \setminus \{\mathbf{0}\} \to \mathbb{C}P^2$ is smooth, we conclude that the map $\widetilde{F}: \mathbb{C}^2 \setminus \{\mathbf{0}\} \to \mathbb{C}P^2$ given by

$$\widetilde{F}(z^0, z^1) = ((z^0)^2 : (z^1)^2 : z^0 z^1)$$

is smooth as well. We now observe that $\widetilde{F}$ is the composition of F with the quotient map $\mathbb{C}^2 \setminus \{\mathbf{0}\} \to \mathbb{C}P^1$, so the claim from page 65 gives us that F is smooth.

* * *

* *

Chapter 4

 # 53 (page 76).
Given an open subset $U \subseteq \mathbb{R}^m$ we want to show that every $\mathbf{y} \in U$ is contained in some rational ε-ball $B_\varepsilon(\mathbf{x})$ with $B_\varepsilon(\mathbf{x}) \subseteq U$. Choose $\varepsilon \in \mathbb{Q}_{>0}$ sufficiently small, so that $B_{2\varepsilon}(\mathbf{y}) \subseteq U$. Let $\mathbf{x} \in \mathbb{Q}^m$ with $||\mathbf{x}-\mathbf{y}|| < \varepsilon$. Then $\mathbf{y} \in B_\varepsilon(\mathbf{x})$. On the other hand, for all $\mathbf{z} \in B_\varepsilon(\mathbf{x})$, we have

$$||\mathbf{z}-\mathbf{y}|| \leq ||\mathbf{z}-\mathbf{x}|| + ||\mathbf{x}-\mathbf{y}|| < 2\varepsilon,$$

so $\mathbf{z} \in B_{2\varepsilon}(\mathbf{y}) \subseteq U$. This shows $B_\varepsilon(\mathbf{x}) \subseteq U$.

 # 54 (page 78).
We consider the map

$$\varphi(U) \times \psi(V) \to \kappa(W), \;\; (u,v) \mapsto (u, v - \widetilde{F}(u))$$

obtained from (4.5) by composition with the diffeomorphism $(v,u) \mapsto (u,v)$. This is a smooth map between open subsets of Euclidean spaces, and is easily seen to be invertible. To show that it is a diffeomorphism, it suffices to show that the Jacobian is invertible. The Jacobian is a block matrix of the form

$$\begin{pmatrix} I_{\dim M} & 0 \\ D\widetilde{F} & I_{\dim N} \end{pmatrix}$$

which is invertible.

 # 55 (page 79).
On each of the four components of the subset where $x^2+y^2 \neq R^2$ and $x \neq 0$ (respectively, $y \neq 0$), one can solve the equation $f(x,y,z) = 0$ uniquely for x (respectively, for y), expressing S as the graph of a smooth function of y and z (respectively, of x and z).

 # 56 (page 80).
One direction is immediate: If F is smooth, then $i \circ F$ is smooth as the composition of smooth maps. For the other direction, suppose $i \circ F$ is smooth. To show that F is smooth, let $q \in Q$ and $F(q) \in S \subseteq M$. Since $i \circ F$ is smooth, there exist charts (U, φ) about q, and (V, ψ) about $i(F(q))$ such that $\psi \circ F \circ \varphi^{-1} : \varphi(U) \to \psi(V)$ is smooth. We may choose (V, ψ) to be a submanifold chart for S; so that the chart $(S \cap V, \psi')$ for S is obtained by restricting and composing ψ with the projection $\pi : \mathbb{R}^m \to \mathbb{R}^s$ onto the first s coordinates (where $s = \dim S$). Then, ψ' is smooth as composition of smooth maps, and so is $\psi' \circ F \circ \varphi^{-1}$; proving that F is smooth.

 # 57 (page 80).
For any submanifold chart (V, ψ), we have that $\psi(S \cap V) = \psi'(S \cap V) \times \{\mathbf{0}\}$. Using this property, we have:

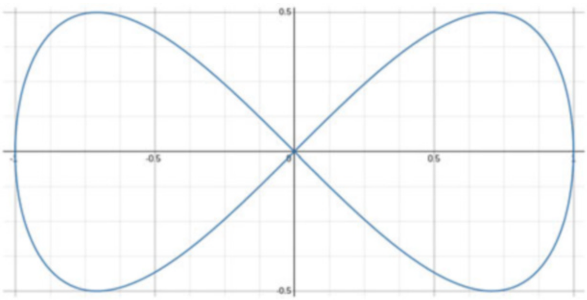
$$
\begin{aligned}
U \cap S &= \bigcup_V \psi^{-1}(\psi'(U' \cap V) \times \mathbb{R}^{m-k}) \cap S \\
&= \bigcup_V \psi^{-1}\big(\psi'(U' \cap V) \times \mathbb{R}^{m-k} \cap \psi(S \cap V)\big) \\
&= \bigcup_V \psi^{-1}\big(\psi'(U' \cap V \cap S) \times \{\mathbf{0}\}\big) \\
&= \bigcup_V \psi^{-1}\big(\psi(U' \cap V \cap S)\big) \\
&= \bigcup_V (U' \cap V \cap S) \\
&= U' \cap S \\
&= U'.
\end{aligned}
$$

 # 58 (page 83).
Noting that $\sin\theta\cos\theta = \frac{1}{2}\sin 2\theta$, we compute

$$
D_\theta F = \begin{pmatrix} -\sin\theta \\ \cos 2\theta \end{pmatrix}.
$$

Since the domain is one-dimensional, a critical point is one where $D_\theta f$ has rank 0. Now, $\sin\theta$ and $\cos 2\theta$ are never both zero, therefore F has no critical points. It is of maximal rank: $\operatorname{rank}_p(F) = 1$ for all points $p \in \mathbb{R}$.

The graph is a figure eight:



 # 59 (page 83).
For $p := (x, y, z)$ we compute:

$$
D_p F = \begin{pmatrix} 0 & z & y \\ y & x & 0 \\ z & 0 & x \\ 2x & 4y & 6z \end{pmatrix}.
$$

The dimension of the domain is 3, and that of the codomain is 4. Therefore, the critical points are those for which the $\operatorname{rank}_p(F) = \operatorname{rank}(D_p F) < 3$. Clearly,

$p=(0,0,0)$ is a critical point, in which case $\text{rank}_p F=0$. No other point is critical. For example, if $z\neq 0$ then the first, third, and last row are linearly independent. Similarly if one of x,y is non-zero. Thus, for $p\neq(0,0,0)$ we have that $\text{rank}_p(F)=3$.

 # 60 (page 85).
First note that $\dim S^1=\dim\mathbb{R}=1$. Next, $S^1\subseteq\mathbb{R}^2$ is a submanifold. If $f:\mathbb{R}\to\mathbb{R}^2$ given by

$$t\mapsto(\cos(2\pi t),\sin(2\pi t))$$

has rank 1, then it will also have rank 1 when viewed as a map $\mathbb{R}\to S^1$. Thus, to prove that it is a local diffeomorphism, suffice it to observe that

$$D_t f=\begin{pmatrix}-2\pi\sin(2\pi t)\\ 2\pi\cos(2\pi t)\end{pmatrix}$$

and sin and cos are never zero together. It cannot be a (global) diffeomorphism since it is non-bijective.

 # 61 (page 85).
The countability assumption is needed to guarantee that the disjoint union $Q=\bigsqcup U_\alpha$ is a manifold. (Recall that manifolds are required to admit a countable atlas.)

 # 62 (page 85).
Each U_α is the domain of a diffeomorphism $\varphi_\alpha: U_\alpha\to\widetilde{U}_\alpha$, for some open $\widetilde{U}_\alpha\subseteq\mathbb{R}^m$. By composing φ_α with a suitable diffeomorphism of $\mathbb{R}^m$, we can arrange that the $\widetilde{U}_\alpha$ are all disjoint. (Here we are using the countability assumption: For example, we can enumerate the U_α's as $U_1,U_2,\ldots$, and then arrange that $\widetilde{U}_k$ is contained inside a ball of radius $1/3$ around the point $(k,0,\ldots,0)\in\mathbb{R}^m$.) Let $\widetilde{Q}\subseteq\mathbb{R}^m$ be the open subset given as the union of $\widetilde{U}_\alpha$'s, then the map $Q\to\widetilde{Q}$ given on $U_\alpha\subseteq Q$ by φ_α is the desired diffeomorphism.

 # 63 (page 88).
Let $p:=(x,y,z)$. We start by computing

$$D_pF=\left(2x\left(\frac{\sqrt{x^2+y^2}-R}{\sqrt{x^2+y^2}}\right),\ 2y\left(\frac{\sqrt{x^2+y^2}-R}{\sqrt{x^2+y^2}}\right),\ 2z\right).$$

We now need to show that for $p\in F^{-1}(r^2)$ we have $D_pF\neq 0$. Note that $D_pF=0$ if and only if one of the following: $x=y=z=0$, or $z=0$ and $\sqrt{x^2+y^2}=R$. In the former case we obtain

$$F(p)=(\sqrt{x^2+y^2}-R)^2+z^2=R^2,$$

and in the latter

$$F(p)=(\sqrt{x^2+y^2}-R)^2+z^2=0.$$

In either case, $F(p)\neq r^2$, so that $p\notin F^{-1}(r^2)$. We therefore conclude that p is a regular value. By Example 4.11, the corresponding submanifold is the 2-torus.

 # 64 (page 90).
If $xyz \neq 0$ we may divide the equation

$$y^2z^2 + x^2z^2 + x^2y^2 = xyz$$
$$x^2(z^2 + y^2) = yz(x - yz)$$

by $(xyz)^2$ to obtain

$$\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2} = \frac{1}{xyz}$$

and by taking absolute values we may assume without loss of generality that x, y, z are positive. Assume now that $x > 1$, so that $0 < 1/x < 1$. Then,

$$\frac{1}{y^2} + \frac{1}{z^2} < \frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2} = \frac{1}{xyz} = \frac{1}{x} \cdot \frac{1}{yz} < \frac{2}{yz}.$$

This is a contradiction since

$$\frac{1}{y^2} + \frac{1}{z^2} - \frac{2}{yz} = \left(\frac{1}{y} - \frac{1}{z}\right)^2 \geq 0.$$

 # 65 (page 91).
The differential (which in this case is the same as the gradient) is the (1×3)-matrix

$$D_{(x,y,z)}f = \begin{pmatrix} 2x(y^2 + z^2) - yz & 2y(z^2 + x^2) - zx & 2z(x^2 + y^2) - xy \end{pmatrix}.$$

This vanishes if and only if all three entries are zero. Vanishing of the first entry gives, after dividing by $2xy^2z^2$, the condition

$$\frac{1}{z^2} + \frac{1}{y^2} = \frac{1}{2xyz};$$

the vanishing of the second and third entry gives similar conditions upon cyclic permutation of x, y, z. Thus we have

$$\frac{1}{z^2} + \frac{1}{y^2} = \frac{1}{x^2} + \frac{1}{z^2} = \frac{1}{y^2} + \frac{1}{x^2} = \frac{1}{2xyz},$$

with a unique solution $x = y = z = \frac{1}{4}$. Thus, $D_{(x,y,z)}f$ has maximal rank (i.e., it is non-zero) except at this point $\left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right)$. But this point does not lie on S. We conclude that $S \cap U$ is a submanifold.

 # 66 (page 93).
Let

$$\mathrm{Graph}(\pi) = \{(\pi(q), q) |\ q \in M\} \subseteq (M/\sim) \times M$$

be the graph of the quotient map. Since π is a smooth map, this is a submanifold of $(M/\sim)\times M$, of dimension equal to the dimension of M. The relation $R\subseteq M\times M$ is the preimage of $\mathrm{Graph}(\pi)$ under the submersion $\pi\times \mathrm{id}_M:\ M\times M\to (M/\sim)\times M$, and is hence a submanifold of $M\times M$. The dimension equals the dimension of $\mathrm{Graph}(\pi)$ (i.e., $\dim M$) plus the dimension of the fibers of the submersion (i.e., $\dim M-\dim(M/\sim)$), thus

$$\dim R = 2\dim M - \dim(M/\sim).$$

It remains to show that $M/\sim$ is Hausdorff if and only if R is closed. Suppose $M/\sim$ is a Hausdorff manifold. To show that R is closed, we have to show that every point $(p,q)\in(M\times M)\backslash R$ has an open neighborhood not meeting R. Since $(p,q)\notin R$, the elements $[p],[q]\in M/\sim$ are distinct. By the Hausdorff condition, there are disjoint open neighborhoods U' of $[p]$ and V' of $[q]$. $U'\cap V'=\emptyset$ means that elements of $U=\pi^{-1}(U')$ are never related to elements of $V=\pi^{-1}(V)$, that is, $U\times V$ is an open neighborhood of (p,q) not meeting R.

Conversely, if $M/\sim$ is a non-Hausdorff manifold, we can find distinct $[p],[q]\in M/\sim$ not having disjoint open neighborhood. Hence, $(p,q)\in(M\times M)\backslash R$ is such that every open neighborhood of (p,q) of the form $U\times V$ meets R. But every open neighborhood of (p,q) contains a neighborhood of the form $U\times V$; hence we conclude that *every* open neighborhood of (p,q) meets R. Hence R is not closed.

 # 67 (page 96).
The surface Σ is a Klein bottle. For an informal argument, let Σ_+ be the open subset swept out by the segment $-\pi/2<t<\pi/2$ of the figure eight. This open segment is just a ('curled up') open interval, and rotating about the z-axis while rotating the segment at half the speed creates a Möbius strip. Similarly, the open subset Σ_- swept out by the segment $\pi/2<t<3\pi/2$ is a Möbius strip. We thus have two Möbius strips glued along their boundary, which is a Klein bottle. (Remember, it is possible to remove a circle from the Klein bottle to create two Möbius strips!)

Alternatively, we could also consider the larger segment $0<t<2\pi$ of the figure eight. This is an (immersed) open interval, which again generates a Möbius strip. (Remember, it is possible to remove a circle from a Klein bottle to create one Möbius strip.)

Yet another approach is to remove from Σ a circle corresponding to one copy of the figure eight itself. Here, the "rotation" no longer matters, and so the resulting surface is an open cylinder. Thus, Σ is obtained from a cylinder by gluing the two boundary circles with a twist, resulting in a Klein bottle. (Remember, it is possible to remove a circle from a Klein bottle to create a cylinder.)

 # 68 (page 97).
Consider the formula for the immersion $F:\ \mathbb{R}^2\to\mathbb{R}^3$ given in the example. Since the right-hand side of the equation for F does not change under the transformations

$$(t,\varphi)\mapsto(t+2\pi,\varphi),\quad (t,\varphi)\mapsto(-t,\varphi+2\pi),$$

and the set of equivalence classes $\Sigma=\mathbb{R}^2/\sim$ for the equivalence relation generated by these transformations is a Klein bottle, the map F descends to an immersion of the Klein bottle. It is straightforward to check that this immersion of the Klein bottle is injective, except over the "central circle" corresponding to $t=0$, where it is 2-to-1.

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*   *

Chapter 5

 # 69 (page 106).
Suppose first that f is the restriction $f = F|_M$ of a function $F \in C^\infty(\mathbb{R}^n)$. Then $f(\gamma(t)) = F(\gamma(t))$ for all $t \in J$, and the derivative may be computed using the chain rule:

$$\left.\frac{d}{dt}\right|_{t=0} f(\gamma(t)) = \left.\frac{d}{dt}\right|_{t=0} F(\gamma(t)) = \dot{\gamma}(0) \cdot (\nabla F)|_p = \mathbf{v} \cdot (\nabla F)|_p.$$

The rightmost expression depends only on p and on $\mathbf{v}$, not on the choice of curve γ with $\gamma(0) = p,\ \dot{\gamma}(0) = \mathbf{v}$.

In general, since $M \subseteq \mathbb{R}^n$ is a submanifold, we may always choose an open neighborhood U of p in $\mathbb{R}^n$ such that $f|_{U \cap M}$ extends to a smooth function $F \in C^\infty(U)$. For t close to 0, we have that $\gamma(t) \in U$, and the calculation above goes through.

 # 70 (page 108).
The maps

$$f \mapsto f(0), \quad f \mapsto f''(0), \quad f \mapsto f'(1), \quad f \mapsto \int_0^1 f(t)\mathrm{d}t$$

are all linear, but none of them satisfies the product rule for $p = 0$.

 # 71 (page 109).
Part (b) follows directly from the product-rule formula. For part (a), let $f \in C^\infty(M)$ be the constant map 1. Then, at each point $p \in M$ we have

$$v(f) = v(f \cdot f) = 2f(p)v(f) = 2v(f)$$

so that $v(f) = 0$. Linearity implies that v vanishes on any other constant map.

 # 72 (page 112).
Let $g, h \in C^\infty(N)$. Note that $g \circ F$ and $h \circ F$ are both elements of $C^\infty(M)$. Then, since $v \in T_pM$ satisfies the product rule,

$$\begin{aligned}
(T_pF(v))(gh) &= v((gh) \circ F) \\
&= v((g \circ F)(h \circ F)) \\
&= (h \circ F)(p)v(g \circ F) + (g \circ F)(p)v(h \circ F) \\
&= h(q)(T_pF(v))(g) + g(q)(T_pF(v))(h).
\end{aligned}$$

 # 73 (page 113).
Suppose that $v \in T_pM$ is represented by the curve $\gamma : J \to M$. Then $(T_pF)(v)$ is represented by the curve $F \circ \gamma$, and $(T_pG)((T_pF)(v))$ is represented by the curve $G \circ (F \circ \gamma)$. But this coincides with the curve $(G \circ F) \circ \gamma$ representing $(T_p(G \circ F))(v)$.

 # 74 (page 113).

(a) Let $v \in T_pM$ be arbitrary, and suppose it is represented by the curve $\gamma : J \to M$. Then, $(T_p\mathrm{id})(v)$ is represented by the curve $\mathrm{id} \circ \gamma = \gamma$. That is, $(T_p\mathrm{id})(v) = v$.

(b) Suppose that $F \in C^\infty(M,N)$ is a diffeomorphism, and let $G \in C^\infty(N,M)$ be its inverse. Then, $G \circ F \in C^\infty(M,M)$ is the identity map id_M. By part (a) and the chain rule:

$$\mathrm{id}_{T_pM} = T_p(G \circ F) = T_{F(p)}G \circ T_pF.$$

Similarly,

$$\mathrm{id}_{T_{F(p)}N} = T_{F(p)}(F \circ G) = T_pF \circ T_{F(p)}G.$$

We therefore see that T_pF is invertible, with inverse

$$(T_pF)^{-1} = T_{F(p)}F^{-1},$$

and conclude that $T_pF : T_pM \to T_{F(p)}N$ is a linear isomorphism.

(c) Let $v \in T_pM$ be arbitrary, and suppose it is represented by the curve $\gamma : J \to M$. Then, $(T_pF)(v)$ is represented by the constant curve $F \circ \gamma$, and is therefore the zero vector.

Observe that this also follows from ✎71, since for any $g \in C^\infty(N)$ the composition $g \circ F$ is a constant function, and so

$$(T_pF)(v)(g) = v(g \circ F) = 0.$$

 # 75 (page 115).
Let $p = (x,y) = (r,\theta)$. Then,

$$\left.\frac{\partial}{\partial r}\right|_p = \left.\frac{\partial x}{\partial r}\right|_p \left.\frac{\partial}{\partial x}\right|_p + \left.\frac{\partial y}{\partial r}\right|_p \left.\frac{\partial}{\partial y}\right|_p = \cos\theta \left.\frac{\partial}{\partial x}\right|_p + \sin\theta \left.\frac{\partial}{\partial y}\right|_p.$$

$$\left.\frac{\partial}{\partial \theta}\right|_p = \left.\frac{\partial x}{\partial \theta}\right|_p \left.\frac{\partial}{\partial x}\right|_p + \left.\frac{\partial y}{\partial \theta}\right|_p \left.\frac{\partial}{\partial y}\right|_p = -r\sin\theta \left.\frac{\partial}{\partial x}\right|_p + r\cos\theta \left.\frac{\partial}{\partial y}\right|_p.$$

 # 76 (page 116).
Using the chain rule $T_p(G \circ F) = T_{F(p)}G \circ T_pF$. The result follows since the composition of surjective maps is surjective and the composition of injective maps is injective.

 # 77 (page 118).

(a) For $p = (x,y,z) \in \mathbb{R}^3$, the Jacobian matrix is

$$T_ph = D_ph = \begin{pmatrix} y & x & 0 \end{pmatrix},$$

a (1×3)-matrix regarded as a linear map $\mathbb{R}^3 \to \mathbb{R}$. For the kernel of this map, we consider two cases:

(i) $x = 0, y = 0$. In this case, T_ph is the zero map, hence $\ker(T_ph) = \mathbb{R}^3$.

(ii) $x \neq 0$ or $y \neq 0$. In this case, $T_p h \neq 0$, hence its kernel is two-dimensional and we can write

$$\ker(T_p h) = \operatorname{span}\left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} x \\ -y \\ 0 \end{pmatrix} \right\}.$$

(b) Suppose now that $p = (x,y,z) \in S^2$. Case (i) above happens for the north pole $p = (0,0,1)$ or south pole $p = (0,0,-1)$. Since $\ker(T_p h) = \mathbb{R}^3$, the condition $T_p S \subseteq \ker(T_p h)$ is automatic in this case.
Case (ii) happens for all points of S^2 except the north and south poles. Since both $T_p S$ and $\ker(T_p h)$ are two-dimensional, the condition $T_p S \subseteq \ker(T_p h)$ holds if and only if $T_p S = \ker(T_p h)$. For $p = (x,y,z)$ we get the two conditions:

$$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0 \Leftrightarrow z = 0,$$

$$\begin{pmatrix} x \\ -y \\ 0 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0 \Leftrightarrow x^2 - y^2 = 0.$$

Since we must also have $x^2 + y^2 + z^2 = 1$ we obtain the four solutions

$$\left(\pm \frac{1}{\sqrt{2}}, \pm \frac{1}{\sqrt{2}}, 0 \right)$$

(with all possible sign combinations).
In summary, we have found six critical points:

$$(0,0,\pm 1), \quad \left(\pm \frac{1}{\sqrt{2}}, \pm \frac{1}{\sqrt{2}}, 0 \right).$$

 # 78 (page 119).
Recall that the determinant of a complex $(n \times n)$-matrix is the product of its eigenvalues, while the trace is the sum of the eigenvalues. Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of X (repeated according to multiplicity). Then $I + tX$ has eigenvalues $1 + t\lambda_1, \ldots, 1 + t\lambda_n$, and therefore

$$\det(I + tX) = \prod_{i=1}^{n} (1 + t\lambda_i).$$

Taking the derivative at $t = 0$ we obtain, using the product rule,

$$\left. \frac{d}{dt} \right|_{t=0} \det(I + tX) = \lambda_1 + \cdots + \lambda_n = \operatorname{tr}(X).$$

Now, the group $\mathrm{SL}(n,\mathbb{R}) = \{A \in \mathrm{Mat}_{\mathbb{R}}(n) | \det(A) = 1\}$ is given as the level set $F^{-1}(1)$ of the function $\det : \mathrm{Mat}_{\mathbb{R}}(n) \to \mathbb{R}$. We can therefore use Proposition 5.18. We calculate

$$D_AF(X) = \left.\frac{d}{dt}\right|_{t=0} F(A+tX) = \left.\frac{d}{dt}\right|_{t=0} \det(A+tX) = \left.\frac{d}{dt}\right|_{t=0} \det(I+tA^{-1}X)$$
$$= \operatorname{tr}(A^{-1}X).$$

In particular, for $A = I$ the kernel of this map is

$$\mathfrak{sl}(n,\mathbb{R}) = \{X \in \mathrm{Mat}_{\mathbb{R}}(n) |\ \operatorname{tr}(X) = 0\}.$$

 # 79 (page 119).

(a) If $\gamma(t) \in G$ is a curve through $\gamma(0) = I$, representing $X = \frac{d}{dt}\big|_{t=0} \gamma(t) \in T_IG = \mathfrak{g}$, then $t \mapsto \lambda(t) = A\gamma(t)$ is a curve through $\lambda(0) = A$, representing the tangent vector $\dot{\lambda}(0) \in T_AG$. But
$$\left.\frac{d}{dt}\right|_{t=0} \lambda(t) = AX.$$
This shows that the isomorphism $\mathrm{Mat}_{\mathbb{R}}(n) \to \mathrm{Mat}_{\mathbb{R}}(n),\ X \mapsto AX$ restricts to an isomorphism $\mathfrak{g} \to T_AG$, hence that T_AG is obtained by "left translation" from $\mathfrak{g}$. The argument for right translation is similar.

(b) By the previous part, AX can be written $X'A$ for some $X' \in \mathfrak{g}$. But $AX = X'A$ is equivalent to $AXA^{-1} = X' \in \mathfrak{g}$.

(c) Represent Y by a curve $\lambda(t)$. By part (b),
$$\lambda(t)X\lambda(t)^{-1} \in \mathfrak{g}$$
for all t. Taking the t-derivative, we get
$$\dot{\lambda}(t)X\lambda(t)^{-1} - \lambda(t)X\lambda^{-1}(t)\dot{\lambda}(t)\lambda(t)^{-1} \in \mathfrak{g},$$
where we used the usual formula for the derivative of an inverse matrix,
$$\frac{d\lambda^{-1}}{dt} = -\lambda(t)^{-1}\frac{d\lambda}{dt}\lambda(t)^{-1}.$$
(Recall that this formula is proved by taking the derivative of both sides of $\lambda(t)\lambda(t)^{-1} = I$.) Setting $t = 0$, we obtain
$$YX - XY \in \mathfrak{g}.$$

 # 80 (page 120).

Let $F : \mathbb{R}^3 \to \mathbb{R}^3$ be as in the question

$$F(x,y,z) = (yz, xz, xy).$$

(a) For $p = (x, y, z)$ we compute

$$D_pF = \begin{pmatrix} 0 & z & y \\ z & 0 & x \\ y & x & 0 \end{pmatrix}.$$

The determinant is $2xyz$, so the kernel is trivial unless at least one of $x = 0$, $y = 0$, or $z = 0$.

(b) Suppose $x = 0$, i.e., that p lies on the yz-plane, so that $p = (0, y, z)$. The Jacobian is then

$$D_pF = \begin{pmatrix} 0 & z & y \\ z & 0 & 0 \\ y & 0 & 0 \end{pmatrix}.$$

Unless p is the origin, the kernel is one-dimensional

$$\ker D_pF = \operatorname{span}\left\{ \begin{pmatrix} 0 \\ -y \\ z \end{pmatrix} \right\}.$$

This vector will lie in $T_pS^2 = \operatorname{span}(p)^\perp$ if and only if

$$\begin{pmatrix} 0 \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 0 \\ -y \\ z \end{pmatrix} = 0 \iff y^2 = z^2.$$

Together with the condition $x^2 + y^2 + z^2 = 1$ (since $p \in S^2$) we get the four points

$$p = \left(0, \pm\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}}\right).$$

Similarly, if p lies on the xz-plane we get

$$p = \left(\pm\frac{1}{\sqrt{2}}, 0, \pm\frac{1}{\sqrt{2}}\right)$$

and if p lies on the xy-plane we get

$$p = \left(\pm\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}}, 0\right).$$

(c) We conclude that the map $\pi : S^2 \to \mathbb{R}^3$ given by (5.9) has exactly twelve critical points, hence the map $\mathbb{R}\mathrm{P}^2 \to \mathbb{R}^3$ defining Steiner's surface has exactly six critical points

$$\left(0 : \frac{1}{\sqrt{2}} : \pm\frac{1}{\sqrt{2}}\right), \left(\frac{1}{\sqrt{2}} : 0 : \pm\frac{1}{\sqrt{2}}\right), \left(\frac{1}{\sqrt{2}} : \pm\frac{1}{\sqrt{2}} : 0\right).$$

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Chapter 6

 # 81 (page 123).
For vector fields $X, Y \in \mathfrak{X}(M)$ and scalars $a \in \mathbb{R}$ we define sums $X+Y$ and scalar multiples aX by

$$(X+Y)_p = X_p + Y_p, \quad (aX)_p = aX_p.$$

These are again vector fields; for example, if $f \in C^\infty(M)$, the function

$$p \mapsto (X+Y)_p(f) = X_p(f) + Y_p(f)$$

is smooth because it is the sum of the smooth functions $p \mapsto X_p(f)$ and $p \mapsto Y_p(f)$

 # 82 (page 124).
Suppose $\{X_p,\ p \in M\}$ is a collection of tangent vectors such that for all $f \in C^\infty(M)$ the function $X(f)$ defined by $X(f)|_p = X_p(f)$ is smooth. Then $f \mapsto X(f)$ is a linear map $C^\infty(M) \to C^\infty(M)$ satisfying the product rule (6.2), so it is a vector field in the sense of the second definition.

Conversely, if $X \in \mathrm{Hom}(C^\infty(M), C^\infty(M))$ is a linear map satisfying the product rule (6.2), define $X_p :\ C^\infty(M) \to \mathbb{R}$ by $X_p(f) = X(f)|_p$. Then $v = X_p$ satisfies the product rule (Equation 5.3) at p, hence it is an element of T_pM, and for any given $f \in C^\infty(M)$ the map $p \mapsto X_p(f)$ is the function $X(f)$, hence is smooth. Thus X defines a vector field in the sense of the first definition.

 # 83 (page 124).
In terms of the first definition, observe that the map $p \mapsto (hX)_p(f) = h|_p X_p(f)$ is smooth, since it is the product of the smooth functions h and $X(f)$. In terms of the second definition, one checks that hX satisfies the product rule.

 # 84 (page 125).
Let $\widetilde{g} = g \circ \varphi^{-1} \in C^\infty(\widetilde{U})$ be the local coordinate expression of g. Choose a ‘bump function’ χ supported on an open ball $B_\varepsilon(\widetilde{p}) \subseteq \widetilde{U}$ and equal to 1 on a smaller ball around $\widetilde{p}$ (see Lemma C.8 in the appendix). Let $\widetilde{f} = \chi\widetilde{g}$. Then $f = \widetilde{f} \circ \varphi \in C^\infty(U)$ coincides with g near p and, since it has compact support in U, it extends by zero to a smooth function on all of M (cf. Lemma 3.5).

 # 85 (page 125).

(a) We have

$$v = \ln y, \quad u = \ln x - \ln\ln y, \quad w = \frac{z}{(\ln x - \ln\ln y)\ln^2 y}$$

which is well-defined since $x \geq y > 1$.

(b) We have

$$\begin{aligned} \frac{\partial}{\partial u} &= \frac{\partial x}{\partial u}\frac{\partial}{\partial x} + \frac{\partial y}{\partial u}\frac{\partial}{\partial y} + \frac{\partial z}{\partial u}\frac{\partial}{\partial z} \\ &= (e^u v)\frac{\partial}{\partial x} + (v^2 w)\frac{\partial}{\partial z} \\ &= x\frac{\partial}{\partial x} + \frac{z}{\ln x - \ln\ln y}\frac{\partial}{\partial z}. \end{aligned}$$

By similar calculations,

$$\frac{\partial}{\partial v} = \frac{x}{\ln y}\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + \frac{2z}{\ln y}\frac{\partial}{\partial z}$$

and

$$\frac{\partial}{\partial w} = (\ln x - \ln\ln y)\ln^2 y\frac{\partial}{\partial z}.$$

 # 86 (page 127).
Given a smooth function $h \in C^\infty(M)$ we have

$$\begin{aligned}
[X, fY](h) &= (X \circ (fY))(h) - ((fY) \circ X)(h) \\
&= X(f \cdot Yh) - f((Y \circ X)(h)) \\
&= (Xf)(Yh) + f((X \circ Y)(h)) - f((Y \circ X)(h)) \\
&= ((Xf)Y + f[X,Y])(h)
\end{aligned}$$

which proves the formula.

Using the first part we can easily calculate:

$$\begin{aligned}
[fX, gY] &= f(Xg)Y + g[fX, Y] \\
&= f(Xg)Y - g[Y, fX] \\
&= f(Xg)Y - g(Yf)X - fg[Y,X] \\
&= f(Xg)Y - g(Yf)X + fg[X,Y].
\end{aligned}$$

 # 87 (page 129).

(a) Since

$$\frac{\partial 2\sqrt{xy}}{\partial x} = \sqrt{\frac{y}{x}}, \quad \frac{\partial 2\sqrt{xy}}{\partial y} = \sqrt{\frac{x}{y}},$$

we have

$$X \circ Y = 2\sqrt{\frac{x}{y}}\frac{\partial}{\partial x} + \text{higher order terms.}$$

Similarly, $\frac{\partial}{\partial x}\left(\frac{x}{y}\right) = \frac{1}{y}$ and so

$$Y \circ X = 2\sqrt{\frac{x}{y}}\frac{\partial}{\partial x} + \text{higher order terms.}$$

In conclusion, $[X,Y]$ vanishes.

(b) It may be not so easy to "guess" the right change of coordinates. But after experimenting one finds that

$$x = uv^2, \quad y = u$$

works:

$$\begin{aligned}
\frac{\partial}{\partial u} &= \frac{\partial x}{\partial u}\frac{\partial}{\partial x} + \frac{\partial y}{\partial u}\frac{\partial}{\partial y} = v^2\frac{\partial}{\partial x} + \frac{\partial}{\partial y} = \frac{x}{y}\frac{\partial}{\partial x} + \frac{\partial}{\partial y} = X, \\
\frac{\partial}{\partial v} &= \frac{\partial x}{\partial v}\frac{\partial}{\partial x} + \frac{\partial y}{\partial v}\frac{\partial}{\partial y} = 2uv\frac{\partial}{\partial x} = 2\sqrt{xy}\frac{\partial}{\partial x} = Y.
\end{aligned}$$

 # 88 (page 129).
We have found in Example 5.1 that

$$T_pS^2 = (\mathbb{R}p)^\perp.$$

(a) For $p = (a,b,c) \in S^2$, the vectors

$$X_p = (0,-c,b), \;\; Y_p = (c,0,-a), \;\; Z_p = (-b,a,0),$$

all have zero dot product with $p = (a,b,c)$.

(b) We calculate

$$[X,Y] = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} = -Z,$$

and similarly $[Y,Z] = -X$ and $[Z,X] = -Y$. In particular, the brackets are again tangent to S^2, as implied by Proposition 6.11.

 # 89 (page 129).
By straightforward calculation, we have the following equalities of operators on $C^\infty(M)$:

$$\begin{aligned}
[X,[Y,Z]] &= X\circ Y\circ Z - X\circ Z\circ Y - Y\circ Z\circ X + Z\circ Y\circ X,\\
[Z,[X,Y]] &= Z\circ X\circ Y - Z\circ Y\circ X - X\circ Y\circ Z + Y\circ X\circ Z,\\
[Y,[Z,X]] &= Y\circ Z\circ X - Y\circ X\circ Z - Z\circ X\circ Y + X\circ Z\circ Y.
\end{aligned}$$

It is evident that summing the rows results in 0.

 # 90 (page 130).

(a) If F is not surjective there is no candidate for Y_q at points q that are not in the range of F.

(b) If F is not injective, then at points q having more than one preimage p there may be more than one candidate for $T_pF(X_p)$.

(c) If F is bijective, then we can define $(F_*X)_q = T_pF(X_p)$, where p is the unique preimage $F^{-1}(q)$. However, this collection of tangent vectors need not be smooth. For example, let $F:\ \mathbb{R} \to \mathbb{R}$ be the map $F(x) = x^3$. This is a smooth bijection, but not a diffeomorphism (cf. ✎ 48). One finds that

$$F_*\Big(\frac{\partial}{\partial x}\Big) = 3y^{2/3}\frac{\partial}{\partial y}$$

which is not a smooth vector field.

 # 91 (page 131).
We have $X \sim_i Y$ if and only if for all $p \in S$

$$T_p i(X_p) = Y_p.$$

Recall that the image $T_p i(T_p S)$ is identified with $T_p S$ itself, via the injectivity of $T_p i$. Thus, $T_p i(X_p) = X_p$, so that $X \sim_i Y$ if and only if for all $p \in S$

$$X_p = T_p i(X_p) = Y_p.$$

That is, Y is tangent to S, with X as its restriction. In the special case $X = 0$, this just says $Y_p = 0$ for $p \in S$, that is, Y vanishes along S.

 # 92 (page 132).
This uses the previous ✐91. Let $S \subseteq M$ be a submanifold, with $i: S \hookrightarrow M$ the inclusion map. Let $Y_1, Y_2 \in \mathfrak{X}(M)$ be tangent to S, with $X_1, X_2 \in \mathfrak{X}(S)$ their restrictions. Then,

$$X_1 \sim_i Y_1, \quad X_2 \sim_i Y_2,$$

hence $[X_1, X_2] \sim_i [Y_1, Y_2]$, which means that $[Y_1, Y_2]$ is tangent to S, with $[X_1, X_2]$ its restriction.

 # 93 (page 134).
Suppose $\gamma(t) = (x(t), y(t))$ is a solution curve. Then it must satisfy the system of equations

$$\begin{aligned} \frac{dx}{dt} &= -y(t), \\ \frac{dy}{dt} &= x(t). \end{aligned}$$

In particular, $\frac{d^2x}{dt^2} = -x(t)$ which is a second-order linear ODE with solution

$$x(t) = c_1 \cos(t) + c_2 \sin(t).$$

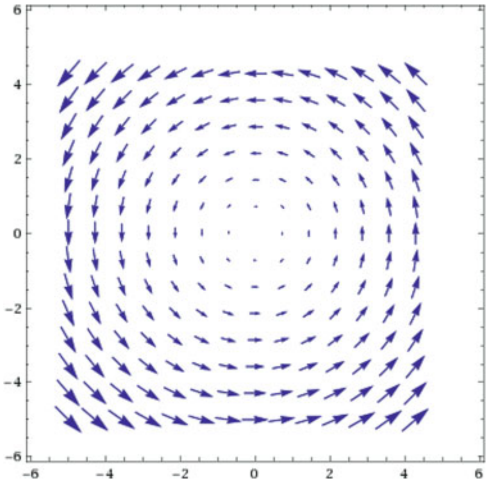
Similarly,

$$y(t) = c_1 \sin(t) - c_2 \cos(t).$$

Any particular "initial condition" (x_0, y_0) at $t = 0$ will give us the values of c_1 and c_2. Namely, $c_1 = x_0$ and $c_2 = -y_0$. We therefore have

$$\gamma(t) = (x_0 \cos t - y_0 \sin t, x_0 \sin t + y_0 \cos t).$$

The picture with $(x_0, y_0) = (1, 1)$ is as follows:



 # 94 (page 136).

(a) The solution to the initial value problem $x(0) = x_0$ is $x(t) = t + x_0$. Since the ODE was only defined on the open interval $(0,1)$, the interval of definition for this solution is determined from the condition $x(t) \in (0,1)$, that is $J_{x_0} = \{t| -x_0 < t < 1 - x_0\}$. It follows that

$$\Phi(t,x) = x + t, \quad \mathscr{J} = \{(x,t)| -x < t < 1-x,\ 0 < x < 1\}.$$

This region forms a parallelogram in $\mathbb{R}^2$. Note that this is a very simple example of a vector field whose solution curves exist only for finite time.

(b) This ODE is solved by separation of variables; the solution is

$$x(t) = \tan(t + \arctan x_0).$$

The solution is only defined for $t + \arctan x_0 \neq \pi/2 + k\pi$ where $k \in \mathbb{Z}$, and since $t_0 = 0$ must be in the interval of definition we obtain the condition,

$$J_{x_0} = \{t \in \mathbb{R}| -\pi/2 < t + \arctan x_0 < \pi/2\}.$$

Consequently

$$\Phi(t,x) = \tan(t + \arctan x), \quad \mathscr{J} = \{(t,x)| -\pi/2 < t + \arctan x < \pi/2\}.$$

 # 95 (page 140).

The idea is to "sew together" the solutions for finite time intervals to obtain a solution defined for all t.

By assumption, $(\delta, p) \in \mathscr{J}$ for all $p \in M$. In particular, also $(\delta, \Phi_\delta(p)) \in \mathscr{J}$. The flow property shows $(2\delta, p) \in \mathscr{J}$. Applying the flow property to this element and to $(\delta, \Phi_{2\delta}(p)) \in \mathscr{J}$ we obtain $(3\delta, p) \in \mathscr{J}$, and proceeding in this manner we get $(k\delta, p) \in \mathscr{J}$. That is, $k\delta \in J_p$ for all $k \in \mathbb{N}$. Likewise $-k\delta \in J_p$ for all $k \in \mathbb{N}$. Since J_p is an interval, this means $J_p = \mathbb{R}$.

 # 96 (page 144).
In general, for any smooth function $F : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ we have by the chain rule

$$\frac{d}{dt} F(t,t) = \left.\frac{\partial F}{\partial u}\right|_{(t,t)} + \left.\frac{\partial F}{\partial v}\right|_{(t,t)}.$$

In our case, $F(u,v) = (\Phi_u^* Y)(\Phi_v^* f)(p)$.

 # 97 (page 146).
Let X and Y be two vector fields such that $[X,Y] = 0$, and let Φ_t and Ψ_s be their flows, respectively. If X and Y are incomplete, then it need not be true that $\Phi_t \Psi_s(p) = \Psi_s \Phi_t(p)$ in general, even if both sides are defined.

For such an example, let M be the open subset of the plane obtained by removing a vertical line segment, e.g.,

$$M = \mathbb{R}^2 \setminus \{(x,y) |\ |x| = 0,\ |y| \leq 1/2\},$$

and put $X = \frac{\partial}{\partial x}$. A vector field of the form

$$Y = f(x,y) \frac{\partial}{\partial y}$$

with $f \in C^\infty(M)$ commutes with X if and only if $\frac{\partial f}{\partial x} = 0$ everywhere. This means that *locally*, f does not depend on x. But it is still possible for f to take on different values to the left and right of the line segment.

Concretely, we may pick $\chi \in C^\infty(\mathbb{R})$ with $\chi(t) = 1$ for $|t| \geq 1/2$ and $\chi(t) > 1$ for $|t| \leq 1/2$, and let

$$f(x,y) = \begin{cases} \chi(y) & x < 0,\ |y| < \frac{1}{2} \text{ (the region to the left of the line segment)}, \\ 1 & \text{elsewhere.} \end{cases}$$

(You may want to draw a picture of the vector field Y.) Letting $p = (-1,-1) \in M$, we then have that

$$\Psi_2(\Phi_2(p)) = \Psi_2((1,-1)) = (1,1),$$

but

$$\Phi_2(\Psi_2(p)) = \Phi_2(-1,R) = (1,R)$$

for some $R > 1$ (depending on the choice of χ).

 # 98 (page 147).
Recall that for any vector field X and any diffeomorphism F,

$$F^*X \sim_F X.$$

Since brackets of related vector fields are again related (Theorem 6.16),

$$[F^*X, F^*Y] \sim_F [X,Y]$$

for all X,Y. But also $F^*[X,Y] \sim_F [X,Y]$; since F is a diffeomorphism this shows $F^*[X,Y] = [F^*X, F^*Y]$. Here we apply this fact to $F = \Phi_t$.

 # 99 (page 147).
The Lie bracket of X,Y is given by

$$Z = [X,Y] = \frac{\partial}{\partial z}.$$

Note that for all $p \in \mathbb{R}^3$, the tangent vectors X_p, Y_p, Z_p are linearly independent. Hence, there cannot exist a two-dimensional submanifold S passing through p with X,Y,Z all tangent to S at p.

 # 100 (page 147).
For $p = (x,y,z)$, note that

$$p \cdot X_p = 0, \;\; p \cdot Y_p = 0, \;\; p \cdot Z_p = 0.$$

Thus, $X_p, Y_p, Z_p \in (\mathbb{R}p)^\perp = T_pS$ where $S \subseteq \mathbb{R}^3$ is the 2-sphere of radius $||p||$, which we knew from ✐ 88. The existence of such an integral submanifold is possible, since the bracket of any two of X,Y,Z lies in the span of these vector fields; for example, $[X,Y] = -Z$.

 # 101 (page 148).
We check smoothness near any given $p \in M$. Choosing a chart around p, we may assume that M is an open subset $U \subseteq \mathbb{R}^m$. Taking U smaller if necessary, we may complete $X_1, \ldots, X_r$ to a collection of vector fields $X_1, \ldots, X_m$ which are a basis everywhere. Write

$$X_i = \sum_j c_i^j \frac{\partial}{\partial x^j}$$

with $c_i^j \in C^\infty(U)$. Since $X_1, \ldots, X_m$ form a basis everywhere, the coefficient matrix is invertible, and by the formula for the inverse matrix (Cramer's rule) the entries of the latter are smooth. Thus,

$$\frac{\partial}{\partial x^j} = \sum_i d_j^i X_i,$$

where $d_j^i \in C^\infty(U)$. Since $Y|_U$ is a linear combination of the coordinate vector fields, with smooth coefficients, it follows that $Y|_U = \sum_{i=1}^m a_i X_i$ where $a_i \in C^\infty(U)$. (It just so happens that $a_{r+1} = \cdots = a_m = 0$.)

 # 102 (page 148).
The assumption that the $\{X_i\}$ and the $\{X_i'\}$ span the same subspace of the tangent space everywhere implies that

$$X_l' = \sum_i a_l^i X_i,$$

where $a_i^j \in C^\infty(M)$ (smoothness holds by the previous ✐ 101). Suppose the $\{X_i\}$ satisfy the Frobenius condition. Then, by ✐ 86,

$$[X_l', X_m'] = \sum_{ij} [a_l^i X_i, a_m^j X_j] = \sum_{ij} \left(a_l^i a_m^j [X_i, X_j] + a_l^i L_{X_i}(a_m^j) X_j - a_m^j L_{X_j}(a_l^i) X_i \right).$$

On the right-hand side, each term is linear combination of X_i's with coefficients in $C^\infty(M)$; hence the $\{X_i'\}$ also satisfy the Frobenius condition.

 # 103 (page 149).
The choice of local coordinates $x_1, \ldots, x_m$ around p, with coordinate map $\varphi : U \to \mathbb{R}^m$ taking p to the origin in $\mathbb{R}^m$, gives an isomorphism $T_p\varphi : T_pM \to T_0\mathbb{R}^m = \mathbb{R}^m$, taking E to a subspace $\widetilde{E} \subseteq \mathbb{R}^m$. Let $\widetilde{F} \subseteq \mathbb{R}^m$ be a complement to $\widetilde{E}$ (for example, the orthogonal space $\widetilde{F} = \widetilde{E}^\perp$). Then $N = \varphi^{-1}(\widetilde{F})$ has the desired property.

 # 104 (page 153).
By Example 6.13, $\mathscr{E}$ is the space of all vector fields X that are tangent to the fibers of Φ, that is, which satisfy $X_p \in \ker(T_p\Phi)$. Any fiber $\Phi^{-1}(n),\ n \in N$ is an integral submanifold.

 # 105 (page 153).
In short, this follows because the statement is a local one, but locally $\mathscr{E}$ is spanned by r vector fields as in Theorem 6.49. Below are more details.

The condition is necessary, since the bracket of two vector fields tangent to S is again tangent to S. On the other hand, if the condition holds, we have to show the existence of an integral submanifold through every given $p \in M$. By definition, the subspace $\mathscr{E}_p = \{X_p |\ X \in \mathscr{E}\} \subseteq T_pM$ is r-dimensional. Hence we may choose $X_1, \ldots, X_r \in \mathscr{E}$ so that their values at p are a basis of $\mathscr{E}_p$. The vector fields remain linearly independent on a neighborhood U of p, hence their values at points $p' \in U$ also span $\mathscr{E}_{p'}$. Since $\mathscr{E}$ is a Lie subalgebra by assumption, the brackets $[X_i, X_j]$ are in $\mathscr{E}$, hence over U they can be written in the form $\sum_k c_{ij}^k X_k$ with functions c_{ij}^k. We are now in the setting of Theorem 6.49; hence there is a submanifold $S \subseteq U$ through p such that $X_1, \ldots, X_r$ are all tangent to S. Since these vector fields span $\mathscr{E}$ over U, the claim follows.

* * *
* *

Chapter 7

 # 106 (page 158).
By definition, $\Omega^k(U)$ is spanned by expressions $\sum_I \omega_I \mathrm{d}x^I$ where I is a sequence of indices $1 \leq i_1 < \ldots < i_k \leq m$.

(a) If $k > m$ there is no such sequence, hence $\Omega^k(U) = \mathrm{span}\{0\}$.
(b) For $k = m$, there is a *unique* such sequence namely $i_1 = 1, \ldots, i_m = m$. Hence, $\Omega^m(U)$ is given by expressions

$$\omega = f \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^m$$

with $f \in C^\infty(U)$. We have $\mathrm{d}\omega = 0$ since $\Omega^{m+1}(U) = \{0\}$.

 # 107 (page 158).
In each case we use the formula together with the knowledge that (for example) $\mathrm{d}x \wedge \mathrm{d}x = 0$ to compute more efficiently.

(a) We observe that the z-partial derivative of the coefficients is 0, so that

$$\mathrm{d}\alpha = \frac{\partial(y^2 e^x)}{\partial x} \mathrm{d}x \wedge \mathrm{d}y + \frac{\partial(2ye^x)}{\partial y} \mathrm{d}y \wedge \mathrm{d}x = (y^2 - 2)e^x \mathrm{d}x \wedge \mathrm{d}y.$$

(b) As with the previous computation:

$$\mathrm{d}\beta = \frac{\partial(y^2 e^x)}{\partial y} \mathrm{d}y \wedge \mathrm{d}x + \frac{\partial(2ye^x)}{\partial x} \mathrm{d}x \wedge \mathrm{d}y = (-2ye^x + 2ye^x) \mathrm{d}x \wedge \mathrm{d}y = 0.$$

Indeed, we could have saved some computation by noticing that

$$\beta = \mathrm{d}(y^2 e^x)$$

so the result follows immediately from $\mathrm{d} \circ \mathrm{d} = 0$.

(c) Using the observations at the beginning of the answer we have

$$\begin{aligned} \mathrm{d}\rho &= \frac{\partial(e^{x^2 y} \sin z)}{\partial z} \mathrm{d}z \wedge \mathrm{d}x \wedge \mathrm{d}y + \frac{\partial(2\cos(z^3 y))}{\partial y} \mathrm{d}y \wedge \mathrm{d}x \wedge \mathrm{d}z \\ &= (e^{x^2 y} \cos z + 2z^3 \sin(z^3 y)) \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z. \end{aligned}$$

(d) We may use the formula to compute $\mathrm{d}\omega$. However, since ω is an element of $\Omega^3(\mathbb{R}^3)$ the result $\mathrm{d}\omega$ is an element of $\Omega^4(\mathbb{R}^3)$ which is simply $\{0\}$ (cf. ✐ 106). Thus, $\mathrm{d}\omega = 0$.

 # 108 (page 160).
Note first that the map $v \mapsto A_v = (\cdot, v)$ is linear, since $(\cdot, \cdot)$ is bilinear. To see that this map is injective, let $v \in E$ with $A_v = 0$. Then $0 = A_v(v) = (v, v) = ||v||^2$, hence $v = 0$. Since $\dim E^* = \dim E < \infty$, it follows that the map is also surjective.

Note: In the second part, we used that the inner product is positive definite. More generally, if $(\cdot, \cdot) : E \times E \to \mathbb{R}$ is a *non-degenerate* symmetric bilinear form, in the sense that $(w, v) = 0$ for all $w \in E$ implies that $v = 0$, then the map $v \mapsto A_v = (\cdot, v)$ is still injective (by the very definition of non-degeneracy), and hence defines an isomorphism between E and E^*.

 # 109 (page 162).
The identification $T_a\mathbb{R} = \mathbb{R}$ is given by the basis vector $\frac{\partial}{\partial t}\Big|_{t=a}$; hence it takes $w \in T_a\mathbb{R}$ to the scalar λ such that

$$w(g) = \lambda \left.\frac{\partial g}{\partial t}\right|_{t=a}.$$

Taking $g = \mathrm{id}_{\mathbb{R}} : t \mapsto t$, we see that the identification takes the tangent vector w to the number $w(\mathrm{id}_{\mathbb{R}})$. We calculate, for $v \in T_pM$ and $f \in C^\infty(\mathbb{R})$,

$$\big((T_pf)(v)\big)(\mathrm{id}_{\mathbb{R}}) = v(\mathrm{id}_{\mathbb{R}} \circ f) = v(f),$$

which shows that $(T_pf)(v)$ corresponds to $v(f) = \langle (\mathrm{d}f)_p, v\rangle$.

 # 110 (page 163).
The tangent map at a point (x,y,z) is represented by the Jacobian:

$$D_{(x,y,z)}F = \begin{pmatrix} 2xy & x^2 & e^z \\ -1 & z & y \end{pmatrix}$$

and at $p = (1,1,1)$ it is

$$D_{(1,1,1)}F = \begin{pmatrix} 2 & 1 & e \\ -1 & 1 & 1 \end{pmatrix}.$$

The coefficients of $T_pF\left(\frac{\partial}{\partial x}\Big|_p\right)$ are given by the first column of this matrix:

$$T_pF\left(\left.\frac{\partial}{\partial x}\right|_p\right) = 2\left.\frac{\partial}{\partial u}\right|_{F(p)} - \left.\frac{\partial}{\partial v}\right|_{F(p)}.$$

The dual map at $F(p) = (1+e,0)$ is represented by the transpose of $D_{(1,1,1)}F$:

$$D_{(1,1,1)}F^\top = \begin{pmatrix} 2 & -1 \\ 1 & 1 \\ e & 1 \end{pmatrix}.$$

Since $(\mathrm{d}v)_{F(p)}$ is the second standard basis vector, the coefficients of $(T_p^*F)(dv)_{F(p)}$ are the second column of this matrix:

$$(T_p^*F)(dv)_{F(p)} = -\left.\mathrm{d}x\right|_p + \left.\mathrm{d}y\right|_p + \left.\mathrm{d}z\right|_p.$$

 # 111 (page 165).
For any vector field $X \in \mathfrak{X}(M)$ we have

$$\langle \mathrm{d}(fg), X\rangle = X(fg) = X(f)g + fX(g) = g\langle \mathrm{d}f, X\rangle + f\langle \mathrm{d}g, X\rangle = \langle g\mathrm{d}f + f\mathrm{d}g, X\rangle.$$

 # 112 (page 165).
For $Y \in \mathfrak{X}(U)$, we have to define a smooth function $\alpha|_U(Y) \in C^\infty(U)$. (Note that Y need not be of the form $X|_U$.) To define $(\alpha|_U)(Y)$ near a given point $p \in U$, choose any bump function χ with compact support supported in U, with χ equal to 1 on a smaller open neighborhood U_1 of p. Then χY has compact support, hence it extends by zero to a vector field on M, and it agrees with Y on U_1. We put

$$(\alpha|_U)(Y)\big|_{U_1} = \alpha(\chi Y)|_{U_1} \in C^\infty(U_1).$$

This does not depend on the choice of χ, since the value of this function at $q \in U_1$ is $\alpha_q((\chi Y)_q) = \alpha_q(Y_q)$. Furthermore, $\alpha|_U$ defined in this way is $C^\infty(U)$-linear. $\square$

 # 113 (page 165).
"$\Leftarrow$". Suppose the coefficient functions for α in any chart(U, φ) are smooth. Using the chart to identify U with an open subset of $\mathbb{R}^m$ (rather than writing out φ everywhere) we have

$$\alpha|_U = \sum_{i=1}^m \alpha_i \mathrm{d}x^i$$

with $\alpha_i \in C^\infty(U)$. Similarly, for all vector fields X,

$$X|_U = \sum_{i=1}^m X^i \frac{\partial}{\partial x^i}$$

with $X^i \in C^\infty(U)$. It follows that $\alpha(X)|_U = \sum_{i=1}^m \alpha_i X^i$ is smooth. Since this holds for all charts, we conclude that $\alpha(X)$ is smooth, hence α is a 1-form.

"$\Rightarrow$". Suppose that $\alpha \in \Omega^1(M)$, and (U, φ) an arbitrary chart. By the previous 112, $\alpha|_U$ is a 1-form, with $(\alpha|_U)_p = \alpha_p$ for $p \in U$. Using the chart to identify U with an open subset of $\mathbb{R}^m$,

$$\alpha|_U = \sum_{i=1}^m \alpha_i \mathrm{d}x^i.$$

The coefficient functions are obtained as $\alpha_i = (\alpha|_U)(\frac{\partial}{\partial x^i})$, hence are smooth. $\square$

 # 114 (page 166).
We simply observe

$$F^*(f+g) = (f+g)\circ F = f \circ F + g \circ F = F^* f + F^* g.$$

Similarly,

$$F^*(f \cdot g) = (f \cdot g)\circ F = (f \circ F)\cdot(g \circ F) = (F^* f)\cdot(F^* g).$$

Next, suppose $F : M \to N$ and $G : N \to M$ are smooth, and let $h \in C^\infty(M)$ be arbitrary. Then, since composition of functions is associative,

$$(G \circ F)^*(h) = h \circ (G \circ F) = (h \circ G)\circ F = F^*(h \circ G) = F^*(G^* h).$$

We conclude that $(G \circ F)^* = F^* \circ G^*$ as we wanted to show. (Observe also that this expression indeed makes sense in terms of the domain and range.)

 # 115 (page 167).

(a) We have

$$F^*(\mathrm{d}u) = \mathrm{d}(x^3e^{yz}) = 3x^2e^{yz}\mathrm{d}x + x^3ze^{yz}\mathrm{d}y + x^3ye^{yz}\mathrm{d}z$$

and

$$F^*(v\cos u\mathrm{d}v) = F^*(v\cos u)F^*(\mathrm{d}v) = (\sin x\cos(x^3e^{yz})\cos x)\mathrm{d}x.$$

(b) The function $F^*g = g\circ F = x^3e^{yz}\sin x$ has differential

$$\mathrm{d}(F^*g) = \mathrm{d}(x^3\sin xe^{yz}) = (3x^2e^{yz}\sin x + x^3e^{yz}\cos x)\mathrm{d}x + x^3ze^{yz}\sin x\mathrm{d}y + x^3ye^{yz}\sin x\mathrm{d}z.$$

On the other hand, the pullback of $\mathrm{d}g = v\mathrm{d}u + u\mathrm{d}v$ is

$$\begin{aligned} F^*(dg) &= \sin x\,\mathrm{d}(x^3e^{yz}) + x^3e^{yz}\,\mathrm{d}(\sin x) \\ &= (3x^2e^{yz}\sin x + x^3e^{yz}\cos x)\mathrm{d}x + x^3\,ze^{yz}\sin x\,\mathrm{d}y + x^3ye^{yz}\sin x\,\mathrm{d}z. \end{aligned}$$

We observe that $F^*(dg) = d(F^*g)$, as expected.

 # 116 (page 168).
We compute

$$\begin{aligned} \gamma^*\beta &= \mathrm{d}(\cos t) - \mathrm{d}(\sin t) = -(\sin t + \cos t)\mathrm{d}t; \\ \left\langle \gamma^*\beta, \frac{\partial}{\partial t} \right\rangle &= -(\sin t + \cos t); \\ \langle \beta, X\rangle &= -(x+y); \\ \gamma^*\langle \beta, X\rangle &= -(\cos t + \sin t). \end{aligned}$$

We see that $\left\langle \gamma^*\beta, \frac{\partial}{\partial t} \right\rangle = \gamma^*\langle \beta, X\rangle$, as expected.

 # 117 (page 169).

(a) We may use Lemma 7.12 to conclude

$$\int_\gamma f\mathrm{d}x = \int_\gamma \mathrm{d}F = F(\gamma(b)) - F(\gamma(b)) = \int_{\gamma(a)}^{\gamma(b)} F'(x)\mathrm{d}x = \int_{\gamma(a)}^{\gamma(b)} f(x)\mathrm{d}x.$$

(b) By part (a) above, we have

$$\int_{\gamma(a)}^{\gamma(b)} f(x)\mathrm{d}x = \int_\gamma f\mathrm{d}x = \int_a^b \gamma^*(f\mathrm{d}x).$$

But $\gamma^*(f\mathrm{d}x) = f(\gamma(t))\mathrm{d}\gamma = f(\gamma(t))\,\gamma'(t)\mathrm{d}t$. Thus,

$$\int_{\gamma(a)}^{\gamma(b)} f(x)\mathrm{d}x = \int_a^b f(\gamma(t))\,\gamma'(t)\mathrm{d}t$$

as we wanted to show.

 # 118 (page 169).
By Proposition 7.15 we have

$$\int_\gamma \mathrm{d}(fg) = f(q)g(q) - f(p)g(p).$$

On the other hand (cf. ✐ 111),

$$\mathrm{d}(fg) = g\mathrm{d}f + f\mathrm{d}g.$$

Thus,

$$\int_\gamma f\mathrm{d}g = f(q)g(q) - f(p)g(p) - \int_\gamma g\mathrm{d}f.$$

 # 119 (page 170).
One can compute $\gamma^*\alpha$ directly via the formula from Lemma 7.12 and then compute the resulting integral, but this would be too much work. It is simpler to note that

$$\alpha = \mathrm{d}(y^2e^x),$$

so Proposition 7.15 implies

$$\int_\gamma \alpha = \int_\gamma \mathrm{d}(y^2e^x) = y^2e^x\big|_{\gamma(1)} - y^2e^x\big|_{\gamma(0)} = e.$$

 # 120 (page 171).
By the definition of 1-forms, it is clear that $\alpha\wedge\beta$ is a well-defined map $\mathfrak{X}(M)\times\mathfrak{X}(M)\to C^\infty(M)$. It is skew-symmetric since

$$(\alpha\wedge\beta)(Y,X) = \alpha(Y)\beta(X) - \alpha(X)\beta(Y) = -(\alpha\wedge\beta)(X,Y).$$

It remains to verify $C^\infty(M)$-linearity in the first argument, for fixed Y. But for fixed Y, the map $X\mapsto(\alpha\wedge\beta)(X,Y)$ is the 1-form

$$\beta(Y)\alpha - \alpha(Y)\beta:\ \mathfrak{X}(M)\to C^\infty(M)$$

which is $C^\infty(M)$-linear by definition.

 # 121 (page 172).
The expression for $(\mathrm{d}\alpha)(X,Y)$ changes sign under exchange of X,Y, hence it is skew-symmetric. To verify $C^\infty(M)$-linearity in the first argument, let $f\in C^\infty(M)$ and compute

$$(\mathrm{d}\alpha)(fX,Y) = L_{fX}(\alpha(Y)) - L_Y(\alpha(fX)) - \alpha([fX,Y]).$$

The first term is $fL_X(\alpha(Y))$ (since $L_{fX}(g) = fL_X(g)$ for $g\in C^\infty(M)$). The second term is $-L_Y(f\alpha(X)) = -L_Y(f)\alpha(X) - fL_Y(\alpha(X))$, while the last term is $-\alpha(f[X,Y] - L_Y(f)X) = -f\alpha([X,Y]) + L_Y(f)\alpha(X)$. Putting everything together, and observing the cancelation of terms involving Lie derivatives of f, we obtain

$$(\mathrm{d}\alpha)(fX,Y) = f\big(L_f(\alpha(Y)) - L_Y(\alpha(X)) - \alpha([X,Y])\big) = f(\mathrm{d}\alpha)(X,Y).$$

Suppose now that $\alpha = \mathrm{d}f$. Then (using that $(\mathrm{d}f)(X) = X(f) = L_X f$ by definition of the exterior differential on functions)

$$\begin{aligned}(\mathrm{d}(\mathrm{d}f))(X,Y) &= L_X(\mathrm{d}f(Y)) - L_Y(\mathrm{d}f(X)) - \mathrm{d}f([X,Y]) \\ &= L_X L_Y f - L_Y L_X f - L_{[X,Y]} f \\ &= 0,\end{aligned}$$

where the last equality holds by the definition of the Lie bracket.

 # 122 (page 172).
For $i < j$ we find, using $\left[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right] = 0$,

$$\begin{aligned}(\mathrm{d}\alpha)_{ij} &= (\mathrm{d}\alpha)\Big(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\Big) \\ &= L_{\frac{\partial}{\partial x^i}} \alpha\left(\frac{\partial}{\partial x^j}\right) - L_{\frac{\partial}{\partial x^j}} \alpha\left(\frac{\partial}{\partial x^j}\right) \\ &= \frac{\partial \alpha^j}{\partial x^i} - \frac{\partial \alpha^i}{\partial x^j},\end{aligned}$$

as desired.

 # 123 (page 173).
The map $\mathrm{Sk}(\gamma)$ is $C^\infty(M)$-multilinear. To check skew-symmetry, let $\tau \in S_k$. We have

$$\mathrm{Sk}(\gamma)(X_{\tau(1)}, \ldots, X_{\tau(k)}) = \sum_{s \in S_k} \mathrm{sign}(s)\gamma(X_{s\tau(1)}, \ldots, X_{s\tau(k)}).$$

As s runs through S_k, so does $s' = s\tau$. Since $\mathrm{sign}(s\tau) = \mathrm{sign}(s)\mathrm{sign}(\tau)$ we obtain

$$\mathrm{sign}(\tau) \sum_{s' \in S_k} \mathrm{sign}(s')\gamma(X_{s'(1)}, \ldots, X_{s'(k)}) = \mathrm{sign}(\tau)\mathrm{Sk}(\gamma)(X_1, \ldots, X_k).$$

If $\alpha \in \Omega^k(M)$, then for each $s \in S_k$ we have

$$\mathrm{sign}(s)\alpha(X_{s(1)}, \ldots, X_{s(k)}) = \alpha(X_1, \ldots, X_k)$$

so that the sum is simply $\mathrm{card}(S_k)\alpha = k!\alpha$.

 # 124 (page 175).
S_2 has only two permutations, $(1\ 2)$ (even) and $(2\ 1)$ (odd). Thus, for $\alpha, \beta \in \Omega^1(M)$,

$$(\alpha \wedge \beta)(X_1, X_2) = \alpha(X_1)\beta(X_2) - \alpha(X_2)\beta(X_1)$$

which agrees with our definition in Equation (7.10).

 # 125 (page 175).
Since $(3,2)$-shuffles have their first three elements in order, they are determined by which 2 elements do not appear among the first three. There will be

$$\binom{5}{2} = \frac{5!}{2!3!} = 10$$

such shuffles. Here they are:

$$(1\ 2\ 3\ 4\ 5), (1\ 2\ 4\ 3\ 5), (1\ 3\ 4\ 2\ 5), (2\ 3\ 4\ 1\ 5), (1\ 2\ 5\ 3\ 4),$$

$$(1\ 3\ 5\ 2\ 4), (2\ 3\ 5\ 1\ 4), (1\ 4\ 5\ 2\ 3), (2\ 4\ 5\ 1\ 3), (1\ 5\ 5\ 2\ 3).$$

 # 126 (page 176).
The

$$\binom{4}{2} = \frac{4!}{2!2!} = 6$$

$(2,2)$-shuffles of $(1\ 2\ 3\ 4)$ are

$$(1\ 2\ 3\ 4),\ \ (1\ 4\ 2\ 3),\ \ (2\ 3\ 1\ 4),\ \ (3\ 4\ 1\ 2),\ \ (1\ 3\ 2\ 4),\ \ (2\ 4\ 1\ 3).$$

Here, the first four are even permutations whereas the last two are odd. Therefore, $(\alpha \wedge \beta)(X_1, X_2, X_3, X_4)$ is the sum of six terms,

$$\begin{aligned}&\alpha(X_1,X_2)\beta(X_3,X_4) + \alpha(X_1,X_4)\beta(X_2,X_3) + \alpha(X_2,X_3)\beta(X_1,X_4)\\ &+ \alpha(X_3,X_4)\beta(X_1,X_2) - \alpha(X_1,X_3)\beta(X_2,X_4) - \alpha(X_2,X_4)\beta(X_1,X_3).\end{aligned}$$

 # 127 (page 176).
Given $s \in S_{k+l}$, the transformation $s'' \in S_k \times S_l$ is uniquely determined by the condition that

$$s''(i) < s''(j) \Leftrightarrow s(i) < s(j),$$

for all indices i, j that are either both $\leq k$ or both $> k$. One then defines s' by

$$s'(s''(i)) = s(i)$$

for all i; the condition defining s'' guarantees that this is a (k,l)-shuffle.

As an example, with $k = 3, l = 2$, let us write

$$s = (4\ 1\ 2\ 5\ 3)$$

as a product $s's''$ with $s'' \in S_k \times S_l$ and $s' \in S_{k,l}$. The element s'' has $1,2,3$ among the first three entries and $4,5$ among the last two, appearing in a similar order as the first three and last two entries of s. Thus, we replace the string 412 in s by 312 and the string 53 by 54, arriving at

$$s'' = (3\ 1\ 2\ 5\ 4) \in S_3 \times S_2.$$

To obtain s from s'', we need to map $1 \mapsto 1,\ 2 \mapsto 2,\ 3 \mapsto 4,\ 4 \mapsto 3,\ 5 \mapsto 5$, so

$$s' = (1\ 2\ 4\ 3\ 5) \in S_{3,2}.$$

Let us now use this decomposition $s = s's''$ in Formula (7.14) for the wedge product. We obtain

$$\frac{1}{k!l!} \sum_{s' \in S_{k,l}} \sum_{s'' \in S_k \times S_l} \mathrm{sign}(s')\mathrm{sign}(s'')\alpha(X_{s's''(1)}, \ldots, X_{s's''(k)})\ \beta(X_{s's''(k+1)}, \ldots, X_{s's''(k+l)}).$$

For fixed s', the action of s'' amounts to a permutation of the entries of α, and likewise for β. We have

$$\begin{aligned} &\mathrm{sign}(s'')\alpha(X_{s's''(1)}, \ldots, X_{s's''(k)})\ \beta(X_{s's''(k+1)}, \ldots, X_{s's''(k+l)}) \\ &= \alpha(X_{s'(1)}, \ldots, X_{s'(k)})\ \beta(X_{s'(k+1)}, \ldots, X_{s'(k+l)}). \end{aligned}$$

Hence, for fixed s', the terms in the sum over s'' are all equal to the term where $s'' = 1$. Since there are $|S_k \times S_l| = k!l!$ such terms, we arrive at

$$\sum_{s' \in S_{k,l}} \mathrm{sign}(s')\alpha(X_{s'(1)}, \ldots, X_{s'(k)})\ \beta(X_{s'(k+1)}, \ldots, X_{s'(k+l)})$$

as in (7.15).

 # 128 (page 177).
Consider $S_{k_1+k_2}$ as the subgroup of $S_{k_1+k_2+k_3}$ fixing the last k_3 indices. With this understanding, there is a well-defined bijection (as sets)

$$S_{k_1+k_2,k_3} \times S_{k_1,k_2} \to S_{k_1,k_2,k_3},\quad (s', s'') \mapsto s's''.$$

Using this decomposition we compute

$$\begin{aligned} &\sum_{s \in S_{k_1,k_2,k_3}} \mathrm{sign}(s)\alpha_1(X_{s(1)}, \ldots)\alpha_2(X_{s(k_1+1)}, \ldots)\alpha_3(X_{s(k_1+k_2+1)}, \ldots) \\ &= \sum_{s' \in S_{k_1+k_2,k_3}} \mathrm{sign}(s') \sum_{s'' \in S_{k_1,k_2}} \mathrm{sign}(s'')\alpha_1(X_{s's''(1)}, \ldots)\alpha_2(X_{s's''(k_1+1)}, \ldots)\alpha_3(X_{s'(k_1+k_2+1)}, \ldots) \\ &= \sum_{s' \in S_{k_1+k_2,k_3}} \mathrm{sign}(s')(\alpha_1 \wedge \alpha_2)(X_{s'(1)}, \ldots)\alpha_3(X_{s'(k_1+k_2+1)}, \ldots) \\ &= ((\alpha_1 \wedge \alpha_2) \wedge \alpha_3)(X_1, \ldots, X_{k_1+k_2+k_3}). \end{aligned}$$

Similarly, we have a decomposition of S_{k_1,k_2,k_3} as $S_{k_1,k_2+k_3} \times S_{k_2,k_3}$, leading to the bracketing $\alpha_1 \wedge (\alpha_2 \wedge \alpha_3)$.

 # 129 (page 179).
To show that α is closed we compute its differential

$$\begin{aligned} \mathrm{d}\alpha &= \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right)\mathrm{d}x\wedge\mathrm{d}y + \frac{\partial}{\partial y}\left(\frac{-y}{x^2+y^2}\right)\mathrm{d}y\wedge\mathrm{d}x \\ &= \left(\frac{(x^2+y^2)-2x^2}{(x^2+y^2)^2} + \frac{(x^2+y^2)-2y^2}{(x^2+y^2)^2}\right)\mathrm{d}x\wedge\mathrm{d}y = 0. \end{aligned}$$

To show that it is not exact consider the smooth path $\gamma\colon\mathbb{R}\to\mathbb{R}^2$ given by $\gamma(t) = (\cos t, \sin t)$ for $t\in[0,2\pi]$. Since $\gamma(0)=\gamma(2\pi)$, this is a loop. If α were exact the integral $\int_\gamma \alpha$ would have to be zero. But

$$\gamma^*\alpha = \frac{\cos t\cos t\,\mathrm{d}t - \sin t(-\sin t)\mathrm{d}t}{\cos^2 t + \sin^2 t} = \mathrm{d}t$$

so that

$$\int_\gamma \alpha = \int_0^{2\pi}\mathrm{d}t = 2\pi \neq 0.$$

 # 130 (page 179).
The exact k-forms are the differentials of $(k-1)$-forms, but for $k=0$ there are none since $\Omega^{-1}(M)=\{0\}$.

 # 131 (page 180).
This is one of those cases where a formula seems reasonably "obvious" after one tries out a couple of examples, whereas the actual proof makes it appear more complicated.

For convenience we denote the vector field in the contraction by $X = X_0$, and think of $S_{k,l}$ as acting on $0,\dots,k+l-1$. The summation in

$$\begin{aligned} \iota_{X_0}(\alpha\wedge\beta)(X_1,\dots,X_{k+l-1}) &= (\alpha\wedge\beta)(X_0,\dots,X_{k+l-1}) \\ &= \sum_{s\in S_{k,l}} \operatorname{sign}(s)\alpha(X_{s(0)},\dots,X_{s(k-1)})\beta(X_{s(k)},\dots,X_{s(k+l-1)}) \end{aligned}$$

splits into two parts: $S_{k,l} = S'_{k,l}\sqcup S''_{k,l}$ where $S'_{k,l}$ consists of shuffles with $s(0)=0$, and $S''_{k,l}$ those with $s(k)=0$. We may identify $S'_{k,l}$ with $S_{k-1,l}$ (acting on indices $1,\dots,k+l-1$ in the natural way) and hence obtain

$$\begin{aligned} &\sum_{s\in S_{k-1,l}} \operatorname{sign}(s)\alpha(X_0,X_{s(1)},\dots,X_{s(k-1)})\beta(X_{s(k)},\dots,X_{s(k+l-1)}) \\ &= (\iota_{X_0}\alpha\wedge\beta)(X_1,\dots,X_{k+l-1}). \end{aligned}$$

Similarly, $S''_{k,l}$ is identified with $S_{k,l-1}$, as follows: Every $\tau\in S_{k,l-1}$ (acting as usual on the set $\{1,\dots,k+l-1\}$) determines an element $s\in S''_{k,l}$ (acting on $\{0,\dots,k+l-1\}$) by

$$\begin{aligned} &s(0)=\tau(1),\dots,s(k-1)=\tau(k),\ \ s(k)=0, \\ &s(k+1)=\tau(k+1),\dots,s(k+l-1)=\tau(k+l-1). \end{aligned}$$

Note that $\mathrm{sign}(s) = (-1)^k \mathrm{sign}(\tau)$. Hence, the second sum is

$$\sum_{s\in S''_{k,l}} \mathrm{sign}(s)\alpha(X_{s(0)},X_{s(1)},\ldots,X_{s(k-1)})\beta(X_0, X_{s(k+1)},\ldots,X_{s(k+l-1)})$$
$$= (-1)^k \sum_{\tau\in S_{k,l-1}} \mathrm{sign}(\tau)\alpha(X_{\tau(1)},\ldots,X_{\tau(k)})\beta(X_0,X_{\tau(k+1)},\ldots,X_{\tau(k+l-1)})$$
$$= (-1)^k (\alpha\wedge\iota_{X_0}\beta)(X_1,\ldots,X_{k+l-1}).$$

In conclusion, $\iota_X(\alpha\wedge\beta) = \iota_X\alpha\wedge\beta + (-1)^k\alpha\wedge\iota_X\beta$.

 # 132 (page 180).

$$\iota_Z(\sin(x)\mathrm{d}x\wedge\mathrm{d}y) = \sin(x)(\iota_Z\mathrm{d}x)\mathrm{d}y - \sin(x)\mathrm{d}x(\iota_Z\mathrm{d}y) = e^x\sin(x)\mathrm{d}y + \sin(x)\mathrm{d}x.$$

 # 133 (page 181).
For α of degree k and β of degree l,

$$\begin{aligned}(D_1\circ D_2)(\alpha\wedge\beta) &= D_1(D_2\alpha\wedge\beta + (-1)^{kr_2}\alpha\wedge D_2\beta)\\ &= (D_1\circ D_2)\alpha\wedge\beta + (-1)^{r_1(k+r_2)}D_2\alpha\wedge D_1\beta\\ &\quad + (-1)^{kr_2}D_1\alpha\wedge D_2\beta + (-1)^{k(r_1+r_2)}\alpha\wedge(D_1\circ D_2)\beta.\end{aligned}$$

Similarly,

$$\begin{aligned}(D_2\circ D_1)(\alpha\wedge\beta) &= D_2(D_1\alpha\wedge\beta + (-1)^{kr_1}\alpha\wedge D_1\beta)\\ &= (D_2\circ D_1)\alpha\wedge\beta + (-1)^{r_2(k+r_1)}D_1\alpha\wedge D_2\beta\\ &\quad + (-1)^{kr_1}D_2\alpha\wedge D_1\beta + (-1)^{k(r_2+r_1)}\alpha\wedge(D_2\circ D_1)\beta.\end{aligned}$$

Hence, in the resulting expression for $[D_1,D_2](\alpha\wedge\beta) = (D_1\circ D_2 - (-1)^{r_1r_2}D_2\circ D_1)(\alpha\wedge\beta)$ all "mixed terms" (such as $D_1\alpha\wedge D_2\beta$) cancel out, and the rest combines to

$$[D_1,D_2](\alpha\wedge\beta) = [D_1,D_2]\alpha\wedge\beta + (-1)^{(r_1+r_2)k}\alpha\wedge[D_1,D_2]\beta.$$

 # 134 (page 182).
In each case we use the formula $L_X = \mathrm{d}\circ\iota_X + \iota_X\circ\mathrm{d}$.

(a) We compute

$$\iota_X\alpha = \alpha(X) = -y^2 - x^2 - z^2$$

so that

$$d(\iota_X\alpha) = -2x\mathrm{d}x - 2y\mathrm{d}y - 2z\mathrm{d}z.$$

Moreover,

$$d\alpha = -\mathrm{d}y\wedge\mathrm{d}x - \mathrm{d}x\wedge\mathrm{d}y - \mathrm{d}z\wedge\mathrm{d}z = 0.$$

Therefore,

$$L_X\alpha = -2x\mathrm{d}x - 2y\mathrm{d}y - 2z\mathrm{d}z.$$

Note that we could have saved some computation by noticing that

$$\alpha = d(-xy - \frac{1}{2}z^2).$$

(b) Since α is a 0-form (i.e., a function) we have $L_X\alpha = X(\alpha)$ so that

$$L_X\alpha = X(\alpha) = (x^4 + y\cos z)\cos(xy) + 2(x+y+z)(x^3 + yz^3 + \cos z).$$

(c) We have

$$\iota_X\alpha = \alpha(X) = \cos z(z+y^2)\mathrm{d}z$$

so that

$$d(\iota_X\alpha) = \frac{\partial \cos z(z+y^2)}{\partial y}\mathrm{d}y\wedge\mathrm{d}z = 2y\cos z\mathrm{d}y\wedge\mathrm{d}z.$$

Moreover,

$$\mathrm{d}\alpha = \frac{\partial(z+y^2)}{\partial y}\mathrm{d}y\wedge\mathrm{d}x\wedge\mathrm{d}z = -2y\mathrm{d}x\wedge\mathrm{d}y\wedge\mathrm{d}z$$

so that

$$\iota_X(\mathrm{d}\alpha) = -2y\cos z\mathrm{d}y\wedge\mathrm{d}z - 2xy^2z\mathrm{d}x\wedge\mathrm{d}z.$$

Therefore,

$$L_X\alpha = \mathrm{d}(\iota_X\alpha) + \iota_X(\mathrm{d}\alpha) = -2xy^2z\mathrm{d}x\wedge\mathrm{d}z.$$

 # 135 (page 183).

(a) The proof of the locality of D is analogous to that for the differential d. We have to show that if $\alpha|_U = 0$ then $D\alpha|_U = 0$. Given $p\in U$, choose a bump function $\chi\in C^\infty(M)$ with $\mathrm{supp}(\chi)\subseteq U$ and $\chi = 1$ on a smaller neighborhood $U_1\subseteq U$ of p. Then $\chi\alpha = 0$ and the derivation property gives

$$0 = D(\chi\alpha) = D(\chi)\wedge + \chi\wedge D\alpha.$$

Restricting to U_1, this shows (using $\alpha|_{U_1} = 0, \chi|_{U_1} = 1$) that $(D\alpha)|_U = 0$.

(b) By considering the difference of D and D', suffice it to show that if a degree r superderivation D vanishes on functions and on their differentials, then $D = 0$. Suppose $Df = 0$ and $D(\mathrm{d}f) = 0$ for all f. Lemma 7.21 shows that near a given point $p\in M$, any k-form α is a linear combination of expressions of the form

$$f_0\mathrm{d}f_1\wedge\cdots\wedge\mathrm{d}f_k$$

with functions $f_0,\ldots,f_k\in C^\infty(M)$. Using the derivation property, we obtain

$$D(f_0\mathrm{d}f_1\wedge\cdots\wedge\mathrm{d}f_k) = 0.$$

This shows that $D\alpha$ vanishes on a neighborhood of p. Since p and α were arbitrary, we conclude that $D = 0$.

 # 136 (page 183).

One can follow the pattern of proof of (7.29), by establishing the identity on functions and their differentials, or directly obtain it from (7.29), (7.28), and (7.30):

$$L_{[X,Y]} = [\mathrm{d},\iota_{[X,Y]}] = [\mathrm{d},[L_X,\iota_Y]] = [[\mathrm{d},L_X],\iota_Y] + [L_X,[\mathrm{d},\iota_Y]] = [L_X,L_Y].$$

(The third equality may be confirmed by expanding both sides, keeping track of all the signs; it may also be seen as an instance of a super-Jacobi identity.)

 # 137 (page 183).
Using identity (7.27) from the Cartan calculus, with $Y = X$, gives $[\iota_X, \iota_X] = 0$. But the left hand side is a *graded* super-commutator, hence is $\iota_X \circ \iota_X + \iota_X \circ \iota_X = 2\iota_X \circ \iota_X$.

 # 138 (page 183).
We encourage the reader to carry out the relevant calculations on their own and only glance at the following discussion for inspiration (if needed).

(a) Let $\alpha \in \Omega^1(M)$, and $X, Y \in \mathfrak{X}(M)$. The strategy in computing $(\mathrm{d}\alpha)(X,Y) = \iota_Y \iota_X \mathrm{d}\alpha$ is to move contraction operators to the right, using the Cartan calculus. We compute

$$\begin{aligned}\iota_Y \iota_X \mathrm{d}\alpha &= \iota_Y L_X \alpha - \iota_Y \mathrm{d}\iota_X \alpha \\ &= L_X \iota_Y \alpha - \iota_{[X,Y]}\alpha - L_Y \iota_X \alpha + \mathrm{d}\iota_Y \iota_X \alpha.\end{aligned}$$

But $\iota_Y \iota_X \alpha = 0$ since α is a 1-form. We arrive at

$$(\mathrm{d}\alpha)(X,Y) = L_X(\alpha(Y)) - L_Y(\alpha(X)) - \alpha([X,Y]).$$

(b) Let $\alpha \in \Omega^2(M)$, and $X, Y, Z \in \mathfrak{X}(M)$. By the same calculation as in the case of 1-forms we have

$$\iota_Y \iota_X \mathrm{d}\alpha = L_X \iota_Y \alpha - \iota_{[X,Y]}\alpha - L_Y \iota_X \alpha + \mathrm{d}\iota_Y \iota_X \alpha.$$

Applying ι_Z, we obtain

$$\begin{aligned}\iota_Z \iota_Y \iota_X \mathrm{d}\alpha &= \iota_Z L_X \iota_Y \alpha - \iota_Z \iota_{[X,Y]}\alpha - \iota_Z L_Y \iota_X \alpha + \iota_Z \mathrm{d}\iota_Y \iota_X \alpha \\ &= L_X \iota_Z \iota_Y \alpha + \iota_{[Z,X]}\iota_Y \alpha - \iota_Z \iota_{[X,Y]}\alpha - L_Y \iota_Z \iota_X \alpha - \iota_{[Z,Y]}\iota_X \alpha + L_Z \iota_Y \iota_X \alpha \\ &\quad - \mathrm{d}\iota_Z \iota_Y \iota_X \alpha.\end{aligned}$$

But $\iota_Z \iota_Y \iota_X \alpha = 0$ since α is a 2-form. Rearranging the terms, and putting $\iota_Y \iota_X \alpha = \alpha(X,Y)$ and so on, we arrive at

$$\begin{aligned}(\mathrm{d}\alpha)(X,Y,Z) = {}& L_X(\alpha(Y,Z)) - L_Y(\alpha(X,Z)) + L_Z(\alpha(X,Y)) \\ & - \alpha([X,Y],Z) + \alpha([X,Z],Y) - \alpha([Y,Z],X).\end{aligned}$$

 # 139 (page 184).
The formula $(G \circ F)^* = F^* \circ G^*$ holds for functions $h \in C^\infty(Q) = \Omega^0(Q)$, as well as for exact 1-forms $\mathrm{d}h$, since

$$(F^* \circ G^*)\mathrm{d}h = F^*\mathrm{d}(G^*h) = \mathrm{d}(F^*(G^*h)) = \mathrm{d}((G \circ F)^*h) = (G \circ F)^*\mathrm{d}h.$$

The general case follows since every k-form is locally (near any $p \in Q$) a linear combination of forms

$$h_0 \mathrm{d}h_1 \wedge \cdots \wedge \mathrm{d}h_k \in \Omega^k(Q),$$

and since the pullback of a wedge product is the wedge product of pullbacks.

 # 140 (page 185).
Using the properties of the pullback,

$$F^*(\mathrm{d}u \wedge \mathrm{d}v) = F^*(\mathrm{d}u) \wedge F^*(\mathrm{d}v) = \mathrm{d}F^*u \wedge \mathrm{d}F^*v.$$

This is consistent with the heuristic of "putting $(u,v) = F(x,y,z)$." The final computation is therefore:

$$F^*(\mathrm{d}u \wedge \mathrm{d}v) = \mathrm{d}(y^2 z) \wedge \mathrm{d}(x) = 2yz\mathrm{d}y \wedge \mathrm{d}x + y^2\mathrm{d}z \wedge \mathrm{d}x.$$

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Chapter 8

 # 141 (page 201).
Let $\widetilde{\gamma}\colon \mathbb{R} \to \mathbb{R}^2 \setminus \{\mathbf{0}\}$ be the map covering γ. We have $\widetilde{\gamma}^*\omega = n\,\mathrm{d}t$; integrating from 0 to 1 we obtain $w(\gamma) = n$.

 # 142 (page 201).
It cannot be exact, since $\omega = \mathrm{d}\alpha$ would give $\int_M \omega = 0$ by Stokes' theorem.

 # 143 (page 203).
Consider the map $\pi\colon\ \mathbb{R}^2 \setminus \{\mathbf{0}\} \to S^1,\ \mathbf{x} \mapsto \frac{\mathbf{x}}{||\mathbf{x}||}$. The winding number of $\gamma\colon\ S^1 \to \mathbb{R}^2 \setminus \{\mathbf{0}\}$ is the mapping degree of the map $\pi \circ \gamma\colon\ S^1 \to S^1$.

 # 144 (page 203).
Let $\eta \in \Omega^n(Q)$ be an n-form on Q with $\int_Q \eta = 1$. We have

$$\deg(G) = \int_N G^*\eta.$$

If $\deg(G) = 0$, then Theorem 8.13 implies that $G^*\eta$ is exact, i.e., $G^*\eta = \mathrm{d}\omega$ for some $\omega \in \Omega^{n-1}(N)$. Then,

$$(G \circ F)^*\eta = F^*(G^*\eta) = F^*(\mathrm{d}\omega) = \mathrm{d}(F^*\omega)$$

is also exact, so that $\int_M (G \circ F)^*\eta = 0$ and the formula holds.

If $\deg(G) \neq 0$, then $\omega = \frac{1}{\deg(G)} G^*\eta$ is a top-degree form $\omega \in \Omega^n(N)$ and $\int_N \omega = 1$. Therefore,

$$\deg(F) = \int_M F^*\omega.$$

Now,

$$\deg(G \circ F) = \int_M F^*(G^*\eta) = \int_M F^*(\deg(G)\omega) = \deg(G) \int_M F^*\omega = \deg(G) \cdot \deg(F).$$

 # 145 (page 205).
Given any point $p \in S$, let $v_1, \ldots, v_{m-1}$ be a basis of T_pS. Then $v_1, \ldots, v_{m-1}$ together with X_p is a basis of T_pM. Since Γ is a volume form, we have that

$$\Gamma_p(X_p, v_1, \ldots, v_{m-1}) \neq 0.$$

But this is the same as

$$(i^*(\iota_X\Gamma))_p(v_1, \ldots, v_{m-1}),$$

hence we conclude that $i^*(\iota_X\Gamma)$ does not vanish at p. Since this is true for all $p \in S$, we conclude that $i^*(\iota_X\Gamma)$ is a volume form.

 # 146 (page 208).
Consider first the case $g = 0$. If $g = 0,\ r = 0$, then Σ is a 2-sphere. Cutting along the equator gives two disks D_+, D_- the upper and lower hemispheres. Hence $\chi(S^2) = \chi(D_+) + \chi(D_-) = 1 + 1 = 2$. If $g = 0,\ r = 1$, then Σ is a disk, so $\chi(\Sigma) = 1 = 2 - 1$ by Axiom (E3). If $g = 0,\ r > 1$, then Σ is the surface obtained from a disk D by removing $r - 1$ smaller disks $D_1, \dots, D_{r-1}$. Hence, by Axiom (E2) and Axiom (E1),

$$\chi(D) = \chi(\Sigma) + \chi(D_1) + \cdots + \chi(D_{r-1}).$$

Since $\chi(D) = \chi(D_1) = \cdots = \chi(D_{r-1}) = 1$ by Axiom (E3), this gives $\chi(\Sigma) = 1 - (r - 1) = 2 - r$ as desired.

The case $g > 0$ can be reduced to the case $g = 0$ by cutting the handles. According to Axiom (E2), this does not affect the Euler characteristic. On the other hand, the cut decreases g by 1, but increases r by 2, hence it does not change the expression $2 - 2g - r$.

 # 147 (page 209).
Recall that the Klein bottle may be obtained by gluing the boundary circles of a cylinder ($g = 0, r = 2$) by an orientation reversing diffeomorphism. Hence, the Euler characteristic of a Klein bottle is equal to that of a cylinder, and so is equal to zero.

By Problem 2 from Chapter 1, it is possible to cut the Klein bottle along an embedded circle to produce a Möbius strip, it is also possible to cut the Klein bottle along an embedded circle to produce *two* Möbius strips. Both of these facts show that the Möbius strip has Euler characteristic 0.

The projective plane is obtained by gluing a Möbius strip with a disk, hence its Euler characteristic is $\chi(\mathbb{R}\mathrm{P}^2) = 0 + 1 = 1$.

 # 148 (page 210).
Note that

$$(-X) \circ \gamma = I \circ X \circ \gamma,$$

where $I\colon\ (x, y) \mapsto (-x, -y)$. But the 'winding number form'

$$\omega = \frac{1}{2\pi(x^2 + y^2)}(x\mathrm{d}y - y\mathrm{d}x)$$

does not change under this involution: $I^*\omega = \omega$. Similarly, $(x, y) \mapsto (cx, cy)$ with $c \neq 0$ will not affect ω.

Alternatively, one may argue that the vector field X can be smoothly deformed into $-X$ by a "pointwise rotation," and similarly into cX for $c > 0$ by pointwise "dilation."

 # 149 (page 210).
We have

$$f_X \circ \gamma = (\cos(2\pi t), \sin(2\pi t)), \quad f_Y \circ \gamma = (-\sin(2\pi t), \cos(2\pi t))$$

which gives $(f_X \circ \gamma)^*\omega = (f_Y \circ \gamma)^*\omega = \mathrm{d}t$. Integrating, it follows that both of these maps have winding number 1, hence the rotation number of each of X and Y is 1.

 # 150 (page 210).
For the first claim, we verify that X and Y are pointwise linearly independent away from the origin. The coefficient functions of the vector fields W_{s_1,s_2} are

$$f_{W_{s_1,s_2}}(x,y) = (s_1x - s_2y,\ s_1y + s_2x).$$

This vanishes if and only if

$$\begin{pmatrix} x & -y \\ y & x \end{pmatrix} \begin{pmatrix} s_1 \\ s_2 \end{pmatrix} = 0.$$

Since the coefficient matrix has determinant x^2+y^2, which does not vanish except at the origin, this only holds if $s_1 = s_2 = 0$.

In particular, if $(s_1,s_2) \neq (0,0)$, the vector fields W_{s_1,s_2} are non-zero along γ, and so $\mathrm{rot}_\gamma(W_{s_1,s_2})$ does not depend on s_1, s_2 (since any two points in the s_1s_2-plane may be connected by a path not meeting the origin). The vector field X corresponds to $s_1 = 1, s_2 = 0$, while Z corresponds to $s_1 = -1, s_2 = -1$.

 # 151 (page 211).
In terms of the embedding $\Sigma \subseteq \mathbb{R}^2$, let γ_1, γ_2 be counterclockwise loops around the inner boundaries, and γ a counterclockwise loop around the outer boundary. If X is tangent to all three boundaries components, then $\mathrm{rot}_\gamma(X) = \mathrm{rot}_{\gamma_1}(X) = \mathrm{rot}_{\gamma_2}(X) = 1$. But this contradicts Equation (8.5).

A similar argument shows, more generally, that if Σ is a compact, oriented surface of genus $g = 0$, with $r \neq 2$ boundary components, then Σ does not admit a vector field that is tangent to all boundary components.

 # 152 (page 212).
The new frame amounts to a change of orientation, which changes the sign of the rotation number. Computationally, this comes from the fact that the 1-form ω from the definition of winding numbers changes sign under the diffeomorphism $(x,y) \mapsto (y,x)$.

 # 153 (page 213).
Given charts (U,φ), (U',φ') centered at p with $U, U' \cong \mathbb{R}^2$, we may choose an open neighborhood $U'' \subseteq U \cap U'$ of p, also with $U'' \cong \mathbb{R}^2$. Hence, by restricting the frames and shrinking the paths, we may assume $U = U'$. Next, any two frames (X_1,X_2) and (X_1',X_2') on $U \cong \mathbb{R}^2$ defining the same orientation may be deformed into each other, by "rotation and rescaling," and two loops γ, γ' in $U \setminus \{p\}$, of the same winding number $+1$, may be deformed into each other. Passing to a frame with opposite orientation (for example, replacing X_1, X_2 with $X_1, -X_2$) amounts to composing f with an orientation reversing map, but this also requires reversing the orientation of γ (since it is defined relative to the coordinate frame) and does not change the winding number of $f \circ \gamma$.

 # 154 (page 214).
The vector field $(x^2+y^2)\frac{\partial}{\partial x}$ is such an example. Note that the vector field does not change its direction, hence its rotation number relative to the coordinate frame is zero.

 # 155 (page 214).

(a) Since S^2 is the zero level set of $f(x,y,z) = x^2+y^2+z^2-1$, a vector field $X \in \mathfrak{X}(\mathbb{R}^3)$ is tangent to S^2 if and only if $L_X f$ vanishes along $f^{-1}(0)$. For the given vector field, one actually has $L_X f = 0$ everywhere.

(b) $Y = X|_{S^2}$ vanishes exactly at the north and south poles. To compute the index at those points, we may use x,y coordinates near the poles. In such coordinates, Y is given simply by $-y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$ (just as X itself), and the corresponding rotation numbers are 1.

 # 156 (page 217).

One can take the vector field tangent to one of the factor circles of the torus. For example, thinking of T^2 as the level set $f^{-1}(0)$ of

$$f(x,y,z) = (\sqrt{x^2+y^2}-R)^2 + z^2 - r^2$$

(cf. Example 4.11; here we assume $0 < r < R$) we can take $-y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$. This vector field in $\mathbb{R}^3$ restricts to a vector field on T^2 by Proposition 5.18.

 # 157 (page 219).

Since $\alpha_1 \wedge \alpha_1 = \alpha_2 \wedge \alpha_2 = 0$ we have

$$\alpha_1' \wedge \alpha_2' = \cos^2(\theta)\alpha_1 \wedge \alpha_2 - \sin^2(\theta)\alpha_2 \wedge \alpha_1 = \alpha_1 \wedge \alpha_2.$$

 # 158 (page 220).

The dual (oriented, orthonormal) coframe is $\alpha_1 = \mathrm{d}x$, $\alpha_2 = \mathrm{d}y$, and the area form is the usual volume form $\mathrm{d}A = \mathrm{d}x \wedge \mathrm{d}y$. Since $\mathrm{d}\mathrm{d}x = \mathrm{d}\mathrm{d}y = 0$ we have $\omega = 0$.

 # 159 (page 220).

Bilinearity shows that $y\frac{\partial}{\partial x}, y\frac{\partial}{\partial y}$ is an orthonormal frame; so the dual (oriented, orthonormal) coframe is $\alpha_1 = y^{-1}\mathrm{d}x$, $\alpha_2 = y^{-1}\mathrm{d}y$, which indeed gives g via (8.9). The area form is $\mathrm{d}A = y^{-2}\mathrm{d}x \wedge \mathrm{d}y$. Using the structure equations we compute

$$\mathrm{d}\alpha_1 = y^{-2}\mathrm{d}x \wedge \mathrm{d}y = \alpha_1 \wedge \alpha_2$$

so that $\omega = -\alpha_1$.

 # 160 (page 220).

Using the structure equations we compute

$$\begin{aligned}
\mathrm{d}\alpha_1' &= \mathrm{d}(\cos(\theta)\alpha_1 + \sin(\theta)\alpha_2) \\
&= -\sin(\theta)\mathrm{d}\theta \wedge \alpha_1 + \cos(\theta)\mathrm{d}\alpha_1 + \cos(\theta)\mathrm{d}\theta \wedge \alpha_2 + \sin(\theta)\mathrm{d}\alpha_2 \\
&= (\sin(\theta)\omega - \sin(\theta)\mathrm{d}\theta) \wedge \alpha_1 + (\cos(\theta)\mathrm{d}\theta - \cos(\theta)\omega) \wedge \alpha_2 \\
&= (-\omega + \mathrm{d}\theta) \wedge (-\sin(\theta)\alpha_1 + \cos(\theta)\alpha_2) \\
&= (-\omega + \mathrm{d}\theta) \wedge \alpha_2'.
\end{aligned}$$

Thus,

$$\omega' = \omega - \mathrm{d}\theta.$$

 # 161 (page 221).
In ✎ 158 we calculated that $\omega = 0$, so that $K = 0$.

Conversely, suppose that Σ is a surface with a Riemannian metric such that $K = 0$. Hence, if U is an open neighborhood of a point $p \in \Sigma$, and $\alpha_1, \alpha_2 \in \Omega^1(U)$ is a local coframe with spin connection ω, then $\mathrm{d}\omega = K|_U\ \alpha_1 \wedge \alpha_2 = 0$. Thus ω is closed, and taking U smaller if necessary we may assume that ω is exact: $\omega = \mathrm{d}f$.

Using a coframe rotation by $\theta = f$, we obtain a new orthonormal coframe α_1', α_2' such that $\omega' = \omega - \mathrm{d}\theta = 0$. This means that α_1', α_2' are closed: $\mathrm{d}\alpha_1' = -\omega' \wedge \alpha_2' = 0$, $\mathrm{d}\alpha_2' = \omega' \wedge \alpha_1' = 0$. Taking U smaller if needed, we may assume that α_1', α_2' are exact:

$$\alpha_i' = \mathrm{d}x_i$$

for functions $x_i \in C^\infty(U)$. The functions x_1, x_2 are the desired coordinate functions for which the metric is just the standard flat metric of the Euclidean plane.

 # 162 (page 221).
In ✎ 159 we calculated that $\omega = -\alpha_1 = -y^{-1}\mathrm{d}x$ so that $\mathrm{d}\omega = -y^{-2}\mathrm{d}x \wedge \mathrm{d}y = -\mathrm{d}A$ which gives $K = -1$.

 # 163 (page 223).
The standard frame $\alpha_1 = y^{-1}\mathrm{d}x$, $\alpha_2 = y^{-1}\mathrm{d}y$ is adapted to C. The spin connection is $\omega = -\alpha_1$, as we calculated in ✎ 159. We therefore have $-i^*\omega = i^*\alpha_1$ so that the geodesic curvature is $k_g = 1$ everywhere.

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Chapter 9

 # 164 (page 232).
Let $v \in T_pM$ and $w \in T_qM$ be two tangent vectors, regarded as points of TM. If $p \neq q$, we can use the Hausdorff property of M to choose disjoint coordinate charts (U, φ) and (V, ψ) around p and q; the corresponding coordinate charts $(TU, T\varphi)$, $(TV, T\psi)$ are then disjoint charts around v, w. On the other hand, if $p = q$, and (U, φ) is any coordinate chart around p, then v, w are both contained in the coordinate chart $(TU, T\varphi)$. The preimages of disjoint open neighborhoods of $T\varphi(v), T\varphi(w)$ under $T\varphi$ are then disjoint open neighborhoods of v, w.

 # 165 (page 234).
Let $F(x,y) = x^2 + y^2$ so that $S^1 = F^{-1}(1)$. The tangent map to F at (x,y) is given by the Jacobian matrix

$$D_{(x,y)}F = (2x\ 2y) :\ \mathbb{R}^2 \to \mathbb{R}.$$

Its kernel is the set of column vectors $(r,s)^\top$ with $xr + ys = 0$. This gives the description of TS^1 as the set of all $(x,y,r,s) \in \mathbb{R}^4$ such that $x^2 + y^2 = 1, xr + ys = 0$.

The Jacobian of the map $\Phi(x,y,r,s) = (x^2 + y^2,\ xr + ys)$ is the map $\mathbb{R}^4 \to \mathbb{R}^2$,

$$D_{(x,y,r,s)}\Phi = \begin{pmatrix} 2x & 2y & 0 & 0 \\ r & s & x & y \end{pmatrix}.$$

This has rank 2 unless one row is a multiple of the other row, which only happens if $x = y = 0$. Since $(0,0)$ is not a point on S^1, it follows that Φ has maximal rank on $\Phi^{-1}(1,0)$.

 # 166 (page 235).
This is really just restating the definitions: We have $X \sim_F Y$ if and only if for all $p \in M$,

$$Y_{F(p)} = (T_pF)(X_p).$$

 # 167 (page 236).
By construction, the transition maps covering $\psi \circ \varphi^{-1}$ are

$$(T^*\psi)^{-1} \circ T^*\varphi = T^*(\varphi \circ \psi^{-1}) = \big(T^*(\psi \circ \varphi^{-1})\big)^{-1}.$$

The matrix for $T(\psi \circ \varphi^{-1})$ is the Jacobian matrix $D_{\mathbf{x}}(\psi \circ \varphi^{-1})$. The matrix for the dual map $T^*(\psi \circ \varphi^{-1})$ is its transpose $\big(D_{\mathbf{x}}(\psi \circ \varphi^{-1})\big)^\top$, and finally the matrix for $\big(T^*(\psi \circ \varphi^{-1})\big)^{-1}$ is the inverse,

$$\Big(\big(D_{\mathbf{x}}(\psi \circ \varphi^{-1})\big)^\top\Big)^{-1}.$$

 # 168 (page 236).
Suffice it to show that T^*F is smooth; since F is a diffeomorphism it then follows that $T^*(F^{-1}) = (T^*F)^{-1}$ is smooth also. The argument is analogous to that in the

proof of Proposition 9.2. Using the notation from that proof, letting $\widetilde{F} = \psi \circ F \circ \varphi^{-1}$ be the local coordinate expression for F, the coordinate expression for the tangent map is

$$(T\widetilde{F})(\mathbf{x},\mathbf{a}) = \big(\widetilde{F}(\mathbf{x}),\ (D_{\mathbf{x}}\widetilde{F})(\mathbf{a})\big),$$

hence the expression for the cotangent map is given by the transpose,

$$(T^*\widetilde{F})(\mathbf{y},\mathbf{b}) = \big(\mathbf{x},\ (D_{\mathbf{x}}\widetilde{F})^\top(\mathbf{b})\big),$$

with $\mathbf{x} = \widetilde{F}^{-1}(\mathbf{y})$, and so is a smooth map of $\mathbf{y},\mathbf{b}$.

 # 169 (page 238).
The argument is the same as for TM, simply replace TM with E in the solution to ✐164.

 # 170 (page 239).
Consider the transition map between charts U_i, U_j with $i < j$. The intersection $U_i \cap U_j$ consists of elements $p = (x^0 : \cdots : x^n)$ where both x^i and x^j are non-zero. Let $\mathbf{v} \in E_p$. Suppose the coordinates of $\mathbf{v}$ in the chart $\widehat{U}_j$ are

$$(u^1,\dots,u^n;\lambda),$$

and those in the chart $\widehat{U}_i$ are

$$(v^1,\dots,v^n;\mu).$$

The assumption $p \in U_i \cap U_j$ means that $u^{i+1} \neq 0$, and $(v^1,\dots,v^n)$ are given by

$$\varphi_i \circ \varphi_j^{-1}(u^1,\dots,u^n) = \Big(\frac{u^1}{u^{i+1}},\dots,\frac{u^i}{u^{i+1}},\frac{u^{i+2}}{u^{i+1}},\dots,\frac{u^j}{u^{i+1}},\frac{1}{u^{i+1}},\frac{u^{j+1}}{u^{i+1}},\dots,\frac{u^n}{u^{i+1}}\Big)$$

while the relationship between λ and μ is

$$\mu = u^{i+1}\lambda.$$

This shows smoothness of the transition map for $i < j$; the case $i > j$ is similar.

 # 171 (page 241).
Assume for contradiction that we have a trivialization $(\mathbb{R}\times\mathbb{R})/\!\sim\ \xrightarrow{\cong} (\mathbb{R}/\!\sim)\times\mathbb{R}$. By composition with the quotient map, we obtain a map

$$\mathbb{R}\times\mathbb{R} \to (\mathbb{R}/\!\sim)\times\mathbb{R},\quad (t,\tau)\mapsto([t], f(t,\tau)),$$

where $f(t,\tau)$ is, for all t, a non-zero linear functional with respect to τ. We may thus write $f(t,\tau) = \tau f(t,1) = \tau g(t)$, with $g(t) \neq 0$ for all t. For this map

$$(t,\tau)\mapsto([t],\tau g(t))$$

to descend to equivalence classes, we would need that $g(t+1) = -g(t)$. But this is only possible if g changes sign somewhere, contradicting $g(t) \neq 0$ for all t.

 # 172 (page 243).
By definition, we have canonical isomorphisms $E_p \oplus E'_p = \mathbb{R}^n$ for all p. These combine into the desired isomorphism $E \oplus E' \cong \mathrm{Gr}(k,n) \times \mathbb{R}^n$.

 # 173 (page 244).
The standard basis vectors e_i of $\mathbb{R}^r$ define a frame $\tau_1,\dots,\tau_r$ of the trivial bundle $U \times \mathbb{R}^r$, by the map $\tau_i(p) = (p, e_i)$. Given an isomorphism $E|_U \cong U \times \mathbb{R}^r$, one obtains a frame $\sigma_1,\dots,\sigma_r$ of $E|_U$ by taking preimages of $\tau_1,\dots,\tau_r$. Conversely, any frame $\sigma_1,\dots,\sigma_r$ of $E|_U$ gives an isomorphism $E|_U \to U \times \mathbb{R}^r$, $\sum_i a_i \sigma_i(p) \mapsto \sum_i a_i \tau_i(p)$.

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